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THÈME 1



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Upper bound for the span of $(s, 1)$ -total labelling of graphs

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Abstract: An $(s, 1)$ -total labelling of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) incident vertex and edge receive integers that differ by at least s in absolute value. The *span* of a $(s, 1)$ -total labelling is the maximum difference between two labels. The minimum span of a $(s, 1)$ -total labelling of G is denoted by $\lambda_s^T(G)$.

In [4], it is conjectured that $\lambda_s^T \leq \Delta + 2s - 1$, where Δ is the maximum degree of a vertex in a graph. This is an extension of the Total Colouring Conjecture. It is also shown that $\lambda_s^T \leq 2\Delta + s - 1$ and $\lambda_2^T(G) \leq 6$ if $\Delta(G) \leq 3$ and $\lambda_2^T(G) \leq 8$ if $\Delta(G) \leq 4$.

In this paper, we prove that $\lambda_s^T \leq 2\Delta - \log(\Delta + 2) + s - 1 + 2\log(16s - 10)$. The proof is an induction based on the maximal cut of a graph. We use the same technique to improve a little bit this result in the case of $(2, 1)$ -total labelling. We prove that if $\Delta(G) \geq 3$, then $\lambda_2^T(G) \leq 2\Delta(G)$ and that if $\Delta(G) \geq 5$ is odd then $\lambda_2^T(G) \leq 2\Delta(G) - 1$.

Key-words: $(s, 1)$ -total labelling, maximal cut

Borne supérieure pour l'écart d'une coloration ($s, 1$)-totale

Résumé : Une *coloration* ($s, 1$)-totale d'un graphe G est une affectation d'entiers aux sommets et arêtes de G telle que: (i) deux sommets adjacents reçoivent des entiers distincts, (ii) deux arêtes adjacentes reçoivent des entiers distincts, et (iii) un sommet et une arête incidents reçoivent des entiers qui diffèrent d'au moins s en valeur absolue. L'*écart* d'une coloration ($s, 1$)-totale est la différence maximum entre deux entiers affectés. L'écart minimum d'une coloration ($s, 1$)-totale de G est noté $\lambda_s^T(G)$.

Dans [4], il est conjecturé que $\lambda_s^T \leq \Delta + 2s - 1$, avec Δ le degré maximum d'un sommet du graphe. C'est une généralisation de la Conjecture de la Coloration Totale. Il est également montré que $\lambda_s^T \leq 2\Delta + s - 1$ et $\lambda_2^T(G) \leq 6$ si $\Delta(G) \leq 3$ et $\lambda_2^T(G) \leq 8$ si $\Delta(G) \leq 4$.

Nous montrons ici que $\lambda_s^T \leq 2\Delta - \log(\Delta + 2) + s - 1 + 2 \log(16s - 10)$. La preuve est une récurrence basée sur la coupe maximale d'un graphe. Nous utilisons la même technique pour améliorer un peu cette borne dans le cas des colorations ($2, 1$)-totales. Nous prouvons que si $\Delta(G) \geq 3$, alors $\lambda_2^T(G) \leq 2\Delta(G)$ et si $\Delta(G) \geq 5$ est impair alors $\lambda_2^T(G) \leq 2\Delta(G) - 1$.

Mots-clés : coloration ($s, 1$)-totale, coupe maximale

1 Introduction

Let G be a graph. The degree of a vertex v is denoted by $d_G(v)$ or $d(v)$ if G is clearly understood. The maximum degree of G is denoted by $\Delta(G)$. An $(s, 1)$ -total labelling of a graph G is an assignment of integers to $V(G) \cup E(G)$ such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) incident vertex and edge receive integers that differ by at least s in absolute value. The *span* of an $(s, 1)$ -total labelling is the maximum difference between two labels. The minimum span of a $(s, 1)$ -total labelling of G is denoted by $\lambda_s^T(G)$. Note that a $(1, 1)$ -total labelling is a total colouring and that $\lambda_1^T(G) = \chi^T - 1$ where χ^T is the total colouring number.

