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A relative compactness criterion in Wiener-Sobolev spaces and application to semi-linear Stochastic P.D.Es

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Abstract: We prove a relative compactness criterion in Wiener-Sobolev space which represents a natural extension of the compact embedding of sobolev space H^1 into $\mathbb{L}^2$, at the level of random fields. Then we give a specific statement of this criterion for random fields solutions of semi-linear Stochastic Partial Differential Equations with coefficients bounded in an appropriate way. Finally, we employ this result to construct solutions for semi-linear Stochastic Partial Differential Equations with distribution as final condition. We also give a probabilistic interpretation of this solution in terms of Backward Doubly Stochastic Differential Equations formulated in a weak sense.

Key-words: Backward Doubly SDEs, Malliavin calculus, Stochastic partial differential equations, Weak solutions, Wiener chaos decomposition

Mathematics Subject Classification: 60H10, 60H15, 60H30, 60H07, 35R05.

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Un critère de relative compacité sur l'espace de Wiener-Sobolev et application à l'étude d'EDP Stochastiques semi-linéaires

Résumé : On montre un critère de relative compacité dans l'espace de Wiener-Sobolev qui représente une extension naturelle pour les champs aléatoires du critère d'injection compacte de l'espace de Sobolev H^1 dans $\mathbb{L}^2$. On donne alors une forme spécifique de ce critère pour des champs aléatoires solutions d'EDP Stochastiques dont les coefficients appartiennent à une certaine famille. On emploie finalement ce résultat pour construire la solution d'une EDP Stochastique dont la condition terminale est une distribution. On montre de plus qu'une telle solution a une représentation probabiliste par une Équation Différentielle Doublement Stochastique Rétrograde formulée dans un sens faible

Mots-clés : Calcul de Malliavin, décomposition en chaos de Wiener, EDP stochastiques, équations différentielles doublement stochastiques rétrogrades, solutions faibles

1 Introduction

The aim of this paper is to give a relative compactness criterion on the Wiener-Sobolev space which will be used in the frame of Stochastic PDEs. A relative compactness criterion on the Wiener space has already been given in Da Prato *et al.*(1992) but this criterion is not appropriate for SPDEs because it takes care only of the underlying Brownian noise and not on the space variable. On the other hand classical compact embedding theorems represent useful tools for constructing solutions of PDEs. But of course, these theorems are deterministic and so do not take care of the stochastic part. So we need a relative compactness criterion which deals with both variables in the same time. More precisely, the solution of a SPDE is a random field $u(t,x,\omega)$ such that for each fixed t, $(x,\omega) \mapsto u(t,x,\omega)$ is an element of some Sobolev space, as a function of x, and of the Wiener space, as a function of ω. So if one constructs the solution of a SPDE using some approximation procedure (as it is the case in the problem presented in this paper), then one has a sequence $(u_n)_{n\in\mathbb{N}}$ and one wants to obtain u as the limit of this sequence, say in $\mathbb{L}^2(\mathcal{O}\times\Omega)$, with $\mathcal{O}$ a bounded domain in $\mathbb{R}^d$. Roughly speaking the deterministic criterion says that if the derivatives are bounded (for a bounded sequence of $\mathbb{L}^2(\mathcal{O})$), then the sequence in relatively compact in $\mathbb{L}^2(\mathcal{O})$. The stochastic criterion says that if the Malliavin's derivatives are bounded, then the sequence is relatively compact in the Wiener space. In fact, one needs one more condition for obtaining compactness and we come back further on this point. Our criterion takes care of both in the same time, and it turns out that it is the appropriate tool in order to deal with random fields. The abstract criterion is given in Section 2 and then in Section 3 we specify this criterion in the case of families of solutions of SPDEs. Finally in Section 4 we use it in order to construct a solution for a SPDE with a distribution Λ as final condition. This situation may appear for example in the filtering theory (see Pardoux (1989), Rozovskii (1990)) when one discuss the Zakaï equation of the conditional law density. We have to consider only weak solutions of our SPDE and there are two ways of defining such solutions. On one hand one may define solutions in viscosity sense and there are several recent papers on this subject: Lions and Souganidis (2000), Buckdahn and Ma (2001), Buckdahn and Ma (2002). On the other hand one may consider the variational formulation of SPDEs and look for solution in some Sobolev spaces and this is the approach that we adopt. In the case of determinist semi-linear PDEs, this approach appears in Barles and Lesigne (1997) and afterward it has been extended to Stochastic PDEs in Bally and Matoussi (2001). In order to construct the solution, we approximate the distribution by a sequence of smooth functions $(g_n)_{n\in\mathbb{N}}$ and then we use the relative compactness criterion in order to prove that the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of the standard SPDEs with final condition g_n is relatively compact, and so to obtain the solution corresponding to the distribution as the limit of this sequence. We also prove uniqueness of the solution but this employees ad-hoc arguments which are of course not related to relative compactness. Finally we give the probabilistic interpretation of the solution in terms of BDSDEs (Backward Doubly Stochastic Differential Equations) formulated in a weak sense.

Let us give an idea about the abstract relative compactness criterion. We consider a bounded domain $\mathcal{O}$ of $\mathbb{R}^d$ and we assume that u_n is measurable, $x \mapsto u_n(t,x,\omega)$ is in $H^1(\mathcal{O})$, $dt\,\times$

$dP-$almost surely and $\omega \mapsto u_n(t,x,\omega)$ is in $\mathbb{D}^{1,2}$ (one time differentiable in Malliavin's sense) $dt \times dx-$almost surely. Then we assume three types of condition. First of all we assume that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2([0,T]\times\Omega;H^1(\mathcal{O}))$. This is the stochastic variant of the standard boundedness assumption in $H^1(\mathcal{O})$. We consider then a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ we denote $u_n^\varphi := \int_{\mathbb{R}^d}\varphi(x)u_n(t,x,\omega)dx$ and we assume that for each fixed φ the sequence $(u_n^\varphi)_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^2([0,T];\mathbb{D}^{1,2})$. Note that the integration against φ has a regularization effect and the second hypothesis concerns only the stochastic behavior. Finally, the third hypothesis is a Kolmogorov type condition on u_n^φ and their Malliavin's derivatives. We denote $h=(h_1,h_2)\in[0,T]\times[0,T]$ and we assume that

$$\int_0^T \left|\mathbb{E}u_n^\varphi(t+h_1,\omega)-\mathbb{E}u_n^\varphi(t,\omega)\right|^2 dt+\int_0^T\int_0^T \mathbb{E}\left|D_{\theta+h_2}u_n^\varphi(t+h_1,\omega)-D_\theta u_n^\varphi(t,\omega)\right|^2 dtd\theta \xrightarrow[|h|\to 0]{} 0\,,$$

where the functions are assumed to be continued by 0 outside $[0,T]$. Under theses hypotheses, we prove that $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathbb{L}^2([0,T]\times\mathcal{O}\times\Omega)$. Note that, if u_n depends on x only, our hypotheses reduce to A) and this is the standard compact embedding of $H^1(\mathcal{O})$ in $\mathbb{L}^2(\mathcal{O})$. On the other hand, if u_n depends on t only, our hypotheses reduce to C) and this is the classical relative compactness criterion of a bounded sequence of $\mathbb{L}^2([0,T])$. Finally, if u_n depends on ω, a relative compactness criterion of this kind has been given by Da Prato *et al.*(1992).

Let us now see the counterpart of this criterion in the frame of SPDEs. We consider the SDPEs

$$\begin{aligned} u(t,x) = g(x) + \int_t^T \mathcal{L}u(s,x)ds + \int_t^T f(s,x,u(s,x),\nabla u(s,x)\sigma(x))ds \\ + \int_t^T h(s,x,u(s,x))\overleftarrow{dB}_s, \end{aligned} \quad (1)$$

where $u=(u_1,\ldots,u_N)$, ∇u designates the matrix of first order derivatives with respect to x and $\mathcal{L}u=(Lu_1,\ldots,Lu_N)$ with $L=\frac{1}{2}\sum_{i,j=1}^d(\sigma\sigma^*)_{i,j}\frac{\partial^2}{\partial x_i x_j}+\sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}$. The stochastic integral with respect to the standard Brownian motion $(B_t^1,\ldots,B_t^l)_{0\le t\le T}$ is a backward stochastic integral (see Kunita (1982) or Pardoux and Peng (1994)). The solution $(t,x,\omega)\mapsto u(t,x,\omega)$ will be a random field $\mathcal{F}_{t,T}^B-$ measurable with $\mathcal{F}_{t,T}^B=\sigma\{B_T-B_t;\, t\le s\le T\}\vee\mathcal{N}$ ($\mathcal{N}$ is the collection of negligible sets with respect to the Wiener measure P^B on the canonical space $\Omega_2=C_0([0,T];\mathbb{R}^l)$).

Our aim is to prove that, if the coefficients σ, b, f, g and h are bounded in an appropriate way, then the corresponding solutions represent a relative compact set of random fields. So, given $\bar{L}$, ϵ and $\varsigma>0$, we define $\Gamma_{\bar{L},\epsilon,\varsigma}$ to be the class of coefficients (b,σ,f,g,h) such that

A) a) b, σ and their derivatives up to order 3 are bounded by $\bar{L}$.
   b) $\sigma\sigma^*\ge\epsilon I$

B) f, h and their derivatives of first and second order with respect to x, y and z are bounded by $\bar{L}$

C) for every $0 < t < T$, $\|P_{T-t}g\|_\infty \le \bar{L}/(T-t)^\varsigma$. Here, P_t is the semigroup associated to L.

Moreover, we denote by $\mathcal{U}_{\bar{L},\epsilon,\varsigma}$ the class of the solutions of (1) with coefficients (b,σ,f,g,h) in $\Gamma_{\bar{L},\epsilon,\varsigma}$. Then our result (see Theorem 3, Section 3) says that $\mathcal{U}_{\bar{L},\epsilon,\varsigma}$ is relatively compact in $\mathbb{L}^2([0,\tau]\times\mathcal{O}\times\Omega)$, for every $\tau > T$ and $\mathcal{O}\in\mathbb{R}^d$. This is done in Theorem 3.
The difficult point in our Theorem is that the final condition is allowed to blow up as $t\uparrow T$ and there is a significant effort needed in order to handel this difficulty. We note that under the uniform ellipticity assumption $A)b)$, the semigroup of the diffusion process of infinitesimal generator operator L has a smooth density and the derivative of this density may be controlled. So the hypothesis $C)$ on $P_{T-t}g$ will be verified as soon as g is bounded in a distribution sense. This is the starting point the result presented in Subsection 4.2 where we approximate the distribution by a sequence of smooth functions $(g_n)_{n\in\mathbb{N}}$ which will be bounded in a distribution sense (see Remark 6).

2 A relative compactness criterion in Wiener-Sobolev spaces

We prove in this Section a relative compactness criterion which will use the Wiener chaos decomposition. We introduce some notation (for more details, see Nualart (1996) or Zakaï (1985)). Let $(\Omega,\mathcal{F},\mathbb{P})$ a probability space. We consider an arbitrary measured space $(\mathbf{T},\mathcal{B},\mu)$ and we denote $H := \mathbb{L}^2(\mathbf{T},\mathcal{B},\mu))$. The scalar product on H is denoted by $<.,.>_H$ and $\|h\|_H$ denotes the norm of an element $h\in H$. We consider a Gaussian process on H, $W=\{W(h),h\in H\}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$, i.e. W is a centered Gaussian family of random variables such that $\mathbb{E}(W(h)W(g)) = <h,g>_H$ for all $h,g\in H$. For each $n\ge 1$, we denote $\mathcal{H}_n$ the closed linear subspace of $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P})$ (Wiener chaos of order n) generated by the random variables $\{H_n(W(h)),h\in H,\|h\|_H=1\}$ where H_n is the $n-$th Hermite polynomial. We denote by $\mathcal{G}$ the $\sigma-$field generated by the random variables $\{W(h),h\in H\}$. We have the orthogonal decomposition $\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P}) = \bigoplus_{n=0}^\infty \mathcal{H}_n$. We denote I_m the multiple stochastic integral $\int_{\mathbf{T}^m} f(t_1,\dots,t_m)W(dt_1)\dots W(dt_m)$ of a symmetric elements of $\mathbb{L}^2(\mathbf{T}^m)$. This multiple integral is a map from the symmetric elements of $\mathbb{L}^2(\mathbf{T}^m)$ onto the Wiener chaos $\mathcal{H}_m$ and any square integrable random variable $F\in\mathbb{L}^2(\Omega,\mathcal{G},\mathbb{P})$ can be expanded into a series of multiple stochastic integrals:

$$F = \sum_{m=0}^\infty I_m(f_m)\ , \qquad (2)$$

where $f_0 = \mathbb{E}(F)$, I_0 is the identity mapping on the constants, and f_m are symmetric elements of $\mathbb{L}^2(\mathbf{T}^m)$. We recall that

$$\|F\|_2^2 = \sum_{m=0}^\infty m!\|f_m\|^2_{\mathbb{L}^2(\mathbf{T}^m)}\ . \qquad (3)$$

Remark 1. *Later, we will take* $\mathbf{T} = [0,T] \times \{1,\ldots,d\}$ *and* μ *the product of the Lebesgue measure times the uniform measure on* $1,\ldots,d$*, then* $H = \mathbb{L}^2([0,T] \times \{1,\ldots,d\},\mu)$ *is isomorphic to* $\mathbb{L}^2([0,T];\mathbb{R}^d)$ *and* $\big\{W^i(t) := W([0,t] \times \{i\}), 0 \le t \le T, 1 \le i \le d\big\}$ *is a standard* $d-$*dimensional Brownian motion. Furthermore, for any* $h \in H$*, we denote* $h_t^i = h(t,i)$ *and* $W(h) = \sum_{i=1}^d \int_0^T h_t^i dW_t^i$*. The multiple stochastic integral* $I_m(f_m)$*, defined for square integrable symmetric kernels* $f_m\big((t_1,i_1),\ldots,(t_m,i_m)\big)$*, can be expressed as a sum of iterated Itô's integrals:*

$$I_m(f_m) = m! \sum_{i_1,\ldots,i_m=1}^{d} \int_0^T \int_0^{t_m} \cdots \int_0^{t_2} f_m\left((t_1,i_1),\ldots,(t_m,i_m)\right) dW_{t_1}^{i_1} \ldots dW_{t_m}^{i_m} \ .$$

We recall briefly some facts concerning the Malliavin derivative of a square integrable random variable $F : \Omega \mapsto \mathbb{R}$ in our general framework. We denote $C_p^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that f and all its partial derivatives have polynomial growth. Let $\mathcal{S}$ denote the class of *smooth* random variables F that is $F = f\big(W(h_1),\ldots,W(h_n)\big)$ with $n \in \mathbb{N}$, $h_1,\ldots,h_n \in H$ and $f \in C_p^\infty(\mathbb{R}^n)$.
The derivative of a smooth random variable F is the stochastic process $\{D_tF, t \in \mathbf{T}\}$ defined by

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(W(h_1),\ldots,W(h_n)\big)h_i(t) \ .$$

We will denote $\mathbb{D}^{1,2}$ the domain of D in $\mathbb{L}^2(\Omega)$, i.e. $\mathbb{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}\|DF\|_{\mathbb{L}^2(\mathbf{T})}^2 \ .$$

Let $F \in \mathbb{L}^2(\Omega)$ having the decomposition (2) with symmetric kernels. Then $F \in \mathbb{D}^{1,2}$ if and only if $\sum_{m=1}^\infty mm!\|f_m\|_{\mathbb{L}^2(\mathbf{T}^m)}^2 < \infty$ and we have

$$D_tF = \sum_{i=1}^{\infty} mI_{m-1}(f_m(.,t)) \text{ and } \mathbb{E}\int_{\mathbf{T}} |D_tF|^2 \mu(dt) = \sum_{m=1}^{\infty} mm!\|f_m\|_{\mathbb{L}^2(\mathbf{T}^m)}^2 \ . \tag{4}$$

We denote Π_k the projection on the $k-$th Wiener Chaos $\mathcal{H}_k$. As an immediate consequence of (4), we have

$$\begin{aligned}\left\|F - \sum_{m=0}^{k} \Pi_m F\right\|_2^2 &= \sum_{m\ge k+1} \|\Pi_m F\|_2^2 = \sum_{m \ge k+1} m!\|f_m\|_{\mathbb{L}^2(\mathbf{T}^m)}^2 \\ &\le \frac{1}{k}\sum_{m\ge k+1} mm!\|f_m\|_{\mathbb{L}^2(\mathbf{T}^m)}^2 \le \frac{1}{k}\|F\|_{1,2}^2.\end{aligned}$$

So we get the following estimate

$$\left\| F - \sum_{m=0}^{k} \Pi_m F \right\|_2 \leq \frac{1}{\sqrt{k}} \|F\|_{1,2}. \tag{5}$$

Consider now a random field $v \in \mathbb{L}^2(\mathbf{T} \times \Omega \ ; \ H^1(\mathcal{O}))$ where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain. The decomposition in Wiener chaos of v is given by

$$v(t,x,\omega) = \sum_{m=0}^{\infty} I_m\big(f^m\big(t,x,.\big)\big)(\omega) \ .$$

We denote $C_c^k(\mathcal{O})$ the class of $k-$times differentiable functions and have a compact support included in $\mathcal{O}$. For a function $\varphi \in C_c^\infty(\mathcal{O})$, we define $v^\varphi(t,\omega) = \int_{\mathcal{O}} \varphi(x) v(t,x,\omega)dx$. Then the kernels of the Wiener chaos decomposition of v^φ are

$$f_\varphi^m(t,t_1,\ldots,t_m) = \int_{\mathcal{O}} \varphi(x) f^m(t,x,t_1,\ldots,t_m)dx \ .$$

Theorem 1. *Consider a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{L}^2(\mathbf{T} \times \Omega \ ; \ H^1(\mathcal{O}))$ and suppose that*

a) $\sup_{n\in\mathbb{N}} \mathbb{E} \int_{\mathbf{T}} \|v_n(t,.,\omega)\|^2_{H^1(\mathcal{O})} \, \mu(dt) < \infty$

b) For all $\varphi \in C_c^\infty(\mathcal{O})$ and $t \in \mathbf{T}$, $v_n^\varphi(t,.)$ belongs to $\mathbb{D}^{1,2}$ and $\sup_{n\in\mathbb{N}} \int_{\mathbf{T}} \|v_n^\varphi(t,.)\|^2_{\mathbb{D}^{1,2}} \, \mu(dt) < \infty$

c) For all $\varphi \in C_c^\infty$, $m \in \mathbb{N}$, the sequence $\big(f_{n,\varphi}^m\big)_{n\in\mathbb{N}}$ is relatively compact in $\mathbb{L}^2(\mathbf{T} \times \mathbf{T}^m)$.

d) There exists $a > 0$ such that $\sup_{n\in\mathbb{N}} \mathbb{E} \int_{\mathcal{O}} \left(\int_{\mathbf{T}} |v_n(t,x)|^2 \mu(dt) \right)^{1+a} dx = K < \infty.$

Then $\{v_n; n \in \mathbb{N}\}$ is relatively compact in $\mathbb{L}^2(\mathbf{T} \times \mathcal{O} \times \Omega)$.
If we moreover suppose that the measure μ is finite, then $d)$ is a consequence of $a)$ and the same conclusion holds.

