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Thème 4 — Simulation et optimisation
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Abstract: We show in this report that any system of Linear Matrix Inequalities depending continuously upon scalar parameters and solvable for any value of the latter in a fixed compact set, admits a branch of solutions *polynomial* with respect to the parameters. This result is useful for studying e.g. parametric robustness or gain-scheduling issues.

Key-words: Linear matrix inequalities, regularity of the solutions

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Un résultat d'existence de solutions polynômiales pour les inégalités linéaires matricielles dépendant de paramètres

Résumé : Nous montrons dans ce rapport que tout système d'inégalités linéaires matricielles dépendant continûment de paramètres scalaires et soluble pour toute valeur de ceux-ci dans un ensemble compact fixé, admet une branche de solutions *polynômiales* par rapport à ces paramètres. Ce résultat est utile pour l'étude par exemple de problèmes de robustesse paramétrique ou de séquencement de gains.

Mots-clés : Inégalités linéaires matricielles, régularité des solutions

1 Introduction

Linear Matrix Inequalities (LMIs) have become a powerful unifying framework for expressing and solving many problems in control theory. This class of convex optimization problems, solved by efficient interior-point methods, has spread widely, in particular since the publication in 1994 of the by now classical monograph by S. Boyd *et al.* [4]. Among other, stability, stabilizability, detectability, H^2 and H^∞ performance analysis, and various related design issues may be stated as LMIs, see e.g. recent progress in [8].

The next important step was to introduce parameter-dependent LMIs, see e.g. the exposition in [1] and the references therein. The latter appear naturally when studying control techniques robust against parametric uncertainty, or gain-scheduling methods, as these issues amount to check solvability of LMIs obtained for different values of some parameters. Generally, these parameters are a priori unknown but assumed to be inside a certain prespecified bounded set, and an attempt to extend the use of LMI solvers to these problems immediatly reaches an obstacle: the impossibility to check an infinite (usually uncountable) number of independent LMI conditions.

An early way to circumvent this difficulty has been to assume prescribed, simple, dependency of the LMI solutions with respect to the parameters, see references in [3] in the context of robust stability analysis. According to the admissible parameter set, in the case of constant or affine dependency, but also for general polynomial dependency [3], the coefficients of the involved polynomials may be found (for given value of the degree) as solutions of standard LMIs.

However, very few results exist, guaranteeing existence of solutions with prespecified dependency, to parameter-dependent LMIs. In other words, nothing is known in general on the conservatism of this simplifying assumption, and thus of the derived approaches. For the Lyapunov inequality $P = P^T > 0$, $A^TP + PA < 0$, for given parameter-dependent matrix A, use of the integral form of the Lyapunov equation $A^TP + PA = -I$ permits to show existence of a polynomial solution, in domains where A is analytical and Hurwitz.

In [6, Lemma 1.1], Delchamps established an analyticity result, which permits to conclude that the LMIs which may be transformed into Riccati inequality by Schur transformation possess, if they are solvable, solutions polynomial with respect to their coefficients. Stated initially in the real case, this result is extended to the complex case in [9, Chapter 4, Lemma p. 134].

Last in [5, Proposition 2.1], Y.-S. Chou *et al.* provided a result ensuring existence of polynomial solution, for LMIs depending upon one complex parameter lying on the unit circle.

In the present short note, we provide a general result, Theorem 1, ensuring existence of polynomial solution to any solvable LMI depending continuously upon scalar parameters lying in a fixed compact set. Direct consequences are stated in Corollaries 2, 3. Applications of these results will be developed in further contributions.

2 Main result

We consider in the sequel the following property:

$$\exists x \in \mathbb{R}^p,\ G(x,\delta) \stackrel{\text{def}}{=} G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) > 0\ . \tag{1}$$

Here, $G_0, G_1, \ldots, G_p$ are mappings defined in a *compact* subset K of $\mathbb{R}^m$, and taking values in the set of symmetric matrices of $\mathbb{R}^{n\times n}$. Formula (1) is to be seen as a *parameter-dependent LMI*, depending upon the parameter $\delta \in K$, and with unknown x. It represents indeed the general form of a LMI depending upon scalar parameters.

The aim of the present note is to establish the following result.

Theorem 1. *Suppose $G_0, G_1, \ldots, G_p$ are continuous. If, for any $\delta \in K$, there exists $x(\delta) \in \mathbb{R}^p$ such that $G(x(\delta), \delta) > 0$, then there exists a* polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for any $\delta \in K$, $G(x^*(\delta), \delta) > 0$.* ■

The following result, on non-strict inequalities, is a straightforward consequence of Theorem 1.

Corollary 2. *Suppose $G_0, G_1, \ldots, G_p$ are continuous. Let E be a continuous function, mapping K into the set of positive definite symmetric matrices in $\mathbb{R}^{n\times n}$. If, for any $\delta \in K$, there exists $x(\delta) \in \mathbb{R}^p$ such that $G(x(\delta),\delta) \geq 0$, then there exists a* polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for any $\delta \in K$, $G(x^*(\delta),\delta) > -E(\delta)$.* ■

To prove Corollary 2, apply Theorem 1 to the LMI:

$$\exists x \in \mathbb{R}^p, G(x,\delta) + E(\delta) > 0 \ .$$

Another direct consequence of Theorem 1 gives more informations on the structure of the solution set of the parameter-dependent LMI $G(x,\delta) > 0$.