An $(s, 1)$ -total labelling of a graph G corresponds to an $L(s, 1)$ -labelling of its incidence graph $I(G)$ which is the bipartite graph defined as follows : $V(I(G)) = V(G) \cup E(G)$ and $ve \in E(I(G))$ if and only if $v \in V(G)$, $e \in E(G)$ and v and e are incident. $L(2, 1)$ -labellings were first introduced in Griggs and Yeh [3] and $L(s, 1)$ -labelling have been studied for several class of graphs, for example chordal graphs [1] or planar graphs [5]. The $(2, 1)$ -total labellings of graphs were first studied by Whittlesey, Georges and Mauro [7] as $L(2, 1)$ -labellings of incidence graphs. In [4], Havet and Yu investigate $(s, 1)$ -total labelling for any s . They derive from Brooks and Vizing’s Theorems that $\lambda_s^T \leq 2\Delta + s - 1$. Generalizing the Total Colouring Conjecture, they conjecture the following :

Conjecture 1 (Havet and Yu [4])

$$\lambda_s^T \leq \Delta + 2s - 1$$

By the previous result, it suffices to prove the conjecture for $s < \Delta$. Rosenfeld [6] established that $\lambda_1^T(G) \leq 4$ if $\Delta(G) \leq 3$. Havet and Yu completed the proof of the conjecture for $\Delta \leq 3$ by proving that $\lambda_2^T(G) \leq 6$ if $\Delta(G) \leq 3$.

In this paper, we improve Havet and Yu’s upper bound by showing $\lambda_s^T \leq 2\Delta - \log(\Delta + 2) + s - 1 + 2\log(16s - 8)$. The proof is an induction based on the maximal cut of a graph. The idea and tools are presented Section 2 and the proof is given Section 3. Finally using the same technique to improve a little bit the result in the case of $(2, 1)$ -total labelling. We prove that if $\Delta(G) \geq 3$, then $\lambda_2^T(G) \leq 2\Delta(G)$ this generalizes results of Havet and Yu [4] who proved it for $\Delta \in \{3, 4\}$. Furthermore, we show that if $\Delta(G) \geq 5$ is odd then $\lambda_2^T(G) \leq 2\Delta(G) - 1$.

2 The tools and the idea

Definition 1 A *cut* $[A, B]$ of a graph G is a set of two induced subgraphs A and B of G such that $(V(A), V(B))$ is a partition of $V(G)$. The bipartite graph (A, B) is the graph with vertex set $V(G)$ and edge set $E(G) \setminus (E(A) \cup E(B))$. The edges of (A, B) are called the *cut edges*. A *maximum cut* $[A, B]$ is a cut with maximum number of cut edges.

Lemma 1 *Let G be a graph with maximum degree $2k + 1$. Then a maximum cut $[A, B]$ satisfies $\Delta(A) \leq k$ and $\Delta(B) \leq k$.*

Proof. Consider a maximum cut $[A, B]$. B contains no vertex b of degree greater than k otherwise $[A + b, B - b]$ is a cut with strictly more cut edges. Analogously A has no vertex of degree greater than k . \square

Lemma 2 *Let G be a graph with maximum degree $2k$. Then G has a cut $[A, B]$ such that $\Delta(A) \leq k$ and $\Delta(B) \leq k$.*

Proof. Consider a maximum cut $[A, B]$ which minimizes the number of vertices with degree k in A . As in the proof of Lemma 1, A and B contain no vertex of degree greater than k . Moreover A has no vertex a of degree k otherwise $[A - a, B + a]$ is a cut with the same number of cut edges as $[A, B]$ and one vertex less of degree k in the first subgraph. \square

Lemma 3 *Let G be a bipartite graph with maximum degree Δ . There is an edge colouring c of G in $[1, \Delta]$ such that $c(e) \geq i$ only if it is incident to a vertex of degree at least i .*

Proof. Let us prove it by induction on Δ , the result holding trivially when $\Delta = 0$. Consider now a graph with maximum degree $\Delta \geq 1$. By König's theorem, it admits an edge colouring c_1 in $[1, \Delta]$. Let M be the set of edges coloured Δ incident to a vertex of degree Δ . Consider G' the graph obtained from G by removing M . Since every vertex of degree Δ is adjacent to an edge of M , $\Delta(G') = \Delta - 1$. Then by induction G' has an edge colouring c of G in $[1, \Delta - 1]$ such that $c(e) \geq i$ only if it is incident to a vertex of degree at least i . Extending c into an edge colouring of G in $[1, \Delta]$ by colouring the edges of M with Δ , we obtain the result. \square

Definition 2 Let G be a graph. A *list assignment* L is an assignment of a set $L(v)$ of integers to every vertex v of G . The graph G is *L -colourable* if it admits an application c called *L -colouring* from its vertex set into the set of integers such that for any vertex v , $c(v) \in L(v)$ and for any edge (u, v) , $c(u) \neq c(v)$.