Proof. We denote $H_0^1(\mathcal{O})$ is the closure of $C_c^1(\mathcal{O})$ in $H^1(\mathcal{O})$.
Step 1: We first prove that a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{L}^2(\mathbf{T} \times \Omega \ ; \ H_0^1(\mathcal{O}))$ which satisfies the hypotheses $a)$, $b)$ and $c)$ is relatively compact in $\mathbb{L}^2(\mathbf{T} \times \mathcal{O} \times \Omega)$.
We use the spectral decomposition of the Laplacian operator: there exists $(e_n)_{n\in\mathbb{N}}$ Hilbertian basis of $\mathbb{L}^2(\mathcal{O})$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real numbers with $\lambda_n > 0$ et $\lambda_n \xrightarrow[n\to\infty]{} \infty$ satisfying $e_n \in H_0^1(\mathcal{O}) \cap C^\infty(\mathcal{O})$ and $-\Delta e_n = \lambda_n e_n$. Moreover $(e_n/\sqrt{\lambda_n})_{n\in\mathbb{N}}$ is an Hilbertian basis of $H_0^1(\mathcal{O})$ with the scalar product defined by

$$< f,g >:= \int_{\mathcal{O}} \nabla f(x) \nabla g(x) dx \ . \tag{6}$$

We develop v_n as an element of $\mathbb{L}^2(\mathcal{O})$:

$$v_n(t,x,\omega) = \sum_{i=1}^{\infty} \big(v_n(t,.,\omega), e_i(.)\big)_{\mathbb{L}^2(\mathcal{O})} e_i(x) \ ,$$

and we evaluate the rest of this sum. Using the fact that e_n is an eigenvector of the Laplacian operator and integrating by parts we get

$$\begin{aligned}
\mathbb{E}\int_{\mathbf{T}} \left\| \sum_{i\geq N} \big(v_n(t,.,\omega), e_i(.)\big)_{\mathbb{L}^2(\mathcal{O})} e_i(x) \right\|^2_{\mathbb{L}^2(\mathcal{O})} \mu(dt) &= \sum_{i\geq N} \mathbb{E}\int_{\mathbf{T}} \big(v_n(t,.,\omega), e_i\big)^2_{\mathbb{L}^2(\mathcal{O})} \mu(dt) \\
&= \sum_{i\geq N} \mathbb{E}\int_{\mathbf{T}} \frac{1}{\lambda_i^2} \left(v_n(t,.,\omega), \Delta e_i\right)^2_{\mathbb{L}^2(\mathcal{O})} \mu(dt) \\
&= \sum_{i\geq N} \mathbb{E}\int_{\mathbf{T}} \frac{1}{\lambda_i^2} \left(\nabla v_n(t,.,\omega), \nabla e_i\right)^2_{\mathbb{L}^2(\mathcal{O})} \mu(dt) \\
&\leq \frac{1}{\lambda_N} \sum_{i\geq N} \mathbb{E}\int_{\mathbf{T}} \left(\nabla v_n(t,.,\omega), \frac{\nabla e_i}{\sqrt{\lambda_i}}\right)^2_{\mathbb{L}^2(\mathcal{O})} \mu(dt).
\end{aligned}$$

Since $(e_n/\sqrt{\lambda_n})_{n\in\mathbb{N}}$ is an Hilbertian basis of $H_0^1(\mathcal{O})$ with the scalar product (6) and using a), we may dominate the above series by means of the eigenvalues of the Laplacian:

$$\frac{1}{\lambda_N}\mathbb{E}\int_{\mathbf{T}} \|\nabla v_n(t,.,\omega)\|^2_{\mathbb{L}^2(\mathcal{O})} \mu(dt) \leq \frac{C}{\sqrt{\lambda_N}} \ .$$

Since this evaluation is uniform with respect to n, we may ignore the rests and so the relative compactness of the sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathbb{L}^2(\mathbf{T}\times\mathcal{O}\times\Omega)$ reduces to the relative compactness of each of the sequences $\big((v_n(t,.,\omega), e_i)_{\mathbb{L}^2(\mathcal{O})}\big)_{n\in\mathbb{N}}$, $i\in\mathbb{N}$, in the space $\mathbb{L}^2(\mathbf{T}\times\Omega)$. Since $C_c^\infty(\mathcal{O})\subset\mathbb{L}^2(\mathcal{O})$ is dense, standard arguments show that it is sufficient to prove the relative compactness of $v_n^\varphi(t,\omega) = ((v_n(t,.,\omega),\varphi)_{\mathbb{L}^2(\mathcal{O})}$, $n\in\mathbb{N}$ for each $\varphi\in C_c^\infty(\mathcal{O})$.
We now develop v_n^φ in Wiener chaos:

$$v_n^\varphi(t,\omega) = \sum_{m=0}^{\infty} I_m\big(f^m_{n,\varphi}(t,.)\big)(\omega) \ ,$$

and we employ (5) and hypothesis b) in order to truncate this series at level k:

$$\int_{\mathbf{T}} \left\| v_n^\varphi(t,\omega) - \sum_{m=0}^{k} I_m\big(f^m_{n,\varphi}(t,.)\big)(\omega) \right\|_{\mathbb{L}^2(\Omega)} \mu(dt) \leq \frac{1}{\sqrt{k}} \int_{\mathbf{T}} \|v_n^\varphi(t,.)\|^2_{\mathbb{D}^{1,2}} \mu(dt) \leq \frac{C}{\sqrt{k}} \ .$$

Now it is clear that the relative compactness of $(v_n^\varphi)_{n\in\mathbb{N}}$ in $\mathbb{L}^2(\mathbf{T}\times\Omega)$ reduces to the relative compactness of each sequence $\big(I_m\big(f^m_{n,\varphi}(t,.)\big)(\omega)\big)_{n\in\mathbb{N}}$, $m\in\mathbb{N}$, in $\mathbb{L}^2(\mathbf{T}\times\Omega)$. In view of the

isometry property, this is equivalent to the relative compactness of $(f^m_{n,\varphi})_{n\in\mathbb{N}}$ in $\mathbb{L}^2(\mathbf{T}\times\mathbf{T}^m)$, i.e. to the hypothesis c).
Step 2: Now, we suppose that v_n belongs to $\mathbb{L}^2(\mathbf{T}\times\Omega\ ;\ H^1(\mathcal{O}))$ for any n and we suppose hypotheses a), b), c) and d).
Let $(\varphi_k)_{k\in\mathbb{N}}$ a sequence of $C_c^\infty(\mathcal{O})$ such that $0\leq\varphi_k\leq 1$ and for any $q\in\mathbb{N}$ this sequence converges in $\mathbb{L}^q(\mathcal{O})$ to $\mathbf{1}_{\mathcal{O}}$ (the indicator function of $\mathcal{O}$). We define $v_{n,k}:=\varphi_k v_n$ so that $v_{n,k}\in H^1_0(\mathcal{O})$.
For each fixed k, the sequence $(v_{n,k})_{n\in\mathbb{N}}$ is in the frame of above first step, so it is relatively compact. We choose a subsequence $(v_{n_p,k})_{p\in\mathbb{N}}$ such that it is a convergent sequence of $\mathbb{L}^2(\mathbf{T}\times\mathcal{O}\times\Omega)$ for any $k\in\mathbb{N}$. Since

$$|v_{n_r}(t,x)-v_{n_r,k}(t,x)|\leq|\mathbf{1}_{\mathcal{O}}(x)-\varphi_k(x)|\times|v_{n_r}(t,x)|,$$

we may use d) and Hölder's inequality in order to get

$$\begin{aligned}\mathbb{E}\int_{\mathbf{T}}\|(v_{n_r,k}-v_{n_r})(t,.)\|^2_{\mathbb{L}^2(\mathcal{O})}\mu(dt)&\leq\mathbb{E}\int_{\mathcal{O}}|\mathbf{1}_{\mathcal{O}}(x)-\varphi_k(x)|^2\int_{\mathbf{T}}|v_{n_r}(t,x)|^2\mu(dt)\ dx\\ &\leq c\left(\int_{\mathcal{O}}|\mathbf{1}_{\mathcal{O}}(x)-\varphi_k(x)|^{\frac{2(1+a)}{a}}dx\right)^{\frac{a}{1+a}}\times\sup_{r\in\mathbb{N}}\mathbb{E}\left(\int_{\mathcal{O}}\left[\int_{\mathbf{T}}|v_{n_r}(t,x)|^2\mu(dt)\right]^{1+a}dx\right)^{\frac{1}{1+a}}\\ &\leq c\times K\ \left(\int_{\mathcal{O}}|\mathbf{1}_{\mathcal{O}}(x)-\varphi_k(x)|^{2\times\frac{1+a}{a}}dx\right)^{\frac{a}{1+a}}.\end{aligned}$$

So we have

$$\sup_{r\in\mathbb{N}}\mathbb{E}\int_{\mathbf{T}}\|(v_{n_r,k}-v_{n_r})(t,.)\|^2_{\mathbb{L}^2(\mathcal{O})}\mu(dt)\xrightarrow[k\to+\infty]{}0\ .$$

Using $\|v_{n_p}-v_{n_r}\|_{\mathbb{L}^2(\mathbf{T}\times\mathcal{O}\times\Omega)}\leq\|v_{n_p}-v_{n_p,k}\|_{\mathbb{L}^2(\mathbf{T}\times\mathcal{O}\times\Omega)}+\|v_{n_r,k}-v_{n_r}\|_{\mathbb{L}^2(\mathbf{T}\times\mathcal{O}\times\Omega)}+\|v_{n_r,k}-v_{n_p,k}\|_{\mathbb{L}^2(\mathbf{T}\times\mathcal{O}\times\Omega)}$, the result follows.
If we suppose that the measure μ is finite, then there exists $q>2$ such that $\|u\|_{\mathbb{L}^q(\mathcal{O})}\leq c\|u\|_{H^1(\mathcal{O})}$. We choose $a:=\frac{1}{2}(q-2)$ and we get

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}}\left(\int_{\mathbf{T}}|v_n(t,x)|^2d\mu(t)\right)^{1+a}dx&\leq\mu(\mathbf{T})^a\ \mathbb{E}\int_{\mathbf{T}}\int_{\mathcal{O}}|v_n(t,x)|^{2+2a}dxd\mu(t)\\ &\leq c\ \mu(\mathbf{T})^a\ \mathbb{E}\int_{\mathbf{T}}\|v_n(t,x)\|^2_{H^1(\mathcal{O})}d\mu(t)\ \leq K\ .\end{aligned}$$

□

Our criterion is still infinite because we say nothing about how to handel the relative compactness assumption c). In the case of a classical $d-$dimensional Brownian motion, we are able to give a Kolmogorov type criterion. The chaos of order m will no more appear and we will able to give a condition involving only the first order Malliavin's derivative. This is

what we are doing now.
We deal with the case of $\mathbf{T} := [0,T] \times \{1, \ldots, d\}$ which correspond to a standard Brownian motion in $\mathbb{R}^d$ (see Remark 1). For any $f \in \mathbb{L}^2((0,T)^m)$ and $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$, we denote

$$(\tau_h f)(t) \triangleq f(t+h) \ .$$

We recall that a bounded sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathbb{L}^2((0,T)^m)$ is relatively compact if the two following conditions hold:

$(C_m 1)$ For any open set $\mathcal{T} \triangleq (\alpha_1,\beta_1) \times \ldots \times (\alpha_m,\beta_m)$ with $0 < \alpha_i < \beta_i < T \ , \ 1 \leq i \leq m$,

$$\sup_{n\in\mathbb{N}} \|(\tau_h f_n) - f_n\|^2_{\mathbb{L}^2(\mathcal{T})} < C \ |h| \ , \ \forall h \in \mathbb{R}^m \quad , \max_i |h_i| < \min_i(\alpha_i, T - \beta_i) \ ,$$

$(C_m 2)$ For any $\varepsilon > 0$, there exists an open set $\mathcal{T} \triangleq (\alpha_1,\beta_1) \times \ldots \times (\alpha_m,\beta_m)$ with $0 < \alpha_i < \beta_i < T \ , \ 1 \leq i \leq m$ such that

$$\sup_{n\in\mathbb{N}} \|f_n\|_{\mathbb{L}^2([0,T]^m \setminus \mathcal{T})} < \varepsilon \ .$$

We can now state the following version of Theorem 1.

Theorem 2. *Let $\mathcal{O}$ be a bounded domain of $\mathbb{R}^d$ and let $(v_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{L}^2(\mathbf{T} \times \Omega \ ; \ H^1(\mathcal{O}))$. Suppose that*

(1) $\displaystyle \sup_{n\in\mathbb{N}} \mathbb{E} \int_0^T \|v_n(t,.,\omega)\|^2_{H^1(\mathcal{O})} \, dt < \infty$

(2) For all $\varphi \in C_c^\infty(\mathcal{O})$ and $t \in [0,T]$, $v_n^\varphi(t,.)$ belongs to $\mathbb{D}^{1,2}$ and $\displaystyle \sup_{n\in\mathbb{N}} \int_0^T \|v_n^\varphi(t,.)\|^2_{\mathbb{D}^{1,2}} \, dt < \infty$

(3) For all $\varphi \in C_c^\infty$, the sequence $(\mathbb{E}v_n^\varphi)_{n\in\mathbb{N}}$ of $\mathbb{L}^2([0,T])$ satisfies

$(3i)$ *For any $0 < \alpha < \beta < T$ and $h \in \mathbb{R}$ such that $|h| < \min(\alpha, T-\beta)$, it holds*

$$\sup_{n\in\mathbb{N}} \int_\alpha^\beta |\mathbb{E}v_n^\varphi(t+h) - \mathbb{E}v_n^\varphi(t)|^2 dt < C \ |h|$$

$(3ii)$ *For any $\varepsilon > 0$, there exists $0 < \alpha < \beta < T$ such that*

$$\sup_{n\in\mathbb{N}} \int_{[0,T]\setminus(\alpha,\beta)} |\mathbb{E}v_n^\varphi(t)|^2 dt < \varepsilon.$$

(4) For all $\varphi \in C_c^\infty$ the following conditions are satisfied :

(4i) ***For any*** $0 < \alpha < \beta < T$, $0 < \alpha' < \beta' < T$ ***and*** $h, h' \in \mathbb{R}$ ***such that*** $|h| \vee |h'| < \min(\alpha, \alpha', T-\beta, T-\beta')$, ***it holds***

$$\sup_{n\in\mathbb{N}} \mathbb{E} \int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_{\theta+h} v_n^\varphi(t+h') - D_\theta v_n^\varphi(t)|^2 d\theta dt < C\ (|h| + |h'|)$$

(4ii) ***For any*** $\varepsilon > 0$, ***there exists*** $0 < \alpha < \beta < T$ ***and*** $0 < \alpha' < \beta' < T$ ***such that***

$$\sup_{n\in\mathbb{N}} \mathbb{E} \int_{[0,T]^2 \setminus (\alpha,\beta)\times(\alpha',\beta')} |D_\theta v_n^\varphi(t)|^2 d\theta dt < \varepsilon.$$

Then $\{v_n; n \in \mathbb{N}\}$ ***is relatively compact in*** $\mathbb{L}^2([0,T] \times \mathcal{O} \times \Omega)$.

Proof. It only remains to prove that hypotheses (3) and (4) of Theorem 2 imply hypothesis c) of Theorem 1. Thanks to the identity (3), we get

$$\sum_{m\geq 0} m! \|f_{n,\varphi}^m\|_{\mathbb{L}^2([0,T]\times[0,T]^m)} \leq \sup_{n\in\mathbb{N}} \mathbb{E} \int_0^T \|v_n^\varphi(t,.)\|^2_{\mathbb{L}^2(\mathcal{O})}\, dt < \infty\ .$$

So for any order m, the sequences $(f_{n,\varphi}^m)_{n\in\mathbb{N}}$ are bounded in $\mathbb{L}^2([0,T] \times [0,T]^m)$.
Hypothesis c) of Theorem 1 is true for the chaos of order 0 thanks to (3) of Theorem 2. It remains to check that $(4i)$ implies the condition $(C_{m+1}1)$ and $(4ii)$ implies $(C_{m+1}2)$. Let $\mathcal{T} \triangleq (\alpha_0,\beta_0) \times (\alpha_1,\beta_1) \times \ldots \times (\alpha_m,\beta_m)$ such that $0 < \alpha_i < \beta_i < T$, $0 \leq i \leq m$, and $h = (h_0, h_1, \ldots, h_m) \in \mathbb{R}^{m+1}$. It holds

$$\begin{aligned} &\left\|(\tau_h f_{n,\varphi}^m) - f_{n,\varphi}^m\right\|^2_{\mathbb{L}^2(\mathcal{T})} \\ &\quad \leq C \sum_{i=1}^m \int_{\alpha_0}^{\beta_0} \int_{\alpha_i}^{\beta_i} \left\|f_{n,\varphi}^m(t_0+h_0, ., t_i+h_i, .) - f_{n,\varphi}^m(t_0, ., t_i, .)\right\|^2_{\mathbb{L}^2([0,T]^{m-1})} dt_0 dt_i\ . \end{aligned}$$

We denote $\alpha' \triangleq \min_{1\leq i\leq m} \alpha_i$ and $\beta' \triangleq \max_{1\leq i\leq m} \beta_i$. Since the kernels are symmetric functions we obtain

$$\begin{aligned} \left\|(\tau_h f_{n,\varphi}^m) - f_{n,\varphi}^m\right\|^2_{\mathbb{L}^2(\mathcal{T})} &\leq C\ m \int_{\alpha_0}^{\beta_0} \int_{\alpha'}^{\beta'} \left\|f_{n,\varphi}^m(t+h_0, \theta+h_1, .) - f_{n,\varphi}^m(t, \theta, .)\right\|^2_{\mathbb{L}^2([0,T]^{m-1})} dt d\theta \\ &= C\ m \int_{\alpha_0}^{\beta_0} \int_{\alpha'}^{\beta'} \mathbb{E}\left|I_{m-1}\left(f_{n,\varphi}^m(t+h_0, \theta+h_1, .) - f_{n,\varphi}^m(t, \theta, .)\right)\right|^2 dt d\theta\ . \end{aligned}$$

Using successively the orthogonality of the iterated multiple stochastic integrals, (4) and $(4i)$ we finally get

$$\begin{aligned}
&\left\|(\tau_h f^m_{n,\varphi}) - f^m_{n,\varphi}\right\|^2_{\mathbb{L}^2(\mathcal{T})} \\
&\quad\le C\,\mathbb{E}\int_{\alpha_0}^{\beta_0}\int_{\alpha'}^{\beta'}\left|\sum_{m'\ge 1} m' I_{m'-1}\big(f^{m'}_{n,\varphi}(t+h_0,\theta+h_1,.) - f^{m'}_{n,\varphi}(t,\theta,.)\big)\right|^2 dt d\theta \\
&\quad= C\int_{\alpha_0}^{\beta_0}\int_{\alpha'}^{\beta'}\mathbb{E}|D_{\theta+h_1}v^{\varphi}_n(t+h_0) - D_\theta v^{\varphi}_n(t)|^2 dt d\theta \le\ C(|h_0|+|h_1|)\le C\ |h|\ .
\end{aligned}$$

Now we prove the property $(C_{m+1}2)$. Let $\varepsilon>0$. We assume $(4ii)$ and we introduce

$$\mathcal{T} \triangleq (\alpha,\beta)\times(\alpha',\beta')^m\ ,$$

where α' and β' are defined above. Clearly

$$[0,T]^{m+1}\setminus\mathcal{T}\subset\bigcup_{i=0}^{m-1}\{(0,\alpha)\cup(\beta,T)\}\times[0,T]^i\times\{(0,\alpha')\cup(\beta',T)\}\times[0,T]^{m-1-i}\ ,$$

with the convention that $[0,T]^0\triangleq\emptyset$. Then it holds that

$$\begin{aligned}
&\int_{[0,T]^{m+1}\setminus\mathcal{T}}|f^m_{n,\varphi}(t_0,t_1,\dots,t_m)|^2 dt_0\dots dt_m \\
&\quad\le\sum_{i=1}^{m}\int_{\{(0,\alpha)\cup(\beta,T)\}\times[0,T]^i\times\{(0,\alpha')\cup(\beta',T)\}\times[0,T]^{m-1-i}}|f^m_{n,\varphi}(t_0,t_1,\dots,t_m)|^2 dt_0\dots dt_m\ .
\end{aligned}$$

Using the same arguments as above, we obtain

$$\int_{[0,T]^{m+1}\setminus\mathcal{T}}|f^m_{n,\varphi}(t_0,t_1,\dots,t_m)|^2 dt_0\dots dt_m\quad\le\sup_{n\in\mathbb{N}}\mathbb{E}\int_{[0,T]^2\setminus(\alpha,\beta)\times(\alpha',\beta')}|D_\theta v^{\varphi}_n(t)|^2 d\theta dt<\varepsilon\ ,$$

and the property $(C_{m+1}2)$ follows. □

3 A relative compact set of random fields solutions of SPDEs

In this section, we are interested in the semilinear SPDEs denoted by $\mathcal{E}(b,\sigma,f,g,h)$:

$$\begin{aligned}
u(t,x) = g(x) + \int_t^T\mathcal{L}u(s,x)ds + \int_t^T f(s,x,u(s,x),\nabla u(s,x)\sigma(x))ds \\
+\int_t^T h(s,x,u(s,x))\overleftarrow{dB}_s\ .
\end{aligned}$$

See (1) for precisions about the notations. We suppose that all the coefficients are smooth (essentially bounded and three times differentiable with bounded derivatives), then the SPDE $\mathcal{E}(b,\sigma,f,g,h)$ has a unique solution $u \in C^{0,2}([0,T]\times\mathbb{R}^d)$ which is $\mathcal{F}^B_{t,T}-$ measurable for any $t \leq T$ (see Pardoux and Peng (1994)). We introduce the probabilistic background associated to the differential operator L. For each $(t,x)\in[0,T]\times\mathbb{R}^d$, we consider the diffusion process $(X_s^{t,x})_{t\leq s\leq T}$ solution of the stochastic differential equation (in short SDE):

$$\begin{cases} dX_s^{t,x} &= b(X_s^{t,x})ds + \sigma(X_s^{t,x})dW_s \ , \ t\leq s\leq T \\ X_t^{t,x} &= x. \end{cases}$$

Here, $\{W_t^i, 0 \leq t \leq T, 1 \leq i \leq m\}$ is a standard Brownian motion defined on its canonical space $\Omega_1 = C_0([0,T];\mathbb{R}^m)$. The diffusion process is a Markov process and L is its infinitesimal generator. For any $x \in \mathbb{R}^d$, this Markov process starting from x generates a semi-group denoted by $(P_t^x)_{t\in[0,T]}$.
We denote $C_b^k(\mathbb{R}^p;\mathbb{R}^q)$ the set of functions of class C^k from $\mathbb{R}^p$ into $\mathbb{R}^q$ whose partial derivatives of order less than or equal to k are bounded (and hence the function itself has linear growth). We introduce the following set of coefficients:

Definition 1. *Given $\bar{L} > 0$, $\epsilon > 0$, $\varsigma > 0$, we say that the set of coefficients (b,σ,f,g,h) belongs to $\Gamma_{\bar{L},\epsilon,\varsigma}$ if:*

- *b (resp. σ) from $\mathbb{R}^d$ to $\mathbb{R}^d$ (resp. to $\mathbb{R}^{d\times m}$) are bounded by $\bar{L}$ and 3 times differentiable with bounded derivatives (again by $\bar{L}$).*

  σ if uniformly elliptic, that is $\sigma\sigma^* \geq \epsilon I_d$.

  f from $[0,T]\times\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^{N\times m}$ to $\mathbb{R}^N$ and h from $[0,T]\times\mathbb{R}^d\times\mathbb{R}^N$ to $\mathbb{R}^N$ satisfy

  i) for any $s \in [0,T]$, $(x,y,z) \mapsto \big(f(s,x,y,z), h(s,x,y)\big)$ is of class C_b^3 and all the derivatives are bounded by $\bar{L}$ on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^{N\times m}$.

  ii) $\sup_{s,x,z} |f(s,x,y,z)| \leq \bar{L}(1+|y|)$

  iii) $\sup_{s,x,y} |h(s,x,y)| \leq \bar{L}$

- *g from $\mathbb{R}^d$ to $\mathbb{R}^N$ is an infinitely differentiable function such that there exists $\varsigma > 0$ such that for any $t < T$:*

  iv) $\sup_{x\in\mathbb{R}^d} \big|\big(P_{T-t}g\big)(x)\big| \leq \dfrac{\bar{L}}{(T-t)^\varsigma}.$

Remark 2. *We recall that we will use this result in order to construct the solution of a SPDE with a tempered distribution as final condition. Then g will be regularization of this distribution and so $P_{T-t}g$ explodes as $t \uparrow T$. The speed of this explosion is given by ς and it will depend of the order of the distribution. This will be precise in Remark 6 hereafter in Section 4.*
Anyway, if g is bounded, then $iv)$ is trivially true with $\varsigma = 0$.