Corollary 3. *Suppose $G_0, G_1, \ldots, G_p$ are continuous. Let $\underline{E}, \overline{E}$ be continuous functions, mapping K into the set of symmetric matrices in $\mathbb{R}^{n\times n}$. If, for any $\delta \in K$, there exists $x(\delta) \in \mathbb{R}^p$ such that $\overline{E}(\delta) > G(x(\delta),\delta) > \underline{E}(\delta)$, then there exists a* polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for any $\delta \in K$, $\overline{E}(\delta) > G(x^*(\delta),\delta) > \underline{E}(\delta)$.* ■

The proof of Corollary 3 consists in applying Theorem 1 to the LMI:

$$\exists x \in \mathbb{R}^p, \begin{pmatrix} G(x,\delta) - \underline{E}(\delta) & 0_n \\ 0_n & \overline{E}(\delta) - G(x,\delta) \end{pmatrix} > 0 \ .$$

Proof of Theorem 1. Under the hypothesis of solvability of (1) for any $\delta \in K$, there exists, by continuity and compactness, a real number $\alpha > 0$ such that

$$\forall \delta \in K, \ \{x \in \mathbb{R}^p \ : \ G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) \geq 2\alpha I_n\} \neq \emptyset \ .$$

Define

$$F \ : \ K \to 2^{\mathbb{R}^p}, \ \delta \mapsto F(\delta) = \{x \in \mathbb{R}^p \ : \ G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) \geq \alpha I_n\} \ . \tag{2}$$

The set-valued map F maps K into the non-void closed convex subsets of $\mathbb{R}^p$.

Let us first establish that F fulfils the following property of *lower semicontinuity*, see e.g. [2].

Definition. *Let X be a topological space, Y a metric space. A set-valued map F from X to Y is said* lower semicontinuous *at $x^0 \in X$ if for any $y^0 \in F(x^0)$ and any neighborhood $N(y^0)$ of y^0, there exists a neighborhood $N(x^0)$ such that*

$$\forall x \in N(x^0), \ F(x) \cap N(y^0) \neq 0 \ .$$

F is said lower semicontinuous *if it is lower semicontinuous at every point $x^0 \in X$.* ■

Let $\delta^0 \in K$, $x^0 \in F(\delta^0)$, $\varepsilon > 0$. To prove lower semicontinuity of F at δ^0, we exhibit $\eta > 0$ such that for any $\delta \in K$ with $\|\delta - \delta^0\|_m < \eta$, there exists $x \in F(\delta)$, $\|x - x^0\|_p < \varepsilon$.

Indeed, by assumption, there exists $x^{\delta^0} \in \mathbb{R}^p$ such that $G(x^{\delta^0}, \delta^0) \geq 2\alpha I_n$. For $\lambda \in [0,1]$ to be defined afterwards, let $x \stackrel{\text{def}}{=} (1-\lambda)x^0 + \lambda x^{\delta^0}$. Then, the fact that G is affine with respect to x implies for any $\eta > 0$, any $\delta \in K$ such that $\|\delta - \delta^0\|_m < \eta$:

$$\begin{aligned} G(x,\delta) &= (1-\lambda)G(x^0,\delta) + \lambda G(x^{\delta^0},\delta) \\ &= (1-\lambda)G(x^0,\delta^0) + \lambda G(x^{\delta^0},\delta^0) + (1-\lambda)\left(G(x^0,\delta) - G(x^0,\delta^0)\right) + \lambda\left(G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\right) \\ &\geq \alpha(1+\lambda)I_n - \left(\sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^0,\delta) - G(x^0,\delta^0)\|_n + \sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\|_n\right) I_n \ . \end{aligned}$$

On the other hand,

$$\|x - x^0\|_p = \lambda \|x^{\delta^0} - x^0\|_p \ .$$

So, take $\lambda \in [0,1]$ such that

$$\lambda \leq \frac{\varepsilon}{2\|x^{\delta^0} - x^0\|_p} \ ,$$

and choose $\eta > 0$ such that

$$\sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^0,\delta) - G(x^0,\delta^0)\|_n + \sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\|_n \leq \alpha\lambda \ .$$

With these choices, one has $\|x - x^0\|_p \leq \varepsilon/2 < \varepsilon$, and $G(x,\delta) \geq \alpha(1+\lambda)I_n - \alpha\lambda I_n = \alpha I_n$, so $x \in F(\delta)$, provided that $\delta \in K$ and $\|\delta - \delta^0\|_m < \eta$. One concludes that F is lower continuous at δ^0. This achieves the proof of lower semicontinuity of F.

We now apply to F defined in (2) Michael's Selection Theorem [10], see also [2].

Theorem (Michael's Selection Theorem). *Let X be a metric space, Y a Banach space. Let F from X into the closed convex subsets of Y be lower semicontinuous. Then there exists $f : X \to Y$, a continuous selection from F.* ■

This yields existence of a continuous selection $f : K \to \mathbb{R}^p$ from F defined in (2). This function is such that

$$\forall \delta \in K,\ G(f(\delta),\delta) \geq \alpha I_n \ .$$

It remains to apply to each of the p^2 coefficients of f the following result, see e.g. [7].

Theorem (Weierstrass Approximation Theorem). *Every continuous real-valued function defined on a compact subset K of $\mathbb{R}^m$, is the limit of a sequence of polynomials, which converges uniformly in K.* ■

Thus, the selection f previously exhibited is uniform limit in K of a sequence of (matrix-valued) polynomials in x. In particular, there exists a polynomial function $x^* : K \to \mathbb{R}^p$ such that

$$\forall \delta \in K,\ G(x^*(\delta),\delta) \geq \frac{\alpha}{2} I_n > 0 \ .$$

One concludes that there exists a polynomial solution to the parameter-dependent LMI (1), and this achieves the proof of Theorem 1. □

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