Let k be a non-negative integer. A *k -list assignment* is an assignment L such that $|L(v)| = k$ for every vertex v . A graph is *k -choosable* if it is L -colourable for any k -list assignment L .

Let v be a vertex of G . A *(d, v) -list assignment* of G is a list assignment L such that $|L(u)| = d(u)$ if $u \neq v$ and $|L(v)| = d(v) + 1$. We say that G is *(d, v) -choosable* if it is L -colourable for any (d, v) -list assignment L .

Proposition 1 *Let G be a connected graph and $v \in V(G)$. Then G is (d, v) -choosable.*

Proof. There is an ordering v_1, v_2, \dots, v_n of the vertices of the graph such for $i < n$ the vertex v_i has a neighbour in $\{v_j, i < j \leq n\}$. Hence by a greedy algorithm, one can find an L -colouring of G for any (d, v) -list assignment L . \square

Using this proposition, we strengthen Havet and Yu result [4] stating that $\lambda_s^T(G) \leq 2\Delta + s - 1$.

Lemma 4 *Let G be a graph with maximum degree $\Delta \leq k$. G admits a $(s, 1)$ -total labelling in $[0, 2k + s - 1]$ such that a vertex v is assigned a label in $[0, d(v)]$ and an edge a label in $[k + s - 1, 2k + s - 1]$.*

Proof. Obviously it suffices to prove it when G is connected.

By Vizing’s Theorem, there is an edge colouring c' of G with colours in $[k + s - 1, 2k + s - 1]$. Let v be a vertex of G . Free to permute the colours of c' , we may assume that for every edge incident to v , $c'(v) \geq k + s$. Let L be the (d, v) -list assignment with $L(u) = [0, d(u) - 1]$ if $u \neq v$ and $L(v) = [0, d(v)]$. By Proposition 1, G has an L -colouring. The union of c and c' is an $(s, 1)$ total labelling of G .

Indeed for every edge $e = xy$, if $x \neq v$ then $c(x) \leq k - 1 \leq c'(e) - s$ and if $x = v$ then $c(v) \leq k \leq c'(e) - s$. □

Analogously, we have the following lemma:

Lemma 5 *Let G be a graph with maximum degree $\Delta \leq k$. G admits a $(s, 1)$ -total labelling in $[0, 2k + s - 1]$ such that an edge is assigned a label in $[0, k]$ and a vertex v a label in $[k + s - 1, k + s - 1 + d(v)]$.*

Theorem 1 (Galvin [2]) *Every bipartite graph G is $\Delta(G)$ -edge choosable.*

The idea of the results is to consider a suitable maximum cut of G given by Lemma 1 or 2 and to label edge and vertices of A and B with Lemma 4 or induction hypothesis and Lemma 5 respectively and then to label the bipartite graph (A, B) using Lemma 3. Some few relabellings are then necessary to obtain the desired $(s, 1)$ -total labelling. Theorem 1 is used for some of the relabellings.

3 Main result

The aim of this section is to prove the following theorem :

Theorem 2 *For any $s \geq 1$,*

$$\lambda_s^T \leq 2\Delta - 2\log(\Delta + 2) + 2\log(16s - 8) + s - 1$$

In order to prove this theorem, we prove by induction a stronger result.

Let G be a graph with maximal degree Δ . An $(s, 1)$ -total labelling in $[0, p]$ is a *s-good labelling* if each vertex is assigned a label in $[0, \Delta + s - 1]$.

Theorem 3 *Let G be a graph with maximal degree Δ . Then G has a s-good labelling in $[0, 2\Delta - 2\log(\Delta + 2) + 2\log(16s - 8) + s - 1]$.*

The idea is to prove this result by induction. Note that Lemma 5 give the result for small value of Δ . We will now give two Lemmas allowing us to do an induction step.