The main result of this section is the following Theorem.

Theorem 3. *We denote by* $\mathcal{U}_{\bar{L},\epsilon,\varsigma}$ *the set of all the solutions of the SPDEs* $\mathcal{E}(b,\sigma,f,g,h)$ *with the coefficients* b,σ,f,g,h *in* $\Gamma_{\bar{L},\epsilon,\varsigma}$.
Then $\mathcal{U}_{\bar{L}}$ *is relatively compact in* $L^2(\mathcal{O}\times[0,\tau]\times\Omega_2)$*, for any* $\tau<T$ *and any open bounded subset* $\mathcal{O}$ *of* $\mathbb{R}^d$.

In order to prove this Theorem, we will use the Theorem 2. So we have to show some a priori estimates for the solution of the SPDEs $\mathcal{E}(b,\sigma,f,g,h)$. For that sake, we mainly use the stochastic representation of SPDEs in term of BDSDEs: for the case of smooth coefficients see Pardoux and Peng (1994), and see Bally and Matoussi (2001) for Lipschitz continuous coefficients.
Let $(\Omega,\mathcal{F},P)$ be the canonical probability space associated with the two mutually independent standard Brownian motions $\{W_t^i, 0\le t\le T, 1\le i\le m\}$ and $\{B_t^i, 0\le t\le T, 1\le i\le l\}$. The $m\times l-$dimensional Wiener measure is denoted by P and defined on $\Omega=\Omega_1\times\Omega_2$ and $\mathcal{F}$ is the completion with respect to P of the Borel $\sigma-$field generated by the projections. If we denote P^W and P^B the Wiener measures associated respectively to W and B so that $dP(\omega_1,\omega_2)=dP^W(\omega_1)\otimes dP^B(\omega_2)$. For $0\le t\le r\le T$ we define $\mathcal{F}_{t,r}^W=\sigma\{W_s-W_t;\, t\le s\le r\}\vee\mathcal{N}$ where $\mathcal{N}$ is the collection of $P-$negligible sets. For $t\in[0,T]$ we let $\mathcal{F}_t=\mathcal{F}_t^W\vee\mathcal{F}_{t,T}^B$ (notice that $(\mathcal{F}_t)_{0\le t\le T}$ is not a filtration). For any $r\in\mathbb{N}$, we denote $M^2(0,T;\mathbb{R}^r)$ the set of $r-$dimensional random processes $\{\varphi_t; t\in[0,T]\}$ which satisfy: $\mathbb{E}\int_0^T|\varphi_t|^2dt<\infty$ and φ_t is $\mathcal{F}_t$ measurable for a.e. $t\in[0,T]$.
The following Theorem is proved in Pardoux and Peng (1994).

Theorem 4. *Suppose that* σ, b, f, h *and* g *are* C_b^3. *Let* $x\in\mathbb{R}^d$ *and* $t\in[0,T]$ *be fixed. Then the BDSDE*

$$Y_s^{t,x}=g(X_T^{t,x})+\int_s^T f(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+\int_s^T h(X_r^{t,x},Y_r^{t,x})\overleftarrow{dB}_r-\int_s^T Z_r^{t,x}dW_r \quad t\le s\le T \tag{7}$$

has a unique solution $\{(Y_s^{t,x},Z_s^{t,x}), t\le s\le T\}$ *such that* $\mathbb{E}\sup_{t\le s\le T}|Y_s^{t,x}|^2<\infty$ *and* $Z_\cdot^{t,x}\in M^2(t,T;\mathbb{R}^{N\times m})$.
Moreover, if we define $u(t,x):=Y_t^{t,x}$*, then* $u\in C^{0,2}([0,T]\times\mathbb{R}^d)$ *and* u *solves the equation*

$$u(t,x)=g(x)+\int_t^T\{\mathcal{L}u(s,x)+f(s,x,u(s,x),\nabla u(s,x)\sigma(x))\}ds+\int_t^T h(s,x,u(s,x))\overleftarrow{dB}_s. \tag{8}$$

Conversely, if $u\in C^{0,2}([0,T]\times\mathbb{R}^d)$ *solves the equation* (8)*, then the couple of processes* $(Y_s^{t,x},Z_s^{t,x}):=(u(s,X_s^{t,x}),\nabla u(s,X_s^{t,x})\sigma(X_s^{t,x}))$ *solves the BDSDE* (7).

Notice that there are two independent Brownian motions coming on : the Brownian motion B which represent the noise in the Stochastic PDE and the Brownian motion W

which is used in order to represent by means of diffusion process the operator L which appears in the Stochastic PDE. We will use Malliavin's calculus with respect to B and never with respect to W. So $DF = (D^1F, \dots, D^lF) \in (\mathbb{L}^2([0,T] \times \Omega_2))^l$ will be the generic notation for the Malliavin's derivative of some functional F with respect to B. We refer to Pardoux and Peng (1994) for further details about this and for the next Proposition.

Proposition 1. *Assume that the coefficients f and h are C_b^2 as functions of y and z, and let (Y_s, Z_s) be the solution of BDSDE (7). The Malliavin derivative of (Y_s,Z_s) exists and satisfies the linear equation*

$$\begin{aligned} D_\theta Y_s^{t,x} &= h(X_\theta^{t,x}, Y_\theta^{t,x}) + \int_s^\theta \partial_y f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Y_r^{t,x} + \partial_z f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Z_r^{t,x} dr \\ &\quad + \int_s^\theta \partial_y h(X_r^{t,x}, Y_r^{t,x}) D_\theta Y_r^{t,x} \overleftarrow{dB}_r - \int_s^\theta D_\theta Z_r^{t,x} dW_r \ . \end{aligned} \tag{9}$$

We suppose now that the set of coefficients (b, σ, f, g, h) belongs to $\Gamma_{\bar{L},\epsilon,\varsigma}$.

Notation 1. *For any $t \in [0,T)$, $\mathcal{C}_t$ will denote a set of constants which depend on $1/(T-t)^\varsigma$ and on $\bar{L}, \epsilon$ and ς. In particular, when we say that a constant $c \in \mathcal{C}_t$, this means that c depends on the above constants only.*

Lemma 1. *Let $\{(Y_s^{t,x}, Z_s^{t,x}), t \le s \le T\}$ be the solution of (7). Then for any $\tau < T$, $p \in \mathbb{N}$ there exists a constant $C \in \mathcal{C}_\tau$ such that*

$$\sup_{x \in \mathbb{R}^d} \sup_{r \in [t,\tau]} \mathbb{E} \left| Y_r^{t,x} \right|^{2p} + \sup_{x \in \mathbb{R}^d} \mathbb{E} \left(\int_t^\tau \left| Z_r^{t,x} \right|^2 \right)^p dr \le C \ . \tag{10}$$

Proof. We write the equation (8) under the mild form, that is:

$$u(t,x) = P_{T-t} g(x) + \int_t^T P_{r-t} f\big(r, ., u(r,.), \nabla u(r,.)\sigma(.)\big) dr + \int_t^T P_{r-t} h\big(r, ., u(r,.)\big) \overleftarrow{dB}_r \ ,$$

where $(P_t)_{t \in [0,T]}$ is the semi-group of the diffusion process. This may be obtained by taking conditional expectation with respect to the $\sigma-$field $\mathcal{F}_T^W$ in the BDSDE (7) and using the representation $Y_s^{t,x} = u(s,x)$, $Z_s^{t,x} = \nabla u(s,x)\sigma(x)$. Since $(b, \sigma, f, g, h) \in \Gamma_{\bar{L},\epsilon,\varsigma}$, we have

$$\left| \int_t^T P_{r-t} f\big(r, ., u(r,.), \nabla u(r,.)\sigma(.)\big) dr \right| \le \int_t^T \bar{L}(1 + P_{r-t}|u(r,.)|) dr \ ,$$

and since h is bounded

$$\mathbb{E}^B \left| \int_t^T P_{r-t} h\big(r, ., u(r,.)\big) \overleftarrow{dB}_r \right|^{2p} \le c_p \bar{L}^{2p} \ .$$

Using $|P_{T-t}g(x)| \le \bar{L}/(T-t)^{\varsigma}$ it follows that there exists $C \in \mathcal{C}_\tau$ such that for any $t < \tau$:

$$\mathbb{E}^B|u(t,x)|^{2p} \le C\left(1+\int_t^T \mathbb{E}^B P_{r-t}|u(r,.)|^{2p}dr\right) .$$

We take $\sup_{x\in\mathbb{R}^d}$ and we use the Gronwall Lemma and obtain $\sup\limits_{t\in[0,\tau]}\sup\limits_{x\in\mathbb{R}^d}\mathbb{E}^B|u(t,x)|^{2p} \le C$. So for any $r \in [t,\tau]$, and $x \in \mathbb{R}^d$ we have

$$\mathbb{E}|Y_r^{t,x}|^{2p} = \mathbb{E}|u(r,X_r^{t,x})|^{2p} = \int_{\mathbb{R}^d} p(r-t,x,y)\mathbb{E}^B|u(r,y)|^{2p}dy \le C . \tag{11}$$

Let us now evaluate Z. Writing the BDSDE (7) between t and τ gives

$$\int_t^\tau Z_r^{t,x}dW_r = u(\tau,X_\tau^{t,x}) - u(t,x) + \int_t^\tau f(X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr + \int_t^\tau h(X_r^{t,x},Y_r^{t,x})\overleftarrow{dB}_r .$$

Using Burkholder's inequality and the fact that f has linear growth and h is bounded we obtain

$$\mathbb{E}\left(\int_t^\tau \left|Z_r^{t,x}dr\right|^2\right)^p dr \le C\Big(\mathbb{E}|u(\tau,X_\tau^{t,x})|^{2p} + \mathbb{E}|u(t,x)|^{2p} + \int_t^\tau \mathbb{E}|Y_r^{t,x}|^{2p}dr\Big) .$$

We deduce from (11) that $\mathbb{E}\big(\int_t^\tau |Z_r^{t,x}|^2\big)^p dr \le C$. □

Remark 3. *The above lemma is standard but usually the constant C depends on $\|g\|_\infty$. What is special here is that the contribution of g comes on in the constant C which depends on $\|P_{T-t}g\|_\infty$.*

Lemma 2. *Let $\{(Y_s^{t,x},Z_s^{t,x}), t \le s \le T\}$ be the solution of the BDSDE (7). Then for any $p \in \mathbb{N}$, there exists a constant C depending on $\|\nabla f\|_\infty$, $\|\nabla h\|_\infty$, $\|h\|_\infty$ and p such that*

$$\sup_{0\le t\le s\le\theta\le T}\sup_{x\in\mathbb{R}^d}\mathbb{E}|D_\theta Y_s^{t,x}|^{2p} + \mathbb{E}\left(\int_t^\theta |D_\theta Z_r^{t,x}|^2dr\right)^p \le C . \tag{12}$$

Proof. Note that the equation (9) does no more depend on g. So using standard estimates (see Pardoux and Peng (1994)), we can get for any $p \in \mathbb{N}$ the existence of c_p independent of θ, t, s, n and x such that :

$$\mathbb{E}|D_\theta Y_s^{t,x}|^{2p} + \mathbb{E}\left(\int_t^\theta |D_\theta Z_r^{t,x}|^2dr\right)^p \le c_p\mathbb{E}|h(X_\theta^{t,x},Y_\theta^{t,x})|^{2p} .$$

Since h is bounded, we get (12). □

We shall state some a priori estimation concerning the increments of the Malliavin's derivative of the solution of (8). The techniques developed below are not standard. We treat a special case as an example in order to emphasize the difficulties which come on. Assume for the moment that B is a one dimensional Brownian motion, $h(r, x, y) := h(y)$ and $f(r, x, y, z) := f(z)$. We write the equation (9) in this simple case:

$$D_\theta Y_s^{t,x} = h(Y_\theta^{t,x}) + \int_s^\theta f'(Z_r^{t,x}) D_\theta Z_r^{t,x} dr + \int_s^\theta h'(Y_r^{t,x}) D_\theta Y_r^{t,x} \overleftarrow{dB}_r - \int_s^\theta D_\theta Z_r^{t,x} dW_r \ .$$

Our aim is (see $(4i)$ of Theorem 2) to prove that $\mathbb{E}|D_\theta Y_s - D_{\theta'} Y_s|^2 \le c|\theta - \theta'|$. We take $\theta \le \theta'$ and we write

$$\begin{aligned} D_\theta Y_s^{t,x} - D_{\theta'} Y_s^{t,x} &= h(Y_\theta^{t,x}) - h(Y_{\theta'}^{t,x}) - \int_\theta^{\theta'} f'(Z_r^{t,x}) D_{\theta'} Z_r^{t,x} dr \\ &- \int_\theta^{\theta'} h'(Y_r^{t,x}) D_{\theta'} Y_r^{t,x} \overleftarrow{dB}_r + \int_\theta^{\theta'} D_{\theta'} Z_r^{t,x} dW_r \\ &+ \int_s^\theta f(Z_r^{t,x}) [D_\theta Z_r^{t,x} - D_{\theta'} Z_r^{t,x}] dr + \int_s^\theta h'(Y_r^{t,x}) \big[D_\theta Y_r^{t,x} - D_{\theta'} Y_r^{t,x}\big] \overleftarrow{dB}_r \\ &- \int_s^\theta \big[D_\theta Z_r^{t,x} - D_{\theta'} Z_r^{t,x}\big] dW_r \ . \end{aligned}$$

There are two interesting terms in the right hand side of the above equation:

$$\begin{aligned} \mathbb{E}\left|\int_\theta^{\theta'} h'(Y_r^{t,x}) D_{\theta'} Y_r^{t,x} \overleftarrow{dB}_r\right|^2 &= \mathbb{E}\int_\theta^{\theta'} |h'(Y_r^{t,x})|^2 \ |D_{\theta'} Y_r^{t,x}|^2 dr \\ \text{and} \quad \mathbb{E}\left|\int_\theta^{\theta'} D_{\theta'} Z_r^{t,x} dW_r\right| &= \mathbb{E}\int_\theta^{\theta'} |D_{\theta'} Z_r^{t,x}|^2 dr \ . \end{aligned}$$

The first term is easy to handel: by (12) we dominate it by $\|h'\|_\infty (\sup_{r,\eta} \mathbb{E}|D_\eta Y_r^{t,x}|^2)|\theta - \theta'|$. The second term is more delicate because (12) gives not an evaluation of $\sup_{r,\eta} \mathbb{E}|D_\eta Z_r^{t,x}|^2$ but of $\int_0^T \mathbb{E}|D_\eta Z_r^{t,x}|^2 dr$ only. This is the motivation of the whole work which starts now. So the main effort (which is done in Lemma 8 from the appendix) is to pass from the "weak" norms in $\mathbb{L}^2([0,T] \times \Omega, dr \otimes \mathbb{P})$ to the "strong" norms which involve a supremum over r. In order to solve our problem we employ the representation $Z_r = \nabla u(r, X_r)\sigma(X_r)$ and we use the SPDE satisfied by ∇u. This motivates the evaluations concerning the process $Y_s^{(1),t,x} := \nabla u(s, X_s^{t,x})$ given below.
We introduce some notation. For a function $v : \mathbb{R}^d \to \mathbb{R}^N$, we denote by ∇v its Jacobian matrix and by $\nabla^2 v$ the tensor of the second order derivatives. For $(t, x, y, z) \mapsto f(t, x, y, z)$, we denote f_x (respectively f_y and f_z) the Jacobian matrix of f with respect to x (respectively y and z). We use analogous notations for the function h.

Proposition 2. *Let u be the solution of (8). Then ∇u solves the SPDE*

$$\nabla u(t,x) = \nabla g(x) + \int_t^T \mathcal{L}\nabla u(s,x)ds + \int_t^T \big[f_x(s,x,u,\nabla u\sigma) + f_y(s,x,u,\nabla u\sigma)\nabla u(s,x) + f_z(s,x,u,\nabla u\sigma)(\nabla^2 u(s,x)\sigma + \nabla u\nabla\sigma) + \nabla^2 u\sigma\nabla\sigma^* + \nabla u\nabla b\big]ds + \int_t^T \big(h_x(s,x,u) + h_y(s,x,u)\nabla u(s,x)\big)\overleftarrow{dB}_s \ .$$

Moreover, for any $(t,x)\in[0,T]\times\mathbb{R}^d$, we have $\nabla u(t,x) = Y_t^{(1),t,x}$, $\nabla^2 u(t,x)\sigma(x) = Z_t^{(1),t,x}$ where the couple of processes $(Y_r^{(1),t,x}, Z_r^{(1),t,x})_{t\le r\le T}$ is the solution of the BDSDE:

$$Y_s^{(1),t,x} = \nabla g(X_T^{t,x}) + \int_s^T \Big[f_x(r,X_r^{t,x},Y_r^{t,x},Y_r^{(1),t,x}) + f_y(r,X_r^{t,x},Y_r^{t,x},Y_r^{(1),t,x})Y_r^{(1),t,x} + f_z(r,X_r^{t,x},Y_r^{t,x},Y_r^{(1),t,x})\big(Y_r^{(1),t,x}\sigma(X_r^{t,x}) + Z_r^{(1),t,x}\big) + Z_r^{(1),t,x}\nabla\sigma^*(X_r^{t,x}) + Y_r^{(1),t,x}\nabla b(X_r^{t,x})\Big]dr + \int_s^T h_x(r,X_r^{t,x},Y_r^{t,x}) + h_y(r,X_r^{t,x},Y_r^{t,x})Y_r^{(1),t,x}\ \overleftarrow{dB}_r - \int_s^T Z_r^{(1),t,x}dW_r \ . \tag{13}$$

Proof. The proof is analogous to that in Pardoux and Peng (1994) so we skep it. ☐

We state now two lemmas which play a crucial part in our evaluations. The proofs of these lemmas are given respectively in the Appendix B and C. We assume that the set of coefficients (b,σ,f,g,h) belongs to $\Gamma_{\bar{L},\epsilon,\varsigma}$. Then the two following Lemmas hold.

Lemma 3. *Let $\tau < T$. For any $p\in\mathbb{N}$, there exists $C_p\in\mathcal{C}_\tau$ such that:*

$$\sup_{t\le s\le\tau}\ \sup_{t\le\theta\le T}\mathbb{E}\left|D_\theta Y_s^{(1),t,x}\right|^{2p} \le C_p \ . \tag{14}$$

Lemma 4. *Let $\tau < T$ and $p\in\mathbb{N}$. There exists $C_p\in\mathcal{C}_\tau$ such that for all $0\le t\le s\le\tau$, θ , θ' in $[t,T)$, it holds*

$$\mathbb{E}\left|D_\theta Y_s^{t,x} - D_{\theta'}Y_s^{t,x}\right|^{2p} \le C_p\times|\theta-\theta'|^p \ .$$

Finally, we shall need the following norm equivalence result.