Lemma 6 *Let $k \geq \max(i + 2s - 1, 2i + 6s - 5)$. If every graph of maximal degree k admits a s -good labelling in $[0, 2k - i]$ then every graph G of maximal degree $\Delta = 2k + 2$ admits a s -good labelling in $[0, 2\Delta - i - 2]$.*

Proof. According to Lemma 2 there is a cut $[A, B]$ of G such that $\Delta(A) \leq k$ and $\Delta(B) \leq k + 1$. Thus by hypothesis, there is a s -good labelling of A in $[0, 2k - i]$. And by Lemma 5, there is an $(s, 1)$ -total labelling of B such that vertices are labelled in $[k + s, k + s + d_B(v)]$ and edges in $[0, k + 1]$.

By Lemma 3, label the edges of (A, B) with $[2k - i + 1, 4k - i + 2]$ so that an edge is labelled $4k - i + 3 - l$ only if it is incident to a vertex of degree at least l in (A, B) .

The obtained labelling is not yet an $(s, 1)$ -total labelling. Indeed for $j \in [0, i + 2s - 1]$, edges (a, b) labelled $2k - i + 1 + j$ when b is labelled in $[2k - i + j - s + 2, 2k - i + j + s]$ violate the constraints. Hence they must be relabelled.

Let us consider the bipartite graph induced by such edges. It has degree at most $i + 2s$. We want to relabel the edges with labels in $[k + 2s - 2, 2k - i]$. According to Theorem 1, we need to find a list of $i + 2s$ available labels for each edge. Let (a, b) be an edge labelled $2k - i + 1 + j$ with b labelled in $[2k - i + j - s + 2, 2k - i + j + s]$. Then $d_B(b) \geq k - i + j - 2s + 2$. So b has degree at most $k + i - j + 2s$ in (A, B) . But by construction (a, b) is incident to a vertex of degree at least $2k + 2 - j$ in (A, B) . Since $k \geq i + 2s - 1$ then this vertex is a and $d_A(a) \leq j$. So at most j labels of $[k + 2s - 2, 2k - i]$ are forbidden because of the edges of A incident to a . Moreover at most $2s - j - 2$ labels of $[k + 2s - 2, 2k - i]$ are forbidden because of b (those of $[2k - i + j - 2s + 3, 2k - i]$). Hence at most $2s - 2$ labels of $[k + 2s - 2, 2k - i]$ are forbidden. So because $k \geq 2i + 6s - 5$, at least $k - i - 2s + 3 - (2s - 2) \geq i + 2s$ labels available on (a, b) .

Since the labels of the vertices are in $[0, 2k + 1 + s]$, we have a s -good labelling of G in $[0, 4k - i + 2]$. \square

Lemma 7 *Let $k \geq \max(i + 4s - 1, 2i + 6s - 3)$. If every graph of maximal degree k admits a s -good labelling in $[0, 2k - i]$ then every graph G of maximal degree $\Delta = 2k + 1$ admits a s -good labelling in $[0, 2\Delta - i - 2]$.*

Proof. Let $[A, B]$ be a maximum cut of G . Then $\Delta(A) \leq k$ and $\Delta(B) \leq k$. Thus by hypothesis, there is a s -good labelling of A in $[0, 2k - i]$. And by Lemma 5, there is an $(s, 1)$ -total labelling of B such that vertices are labelled in $[k + s, k + s + d_B(v)]$ and edges in $[1, k]$.

By Lemma 3, label the edges of (A, B) with $[2k - i, 4k - i]$ so that an edge is labelled $4k - i + 1 - l$ only if it is incident to a vertex of degree at least l in (A, B) .

There are two types of edges of (A, B) violating a constraint of an $(s, 1)$ -total labelling :

- (1) edges (a, b) labelled $2k - i + j$ while b is labelled in $[2k - i + j - s + 1, 2k - i + j + s - 1]$ for some $j \in [0, i + 2s - 1]$;
- (2) edges (a, b) labelled $2k - i$ with a incident to an edge (of A) labelled $2k - i$.

Let us first relabel the edges of type (1) with labels in $[k + 2s - 1, 2k - i - 1]$. Let us consider the bipartite graph induced by them. It has degree at most $i + 2s$. According to Theorem 1, we need to find a list of $i + 2s$ available labels for each edge. Let (a, b) be an edge labelled $2k - i + j$ with b labelled in $[2k - i + j - s + 1, 2k - i + j + s - 1]$. Then $d_B(b) \geq k - i + j - 2s + 1$. So b has degree at most $k + i - j + 2s$ in (A, B) . But by construction (a, b) is incident to a vertex of degree at least $2k + 1 - j$ in (A, B) . Since $k \geq i + 2s$, this vertex is a and $d_A(a) \leq j$. So at most j labels are forbidden because of the edges of A incident to a and at most $2s - j - 2$ are forbidden because of b (those of $[2k - i + j - 2s + 2, 2k - i - 1]$). Hence at most $2s - 2$ labels of $[k + 2s - 1, 2k - i - 1]$ are forbidden. So since $k \geq 2i + 6s - 3$, there are at least $k - i - 2s + 1 - (2s - 2) \geq i + 2s$ labels available on (a, b) .

Let us now relabel the edges of type (2). Since a is incident to an edge of A , it has degree less than $2k + 1$ in (A, B) . Hence b has degree $2k + 1$ in (A, B) and thus is isolated in B . In particular b was not incident to an edge of type (1). Let $l(a)$ be the label of a . There is a label in $[0, k + 2s - 1] \setminus [l(a) - s + 1, l(a) + s - 1]$ that is not assigned to any edge of A incident to a . Relabel (a, b) with l . Since $l + s \leq k + 3s - 1 \leq 2k - i - s$, we can relabel b with $k + 3s - 1$.

Since the labels of the vertices are in $[0, 2k + s]$ we have a good labelling of G in $[0, 4k - i]$.
□

Proof of Theorem 3. Set $c_s = 2 \log(16s - 8) + s - 1$. If $\Delta \leq 16s - 10$, then we have the result by Lemma 4. Suppose now that G is a graph with maximal degree $\Delta \geq 16s - 9$.

If Δ is even, set $\Delta = 2k + 2$. By induction hypothesis $\lambda_s^T(H) \leq 2k - 2 \log(k + 2) + c_s$. And setting $i = 2 \log(k + 2) - c_s$, we have $k \geq \max(i + 2s - 2, 2i + 6s - 5)$. Hence by Lemma 6, $\lambda_s^T(G) \leq 2\Delta - 2 \log(k + 2) + c_s - 2$. Since $\log(k + 2) + 1 = \log(2k + 4) = \log(\Delta + 2)$. We obtain $\lambda_s^T(G) \leq 2\Delta - 2 \log(\Delta + 2) + c_s$.

In the same way, we have the result if Δ is odd. □

4 Better bounds when $s = 2$

4.1 Upper bound 2Δ

Theorem 4 *If $\Delta(G)$ is odd and at least 5 then G has a 2-good labelling in $[0, 2\Delta(G)]$.*

Proof. Set $\Delta(G) = 2k + 1$. Consider a maximum cut $[A, B]$ of G . Then $\Delta(A) \leq k$ and $\Delta(B) \leq k$.

Thus by Lemmas 4 and 5, one may label A and B in $[0, 2k + 1]$ such that a vertex v in A receives a label in $[0, d_A(v)]$ (resp. $[k + 1, k + 1 + d_B(v)]$) and edges labels in $[k + 1, 2k + 1]$ (resp. $[0, k]$).

Now by Lemma 3, label the edges of (A, B) in $[2k + 2, 4k + 2]$ such that an edge is assigned $2k + 2$ only if it is adjacent to a vertex with degree $2k + 1$ in (A, B) and so an isolated vertex in A or B .

The label of an edge (a, b) of (A, B) fulfill the constraints of a $(2, 1)$ -total labelling unless it is labelled $2k + 2$ and b is labelled $2k + 1$. But in this case, a is an isolated vertex of A and thus labelled 0. So we may relabel (a, b) with $k + 1$. This is possible since $k \geq 2$ so $(2k + 1) - (k + 1) \geq 2$.