Proposition 3. *Let $\rho:\mathbb{R}^d\to\mathbb{R}$ be a continuous, positive function such that there exists $M>0$ such that for $|x|\ge M$, $\rho\in C_b^2(\mathbb{R}^d,\mathbb{R})$. We assume moreover that b and σ are bounded by $\bar{L}$ and are 3 times differentiable with bounded (by $\bar{L}$) derivatives. Then there exists two constants κ_1 and κ_2 such that for $0\le t\le s\le T$, φ belongs to $\mathbb{L}^1([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$ and we have:*

$$\kappa_1\|\varphi\|_{\mathbb{L}^1([0,T]\times\mathbb{R}^d,dt\otimes\rho(x)dx)} \le \int_t^T\int_{\mathbb{R}^d}\mathbb{E}|\varphi(s,X_s^{t,x})|\rho(x)dxds \le \kappa_2\|\varphi\|_{\mathbb{L}^1([0,T]\times\mathbb{R}^d,dt\otimes\rho(x)dx)} \ , \tag{15}$$

The constant κ_1 and κ_2 depend on T, on ρ and on the bounds of the derivatives of b and σ.

This result was first proved in Barles and Lesigne (1997) and Kunita (1982) for $\rho = 1$. In the case of a general ρ, which we need here, the proof is given in Bally and Matoussi (2001). This result will allow us to obtain a priori estimates for SPDEs from a prior estimates for BDSDEs. Now, the above Lemmas and Theorem 2 enable us to prove Theorem 3.

Proof of Theorem 3.

We have to prove that all the assumptions of Theorem 2 hold true for fixed $\mathcal{O}$ (an open domain of $\mathbb{R}^d$) and $(b, \sigma, f, g, h) \in \Gamma_{\bar{L},\epsilon,\varsigma}$.

Step 1: We first prove that for any $\tau < T$, there exists $C \in \mathcal{C}_\tau$ such that

$$\mathbb{E}^B \int_{[0,\tau]} \|u(t,.,\omega)\|^2_{H^1(\mathcal{O})}\, dt \leq C\ . \tag{16}$$

We use the fact that $u(t,x) = Y_t^{t,x}$ and (10) in order to get that for any $\tau < T$, there exists $C \in \mathcal{C}_\tau$ such that

$$\sup_{x\in\mathbb{R}^d}\ \sup_{t\in[0,\tau]} \mathbb{E}^B |u(t,x)|^{2p} \leq C\ . \tag{17}$$

We also deduce from (10) that

$$\mathbb{E}\int_t^\tau \left|\nabla u(r, X_r^{t,x})\sigma(X_r^{t,x})\right|^2 dr = \mathbb{E}\int_t^\tau \left|Z_r^{t,x}\right|^2 dr \leq C\ .$$

Now we introduce a nonnegative function $\zeta \in \mathbb{L}^1(\mathbb{R}^d)$ which is equal to one on $\mathcal{O}$ and satisfies the hypotheses of the equivalence of norms result (15). We obtain

$$\begin{aligned}\kappa_1 \int_{\mathbb{R}^d}\int_t^\tau \mathbb{E}^B \left|\nabla u(r,x)\sigma(x)\right|^2 \zeta(x) dr dx &\leq \int_{\mathbb{R}^d} \mathbb{E}^W \int_t^\tau \mathbb{E}^B \left|\nabla u(r, X_r^{t,x})\sigma(X_r^{t,x})\right|^2 \zeta(x) dr dx \\ &\leq C\ \|\zeta\|_{\mathbb{L}^1(\mathbb{R}^d)}.\end{aligned}$$

Since σ is uniformly elliptic, $|\nabla u(r,x)\sigma(x)|^2 \geq \epsilon |\nabla u(r,x)|^2$. Then we have

$$\epsilon\kappa_1 \int_{\mathcal{O}}\int_t^\tau \mathbb{E}^B \left|\nabla u(r,x)\right|^2 dr dx \leq \kappa_1 \int_{\mathbb{R}^d}\int_t^\tau \mathbb{E}^B \left|\nabla u(r,x)\sigma(x)\right|^2 \zeta(x) dr dx\ \ \leq C\ \|\zeta\|_{\mathbb{L}^1(\mathbb{R}^d)}.$$

The above inequality and (17) yield (16).

Step 2: For all $\varphi \in C_c^\infty(\mathcal{O})$ and $t \in [0,\tau]$, $u^\varphi(t,.) := \int_{\mathbb{R}^d} u(t,x)\varphi(x)dx$ belongs to $\mathbb{D}^{1,2}$ and there exists $C \in \mathcal{C}_\tau$ such that

$$\int_{[0,\tau]} \|u^\varphi(t,.)\|^2_{\mathbb{D}^{1,2}}\, dt \leq C\ .$$

Indeed, since $D_\theta u(t,x) = D_\theta Y_t^{t,x}$, we get from (12) that for any $\tau < T$, there exists $C \in \mathcal{C}_\tau$ such that

$$\sup_{0\leq t\leq\tau}\ \sup_{t\leq\theta\leq T}\ \sup_{x\in\mathbb{R}^d} \mathbb{E}^B |D_\theta u(t,x)|^{2p} \leq C\ . \tag{18}$$

Step 3: We check that hypothesis 3) of Theorem 2 is fulfilled.
From (17), we first get that for all $\varphi \in C_c^\infty$, $\mathbb{E}^B u^\varphi \in \mathbb{L}^2([0,\tau])$ and for any $\varepsilon > 0$, there exists $0 < \alpha < \beta < \tau$ such that

$$\int_{[0,\tau]\setminus(\alpha,\beta)} |\mathbb{E}^B u^\varphi(t)|^2 dt < \varepsilon.$$

It remains to prove that there exists $C \in \mathcal{C}_\tau$ such that for any $0 < \alpha < \beta < \tau$ and $h \in \mathbb{R}$ ($|h| < \min(\alpha, \tau - \beta)$), it holds

$$\int_\alpha^\beta |\mathbb{E}^B u^\varphi(t+h) - \mathbb{E}^B u^\varphi(t)|^2 dt < C\ |h|\ .$$

This will be an immediate consequence of the following estimation: there exists $C \in \mathcal{C}_\tau$ such that $\forall t, t' \in [0,\tau]$,

$$\left| \int_{\mathcal{O}} \varphi(x) \mathbb{E}^B u(t,x) dx - \int_{\mathcal{O}} \varphi(x) \mathbb{E}^B u(t',x) dx \right| \leq C\big(1 + \|\varphi\|_{\mathbb{L}^1(\mathbb{R}^d)} + \|\mathcal{L}^*\varphi\|_{\mathbb{L}^1(\mathbb{R}^d)}\big)|t - t'|\ . \tag{19}$$

Let $t, t' < T$. We write the equation (8) between t and t':

$$u(t,x) - u(t',x) = \int_t^{t'} \mathcal{L}u(r,x) + f(r,x,u,\nabla u\sigma)\ dr + \int_t^{t'} h(r,x,u)\overleftarrow{dB}_r.$$

We denote $\mathcal{L}^*$ the adjoint operator of $\mathcal{L}$. Integrating against φ and taking expectation gives

$$\int_{\mathbb{R}^d} \varphi(x) \mathbb{E}^B \big[u(t,x) - u(t',x)\big] dx = \mathbb{E}^B \int_t^{t'} \int_{\mathbb{R}^d} u(r,x) \mathcal{L}^*\varphi(x) + f(r,x,u,\nabla u\sigma)\varphi(x)\ dxdr\ .$$

Now, we can proceed as in the proof of (10). Using (10), we get that there exists $C \in \mathcal{C}_{1,\tau}$ such that

$$\begin{aligned} \left| \mathbb{E}^B \int_{\mathbb{R}^d} \int_t^{t'} f(r,x,u,\nabla u\sigma)\varphi(x)\ drdx \right| &\leq |t - t'|\ \ C\big(1 + \|\varphi\|_{\mathbb{L}^1(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \sup_{r\in[0,\tau]} \mathbb{E}^B|u(r,x)|\big)\ , \\ &\leq C\big(1 + \|\varphi\|_{\mathbb{L}^1(\mathbb{R}^d)}\big)\ |t - t'|\ \ . \end{aligned}$$

In the same way, one have

$$\left| \mathbb{E}^B \int_t^{t'} \int_{\mathbb{R}^d} u(r,x) \mathcal{L}^*\varphi(x) drdx \right| \leq C\|\mathcal{L}^*\varphi\|_{\mathbb{L}^1(\mathbb{R}^d)}\ |t - t'|\ \ ,$$

and (19) follows.
Step 4: We finally check that hypothesis 4) of Theorem 2 is fulfilled.

From (18), we deduce that for any $\varepsilon > 0$, there exists $0 < \alpha < \beta < \tau$ and $0 < \alpha' < \beta' < \tau$ such that

$$\mathbb{E}^B \int_{[0,\tau]^2 \setminus (\alpha,\beta)\times(\alpha',\beta')} |D_\theta u^\varphi(t)|^2 d\theta dt < \varepsilon.$$

It remains to prove that for any $0 < \alpha < \beta < \tau$, $0 < \alpha' < \beta' < \tau$ and $h, h' \in \mathbb{R}$ such that $|h| \vee |h'| < \min(\alpha, \alpha', \tau - \beta, \tau - \beta')$, it holds

$$\mathbb{E}^B \int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_{\theta+h} u^\varphi(t+h') - D_\theta u^\varphi(t)|^2 d\theta dt < C\ (|h| + |h'|)\ . \tag{20}$$

Taking $s = t$ in Lemma 4 we get that

$$\mathbb{E}^B \int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_{\theta+h} u^\varphi(t) - D_\theta u^\varphi(t)|^2 d\theta dt < C\ |h|\ .$$

So (20) will follows from

$$\mathbb{E}^B \int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_\theta u^\varphi(t+h') - D_\theta u^\varphi(t)|^2 d\theta dt < C\ |h'|\ . \tag{21}$$

In order to prove (21), we use BDSDE's representation of the random field u and we get

$$\begin{aligned} D_\theta u^\varphi(t) &= \int_{\mathcal{O}} \varphi(x) D_\theta u(t,x) dx = \int_{\mathcal{O}} \varphi(x) D_\theta Y_t^{t,x} dx = \int_{\mathcal{O}} \varphi(x) \mathbb{E}^W D_\theta Y_t^{t,x} dx \\ &= \int_{\mathcal{O}} \varphi(x) \mathbb{E}^W h(X_\theta^{t,x}, Y_\theta^{t,x}) dx + \int_{\mathcal{O}} \varphi(x) \mathbb{E}^W \left(\int_t^\theta \partial_y h(X_r^{t,x}, Y_r^{t,x}) D_\theta Y_r^{t,x} \overleftarrow{dB}_r \right) dx \\ &+ \int_{\mathcal{O}} \varphi(x) \mathbb{E}^W \left(\int_t^\theta \partial_y f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Y_r^{t,x} + \partial_z f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Z_r^{t,x} dr \right) dx. \end{aligned}$$

We denote $(\hat{X}_s^{t,x})_{t \le s \le T}$ the inverse map of the diffeomorphism $x \to X_r^{t,x}$ and $J(\hat{X}_s^{t,x})$ the determinant of the Jacobian matrix of $\hat{X}_s^{t,x}$ (Kunita (1982) or Ikeda and Watanabe (1990)). Remind that we have the representation $Y_r^{t,x} = u(r, X_r^{t,x})$ and $Z_r^{t,x} = \nabla(r, X_r^{t,x})\sigma(X_r^{t,x})$. By a change of variable we get

$$\begin{aligned} D_\theta u^\varphi(t) &= \int_{\mathcal{O}} \mathbb{E}^W \varphi(\hat{X}_\theta^{t,x}) J(\hat{X}_\theta^{t,x}) h(x, u(\theta, x)) dx \\ &\quad + \int_{\mathcal{O}} \left(\int_t^\theta \mathbb{E}^W \varphi(\hat{X}_r^{t,x}) J(\hat{X}_r^{t,x}) F(r,\theta,x) dr \right) dx \\ &\quad + \int_{\mathcal{O}} \left(\int_t^\theta \mathbb{E}^W \varphi(\hat{X}_r^{t,x}) J(\hat{X}_r^{t,x}) H(r,\theta,x) \overleftarrow{dB}_r \right) dx \quad \text{with} \\ H(r,\theta,x) &= \partial_y h(x, u(r,x)) D_\theta u(r,x) \text{ and} \\ F(r,\theta,x) &= \partial_y f(x, u(r,x), \nabla u(r,x)\sigma(x)) D_\theta u(r,x) \\ &\quad + \partial_z f(x, u(r,x), \nabla u(r,x)\sigma(x)) D_\theta \nabla u(r,x)\sigma(x)\ . \end{aligned}$$

Using the above expression we get

$$
\begin{aligned}
D_\theta u^\varphi(t+h') - D_\theta u^\varphi(t) &= \int_{\mathcal{O}} \mathbb{E}^W \big(\varphi(\hat{X}_\theta^{t+h',x})J(\hat{X}_\theta^{t+h',x}) - \varphi(\hat{X}_\theta^{t,x})J(\hat{X}_\theta^{t,x})\big) h(x,u(\theta,x))dx \\
&+ \int_{\mathcal{O}} \left(\int_t^{t+h'} \mathbb{E}^W \varphi(\hat{X}_r^{t,x})J(\hat{X}_r^{t,x})F(r,\theta,x)dr \right) dx \\
&+ \int_{\mathcal{O}} \left(\int_{t+h'}^{\theta} \mathbb{E}^W \big(\varphi(\hat{X}_\theta^{t+h',x})J(\hat{X}_\theta^{t+h',x}) - \varphi(\hat{X}_\theta^{t,x})J(\hat{X}_\theta^{t,x})\big) F(r,\theta,x)dr \right) dx \\
&+ \int_{\mathcal{O}} \left(\int_t^{t+h'} \mathbb{E}^W \varphi(\hat{X}_r^{t,x})J(\hat{X}_r^{t,x})H(r,\theta,x)\overleftarrow{dB}_r \right) dx \\
&+ \int_{\mathcal{O}} \left(\int_{t+h'}^{\theta} \mathbb{E}^W \big(\varphi(\hat{X}_\theta^{t+h',x})J(\hat{X}_\theta^{t+h',x}) - \varphi(\hat{X}_\theta^{t,x})J(\hat{X}_\theta^{t,x})\big) H(r,\theta,x)\overleftarrow{dB}_r \right) dx.
\end{aligned}
$$

Recall that $u(t,x) = Y_t^{t,x}$ and $\nabla u(t,x) = Y_t^{(1),t,x}$. Then as a consequence of (14) and (18) with $s = t$ we have

$$
\mathbb{E}^B |D_\theta u(r,x)|^{2p} + \mathbb{E}|D_\theta \nabla u(r,x)|^{2p} \le C_p
$$

for any $p \in \mathbb{N}$. So that $\mathbb{E}^B|F(r,\theta,x)|^{2p} + \mathbb{E}^B|H(r,\theta,x)|^{2p} \le C_p$ for any $p \in \mathbb{N}$. This evaluation is uniform with respect to θ and r.
Moreover one can find a non negative integrable function $x \mapsto c(x)$ depending on φ such that

$$
\mathbb{E}^W \left| \varphi(\hat{X}_{\bar{t}}^{t,x})J(\hat{X}_{\bar{t}}^{t,x}) - \varphi(\hat{X}_{\bar{t}}^{t',x})J(\hat{X}_{\bar{t}}^{t',x}) \right|^2 \le c(x)|t-t'| \; .
$$

Using Cauchy-Schwartz inequality we get (21) and (20) follows.

Remark 4. *The constants in the above estimations may depend on the test function φ. But since we are interested in some properties of u^φ for every fixed $\varphi \in C_c^\infty(\mathbb{R}^d)$, these constant are convenient.*

4 Stochastic PDEs with a distribution as final condition.

In this Section we use the relative compactness criterion for the solutions of SPDEs in order to prove existence and uniqueness for the solution of the following system of semi-linear Stochastic PDEs

$$
\begin{aligned}
u(t,x) = \Lambda &+ \int_t^T \mathcal{L}u(s,x)ds + \int_t^T f(s,x,u(s,x),\nabla u(s,x)\sigma(x))ds \\
&+ \int_t^T h(s,x,u(s,x))\overleftarrow{dB}_s \; .
\end{aligned} \tag{22}
$$

The final condition Λ is a distribution so we have to introduce the Sobolev spaces in which the solution of the SPDE is defined. First of all, since we work on the whole space $\mathbb{R}^d$ and not on some bounded domain, we have to consider weighted sobolev spaces. So we consider a weight $\rho\ :\ \mathbb{R}^d \to \mathbb{R}$ which satisfies:

Hypothesis 1. $\rho : \mathbb{R}^d \to \mathbb{R}$ is continuous, positive and there exists $M > 0$ such that for $|x| \geq M$, $\rho \in C_b^2(\mathbb{R}^d, \mathbb{R})$. Moreover we assume that the weight ρ satisfies $\frac{1}{\rho} \in \mathbb{L}^1(\mathbb{R}^d)$.

Remark that the equivalence of norms result (15) holds for such a weight. Our distribution Λ will belong to the dual of a weighted Sobolev space. More precisely, we denote $\mathcal{H}_k^\rho$ the weighted Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$|\varphi|_{k,\rho}^2 = \sum_{0 \leq |\alpha| \leq k} \int_{\mathbb{R}^d} |\partial_\alpha \varphi(x)|^2 \rho(x) dx \ ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_l$ its length and $\partial_\alpha = \dfrac{\partial^l}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_l}}$.
$\mathcal{H}_{k,\rho}^{'}$ is the class of functionals $u : \mathcal{H}_k^\rho \to \mathbb{R}$ defined by

$$(u\ , \varphi) := \sum_{0 \leq |\alpha| \leq k} \int_{\mathbb{R}^d} u_\alpha(x) \partial_\alpha \varphi(x) dx\ , \forall \varphi \in \mathcal{H}_k^\rho\ ,$$

where $u_\alpha \in \mathbb{L}^2\left(\mathbb{R}^d\ ,\ \rho^{-1}(x)dx\right)\ , |\alpha| \leq k$. Note that $\mathcal{H}_{k,\rho}^{'}$ coincides with the topological dual of $\mathcal{H}_k^\rho$. The operator norm on $\mathcal{H}_{k,\rho}^{'}$ is given by

$$\|u\|_{k,\rho} := \sup\left\{|(u, \varphi)|\ :\ \varphi \in \mathcal{H}_k^\rho\ , |\varphi|_{k,\rho} \leq 1\right\}.$$

We assume that

Hypothesis 2. There exists an integer k such that $\Lambda \in \mathcal{H}_{k,\rho}^{'}$. For $\varphi \in \mathcal{H}_k^\rho$, we write

$$(\ \Lambda\ ,\ \varphi\ ) = \sum_{0 \leq \alpha \leq k} \int_{\mathbb{R}^d} \lambda_\alpha(x) \partial_\alpha \varphi(x) dx\ ,\ \text{where } \lambda_\alpha \in \mathbb{L}^2(\mathbb{R}^d\ ,\ \rho^{-1}(x)dx). \tag{23}$$

Remark 5. *Let $\mathcal{S}$ (respectively $\mathcal{S}'$) be the space of rapidly decreasing functions (resp. the tempered distribution space). The spaces $\mathcal{S}$ and $\mathcal{S}'$ are characterized by*

$$\mathcal{S} = \bigcap_{p,k>0} \mathcal{H}_k^{\rho_p}, \qquad \mathcal{S}' = \bigcup_{p,k>0} \mathcal{H}'_{k,\rho_p}$$

where $\rho_p(x) = \left(1 + |x|^2\right)^p$. In particular Λ can be a tempered distribution.

We introduce the following space of random fields :

Definition 2. *$\mathcal{W}_{m,\rho}$ is the space of random fields $u\ :\ [0,T[\times\mathbb{R}^d\times\Omega_2\to\mathbb{R}^N$ such that*

- *$u(t,x,.)$ is $\mathcal{F}^B_{t,T}-$measurable.*
- *For all $\tau<T$, $u\in\mathbb{L}^2\big([0,\tau]\times\mathbb{R}^d\times\Omega_2;ds\otimes\rho^{-1}(x)dx\otimes dP^B\big)$.*
- *$\nabla u(t,.,\omega_2)$ exits $dt\otimes dP^B-$a.e. and satisfies $\nabla u\in\mathbb{L}^2\big([0,\tau]\times\mathbb{R}^d\times\Omega_2;ds\otimes\rho^{-1}(x)dx\otimes dP^B\big)$, for all $\tau<T$.*
- *$\lim\limits_{s\uparrow t}(u(s,.),\varphi)=(u(t,.),\varphi)\ dt\otimes dP^B(\omega_2)-a.e.\ \forall\varphi\in\mathcal{H}^\rho_{k+2}$.*
- *$dt\otimes dP^B-$a.e., $u(t,.,\omega_2)\in\mathcal{H}'_{m,\rho}$ is such that*

$$\mathbb{E}^B\int_0^T\|u(s,.,\omega_2)\|^2_{m,\rho}ds<\infty\ .$$

We are now able to give the definition of a weak solution of the SPDE (22):

Definition 3. *Suppose that $\Lambda\in\mathcal{H}'_{k,\rho}$. A solution of (22) is a random field $u\ :\ [0,T[\times\mathbb{R}^d\times\Omega_2\to\mathbb{R}^N$ such that*

(i) $u\in\mathcal{W}_{k+2,\rho}$

(ii) $\mathbb{E}^B\|u(s,.)-\Lambda\|^2_{k+2,\rho}\underset{s\uparrow T}{\longrightarrow}0$

(iii) for all $\varphi\in C^2_c([0,T]\times\mathbb{R}^d)$, for all $t\ ,\tau\in[0,T[$ with $t<\tau$,

$$\begin{aligned}\int_t^\tau\int_{\mathbb{R}^d}\partial_s\varphi(s,x)u(s,x)dsdx+\int_{\mathbb{R}^d}u(t,x)\varphi(t,x)dx=\int_{\mathbb{R}^d}u(\tau,x)\varphi(\tau,x)dx\\ +\int_t^\tau\int_{\mathbb{R}^d}A(u,\varphi)(s,x)dxds+\int_t^\tau\int_{\mathbb{R}^d}\varphi(s,x)f(s,x,u,\nabla u\sigma)dxds\\ +\int_t^\tau\int_{\mathbb{R}^d}\varphi(s,x)h(s,x,u)dx\overleftarrow{dB}_s\ P^B-a.s.,\end{aligned}\tag{24}$$

where $A(\varphi,\psi):=\dfrac{1}{2}\ \nabla\varphi\sigma\sigma^*\nabla\psi\ +\ \varphi div(b-\tilde{A})\psi$ and $\tilde{A}_i=\dfrac{1}{2}\sum\limits_{k=1}^d\dfrac{\partial(\sigma\sigma^*)_{k,i}}{\partial x_k}$ for $1\leq i\leq d$.

We make the following assumptions.

Hypothesis 3. b and σ belong respectively to $C_b^{k+3}(\mathbb{R}^d;\mathbb{R}^d)$, $C_b^{k+3}(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and are bounded (remind that k is the order of the distribution Λ).

Hypothesis 4. Uniform ellipticity: there exists a constant $\epsilon>0$ such that $\sigma\sigma^*\geq\epsilon I_d$.

Hypothesis 5. The functions f and h are bounded and Lipschitz continuous:

$$\sup_{x\in\mathbb{R}^d, s\in[0,T], y\in\mathbb{R}^N, z\in\mathbb{R}^{N\times m}} (|h(s,x,y)| + |f(s,x,y,z)|) \le \bar{L}$$

and for any $y, y' \in \mathbb{R}^N$, $z, z' \in \mathbb{R}^{N\times m}$:

$$\sup_{x\in\mathbb{R}^d, s\in[0,T]} (|h(s,x,y) - h(s,x,y')| + |f(s,x,y,z) - f(s,x,y',z')|) \le \bar{L}(|y-y'| + |z-z'|) \ .$$

The main result of this Section is the following.

Theorem 5. *We assume Hypotheses 1 to 5. There exists a unique solution of the SPDE* (22).

The proof of this Theorem is given in the following three subsections.

4.1 Proof of Uniqueness

Let u and $\bar{u}$ be two solutions of (22). We denote $w := u - \bar{u}$, $F(t,x) = f(t,x,u,\nabla u\sigma) - f(t,x,\bar{u},\nabla\bar{u}\sigma)$ and $H(t,x) = h(t,x,u) - h(t,x,\bar{u})$. We will first prove that w solves a linear SPDE in distribution sense.
Let $\tau < T$. u and $\bar{u}$ solves (22) so that, taking the difference between the equations verified by u and $\bar{u}$ (in the weak sense (24)), we get

$$\int_t^\tau\int_{\mathbb{R}^d} \partial_s\varphi(s,x)w(s,x)dxds + \int_{\mathbb{R}^d} w(t,x)\varphi(t,x)dx = \int_{\mathbb{R}^d} w(\tau,x)\varphi(\tau,x)dx \tag{25}$$
$$+ \int_t^\tau\int_{\mathbb{R}^d} w(s,x)\mathcal{L}^*\varphi(s,x))dxds + \int_t^\tau\int_{\mathbb{R}^d} \varphi(s,x)F(s,x)dxds + \int_t^\tau\int_{\mathbb{R}^d} \varphi(s,x)H(s,x)dx\overleftarrow{dB}_s.$$

We want to pass to the limit as $\tau \to T$. Since F and H are bounded we get

$$\int_t^\tau\int_{\mathbb{R}^d} F(t,x)\varphi(s,x)dxds \xrightarrow[\tau\uparrow T]{P^B-a.s.} \int_t^T\int_{\mathbb{R}^d} F(t,x)\varphi(s,x)dxds \text{ and}$$
$$\int_t^\tau\int_{\mathbb{R}^d} H(t,x)\varphi(s,x)dx\overleftarrow{dB}_s \xrightarrow[\tau\uparrow T]{P^B-a.s.} \int_t^T\int_{\mathbb{R}^d} H(t,x)\varphi(s,x)dx\overleftarrow{dB}_s \ .$$

It remains to treat the terms where w appears. Here we must take care of the fact that w is integrable only on $[0,\tau]$ (see Definition 2) but not on $[0,T]$. Using the property (ii) in Definition 3, we get

$$\begin{aligned}\mathbb{E}^B\left|\int_{\mathbb{R}^d} w(\tau,x)\varphi(\tau,x)dx\right| &\le \mathbb{E}^B\left|\int_{\mathbb{R}^d} u(\tau,x)\varphi(\tau,x)dx - \Lambda(\varphi(\tau,.))\right| \\ &\quad + \mathbb{E}^B\left|\int_{\mathbb{R}^d} \bar{u}(\tau,x)\varphi(\tau,x)dx - \Lambda(\varphi(\tau,.))\right| \\ &\le \mathbb{E}^B\|u(\tau,.) - \Lambda\|_{k+2,\rho}|\varphi|_{k+2,\rho} + \mathbb{E}^B\|\bar{u}(\tau,.) - \Lambda\|_{k+2,\rho}|\varphi|_{k+2,\rho} \\ &\xrightarrow[\tau\to T]{} 0 \ .\end{aligned}$$

Moreover one has

$$\left|\int_{\mathbb{R}^d} w(s,x)\mathcal{L}^*\varphi(s,x))dx\right| \le C\ \|w(s,.)\|_{k+2,\rho}|\varphi(s,.)|_{k+2,\rho}$$
$$\le C\ \|w(s,.)\|_{k+2,\rho}\ \sup_{s\le T}|\varphi(s,.)|_{k+2,\rho}\ .$$

Since $w \in \mathcal{W}_{k+2,\rho}$ (see *(i)* in Definition 3) , $s \to \|w(s,.)\|_{k+2,\rho}$ is integrable on $[0,T] \times \Omega_2$ so we may use the dominated convergence Theorem in order to get

$$\int_t^\tau\int_{\mathbb{R}^d} w(s,x)\mathcal{L}^*\varphi(s,x)dxds \underset{\tau\uparrow T}{\longrightarrow} \int_t^T\int_{\mathbb{R}^d} w(s,x)\mathcal{L}^*\varphi(s,x)dxds\ P^B - a.s.$$

Since

$$\left|\int_{\mathbb{R}^d} w(s,x)\partial_s\varphi(s,x))dx\right| \le C\ \|w(s,.)\|_{k+2,\rho}\ \sup_{s\le T}|\partial_s\varphi(s,.)|_{k+2,\rho}\ ,$$

we use again the dominated convergence Theorem and obtain

$$\int_t^\tau\int_{\mathbb{R}^d}\partial_s\varphi(s,x)w(s,x)dxds \xrightarrow[\tau\uparrow T]{P^B-a.s.} \int_t^T\int_{\mathbb{R}^d}\partial_s\varphi(s,x)w(s,x)dxds\ .$$

Reporting all these in (25), we get that $w \in \mathcal{W}_{k+2,\rho}$ solves the linear SPDE (in distribution sense): that is for all $\varphi \in C_c^\infty([0,T] \times \mathbb{R}^d)$ one has

$$\int_t^T (w_s,\partial_s\varphi(s,.))ds + (w_t,\varphi(t,.)) = \int_t^T (w_s,\mathcal{L}^*\varphi(s,.))ds + \int_t^T (F(s,.),\varphi(s,.))ds$$
$$+ \int_t^T (H(s,.),\varphi(s,.))\overleftarrow{dB}_s\ .$$

Notice that w satisfies the above variational formulation on the whole $[0,T]$ whereas u and $\bar{u}$ satisfy (24) only on $[0,\tau]$, $\tau < T$. The point here is that the final condition disappears by taking the difference of u and $\bar{u}$. We are now in the frame of Bally and Matoussi (2001), Proposition 2.1 and we get that $w = 0$ almost everywhere.

4.2 Existence: the case of smooth coefficients

We prove the following intermediate result:

Theorem 6. *We assume that Hypotheses 1 to 4 are fulfilled. We moreover assume that the coefficients f and h satisfy i), ii) and iii) of Definition 1 and f is moreover bounded. Then there exists a random field u solution of* (22).
Moreover u has the property $\mathbb{E}^B\|u(\tau,.) - \Lambda\|^2_{k+2,\rho} \le c(T-\tau)$.

Now we present the strategy that we employ in order to construct the solution of (22) and we explain why a relative compactness criterion is the appropriate tool to achieve it. The idea is simple and standard. Since the operator L is uniformly elliptic the associated semi-group has a regularization property so, even if $u_T = \Lambda$ is a distribution, u_t will be a function as soon as $t < T$ and ∇u_t will exists pointwise. This is essential in order to give a sense to $f(t, x, u, \nabla u\sigma)$ and to $h(t, x, u)$. In order to put this idea to work we consider a sequence of smooth functions g_n such that $g_n \to \Lambda$ in distribution. More precisely: let $\eta_n(x)$ a sequence of regularization kernels $\eta_n = n^d \eta(nx)$ where $\eta \in C_c^\infty(\mathbb{R}^d)$ is a non-negative function such that $0 \in supp\ \eta$ and $\int_{\mathbb{R}^d} \eta(x)dx = 1$. For any $n \geq 1$, we define $\lambda_{n,\alpha} := \eta_n * \lambda_\alpha$ where $(\lambda_\alpha)_{1\leq|\alpha|\leq k}$ are the coefficients given by (23).
Our sequence of smooth function is defined by

$$g_n(x) := \sum_{0\leq|\alpha|\leq k} (-1)^{|\alpha|} \partial_\alpha \lambda_{n,\alpha}(x),$$

and for $\varphi \in \mathcal{H}_k^\rho$, we denote

$$\Lambda_n(\varphi) := \int_{\mathbb{R}^d} g_n(x)\varphi(x)dx = \sum_{0\leq|\alpha|\leq k} \int_{\mathbb{R}^d} \lambda_{n,\alpha}(x)\partial_\alpha\varphi(x)dx\ .$$

One can prove using standard arguments that

$$\lim_{n\to\infty} \|\Lambda_n - \Lambda\|_{k,\rho}^2 = 0\ . \tag{26}$$

Then we replace the final condition Λ in (22) by g_n and we denote u_n the solution of this new equation:

$$\begin{aligned} u_n(t,x) = g_n(x) + &\int_t^T \mathcal{L}u_n(s,x)ds + f(s,x,u_n(s,x),\nabla u_n(s,x)\sigma(x))ds \\ &+ \int_t^T h(s,x,u_n(s,x))\overleftarrow{dB}_s. \end{aligned} \tag{27}$$

The result of Pardoux and Peng (1994) insures that u_n exists and it remains to pass to the limit (or at least extracting a convergent subsequence) and to construct $u = \lim_n u_n$. Of course one has to take care of the fact that g_n blows up as $n \to \infty$ but one may hope that uniform ellipticity will enable us to get round this difficulty. The first and natural idea is to use the classical compact embedding theorem of $H^1(\mathcal{O})$ in $\mathbb{L}^2(\mathcal{O})$: if we prove that $\|u_n\|_{H^1(\mathcal{O})}$, $n \in \mathbb{N}$ is bounded then $(u_n(t,.))_{n\in\mathbb{N}}$ will be relatively compact in $\mathbb{L}^2(\mathcal{O})$ and so we may subtract a convergent subsequence u_{n_k}, $k \in \mathbb{N}$. But u_n as a random field depends on ω and so the subsequence $(n_k)_{k\in\mathbb{N}}$ will depend on ω and it is not clear that one may subtract a common subsequence which is convergent for every ω. This is the specific difficulty concerning this problem and this is why we have to employ a relative compactness criterion which takes care of both $x \in \mathcal{O}$ and $\omega \in \Omega$. The adequate result is the relative compactness criterion given by Theorem 2.

Remark 6. *If Λ satisfies Hypothesis 2, then under the uniform ellipticity assumption (Hypothesis 4) and Hypothesis 3, we have $P_{T-t}g_n(x) = \int_{\mathbb{R}^d} p_{T-t}(x,y)g_n(y)dy$, where $p_r(x,.)$ is the density of the law of the diffusion process starting from x. Besides (see Friedman (1990) or Kusuoka and Stroock (1985)), there exists $c_1,\ c_2,\ \varsigma > 0$, depending on the bounds of the derivatives of σ and b and on the ellipticity constant ϵ, such that*

$$\sum_{0\leq|\alpha|\leq k} |\partial_\alpha^y p_s(x,y)| \leq \frac{c_1}{s^\varsigma} \exp\left(\frac{-c_2|x-y|^2}{s}\right) .$$

Using this estimate we get

$$|P_{T-t}g_n(x)| \leq \frac{c\ \|g_n\|_{k,\rho}}{(T-t)^\varsigma} \leq \frac{c\ \|\Lambda\|_{k,\rho}}{(T-t)^\varsigma} .$$

This is the only way to have an estimation of $P^x_{T-t}g_n$ uniformly with respect to n.

Thanks to this Remark, one may find $\bar{L} > 0$, $\epsilon > 0$ and $\varsigma > 0$ (independent of n) such that the set of coefficients (b,σ,f,g_n,h) belong to the set $\Gamma_{\bar{L},\epsilon,\varsigma}$.
Consequently by Theorem 3, for every $\tau < T$ and every bounded open set $\mathcal{O} \subset \mathbb{R}^d$, the sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathbb{L}^2([0,\tau]\ ; H^1(\mathcal{O})\times\mathbb{D}^{1,2})$ solutions of $\mathcal{E}(b,\sigma,f,g_n,h)$ is relatively compact in $\mathbb{L}^2([0,\tau]\times\mathcal{O}\times\Omega_2)$. The relatively compactness is given only on $[0,\tau]$ since if u_n has a convergent subsequence, one may not hope that it converges on $[0,T]$ entirely since $u(T,.)$ must be a distribution. Then, passing to a subsequence (always denoted by $(u_n)_{n\in\mathbb{N}}$ in the following), we may assume that there exists a random field u such that $u_n(t,x,\omega_2) \rightarrow u(t,x,\omega_2)\ dt\otimes dx\otimes d\mathbb{P}^B(\omega_2)$-almost-everywhere on $[0,T]\times\mathbb{R}^d\times\Omega_2$. Moreover, since the estimates (10) and (12) hold true for u_n, uniformly with respect to n, we obtain for every $\tau < T$ and every $p\in\mathbb{N}$:

$$\sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{R}^d}\sup_{t\in[0,\tau]}\left(\mathbb{E}^B|u_n(t,x)|^{2p} + \int_0^\tau \mathbb{E}^B|\nabla u_n(r,x)|^2dr\right) \leq C\ . \tag{28}$$

As a consequence, we also get

$$\left\|u_{n_p} - u\right\|_{\mathbb{L}^2([0,\tau]\times\mathbb{R}^d\times\Omega_2;ds\otimes\rho^{-1}(x)dx\otimes dP^B)} \xrightarrow[p\to\infty]{} 0 \quad \forall\ \tau < T\ .$$

Let us now prove that $\nabla u_n \rightarrow \nabla u$ almost surely and in $\mathbb{L}^2$. We define

$$(Y^{t,x}_{n,s},Z^{t,x}_{n,s}) := (u_n(s,X^{t,x}_s),\nabla u_n(s,X^{t,x}_s)\sigma(X^{t,x}_s))\ ,\ t\leq s\leq T\ .$$

This couple of processes solves the BDSDE (see Theorem 4)

$$\begin{aligned} Y^{t,x}_{n,s} = g_n(X^{t,x}_T) &+ \int_s^T f(r,X^{t,x}_r,Y^{t,x}_{n,r},Z^{t,x}_{n,r})dr + \int_s^T h(r,X^{t,x}_r,Y^{t,x}_{n,r})\overleftarrow{dB}_r \\ &- \int_s^T Z^{t,x}_{n,r}dW_r\ ,t\leq s\leq T\ . \end{aligned} \tag{29}$$

Classical estimations on the BDSDE (29) (see Pardoux and Peng (1994)) yield

$$\int_s^\tau \mathbb{E}|Z^{t,x}_{n,r} - Z^{t,x}_{m,r}|^2 dr \leq \mathbb{E}|Y^{t,x}_{n,\tau} - Y^{t,x}_{m,\tau}|^2 + C \int_s^\tau \mathbb{E}|Y^{t,x}_{n,r} - Y^{t,x}_{m,r}|^2 dr.$$

Integrating against ρ^{-1} and using twice the equivalence of norms result (15), we get

$$\int_t^\tau \int_{\mathbb{R}^d} \mathbb{E}^B |\nabla u_n(s,x)\sigma(x) - \nabla u_m(s,x)\sigma(x)|^2 \rho^{-1}(x) dx ds \leq$$
$$C \int_{\mathbb{R}^d} \mathbb{E}^B |u_n(\tau,x) - u_m(\tau,x)|^2 \frac{dx}{\rho(x)} + C \int_t^\tau \int_{\mathbb{R}^d} \mathbb{E}^B |u_n(s,x) - u_m(s,x)|^2 \frac{dx}{\rho(x)} ds$$

and the above terms converge to 0 as $n,m \to +\infty$. Since $\sigma\sigma^* \geq \epsilon I_d$, we conclude that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{L}^2([0,\tau] \times \mathbb{R}^d \times \Omega_2 \; ; \; ds \otimes \rho^{-1}(x)dx \otimes dP^B(\omega_2))$. Let $v = \lim_n \nabla u_n$. A standard argument shows now that $u \in H^1(\mathcal{O})$ and $v = \nabla u$. Using (28) we get that

$$\sup_{x\in\mathbb{R}^d} \sup_{t\in[0,\tau]} \mathbb{E}^B |u(t,x)|^2 + \int_0^\tau \mathbb{E}^B |\nabla u(r,x)|^2 dr \leq C \; , \; \forall \, \tau < T \; .$$

Since $(u_n \; , \; \nabla u_n) \to (u \; , \; \nabla u)$ almost-surely and in $\mathbb{L}^2([0,\tau] \times \mathbb{R}^d \times \Omega_2 \; ; \; ds \otimes \rho^{-1}(x)dx \otimes dP^B(\omega_2))$ we may pass to the limit in the equation (24) for u_n in order to obtain (24) for $(u \; , \; \nabla u)$.
We shall need in the following, the notion of stochastic test functions. Let $\varphi \in C^2_c(\mathbb{R}^d)$. We define

$$\psi_t : \Omega \times [t,T] \times \mathbb{R}^d \to \mathbb{R} \text{ by } \psi_t(s,x) = \varphi(\hat{X}^{t,x}_s) J(\hat{X}^{t,x}_s) \; , \tag{30}$$

with $J(\hat{X}^{t,x}_s)$ the determinant of the Jacobian matrix of $x \mapsto \hat{X}^{t,x}_s$, the inverse map of $x \mapsto X^{t,x}_s$. It is proved in Bally and Matoussi (2001) (see Kunita (1982) also) that if $\varphi \in C^\infty_c(\mathbb{R}^d)$, then for $\mathbb{P}-$almost every ω, ψ_t belongs to $\mathcal{H}^\rho_k$ and we have the estimation

$$\sup_{t\leq s\leq T} \mathbb{E}\, |\psi_t(s,.)|^2_{k,\rho} \leq c|\varphi|^2_{k,\rho}, \tag{31}$$

where $c > 0$ depends on T and on the weight ρ only.