Since the vertices are labelled in $[0, 2k + 1]$ we have a 2-good labelling. \square

The proof of Theorem 4 does not work when $\Delta = 3$. However, we give an alternative proof of a result of Havet and Yu [4] asserting that a graph with maximum degree 3 has a $(2, 1)$ -total labelling in $[0, 6]$.

Theorem 5 *If $\Delta(G) \leq 3$ then $\lambda_2^T(G) \leq 6$.*

Proof. Let $[V_1, V_2]$ a maximal cut of G . Easily $\Delta(V_i) \leq 1$.

For $i = 1, 2$, let S_i (resp. T_i) be the set of isolated vertices (resp. vertices with degree 1) in G_i .

Label the edges of V_1 (resp. V_2) with 3 (resp. 0) and their endvertices with 0 and 1 (resp. 2 and 3). Label the vertices of S_2 with 2.

By König's Theorem, there is a 3-edge colouring of (V_1, V_2) with colours a, b and c . For each a -coloured edge (u, v) with $u \in G_1$ do the following :

- If $u \in S_1$ and $v \in S_2$, assign 4 to (u, v) and 0 to u .
- If $u \in T_1$ and $v \in S_2$, assign 4 to (u, v) .
- If $u \in S_1, v \in T_2$ and v is labelled 2 then assign 4 to (u, v) and 0 to u .

At this stage the vertices of S_1 whose incident a -coloured edge has an end in T_2 labelled 3 are not yet coloured. We will label them one after another doing the following algorithm :

- (1) If there is a vertex $y \in T_2$ that is adjacent to two non labelled vertices x and z (of S_1), assign 0 to x and z , 3 to (x, y) , 4 to (y, z) and relabel y with 6. Go to (1).
- (2) If there is a vertex $y \in T_2$ that is adjacent to a non-labelled vertex x and a labelled vertex $z \in S_1$, then z is labelled 0 and there is an integer l in $\{2, 3, 4\}$ that label no edge incident to z . Then assign 0 to x , l to (y, z) , an integer of $\{2, 3, 4\} \setminus \{l\}$ to (x, y) and relabel y with 6. Go to (2).
- (3) If there is a vertex $y \in T_2$ that is adjacent to a non-labelled vertex x and a vertex $z \in T_1$. Let e be the edge of B incident to z and distinct from (y, z) .
If e is not labelled yet then assign 4 to (y, z) , 3 to (x, y) and 0 to x . Relabel y with 6. Go to (3).

Otherwise e is already labelled with 4. Let a be the label of z . Assign 6 to (y, z) , 4 to (x, y) and a to x . Relabel y with the integer of $\{0, 1\} \setminus \{a\}$. Go to (3).

Let E' be the set of non labelled edges. It induces a bipartite graph with maximum degree 2. And the vertices incident to edges of E' are labelled in $[0, 3]$.

By König's theorem, E' can be two coloured with label 5 and 6. It is easy to see that we have a $(2, 1)$ -total labelling of G . □

Remark 1 The $(2, 1)$ -total labelling obtained by such a proof is really different from the one obtained by the proof of Havet and Yu [4].

Theorem 6 *If $\Delta(G)$ is even and at least 6 then G has a 2-good labelling in $[0, 2\Delta(G)]$.*

Proof. Set $\Delta = 2k$. Consider a cut $[A, B]$ as in Lemma 2. Following Lemma 5, label A such that a vertex v receives a label in $[k + 1, k + 1 + d_A(v)]$ and an edge a label in $[1, k]$. Following Lemma 4, label B such that a vertex v receives a label in $[0, d_B(v)]$ and an edge a label in $[k + 1, 2k + 1]$.

Now by Lemma 3, label the edges of (A, B) in $[2k + 1, 4k]$ such that an edge is assigned $2k + 1$ only if it is adjacent to a vertex with degree $2k$ in (A, B) and so an isolated vertex in A or B .

The label of an edge (a, b) of (A, B) fullfill the constraints of a $(2, 1)$ -total labelling unless (a, b) is labelled $2k + 1$ and 1) a is labelled $2k$ or 2) b is incident to an edge of B labelled $2k + 1$. Thus we need some relabelling.