Proof of Theorem 6:
It remains to prove that u constructed above satisfies the properties (ii) and (iii) in the Definition 3. We first prove that $\mathbb{E}^B \|u(\tau,.) - \Lambda\|^2_{k+2,\rho} \xrightarrow[\tau\uparrow T]{} 0$.
Remind (see (26)) that $u_n(T,.) = \Lambda_n \xrightarrow[n\to\infty]{} \Lambda$ with respect to the $\|.\|_{k,\rho}-$norm. We fix $\tau < T$ and we write

$$\int_{\mathbb{R}^d} u(\tau,x)\varphi(x)dx - \Lambda(\varphi) = \lim_{n\to\infty} \int_{\mathbb{R}^d} (u_n(\tau,x) - u_n(T,x))\, \varphi(x) dx \; , \; P^B\text{-almost surely.}$$

Let us compute $\int_{\mathbb{R}^d}(u_n(\tau,x)-u_n(T,x))\varphi(x)dx$. Since u_n solves (27), we have the following weak formulation on $[0,T]$ (and not only on $[0,\tau]$ since g_n is a function)

$$\begin{aligned}\int_{\mathbb{R}^d} u_n(t,x)\varphi(t,x)dx &= \int_{\mathbb{R}^d}\varphi(T,x)u_n(T,x)dx+\int_t^T\int_{\mathbb{R}^d}\mathcal{L}u_n(s,x)\varphi(s,x)dxds\\ &+\int_t^T\int_{\mathbb{R}^d}\varphi(s,x)f(s,x,u_n(s,x),\nabla u_n(s,x)\sigma(x))dxds\\ &+\int_t^T\int_{\mathbb{R}^d}\varphi(s,x)h(s,x,u_n(s,x))dx\overleftarrow{dB}_s\ .\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^d} u(\tau,x)\varphi(x)dx-\Lambda(\varphi)=\lim_{n\to\infty}\left((I_n^1,\varphi)+(I_n^2,\varphi)+(I_n^3,\varphi)\right)\ \text{with} \tag{32}$$

$$\begin{aligned}(I_n^1,\varphi) &= \int_\tau^T\int_{\mathbb{R}^d} u_n(s,x)\mathcal{L}^*\varphi(x)dxds\\ (I_n^2,\varphi) &= \int_\tau^T\int_{\mathbb{R}^d} f(s,x,u_n,\nabla u_n\sigma)\varphi(x)dxds\\ (I_n^3,\varphi) &= \int_\tau^T\int_{\mathbb{R}^d} h(s,x,u_n)\varphi(x))dx\overleftarrow{dB}_s\ .\end{aligned}$$

Since f is bounded, $|(I_n^2,\varphi)|\le c(T-\tau)|\varphi|_{0,\rho}$ and then

$$\mathbb{E}^B\|I_n^2\|_{k,\rho}^2\le c\ (T-\tau)^2\ . \tag{33}$$

Moreover one has

$$\int_\tau^T\int_{\mathbb{R}^d} h(s,x,u_n)\varphi(x)dx\overleftarrow{dB}_s=\int_{\mathbb{R}^d}\left(\int_\tau^T h(s,x,u_n)\overleftarrow{dB}_s\right)\frac{1}{\rho(x)}\varphi(x)\ \rho(x)dx,$$

and thanks to the Cauchy-Schwartz inequality with respect to the measure $\rho(x)dx$ we get

$$\begin{aligned}|(I_n^3,\varphi)| &\le \left(\int_{\mathbb{R}^d}\left|\int_\tau^T h(s,x,u_n)\overleftarrow{dB}_s\right|^2\frac{dx}{\rho(x)}\right)^{\frac12}\ \left(\int_{\mathbb{R}^d}|\varphi(x)|^2\rho(x)dx\right)^{\frac12}\\ &\le \left(\int_{\mathbb{R}^d}\left|\int_\tau^T h(s,x,u_n)\overleftarrow{dB}_s\right|^2\frac{dx}{\rho(x)}\right)^{\frac12}\ |\varphi|_{0,\rho}\ ,\ \text{so that}\\ \|I_n^3\|_{k,\rho}^2 &\le \int_{\mathbb{R}^d}\left|\int_\tau^T h(s,x,u_n)\overleftarrow{dB}_s\right|^2\frac{dx}{\rho(x)}\ .\end{aligned}$$

Then, since h is bounded and ρ^{-1} is integrable, we get

$$\mathbb{E}^B \|I_n^3\|_{k,\rho}^2 \leq c(T-\tau) \ . \tag{34}$$

Remind that we have no estimation of $u_n(s,x)$ near T so we have to estimate I_n^1 as follows. We integrate the BDSDE satisfied by $Y_{n,s}^{t,x} = u_n(s, X_s^{t,x})$ (see (29)) against $\psi := \mathcal{L}^*\varphi$:

$$\begin{aligned}\int_{\mathbb{R}^d} Y_{n,s}^{t,x}\psi(x)dx = \int_{\mathbb{R}^d} g_n(X_T^{t,x})\psi(x)dx + \int_{\mathbb{R}^d}\int_s^T f(r, X_r^{t,x}, Y_{n,r}^{t,x}, Z_{n,r}^{t,x})dr\psi(x)dx \\ + \int_s^T h(r, X_r^{t,x}, Y_{n,r}^{t,x})\overleftarrow{dB}_r\psi(x)dx - \int_{\mathbb{R}^d}\int_s^T Z_{n,r}^{t,x}dW_r\psi(x)dx \ , t \leq s \leq T \ .\end{aligned}$$

We write the above BDSDE for $s = t$ and we use the fact that $u_n(t,x) = \mathbb{E}^W Y_{n,t}^{t,x}$ (the last equality is due to the fact that $Y_{n,t}^{t,x}$ is deterministic with respect to W) in order to get

$$\begin{aligned}\int_{\mathbb{R}^d} u_n(t,x)\psi(x)dx = \int_{\mathbb{R}^d}\mathbb{E}^W g_n(X_T^{t,x})\psi(x)dx + \int_{\mathbb{R}^d}\int_t^T \mathbb{E}^W f(r, X_r^{t,x}, Y_{n,r}^{t,x}, Z_{n,r}^{t,x})dr\psi(x)dx \\ + \int_{\mathbb{R}^d}\int_t^T \mathbb{E}^W h(r, X_r^{t,x}, Y_{n,r}^{t,x})\overleftarrow{dB}_r\psi(x)dx \ .\end{aligned}$$

Now we integrate with respect to $t \in [\tau, T]$ and we obtain

$$\begin{aligned}(I_n^1,\varphi) &= (I_n^{1,1},\varphi) + (I_n^{1,2},\varphi) + (I_n^{1,3},\varphi) \text{ with} \\ (I_n^{1,1},\varphi) &= \int_\tau^T\int_{\mathbb{R}^d}\mathbb{E}^W g_n(X_T^{t,x})\psi(x)dxdt \\ (I_n^{1,2},\varphi) &= \int_\tau^T\int_{\mathbb{R}^d}\int_t^T \mathbb{E}^W f(r, X_r^{s,x}, u_n(r, X_r^{s,x}), \nabla u_n\sigma(r, X_r^{s,x}))dr \ \psi(x)dxdt \\ (I_n^{1,3},\varphi) &= \int_\tau^T\int_{\mathbb{R}^d}\int_t^T E^W h(r, X_r^{s,x}, u_n(s, X_r^{s,x}))\overleftarrow{dB}_r \ \psi(x)dxdt \ .\end{aligned}$$

Note that $|\psi|_{0,\rho} = |\mathcal{L}^*\varphi|_{0,\rho} \leq C|\varphi|_{2,\rho}$, so, in the same way that we obtained (33) and (34), we have

$$\mathbb{E}^B\|I_n^{1,2}\|_{k,\rho}^2 + \mathbb{E}^B\|I_n^{1,3}\|_{k,\rho}^2 \leq c(T-\tau) \ . \tag{35}$$

Finally we evaluate $I_n^{1,1}$. We denote $\psi_t(T,x) = \psi(\hat{X}_T^{t,x})J(\hat{X}_T^{t,x})$ (see (30)), by a change of variable we get

$$(I_n^{1,1},\varphi) = \int_\tau^T\int_{\mathbb{R}^d} g_n(x)\mathbb{E}^W\left(\psi(\hat{X}_T^{t,x})J(\hat{X}_T^{t,x})\right)dxdt = \int_{\mathbb{R}^d} g_n(x)\left(\int_\tau^T \mathbb{E}^W\psi_t(T,.)ds\right)dx \ .$$

Since $\sup_{t\le s\le T}\mathbb{E}\,|\psi_t(s,.)|^2_{k,\rho}\le c|\psi|^2_{k,\rho}\le c|\varphi|_{k+2,\rho}$ (see (31)), it follows

$$\begin{aligned}\left|(J_n^{1,1},\varphi)\right| &\le \|g_n\|_{k,\rho}\left|\int_\tau^T \mathbb{E}^W\psi_t(T,.)dt\right|_{k,\rho}\\ &\le \|g_n\|_{k,\rho}\int_\tau^T|\mathbb{E}^W\psi_t(T,.)|_{k,\rho}dt\\ &\le \|g_n\|_{k,\rho}\int_\tau^T|\varphi|_{k+2,\rho}dt\\ &\le (T-\tau)\|g_n\|_{k,\rho}|\varphi|_{k+2,\rho}\ .\end{aligned}$$

Since $\|g_n\|_{k,\rho}\le c\|\Lambda\|_{k,\rho}$,

$$\mathbb{E}^B\|I_n^{1,1}\|^2_{k+2,\rho}\le c(T-\tau)\ ,$$

then using (35), we get that

$$\mathbb{E}^B\|I_n^{1}\|^2_{k+2,\rho}\le c(T-\tau)\ . \tag{36}$$

Combining (33), (34) and (36) with (32) we finally get that

$$\mathbb{E}^B\|u(\tau,.)-\Lambda\|^2_{k,\rho}\le c(T-\tau)\ ,$$

with c depending on $\|f\|_\infty$, $\|h\|_\infty$ and $\|\Lambda\|_{k,\rho}$.

The same technics as above gives easily that

$$\begin{aligned}&\lim_{s\uparrow t}(u(s,.),\varphi)=(u(t,.),\varphi)\ dt\otimes dP^B(\omega_2)-a.e.\ \forall\varphi\in\mathcal{H}^\rho_{k+2}\text{ and}\\ &\sup_{0\le\tau\le T}\mathbb{E}^B\|u(\tau,.)\|^2_{k+2,\rho}<\infty\ .\end{aligned} \tag{37}$$

It follows from (37) that the solution satisfies $\mathbb{E}^B\displaystyle\int_0^T\|u(s,.)\|^2_{m,\rho}ds<\infty$.

4.3 Existence: the case of Lipschitz continuous coefficients

We prove the existence part of Theorem 5. Now we drop the smoothness assumption on the coefficients f and h and just assume Hypothesis 5.
Let $\varrho\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^N,\mathbb{R}^{N\times m};\mathbb{R}^+)$ be a mollifier with

$$\int_{\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^{N\times m}}\varrho(x,y,z)dxdydz=1\ ,$$

and for $l \in \mathbb{N}$,

$$\varrho^l(x,y,z) = l^{d\times N\times N\times m}\varrho(lx,ly,lz)$$
$$f^l(r,x,y,z) = \int_{\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^{N\times m}} f(r,x-x',y-y',z-z')\varrho(x',y',z')dx'dy'dz'$$
$$h^l(r,x,y,z) = \int_{\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^{N\times m}} h(r,x-x',y-y')\varrho(x',y',z')dx'dy'dz' \ .$$

Since f and h are globally Lipschitz continuous

$$K(l) = \|f^l - f\|_\infty + \|h^l - h\|_\infty \xrightarrow[l\to\infty]{} 0 \ . \tag{38}$$

Moreover, $\{f^l, g^l \ ; \ l \in \mathbb{N}^*\}$ satisfy $i)$, $ii)$ and $ii)$ in the definition of $\Gamma_{\bar{L},\epsilon,\varsigma}$ (see definition 1) and are bounded by $\bar{L}$ uniformly with respect to l. So one may introduce for any $l > 0$ the random field $u^l : [0,T[\times\mathbb{R}^d \times \Omega_2 \to \mathbb{R}^N$ which is the unique solution (as constructed in the previous Section) of the SPDE:

$$u^l(t,x) = \Lambda + \int_t^T \mathcal{L}u^l(s,x)ds + \int_t^T f^l(s,x,u^l(s,x),\nabla u^l(s,x)\sigma(x))ds$$
$$+ \int_t^T h^l(s,x,u^l(s,x))\overleftarrow{dB}_s \ . \tag{39}$$

We will construct the solution of 22 (with Lipschitz coefficients) as the limit of the above approximating sequence. We have the following Proposition.

Proposition 4. *Let $\{u^l \ , \ l \in \mathbb{N}^*\}$ be the solutions of SPDEs* (39)*. For any $\tau < T$, there exists a random field u such that*

$$\mathbb{E}^B \int_{\mathbb{R}^d}\int_0^\tau \left(|u^l(r,x) - u(r,x)|^2 + |\nabla u^l - \nabla u|^2\right)\rho^{-1}(x)drdx \xrightarrow[p\to\infty]{} 0 \ .$$

Proof. Let $l, l' > 0$. We denote $w^{l,l'} = u^l - u^{l'}$,

$$F^{l,l'}(r,x) = f^l(r,x,u^l,\nabla u^l\sigma) - f^{l'}(r,x,u^{l'},\nabla u^{l'}\sigma) \text{ and}$$
$$H^{l,l'}(r,x) = f^l(r,x,u^l,\nabla u^l\sigma) - f^{l'}(r,x,u^{l'},\nabla u^{l'}\sigma) \ .$$

Thanks to Bally and Matoussi (2001), the distribution valued process $(Y^{t,\cdot}_{l,l',s}, Z^{t,\cdot}_{l,l',s})_{s\in[0,T]}$ defined by

$$\left(Y^{t,\cdot}_{l,l',s},\varphi\right) := \int_{\mathbb{R}^d} w^{l,l'}(s,x)\psi_t(s,x)dx$$
$$\left(Z^{t,\cdot}_{l,l',s},\varphi\right) := \int_{\mathbb{R}^d} \nabla w^{l,l'}(s,x)\sigma(x)\psi_t(s,x)dx \ ,$$

solves in the sense of distribution

$$Y^{t,\cdot}_{l,l',s} = \int_s^T F^{l,l'}(r, X^{t,\cdot}_r)dr + \int_s^T H^{l,l'}(r, X^{t,\cdot}_r)\overleftarrow{dB}_r - \int_s^T Z^{t,\cdot}_{l,l',r}dW_r \ .$$

Since both $F^{l,l'}$, $H^{l,l'}$ are bounded (by $2\bar{L}$, uniformly in l, l'), we may write this equation pointwise:

$$Y^{t,x}_{l,l',s} = \int_s^T F^{l,l'}(r, X^{t,x}_r)dr + \int_s^T H^{l,l'}(r, X^{t,x}_r)\overleftarrow{dB}_r - \int_s^T Z^{t,x}_{l,l',r}dW_r \ .$$

We write

$$\begin{aligned} F^{l,l'}(r, X_r) &= f^l(r, X_r, u^l(r, X_r), \nabla u^l(r, X_r)\sigma(X_r)) - f^{l'}(r, X_r, u^l(r, X_r), \nabla u^l(r, X_r)\sigma(X_r)) \\ &\quad + f^{l'}(r, X_r, u^l(r, X_r), \nabla u^l(r, X_r)\sigma(X_r)) - f^{l'}(r, X_r, u^{l'}(r, X_r), \nabla u^{l'}(r, X_r)\sigma(X_r)) \ , \\ H^{l,l'}(r, X_r) &= h^l(r, X_r, u^l(r, X_r)) - h^{l'}(r, X_r, u^l(r, X_r)) \\ &\quad + h^{l'}(r, X_r, u^l(r, X_r)) - h^{l'}(r, X_r, u^{l'}(r, X_r)) \ . \end{aligned}$$

Let $\delta > 0$. By (38) we can find n_0 such that $l, l' \geq n_0$ implies

$$\begin{aligned} &|F^{l,l'}(r, X_r)| \leq \delta + \bar{L}|Y^{t,x}_{l,l',r}| + \bar{L}|Z^{t,x}_{l,l',r}| \text{ and} \\ &|H^{l,l'}(r, X_r)| \leq \delta + |Y^{t,x}_{l,l',r}| \ . \end{aligned}$$

Estimates on BDSDE yield

$$\mathbb{E}|Y^{t,x}_{l,l',s}|^2 + \mathbb{E}\int_s^T |Z^{t,x}_{l,l',r}|^2 dr \leq C\ \delta^2 + C\int_t^T \mathbb{E}|Y^{t,x}_{l,l',r}|^2 dr + \tfrac{1}{2}\int_t^T |Z^{t,x}_{l,l',r}|^2 dr.$$

Using the Gronwall's Lemma we get that for $l, l' \geq n_0$,

$$\mathbb{E}|Y^{t,x}_{l,l',s}|^2 + \mathbb{E}\int_s^T |Z^{t,x}_{l,l',r}|^2 dr \leq c\ \delta^2 \ .$$

Taking $s = t$ and integrating against ρ^{-1} we first obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathbb{E}^B|w^{l,l'}(t,x)|^2\rho^{-1}(x)dx \leq c\delta^2 \text{ and then} \\ \int_0^\tau &\int_{\mathbb{R}^d} \mathbb{E}^B|w^{l,l'}(t,x)|^2\rho^{-1}(x)dxdt \leq c\ \tau\ \delta^2 \ . \end{aligned} \tag{40}$$

Using the equivalence of norms result (see (15)) and the uniform ellipticity, we get in the same way

$$\int_{\mathbb{R}^d}\int_0^\tau \mathbb{E}^B|\nabla w^{l,l'}(r,x)|^2\rho^{-1}(x)drdx \leq c\ \int_{\mathbb{R}^d}\int_0^\tau \mathbb{E}|Z^{t,x}_{l,l',r}|^2\rho^{-1}(x)drdx \leq c\ \delta^2 \ .$$

So the sequence $(u^l(.,.))_{l\in\mathbb{N}}$ is a Cauchy sequence of $\mathbb{L}^2\big([0,\tau]\times\mathbb{R}^d\times\Omega_2;dt\otimes\rho^{-1}(x)dx\otimes dP^B\big)$ (remind that u^l belongs to this space thanks to Definition 3) hence it converges to a function u of this space. It is the same situation for the sequence $(\nabla u^l(.,.))_{l\in\mathbb{N}}$ and it is easy to check that its limit in $\mathbb{L}^2\big([0,\tau]\times\mathbb{R}^d\times\Omega_2;dt\otimes\rho^{-1}(x)dx\otimes dP^B\big)$ is ∇u and we get the result. □

Remark 7. *Thanks to* (40)*, we have for any* $\tau < T$*:*

$$\sup_{t\in[0,\tau]} \|w^{l,l'}\|_{\mathbb{L}^2(\mathbb{R}^d\times\Omega_2;\rho^{-1}(x)dx\otimes\Omega_2)} \le c\delta\ .$$

So the sequence $(u^l(.,.))_{l\in\mathbb{N}}$ *is also a Cauchy sequence of* $\mathcal{C}\big([0,\tau];\mathbb{L}^2((\mathbb{R}^d\times\Omega_2;\rho^{-1}(x)dx\otimes dP^B)\big)$*.*

Proof of the existence:
Let u constructed in Proposition 4. We may prove (24) by passing to the limit as in the proof of Theorem 6. Let us now show that $\mathbb{E}^B\|u(\tau,.)-\Lambda\|^2_{k+2,\rho}\xrightarrow[\tau\uparrow T]{}0$. We first write for any $l\in\mathbb{N}$:

$$\int_{\mathbb{R}^d} u(\tau,x)\varphi(x)dx-\Lambda(\varphi)=\int_{\mathbb{R}^d}\big(u(\tau,x)-u^l(\tau,x)\big)\varphi(x)dx+\int_{\mathbb{R}^d}u^l(\tau,x)\varphi(x)dx-\Lambda(\varphi)\ .$$

We first compute

$$\left|\int_{\mathbb{R}^d}\big(u(\tau,x)-u^l(\tau,x)\big)\varphi(x)dx\right|\le\big|u(\tau,x)-u^l(\tau,x)\big|^2_{\mathbb{L}^2(\mathbb{R}^d;\rho^{-1}(x)dx)}\ |\varphi|_{0,\rho}\ \text{so that}$$
$$\mathbb{E}^B\|u(\tau,.)-u^l(\tau,.)\|^2_{k,\rho}\le\sup_{t\in[0,\tau']}\big|u(\tau,x)-u^l(\tau,x)\big|^2_{\mathbb{L}^2(\mathbb{R}^d\times\Omega_2;\rho^{-1}(x)dx\otimes dP^B)}\ ,$$

with $T>\tau'>\tau$.
Let $\delta>0$. By Remark 7, there exits $l_0\in\mathbb{N}$ such that

$$\mathbb{E}^B\|u(\tau,.)-u^{l_0}(\tau,.)\|^2_{k,\rho}\le\delta\ . \tag{41}$$

Since $\mathbb{E}^B\|u^l(\tau,.)-\Lambda\|^2_{k,\rho}\le c(T-\tau)$, we have

$$\begin{aligned}\mathbb{E}^B\|u^l(\tau,.)-\Lambda\|^2_{k,\rho}&\le\tilde{c}_T\big(\|f^l\|_\infty+\|\Lambda\|_{k,\rho}+2\|h^l\|_\infty\big)(T-\tau)\\&=\tilde{c}_T\big(3\bar{L}+\|\Lambda\|_{k,\rho}\big)(T-\tau)\ .\end{aligned}$$

Combining this with (41) yields the result.
One can easily check that $\sup_{t\in[0,T]}\mathbb{E}^B\|u(t,.)\|^2_{k+2,\rho}<\infty$ so that $u\in\mathcal{W}_{k+2,\rho}$.

4.4 Weak formulation of BDSDEs

We recall briefly how we can establish the link between SPDEs and BDSDEs formulated in a weak sense. For more about this we refer to Bally and Matoussi (2001).
The main idea is to use the function ψ_t as a random test function. These functions are defined in (30). The problem is that $s \to \psi_t(s,.)$ is not differentiable so that $\int_t^T (u_s, \partial_s \psi_t) ds$ has no sense. Anyway, it is shown in Bally and Matoussi (2001) that we way replace $\partial_s \psi_t ds$ by the Itô integral with respect to $d\psi_t(s,x)$ thanks to the following semimartingale decomposition of $\psi_t(s,x)$.

Lemma 5. *For every function* $\varphi \in C_c^2(\mathbb{R}^d)$

$$\psi_t(s,x) = \varphi(x) - \sum_{i,j=1}^{d} \int_t^s \frac{\partial}{\partial x_i} (\psi_t(r,x)\sigma_{i,j}(x)) dW_r^j + \int_t^s L^* \psi_t(r,x) dr.$$

Then we may extend the weak formulation of the SPDEs in the following way.

Proposition 5. *If* u *is solution of* (22) *then for all function* $\varphi \in C_c^2(\mathbb{R}^d)$*, for* $t \leq \tau < T$*, the following equality holds* $P-$*a.s.*

$$\begin{aligned} \int_t^\tau \int_{\mathbb{R}^d} u(s,x) d\psi_t(s,x) dx + \int_{\mathbb{R}^d} u(t,x)\psi_t(t,x) dx &= \int_{\mathbb{R}^d} \psi_t(\tau,x) u(\tau,x) dx \\ + \int_t^\tau \int_{\mathbb{R}^d} A\big(u(s,.), \psi_t(s,.)\big) dx ds &+ \int_t^\tau \int_{\mathbb{R}^d} \psi_t(s,x) f(s,x,u,\nabla u \sigma) dx ds \\ + \int_t^\tau \int_{\mathbb{R}^d} \psi_t(s,x) h(s,x,u) dx \overleftarrow{dB}_s \ . \end{aligned}$$

Proof. The proof is the same as that of Proposition 2.3 of Bally and Matoussi (2001). □

Definition 4. *The* $\mathcal{H}'_{0,\rho} \times \mathcal{H}'_{0,\rho}$ *valued process* $(Y_s^{t,\cdot}, Z_s^{t,\cdot})_{t \leq s < \tau}$ *solves the BDSDE*

$$Y_s^{t,\cdot} = \Lambda + \int_s^T f(r, X_r^{t,x}, Y_r^{t,\cdot}, Z_r^{t,\cdot}) dr + \int_s^T h(r, X_r^{t,x}, Y_r^{t,\cdot}) \overleftarrow{dB}_r - \int_s^T Z_r^{t,\cdot} dW_r \ , \tag{42}$$

with final condition Λ *if*

i) $(Y_s^{t,\cdot}, Z_s^{t,\cdot})$ *is* $\mathcal{F}_s-$*mesurable for* $t \leq s \leq T$*.*

ii) For all $\varphi \in C_c^2(\mathbb{R}^d)$ *and for all* $t \leq \tau < T$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) Y_s^{t,x} dx &= \int_{\mathbb{R}^d} Y_\tau^{t,x} \varphi(x) dx + \int_{\mathbb{R}^d} \varphi(x) \left[\int_s^\tau f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \right] dx \\ &+ \int_{\mathbb{R}^d} \varphi(x) \left[\int_s^\tau h(r, X_r^{t,x}, Y_r^{t,x}) \overleftarrow{dB}_r \right] dx - \int_{\mathbb{R}^d} \varphi(x) \left[\int_s^\tau Z_r^{t,x} dW_r \right] dx \ . \end{aligned}$$

iii) For all $\varphi \in C_c^2(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \varphi(x) Y_\tau^{t,x} dx \xrightarrow[\tau \to T]{\mathbb{L}^2(\Omega)} \Lambda\left(\psi_t(T,x)\right)$.

Theorem 7. *Let u be the unique solution of* (22)*. The couple of processes* $(Y_s^{t,x}, Z_s^{t,x})$, $t \le s \le \tau$ *defined by* $Y_s^{t,x} = u(s, X_s^{t,x})$ *and* $Z_s^{t,x} = \nabla u(s, X_s^{t,x})\sigma(X_s^{t,x})$ *satisfies the weak BDSDE* (42).

Proof. The proof of (ii) is the same as in Bally and Matoussi (2001), Theorem 3.1. We check now $iii)$. Let $\varphi \in C_c^2(\mathbb{R}^d)$. Using a change of variable we obtain

$$\begin{aligned}
\mathbb{E}\left|\int_{\mathbb{R}^d} \varphi(x) Y_\tau^{t,x} dx - \Lambda\left(\psi_t(T,.)\right)\right|^2 &= \mathbb{E}\left|\int_{\mathbb{R}^d} \psi_t(\tau,x) u(\tau,x) dx - \Lambda\left(\psi_t(T,.)\right)\right|^2 \\
&\le c\big(I_1(\tau) + I_2(\tau)\big) \text{ with} \\
I_1(\tau) = \mathbb{E}\left|\int_{\mathbb{R}^d} u(\tau,x)\left(\psi_t(\tau,x) - \psi_t(T,x)\right) dx\right|^2 &\text{ and} \\
I_2(\tau) = \mathbb{E}\left|\big(u(\tau,.) - \Lambda, \psi_t(\tau,.)\big)\right|^2 &.
\end{aligned}$$

Since u is the unique solution of (22) it satisfies (37). So we can write

$$\begin{aligned}
I_1(\tau) &\le \mathbb{E}^B \|u(\tau,.)\|_{k+2,\rho}^2 \mathbb{E}^W \left|\psi_t(\tau,.) - \psi_t(T,.)\right|_{k+2,\rho}^2 \\
&\le \sup_{\tau \le T} \mathbb{E}^B \|u(\tau,.)\|_{k+2,\rho}^2 \mathbb{E}^W \left|\psi_t(\tau,.) - \psi_t(T,.)\right|_{k+2,\rho}^2 .
\end{aligned}$$

Since $\sup_{x \in \mathbb{R}^d} \mathbb{E}^W \left|\sum_{0 \le |\alpha| \le k} \partial_\alpha \varphi(\hat{X}_\tau^{t,x}) J(\hat{X}_\tau^{t,x}) - \partial_\alpha \varphi(\hat{X}_T^{t,x}) J(\hat{X}_T^{t,x})\right|^2 \xrightarrow[\tau \uparrow T]{} 0$, it follows that

$$\mathbb{E}^W \left|\psi_t(\tau,.) - \psi_t(T,.)\right|_{k+2,\rho}^2 \xrightarrow[\tau \uparrow T]{} 0 ,$$

and so $I_1(\tau) \to 0$ as $\tau \uparrow T$. Moreover using (31)

$$\begin{aligned}
I_2(\tau) &\le \mathbb{E}^B \|u(\tau,.) - \Lambda\|_{k+2,\rho}^2 \mathbb{E}^W \left|\psi_t(T,.)\right|_{k+2,\rho}^2 \\
&\le \mathbb{E}^B \|u(\tau,.) - \Lambda\|_{k+2,\rho}^2 \left|\varphi\right|_{k+2,\rho}^2 ,
\end{aligned}$$

and using (ii) of the definition 3 we get $I_2(\tau) \to 0$ as $\tau \uparrow T$, and $iii)$ follows. □

A Appendix: A priori estimates for a class of BDSDEs

In this Appendix we prove a priori estimates for a class of BDSDE for which the coefficients have linear growth with respect to y and z but the "constants" which control this growth are random (and not bounded). Independently of our study of SPDE, this kind of estimation are of own interest and non-standard. That is the reason why they are presented here in the following general framework. We consider a couple of adapted square integrable processes $(Y_t, Z_t)_{0\le t\le T}$ which verify the BDSDE

$$Y_t = \xi_t + \int_t^T F(s,\omega,Y_s,Z_s)ds + \int_t^T H(s,\omega,Y_s,Z_s)\overleftarrow{dB}_s - \int_t^T Z_s dW_s\ . \tag{A.1}$$

We do not bother about the existence and uniqueness of the solution of this equation: we just consider a couple of processes which verify the equality, if any such couple exists.

A.1 $\mathbb{L}^p-$estimates for BDSDEs

We consider $(Y_t, Z_t)_{0\le t\le T}$ verifying (A.1). Our hypotheses are:

- $\xi_t \in \mathbb{L}^{2p}(\Omega, \mathcal{F}_T, \mathbb{P})\ \forall\ 0 \le t \le T$. (A.2)
- There are some non-negative adapted processes a, b, d: $[0,T]\times\Omega \to \mathbb{R}^+$ and two real constants c and e such that:

$$\begin{cases} |F(s,\omega,y,z)| \le a(s,\omega) + b(s,\omega)|y| + c|z| \\ |H(s,\omega,y,z)| \le d(s,\omega) + e|y| \\ \Xi_{t,T,p} := \mathbb{E}\left(\int_t^T a(s,\omega)ds\right)^{2p} + \mathbb{E}\left(\int_t^T d(s,\omega)^2 ds\right)^p < \infty \\ \mathbb{E}\left(\int_t^T b(s,\omega)^2 ds\right)^{2p} < \infty \end{cases} \tag{A.3}$$

Lemma 6. *Assume that* (A.2) *and* (A.3) *hold for some* $p \ge 1$. *Then there exists constant* C_p *(which depends only on* p *and* c*) such that*

$$\begin{aligned} \mathbb{E}|Y_t|^{2p} + \mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^p &\le C_p\big(\|\xi_t\|_{2p}^{2p} + \Xi_{t,T,p}\big) \\ &+ C_p\left[e^{2p} + \left(\mathbb{E}\big(\int_t^T |b(s,\omega)|^2 ds\big)^{2p}\right)^{\frac12}\right]\left(\mathbb{E}\big(\int_t^T |Y_s|^2 ds\big)^{2p}\right)^{\frac12}\ . \end{aligned} \tag{A.4}$$

Moreover if the non-negative adapted process $b(s,\omega)$ *is bounded by a constant* $\bar{b}$, *then*

$$\mathbb{E}|Y_t|^{2p} + \mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^p \le M_{t,T,p} \tag{A.5}$$

$$\textit{with}\quad M_{t,T,p} = C_p\left(\|\xi_t\|_{2p}^{2p} + \mathbb{E}\left(\int_t^T a(s,\omega)ds\right)^{2p} + \mathbb{E}\left(\int_t^T d(s,\omega)^2 ds\right)^p\right) e^{T^p C_p\left(\bar{b}^{2p} + e^{2p}\right)}\ .$$

The important point in (A.4) is that we may control the "strong" norm $\sup_{t\in[0,T]} \mathbb{E}|Y_t|^{2p}$ by the "weak" norm $(\mathbb{E}|\int_0^T |Y_s|^2 ds|^{2p})^{\frac12}$. Note anyway that we are obliged to take a higher power inside the expectation in the "weak" norm and so we may not use the Gronwall's Lemma in order to eliminate Y in the right hand side of (A.4). Note also that (A.4) is void of reason if $\mathbb{E}(\int_0^T |Y_s|^2 ds)^{2p} = +\infty$. But in our context, we are able to prove that this quantity is finite for reasons which do not appear in a general framework.

Proof. Here C_p is the generic notation for a constant which depends on p and may vary from line to line. The dependence of c of will be pointed out hereafter.
Proof of (A.4)**:** We fix $t \leq T$ and for $s \in]t,T]$ we define $\Phi(s) := Y_t + \int_t^s Z_r dW_r$. Using Itô's formula we get

$$\mathbb{E}|\Phi(T)|^{2p} = \mathbb{E}|Y_t|^{2p} + 2p(2p-1)\mathbb{E}\int_t^T |\Phi(r)|^{2p-2}|Z_r|^2 dr \geq \mathbb{E}|Y_t|^{2p} .$$

We write the equation (A.1) in the form

$$\Phi(T) = Y_t + \int_t^T Z_s dW_s = \xi_t + \int_t^T F(s,\omega,Y_s,Z_s)ds + \int_t^T H(s,\omega,Y_s,Z_s)\overleftarrow{dB}_s .$$

Then $\mathbb{E}|Y_t|^{2p} \leq \mathbb{E}|\Phi(T)|^{2p}$

$$\leq C_p\ \mathbb{E}|\xi_t|^{2p} + C_p\ \mathbb{E}\left(\int_t^T |F(s,\omega,Y_s,Z_s)|ds\right)^{2p} + C_p\ \mathbb{E}\left(\int_t^T |H(s,\omega,Y_s,Z_s)|^2 ds\right)^{p},$$

and moreover, by (A.3) we get

$$\begin{aligned}
\mathbb{E}|Y_t|^{2p} &\leq C_p\ \mathbb{E}|\xi_t|^{2p} + C_p\ \mathbb{E}\left(\int_t^T \big(a(s,\omega) + b(s,\omega)|Y_s| + c|Z_s|\big)ds\right)^{2p} \\
&\quad + C_p\ \mathbb{E}\left(\int_t^T \big(d(s,\omega) + e|Y_s|\big)^2 ds\right)^{p} \\
&\leq C_p\ \mathbb{E}|\xi_t|^{2p} + C_p\ \Xi_{t,T,p} + C_p\ \mathbb{E}\left(\int_t^T b(s,\omega)|Y_s|ds\right)^{2p} \\
&\quad + C_p\ \ c^{2p}\ (T-t)^p\ \mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^{p} + C_p\ \ e^{2p}\mathbb{E}\left(\int_t^T |Y_s|^2 ds\right)^{p} .
\end{aligned} \tag{A.6}$$

Using the Cauchy-Schwartz inequality we have

$$\mathbb{E}\left(\int_t^T b(s,\omega)|Y_s|ds\right)^{2p} \leq \left(\mathbb{E}\left(\int_t^T b(s,\omega)^2 ds\right)^{2p}\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_t^T |Y_s|^2 ds\right)^{2p}\right)^{\frac{1}{2}} . \tag{A.7}$$

We denote

$$\Gamma_p := C_p\big(\|\xi_t\|_{2p}^{2p} + \Xi_{t,T,p}\big) + C_p\left[e^{2p} + \Big(\mathbb{E}\big(\int_t^T |b(s,\omega)|^2 ds\big)^{2p}\Big)^{\frac{1}{2}}\right]\Big(\mathbb{E}\big(\int_t^T |Y_s|^2 ds\big)^{2p}\Big)^{\frac{1}{2}} .$$

Reporting (A.7) into (A.6), we get

$$\mathbb{E}|Y_t|^{2p} \leq \Gamma_p + C_p\, c^{2p}\, (T-t)^p \mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^p . \tag{A.8}$$

Using Burkholder's inequality and (A.1), we write

$$\begin{aligned}\mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^p &\leq C_p\, \mathbb{E}\left|\int_t^T Z_s dW_s\right|^{2p} \\ &= C_p\, \mathbb{E}\left|\xi_t - Y_t + \int_t^T F(s,\omega,Y_s,Z_s)ds + \int_t^T H(s,\omega,Y_s,Z_s)\overleftarrow{dB}_s\right|^{2p} .\end{aligned}$$

The same computations as before yields

$$\mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^p \leq C_p\,\Gamma_p + C_p\, c^{2p}\,(T-t)^p \mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^p .$$

Then we choose $T-t$ small enough in order to have $C_p\, c^{2p}\,(T-t)^p \leq \frac{1}{2}$ and we obtain

$$\mathbb{E}\left(\int_t^T |Z_s|^2 ds\right)^p \leq C_p\,\Gamma_p .$$

We report the above estimation into (A.8) and we obtain $\mathbb{E}|Y_t|^{2p} \leq C_p\,\Gamma_p$, i.e. (A.4) for $T-t$ small. Extending this result to any arbitrary time interval implies that C_p depends on c.

Proof of (A.5)**:**
Assume now that the process b is bounded by $\bar{b}$. Then we do the same reasoning as above except for the fact that in (A.7) we do no more use of Schwartz's inequality but we dominate directly

$$\mathbb{E}\left|\int_t^T b(s,\omega)Y_s ds\right|^{2p} \leq \bar{b}^{2p}\,(T-t)^p \mathbb{E}\left(\int_t^T |Y_s|^2 ds\right)^p .$$

The same computations as above permit to eliminate the term in Z and we obtain

$$\begin{aligned}\mathbb{E}|Y_t|^{2p} &\leq C_p\big(\|\xi_t\|_{2p}^{2p} + \mathbb{E}\big(\int_t^T a(s,\omega)ds\big)^{2p} + \mathbb{E}\big(\int_t^T d(s,\omega)^2 ds\big)^p\big) \\ &\qquad + C_p(e^{2p}+\bar{b}^{2p})\mathbb{E}\Big(\int_t^T |Y_s|^2 ds\Big)^p \\ &\leq C_p\big(\|\xi_t\|_{2p}^{2p} + \mathbb{E}\big(\int_t^T a(s,\omega)ds\big)^{2p} + \mathbb{E}\big(\int_t^T d(s,\omega)^2 ds\big)^p\big) \\ &\qquad + C_p\, T^{p-1}(e^{2p}+\bar{b}^{2p})\mathbb{E}\int_t^T |Y_s|^{2p} ds.\end{aligned}$$

So we may use the Gronwall's Lemma in order to get (A.5). □

A.2 A priori estimates for BDSDEs far from the final time

Note that in the evaluation (A.4) and (A.5), $\|\xi_t\|_{2p}^{2p}$ appears. As we can see in the previous sections, we use these evaluations in the case when ξ_T is a distribution and this is the delicate point. The following lemma permits us to get round this difficulty. We prove that, at least when we are far from T, we obtain evaluations of the moments of Y and Z which are independent of the final condition.