1) If a is labelled $2k$, then a is not isolated in A . Thus b is isolated in B . Then relabel (a, b) with 0 and b with 2.

2) If b is incident to an edge (b, b') of B which is labelled $2k + 1$, then b is not isolated in B . Thus a is isolated in A . In particular such an edge is disjoint from any edge of type 1). Let $l(b)$ be the label assigned to b . If $l(b) \geq 2$ then relabel (a, b) with 0. If $l(b) \leq 1$ then relabel (a, b) with 3 and a with 5 if $k = 3$. This is valid since $k \geq 3$.

In such a $(2, 1)$ -total labelling a vertex is assigned an integer in $[0, 2k]$, so we have a 2-good labelling. □

One can extend Theorem 6 for $\Delta = 4$. This strengthen a result of Havet and Yu [4] stating that $\lambda_2(G) \leq 8$ if $\Delta \leq 4$.

Theorem 7 *If $\Delta(G) = 4$ then G has a 2-good labelling in $[0, 8]$.*

Proof. By Lemma 2, G has a cut $[A, B]$ such that $\Delta(A) \leq 1$ and $\Delta(B) \leq 2$.

Label the vertices of A with $\{0, 1\}$ and its edges with $\{3\}$ such that the isolated vertices of A receive 0.

Label the vertices and edges of B which do not lay on odd cycle of B as follows :

- (i) the isolated vertices of B are labelled 3;
- (ii) The vertices and edges of an even cycle or a path are labelled alternatively 3 and 4 and 0 and 1 respectively.

According to Lemma 3, label the edges of (A, B) with $[5, 8]$ so that an edge assigned 5 is incident to a vertex of degree 4 in (A, B) which are isolated vertices in A or B .

Some constraints are violated each time an edge (a, b) of (A, B) is labelled 5 and a is labelled 4. But in that case, a is not isolated in A thus b is isolated in B and so is labelled 0. Then relabel (a, b) with 2.

At this stage, it remains to assign labels to vertices and edges of odd cycles of G .

Let $C = (b_0, b_1, \dots, b_{2p}, b_0)$ be an odd cycle of B . Then two consecutive vertices, say b_0 and b_1 are either both incident to an edge labelled 5 or both non incident to an edge labelled 5. Then for $1 \leq i \leq p$, label b_{2i-1} with 3, b_{2i} with 4, (b_{2i-1}, b_{2i}) with 1 and (b_{2i}, b_{2i+1}) with 0. And label b_0 with 2.

If b_0 and b_1 are non incident to an edge labelled 5 then label (b_0, b_1) with 5. Otherwise there is a label $l \in [6, 8]$ such that both b_0 and b_1 are incident to no edge labelled l . Label (b_0, b_1) with l .

Since the vertices are labelled in $[0, 4]$, we have a 2-good labelling of G in $[0, 8]$. \square

Corollary 1 $\lambda_2^T(G) \leq 2\Delta - 2 \log(\Delta + 2) + 8$

4.2 Upper bound $2\Delta - 1$ for odd Δ

Theorem 8 *If $\Delta(G)$ is odd and at least 7 then G has a 2-good labelling in $[0, 2\Delta(G) - 1]$.*

Proof. Set $\Delta(G) = 2k + 1$. Consider a maximum cut $[A, B]$ of G . Then $\Delta(A) \leq k$ and $\Delta(B) \leq k$.

Following Lemma 4, label A such that each vertex v of A is assigned a label in $[0, d_A(v)]$ and each edge e a label in $[k + 1, 2k + 1]$.

Following Lemma 5, label B such that each vertex v of B is assigned a label in $[k + 1, k + 1 + d_B(v)]$ and each edge e a label in $[0, k]$.

By Lemma 3, label the edges of (A, B) with $[2k + 1, 4k + 1]$ so that an edge is labelled $4k + 2 - i$ only if it is incident to a vertex of degree i in (A, B) .