Lemma 7. *Assume* (A.2) *and* (A.3) *for some* $p \geq 1$. *Then for any* $0 \leq t < \tau < T$

$$\mathbb{E}|Y_t|^{2p} + \mathbb{E}\left(\int_t^\tau \tfrac{\tau-s}{\tau-t}|Z_s|^2 ds\right)^p \leq C_p \Xi_{t,\tau,p} + C_p\Big[e^{2p} + \tfrac{1}{(\tau-t)^p} + \Big(\mathbb{E}\big(\textstyle\int_t^\tau |b(s,\omega)|^2 ds\big)^{2p}\Big)^{\frac{1}{2}}\Big]\Big(\mathbb{E}\big(\textstyle\int_t^\tau |Y_s|^2 ds\big)^{2p}\Big)^{\frac{1}{2}}. \tag{A.9}$$

In particular for any $\delta > 0$ *and* $t \leq T - \delta$

$$\mathbb{E}|Y_t|^{2p} + \mathbb{E}\left(\int_t^{T-\delta} |Z_s|^2 ds\right)^p \leq C_p \Xi_{t,T-\delta/2,p} + C_p\Big[e^{2p} + \tfrac{2^p}{\delta^p} + \Big(\mathbb{E}\big(\textstyle\int_t^{T-\delta/2} |b(s,\omega)|^2 ds\big)^{2p}\Big)^{\frac{1}{2}}\Big]\Big(\mathbb{E}\big(\textstyle\int_t^{T-\delta/2} |Y_s|^2 ds\big)^{2p}\Big)^{\frac{1}{2}}. \tag{A.10}$$

Proof. We take $t < s < u \leq \tau$ and we write the BDSDE (A.1) between s and u

$$Y_s = Y_u + \int_s^u F(r,\omega,Y_r,Z_r)dr + \int_s^u H(r,\omega,Y_r,Z_r)\overleftarrow{dB}_r - \int_s^u Z_r dW_r\ .$$

We will integrate between s and τ with respect to u in the above equation. For this we use the stochastic Fubini's Theorem (see Protter (1995)) and we obtain

$$\frac{1}{\tau-s}\int_s^\tau \left(\int_s^u Z_r dW_r\right) du = \frac{1}{\tau-s}\int_s^\tau \left(\int_r^\tau Z_r du\right) dW_r = \int_s^\tau Z_r \frac{\tau-r}{\tau-s} dW_r\ .$$

In the same way, we apply Fubini's Theorem to the other terms and get

$$Y_s = \tfrac{1}{\tau-s}\int_s^\tau Y_u du + \int_s^\tau \tfrac{\tau-r}{\tau-s} F(r,\omega,Y_r,Z_r)dr + \int_s^\tau \tfrac{\tau-r}{\tau-s} H(r,\omega,Y_r,Z_r)\overleftarrow{dB}_r - \int_s^\tau \tfrac{\tau-r}{\tau-s} Z_r dW_r\ . \tag{A.11}$$

Now on s and τ are fixed. We denote

$$\tilde{Y}_r := \tfrac{\tau-r}{\tau-s} Y_r\ ,\ \tilde{Z}_r := \tfrac{\tau-r}{\tau-s} Z_r\ ,\ \tilde{\xi}_s := \tfrac{1}{\tau-s}\int_s^\tau Y_u du\ ,$$

$$\tilde{F}_s(r,\omega,y,z) := \tfrac{\tau-r}{\tau-s} F(r,\omega,\tfrac{\tau-s}{\tau-r}y,\tfrac{\tau-s}{\tau-r}z)\ ,\ \tilde{H}_s(r,\omega,y,z) := \tfrac{\tau-r}{\tau-s} H(r,\omega,\tfrac{\tau-s}{\tau-r}y,\tfrac{\tau-s}{\tau-r}z)\ .$$

With this notations (A.11) becomes (note that $\tilde{Y}_s = Y_s$)

$$\tilde{Y}_s = \tilde{\xi}_s + \int_s^\tau \tilde{F}_s(r,\omega,\tilde{Y}_r,\tilde{Z}_r)dr + \int_s^\tau \tilde{H}_s(r,\omega,\tilde{Y}_r,\tilde{Z}_r)\overleftarrow{dB}_r - \int_s^\tau \tilde{Z}_r dW_r \ . \tag{A.12}$$

We moreover note that

$$\begin{aligned} |\tilde{F}_s(r,\omega,\tilde{Y}_r,\tilde{Z}_r)| &\le a(r,\omega) + b(r,\omega)|\tilde{Y}_r| + c|\tilde{Z}_r| \ , \\ |\tilde{H}_s(r,\omega,\tilde{Y}_r,\tilde{Z}_r)| &\le d(r,\omega) + e|\tilde{Y}_r| \text{ and} \\ \mathbb{E}|\tilde{\xi}_s|^{2p} &\le \tfrac{1}{(\tau-s)^p}\left(\mathbb{E}\int_s^\tau |Y_u|^2 du\right)^p \ . \end{aligned}$$

Since the evaluations which lead to (A.4) are uniform with respect to the time variable, the same calculus for the equation (A.12) give (A.9).
Fix now $\delta > 0$ and take $\tau = T - \delta/2$. Since $\frac{\tau - r}{\tau - t} \ge \delta/2$ for $t \le r \le T - \delta$, we obtain (A.10). □

A.3 Dependence of the solution of BDSDEs on a parameter

In this section we consider a couple of square integrable adapted processes $(Y_t^\theta, Z_t^\theta)_{0\le t\le T}$ which solves

$$Y_t^\theta = \xi^\theta + \int_t^{T_\theta} F(s,\omega,\theta,Y_s^\theta,Z_s^\theta)ds + \int_t^{T_\theta} H(s,\omega,\theta,Y_s^\theta)\overleftarrow{dB}_s - \int_t^{T_\theta} Z_s^\theta dW_s \ . \tag{A.13}$$

Here θ is a parameter with values in a metric space (Θ, d) and $\theta \mapsto T_\theta$ is a deterministic function from Θ to $[0,T]$. We assume

(i) $\sup_{\theta\in\Theta} \mathbb{E}|\xi^\theta|^{2p} < \infty$.

(ii) There are some adapted processes a and d defined on $[0,T]\times\Omega\times\Theta$ and b, c and e defined on $[0,T]\times\Omega$ such that:

$$\begin{cases} F(s,\omega,\theta,y,z) = a(s,\omega,\theta) + b(s,\omega)y + c(s,\omega)z \\ H(s,\omega,\theta,y) = d(s,\omega,\theta) + e(s,\omega)y \\ \Xi_{t,T,p} := \sup_{\theta\in\Theta}\sup_{s\in[0,T]} \mathbb{E}\left(|a(s,\omega,\theta)|^{4p} + |d(s,\omega,\theta)|^{4p}\right) < \infty \\ \sup_\omega \sup_{s\in[0,T]}\left(|b(s,\omega)| + |c(s,\omega)| + |e(s,\omega)|\right) \ = M < \infty \end{cases}$$

(iii) $\mathbb{E}\Big(\int_0^T a(s,\omega,\theta) - a(s,\omega,\theta')ds\Big)^{2p} + \mathbb{E}\Big(\int_0^T |d(s,\omega,\theta) - d(s,\omega,\theta')|^2 ds\Big)^p \le C d(\theta,\theta')^p$

(iv) $\mathbb{E}|\xi^\theta - \xi^{\theta'}|^{2p} \le C d(\theta,\theta')^p$.

Notice that the coefficient H does not depend on the variable z anymore.

Lemma 8. *Assume that $(i), \ldots, (iv)$ hold for some p and assume also that*

$$(v) \quad \sup_{\theta\in\Theta}\sup_{t\le T}\left(\mathbb{E}|Y_t^\theta|^{4p} + \mathbb{E}|Z_t^\theta|^{4p}\right) \le K < \infty \ .$$

Then there exists a constant K_p which depends on p, K, M such that

$$\mathbb{E}|Y_t^\theta - Y_t^{\theta'}|^{2p} + \mathbb{E}\left(\int_t^{T_\theta}|Z_s^\theta - Z_s^{\theta'}|^2 ds\right)^p \le K_p(d(\theta,\theta')^p + |T_\theta - T_{\theta'}|^p) \ . \tag{A.14}$$

Proof. Let $\theta, \theta' \in \Theta$. We suppose $T_\theta \le T_{\theta'}$ and we denote

$$\Upsilon(\theta,\theta') = \xi^\theta - \xi^{\theta'} - \int_{T_\theta}^{T_{\theta'}} F(s, Y_s^{\theta'}, Z_s^{\theta'})ds - \int_{T_\theta}^{T_{\theta'}} H(s, Y_s^{\theta'})\overleftarrow{dB}_s + \int_{T_\theta}^{T_{\theta'}} Z_s^{\theta'} dW_s \ .$$

Thanks to (ii), (iv) and (v) it holds

$$\mathbb{E}|\Upsilon(\theta,\theta')|^{2p} \le C(d(\theta,\theta')^p + |T_\theta - T_{\theta'}|^p) \ .$$

We write

$$\begin{aligned}
Y_t^\theta - Y_t^{\theta'} &= \Upsilon(\theta,\theta') + \int_t^{T_\theta} \big(a(s,\theta) - a(s,\theta')\big) + b(s)(Y_s^\theta - Y_s^{\theta'}) + c(s)(Z_s^\theta - Z_s^{\theta'})\big)ds \\
&\quad + \int_t^{T_\theta} \big(d(s,\theta) - d(s,\theta')\big) + e(s)(Y_s^\theta - Y_s^{\theta'})\big)\overleftarrow{dB}_s - \int_t^{T_\theta} Z_s^\theta - Z_s^{\theta'} dW_s \ , \\
&= \Upsilon(\theta,\theta') \\
&\quad + \int_t^T \mathbf{1}_{[t,T_\theta]}(s)\big((a(s,\theta) - a(s,\theta')) + b(s)(Y_s^\theta - Y_s^{\theta'}) + c(s)(Z_s^\theta - Z_s^{\theta'})\big)ds \\
&\quad + \int_t^T \mathbf{1}_{[t,T_\theta]}(s)\big((d(s,\theta) - d(s,\theta')) + e(s)(Y_s^\theta - Y_s^{\theta'})\big)\overleftarrow{dB}_s \\
&\quad - \int_t^T \mathbf{1}_{[t,T_\theta]}(s)(Z_s^\theta - Z_s^{\theta'})dW_s \ .
\end{aligned}$$

Using (A.5) first and (iii) we get

$$\mathbb{E}|Y_t^\theta - Y_t^{\theta'}|^{2p} + \mathbb{E}\left(\int_t^{T_\theta}|Z_s^\theta - Z_s^{\theta'}|^2 ds\right)^p \le C_p\left(\|\Upsilon(\theta,\theta')\|_{2p}^{2p} + C\ d(\theta,\theta')^p\right)e^{2M^{2p}T^pC_p} \ ,$$

and (A.14) follows. □

B Appendix: Proof of Lemma 3

The aim of this section is to prove Lemma 3, i.e. to evaluate the $\mathbb{L}^p-$norms of the Malliavin derivatives of $Y_t^{(1)}$. There will be two specific difficulties in doing it. The first one is that we want to obtain bounds which are independent of the final condition (which blows up in our frame). The second one is that the coefficients of the equations of $D_\theta Y_t^{(1)}$ (see (B.7) below) have not linear growth but quadratic growth and so standard evaluations does not work. We shall point out these difficulties in a precise way through our proof. The main tools are the lemmas proved in the Appendix A.

We first notice that thanks to (10) and (12), for any $\tau < T$, there exists a constant $C_p \in \mathcal{C}_\tau$ such that

$$\sup_{t\le s\le\tau} \mathbb{E}|Y_s^{t,x}|^{2p} + \mathbb{E}\left(\int_t^\tau |Z_r^{t,x}|^2 dr\right)^p \le C_p \ , \ t \le \tau \ , \ x \in \mathbb{R}^d \tag{B.1}$$

$$\sup_{t\le s\le\tau} \mathbb{E}|D_\theta Y_s^{t,x}|^{2p} + \mathbb{E}\left(\int_t^\tau |D_\theta Z_r^{t,x}|^2 dr\right)^p \le C_p \ , \ t \le \theta \le \tau \ , \ x \in \mathbb{R}^d \ . \tag{B.2}$$

Now we prove analogous estimations for the process $(Y^{(1)}, Z^{(1)})$.

Step 1: We prove that for any $\tau < T$ there exists $C_p \in \mathcal{C}_\tau$ such that

$$\sup_{t\le s\le\tau} \mathbb{E}|Y_s^{(1),t,x}|^{2p} + \mathbb{E}\left(\int_t^\tau |Z_r^{(1),t,x}|^2 dr\right)^p \le C_p \ , \ t \le \tau \ , \ x \in \mathbb{R}^d \ . \tag{B.3}$$

Since $Z_s^{t,x} = Y_s^{(1),t,x}\sigma(X_s^{t,x})$, we get that $|Y_s^{(1),t,x}| \le (\sup_{x\in\mathbb{R}^d}|\sigma(x)|)|Z_s^{t,x}|$ and using (B.1) with $\tau_0 := (T+\tau)/2$ we obtain

$$\mathbb{E}\left(\int_t^{\tau_0} |Y_r^{(1),t,x}|^2 dr\right)^p \le C \in \mathcal{C}_{\tau_0} \ . \tag{B.4}$$

We recall now the BDSDE (13) satisfied by the couple $(Y_r^{(1)}, Z_r^{(1)})$ (in order to relieve the reader of the notation we forget the superscripts t, x):

$$\begin{aligned} Y_s^{(1)} = \nabla g(X_T^t) - \int_s^T Z_r^{(1)} dW_r + \int_s^T \big[f_x(r, X_r, Y_r, Y_r^{(1)}) + F^1(r, X_r, Y_r, Y_r^{(1)}) Y_r^{(1)} \\ + F^2(r, X_r, Y_r, Y_r^{(1)}) Z_r^{(1)} \big] dr + \int_s^T h_x(r, X_r, Y_r) + h_y(r, X_r, Y_r) Y_r^{(1)} \overleftarrow{dB}_r \end{aligned} \tag{B.5}$$

where $F^1(r, X_r, Y_r, Y_r^{(1)}) = \nabla b(X_r) + f_y(r, X_r, Y_r, Y_r^{(1)}) + f_z(r, X_r, Y_r, Y_r^{(1)})\sigma(X_r)$

and $F^2(r, X_r, Y_r, Y_r^{(1)}) = \nabla\sigma^*(X_r) + f_z(r, X_r, Y_r, Y_r^{(1)})$.

Estimations like (B.3) for the BDSDE (B.5) are standard but we point out the fact that in classical computations the final condition is involved and this is unsatisfactory in our frame. So we use the Lemma 7 from the previous Appendix. The BDSDE (B.5) is linear in $Y^{(1)}$

and $Z^{(1)}$ with bounded coefficients. So the hypothesis (A.3) is verified. By (A.10) we have for any $\delta > 0$ and $t \le \tau_0 - \delta$:

$$\mathbb{E}|Y_t^{(1)}|^{2p} + \mathbb{E}\left(\int_t^{\tau_0-\delta} |Z_s^{(1)}|^2 ds\right)^p \le C\Big(\mathbb{E}\left(\int_t^{\tau_0-\delta/2} |Y_r^{(1)}|^2 dr\right)^p\Big)^{1/2} ,$$

and by (B.4), the above quantity is bounded. We choose $\delta = (T-\tau)/2$ and (B.3) is proved.
Step 2: We prove that for any $\tau < T$ there exists $C_p \in \mathcal{C}_\tau$ such that

$$\sup_{t\le s\le \tau} \mathbb{E}|D_\theta Y_s^{(1),t,x}|^{2p} + \mathbb{E}\left(\int_t^{\tau} |D_\theta Z_r^{(1),t,x}|^2 dr\right)^p \le C_p \ , \ t \le \theta \le \tau \ , \ x \in \mathbb{R}^d \ . \tag{B.6}$$

It is clear that if (B.6) is true, we obtain in particular (14) of Lemma 3. We first write the equation satisfied by the Malliavin's derivatives of the couple $(Y^{(1)}, Z^{(1)})$.

$$\begin{aligned} D_\theta Y_s^{(1)} &= \varpi(\theta) - \int_s^\theta D_\theta Z_r^{(1)} dW_r \\ &+ \int_s^\theta \big(\Upsilon(\theta)(r) + \Theta(r, X_r, Y_r, Y_r^{(1)}, Z_r^{(1)}) D_\theta Y_r^{(1)} + F^2(r, X_r, Y_r, Y_r^{(1)}) D_\theta Z_r^{(1)}\big) dr \\ &+ \int_s^\theta \Psi(\theta)(r) + \partial_y h(r, X_r, Y_r) D_\theta Y_r^{(1)} \overleftarrow{dB}_r \ , \end{aligned} \tag{B.7}$$

where we denote:

$$\begin{aligned} \varpi(\theta) &= h_x(\theta, X_\theta, Y_\theta) + h_y(\theta, X_\theta, Y_\theta) Y_\theta^{(1)} \\ \Upsilon(\theta)(r) &= f_{xy}(r, X_r, Y_r, Y_r^{(1)}) D_\theta Y_r + F_y^1(r, X_r, Y_r, Y_r^{(1)}) Y_r^1 D_\theta Y_r \\ &\qquad + F_y^2(r, X_r, Y_r, Y_r^{(1)}) Z_r^{(1)} D_\theta Y_r \\ \Psi(\theta)(r) &= h_{xy}(r, X_r, Y_r) D_\theta Y_r + h_{yy}(r, X_r, Y_r)\big(D_\theta Y_r\big) Y_r^{(1)} \\ \Theta(r, X_r, Y_r, Y_r^{(1)}, Z_r^{(1)}) &= f_{xy}(r, X_r, Y_r, Y_r^{(1)}) + F^1(r, X_r, Y_r, Y_r^{(1)}) + F_y^1(r, X_r, Y_r, Y_r^{(1)}) \\ &\qquad + F_y^2(r, X_r, Y_r, Y_r^{(1)}) Z_r^{(1)} \ . \end{aligned}$$

The important fact in this equation is that Θ is not bounded, so, in order to prove (B.6), we have to apply (A.4) from Lemma 6. Let us check (A.2). In $\xi \triangleq \varpi(\theta)$ is involved only $Y^{(1)}$. Then (A.2) follows from (B.3). Using (B.2) and (B.3) we obtain that $\mathbb{E}(\int_t^\tau |\Psi(\theta)(r)|^2)^p \le C$, hence (A.3) is fulfilled.
We check now that for all θ, $\mathbb{E}(\int_t^\tau \Upsilon(\theta)(r) dr)^{2p} \le C$. Let us detail the most difficult term:

$$\begin{aligned} &\mathbb{E}\big(\textstyle\int_t^\tau F_y^2(r, X_r, Y_r, Y_r^{(1)}) Z_r^{(1)} D_\theta Y_r dr\big)^{2p} \\ &\qquad \le \mathbb{E}\big(\textstyle\int_t^\tau |F_y^2(r, X_r, Y_r, Y_r^{(1)}) D_\theta Y_r|^2 dr\big)^p \big(\textstyle\int_t^\tau |Z_r^{(1)}|^2 dr\big)^p \\ &\qquad \le \Big(\mathbb{E}\big(\textstyle\int_t^\tau |F_y^2(r, X_r, Y_r, Y_r^{(1)}) D_\theta Y_r|^2 dr\big)^{2p} \mathbb{E}\big(\textstyle\int_t^\tau |Z_r^{(1)}|^2 dr\big)^{2p}\Big)^{1/2} . \end{aligned}$$

We obtain the expected estimations using the boundedness of F^2, (B.2) and (B.3).
It remains to evaluate Θ. Clearly Θ is not bounded and that is why we obtain (A.4) but not (A.5). As a consequence of (B.3) we have

$$\mathbb{E}\big(\int_t^\tau |\Theta(r, X_r, Y_r, Z_r^{(1)}, Z_r^{(1)}|^2 dr\big)^{2p} \leq C_p \, \|\nabla^2 f + \nabla f + \nabla b + \sigma\|_\infty \, \mathbb{E}(\int_t^\tau |Z_r^{(1)}|^2 dr)^{2p} \leq C \; .$$

So the hypotheses of Lemma 6 are satisfied and we may use (A.4) and we get

$$\sup_{t \leq s \leq \tau} \mathbb{E}|D_\theta Y_s^{(1)}|^{2p} + \mathbb{E}\left(\int_t^\tau |D_\theta Z_r^{(1)}|^2 dr\right)^p \leq C_p \left(\mathbb{E}(\int_t^\tau |D_\theta Y_r^{(1)}|^2 dr)^{2p}\right)^{1/2} .$$

Since $D_\theta Z_s^{t,x} = D_\theta Y_s^{(1),t,x} \sigma(X_s^{t,x})$ and σ is uniformly elliptic, we get that $|D_\theta Y_s^{(1),t,x}| \leq (1/\epsilon)|D_\theta Z_s^{t,x}|$ (remind that ϵ is the uniform ellipticity constant of σ). We use moreover (B.2) and this yields

$$\mathbb{E}\left(\int_t^\tau |D_\theta Y_r^{(1),t,x}|^2 dr\right)^p \leq C_p \; , \; \forall t \; \leq \theta \leq \tau \; ,$$

and (B.6) is proved.

C Appendix: Proof of Lemma 4

Proof. The result will follows from the Lemma 8 from the Appendix A with

$$a(r,\omega) \triangleq 0 \; , \qquad b(r,\omega) \triangleq f_y(X_r, Y_r, Z_r) \; , \qquad c(r,\omega) \triangleq f_z(X_r, Y_r, Z_r) \; ,$$
$$d(r,\omega) \triangleq 0 \; , \qquad e(r,\omega) \triangleq h_y(X_r, Y_r) \; .$$

Then the equation (9) coincides with the equation (A.13). We check the hypotheses of Lemma 8. Since h is bounded, (i) holds true. Moreover, b, c, and e are bounded and it only remains to verify (iv) and (v).
It is well known that under our assumptions, we can find a constant C_p such that

$$\mathbb{E}|X_s - X_{s'}|^{2p} \leq C_p |s - s'|^p \quad \forall s, s' \in [0,\tau] \; ,$$

and using classical estimation (see Pardoux and Peng (1994)), there exists $C_p \in \mathcal{C}_\tau$ such that

$$\mathbb{E}|Y_s - Y_{s'}|^{2p} + \mathbb{E}\left(\int_0^\tau |Z_r - Z_{r'}|^2 dr\right)^p \leq C_p |s - s'|^p \; \forall s, s' \in [0,\tau] \; ,$$

so (iv) holds true.
Let us check (v). The first part of this condition, namely

$$\sup_{\theta \in [t,T]} \sup_{s \leq \tau} \mathbb{E}|D_\theta Y_s^{t,x}|^{4p} \leq K$$

comes from the estimation (12). Now we prove that

$$\sup_{\theta\in[t,T]}\sup_{s\le\tau}\mathbb{E}|D_\theta Z_s^{t,x}|^{4p}\le K\ .$$

Since $Z_s^{t,x}=\nabla u(s,X_s^{t,x})\sigma(X_s^{t,x})$ and σ is bounded, this estimation will be true as soon as

$$\sup_{\theta\in[t,T]}\sup_{s\le\tau}\mathbb{E}|D_\theta\nabla u(s,X_s^{t,x})|^{4p}\le K\ ,\quad \text{or equivalently}$$

$$\sup_{\theta\in[t,T]}\sup_{s\le\tau}\mathbb{E}|D_\theta Y_s^{(1),t,x}|^{4p}\le K\ ,$$

which is exactly the estimation (14) of Lemma 3. Hence all the hypotheses of Lemma 8 and the conclusion of this Lemma yields the expected result. □

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