This labelling may violate some constraints of a $(2, 1)$ -total labelling in the following cases :

- (1) a vertex $b \in B$ labelled $2k$ or $2k + 1$ is incident to an edge (a, b) of (A, B) labelled $2k + 1$;
- (2) a vertex $b \in B$ labelled $2k + 1$ is incident to an edge (a, b) of (A, B) labelled $2k + 2$;
- (3) a vertex $a \in A$ is incident to two edges labelled $2k + 1$ one (a, a') in A and one (a, b) in (A, B) ;

Therefore, we need to proceed to the following corresponding relabelling :

- (1) Since $k \geq 2$, then $2k > k + 1$ so b is not isolated in B . Thus the vertex a is isolated in A and labelled 0. Then relabel (a, b) with k .

- (2) The vertex b is labelled $2k + 1$ and so $d_B(b) = k \geq 2$. Hence b has degree less than $2k$ in (A, B) and a has degree at least $2k$ in (A, B) . So a has degree at most 1 in A and thus is labelled 0 or 1. One of the two integers $k + 1$ and $k + 2$ does not label the (possible) edge incident to a in A . Then relabel (a, b) with l . This is valid since $k \geq 3$.
- (3) Since a is not isolated in A , then b is isolated in B and thus labelled $k + 1$. If a is labelled 0 or 1 then relabel (a, b) with 3 and b with 5 if $k = 3$. Again this is valid since $k \geq 3$. If a is labelled in $[2, k + 1]$ then relabel (a, b) with 0.

□

The last two relabelling of the previous proof are not valid if $k = 2$. Hence, to get the result when $\Delta = 5$, we need some extra arguments :

Theorem 9 *If $\Delta(G) = 5$ then $\lambda_2^T(G) \leq 9$.*

Proof.

Let $[A, B]$ be a maximum cut of G . Then $\Delta(A) \leq 2$ and $\Delta(B) \leq 2$.

We need a more careful labelling of A than in Theorem 8. Let C be a component of A . If C is not an odd cycle then, following Lemma 4, label C such that each vertex v is assigned a label in $[0, d_A(v)]$ and each edge e a label in $[3, 4]$. If C is an odd cycle $(a_1, a_2, \dots, a_{2p+1}, a_1)$ then for $1 \leq i \leq p$, label a_{2i-1} with 0, (a_{2i-1}, a_{2i}) with 3, a_{2i-1} with 1, and (a_{2i}, a_{2i+1}) with 4. Label a_{2p+1} with 2 and (a_{2p+1}, a_1) with 5. Note that in that case a vertex labelled 1 in A is not incident to an edge labelled 5.

Following Lemma 5, label B such that each vertex v of B is assigned a label in $[3, 3 + d_B(v)]$ and each edge e a label in $[0, 2]$ with B_e .

By Lemma 3, label the edges of (A, B) with $[5, 9]$ so that an edge is labelled $10 - i$ only if it is incident to a vertex of degree at least i in (A, B) .

This labelling may violate the constraints of a $(2, 1)$ -total labelling in the same cases as in Theorem 8 :

- (1) a vertex $b \in B$ labelled 4 or 5 is incident to an edge (a, b) of (A, B) labelled 5;
- (2) a vertex $b \in B$ labelled 5 is incident to an edge (a, b) of (A, B) labelled 6;
- (3) a vertex $a \in A$ is incident to two edges labelled 5 one (a, a') in A and one (a, b) in (A, B) ;

Therefore, we need to proceed to the following corresponding relabelling :

- (1) As in Theorem 8, relabel (a, b) with 2.
- (2) The vertex b is labelled 5 and so $d_B(b) = 2$. Hence b has degree less than 6 in (A, B) and a has degree at least 6 in (A, B) . So a has degree at most 1 in A and thus is labelled 0 or 1. Relabel (a, b) with 3. This may violate a constraint if the edge (a, a') in A incident to a is also labelled 3. If a' is incident to no edge labelled 4 then relabel (a, a') with 4. Otherwise $d_{(A, B)}(a') \leq 2$. Thus there is a label $l \in [5, 7]$ that labels no edge incident to a or a' (since (a, b) is now labelled 3). Relabel (a, a') with l .

- (3) Since a is not isolated in A , then b is isolated in B and thus labelled 3. Moreover, the vertex a is labelled either 0 or 2 because no vertex of A labelled 1 is incident to an edge of A labelled 5. If a is labelled 0 or then relabel (a, b) with 2 and b with 4. If a is labelled 2 then relabel (a, b) with 0.

□

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