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Abstract: The prediction of interaction phenomena between a compressible flow in a moving domain and other models like structural ones requires that some conservation properties need to be satisfied by the numerical schemes. In this paper we investigate the important problem of the work-energy conservation within the fluid for the discrete formulation on moving grids. In the case of a compressible flow, the work performed on the fluid by the moving interface has to be properly translated in a variation of the total fluid energy. We present a numerical model that satisfies this energy conservation property without loosing some other conservation properties such as the Geometric Conservation Law.

Key-words: Arbitrary Lagrangian Eulerian formulation, compressible flow, spatial discretization, energy conservation, fluid-structure interaction

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Conservation de l'énergie dans les discrétisation spatiales d'écoulements compressibles en maillage mobile

Résumé :

La simulation de l'interaction entre un fluide compressible dans un domaine mobile et d'autres modèles comme des structures nécessite l'usage de schémas numériques vérifiant certaines relations de conservation. Dans ce papier est étudiée la conservation en domaine variable de la somme énergie-travail. Dans le cas d'un écoulement avec compressibilité, le travail fourni par le fluide doit provenir intégralement d'une variation de son énergie totale. Nous présentons un schéma numérique qui satisfait cette propriété sans perdre les autres conservations cruciales comme la Loi de Conservation Géométrique.

Mots-clés : Formulation d'Euler Lagrange Arbitraire, écoulement compressible, discrétisation spatiale, conservation de l'énergie, interaction fluide-structure

1 Introduction

The motivation of this study is the improvement of fluid numerical models in moving domain with the purpose of application to complex interaction problems such as fluid-structure unsteady interaction. Let us explain in detail the problems that arise in this kind of compound dynamical systems by concentrating on a basic issue: the numerical prediction of the unsteady interaction between a compressible flow and a structure requires that the global numerical scheme satisfies some conservation properties.

To start with, fixed mesh calculations compulsorily require *conservation formulations* (mass, moments, energy) for the fluid model. Two main motivations for this choice follows. Firstly, physical soundness of many computations requires that important extensive quantities are conserved, during time evolution. Secondly, a certain class of conservative schemes allow the application of the Lax-Wendroff theorem of convergence towards weak solutions. The effect of this convergence appears not only in the capture of shocks but also in very irregular meshes. In the case of moving meshes, the so-called ALE formulation of Hirt, Amsden and Cook gave an answer to the first above conservation issue.

A second notion of conservation arose rather early for moving mesh methods. The *Geometric Conservation Law* (GCL) (introduced by Thomas and Lombard [14]) and its discrete versions ensure that a uniform flow is not perturbed by approximation errors during the mesh motion. Many important arguments can be invoked for the design of schemes respecting this condition.

- It was first shown, for example in [9][7], that its practical impact on accuracy and stability is of paramount importance.

- Since degree zero polynomial solutions are exactly obtained, the GCL can be understood as a consistency condition (*à la* Bramble-Hilbert). A mathematical analysis has demonstrated the role of the GCL for time-space accuracy order for ALE schemes [6].

- Also the CGL guarantees that the uniform solution will not show spurious undershoot or overshoot, that is erroneous maxima/minima. In [3], the authors prove that the discrete CGL is a sufficient condition in order that some schemes satisfy the maximum principle for passive species. This is then a nonlinear stability condition.

A third important conservation aspect in the discretization of fluid-structure interaction is the *energy conservation*. A theoretical review for different models is, for instance, that of [5]. From a numerical point of view, the global energy conservation issue can be crucial, particularly in problems where the energy transfered to the structure is produced by thermodynamic effects. Two main sources of spurious loss of energy can be identified and distinguished:

a. The time advancing method. Basically, the time integration can be done in three different ways. Explicitly (1), implicitly and loosely coupled (2) or implicitly and (3) strongly coupled, which is also called a monolithic approach (see for example [10].

Many times, case (1) has proven to be too expensive in terms of CPU time. On the implicit side, up to our knowledge, only a small subset of monolithic schemes (3) allows a strictly energy conservation integration. The drawback is again their very high computational cost. For that reason, loosely coupled implicit schemes (2) are the most widespread choice. Among them, recently proposed *staggered* algorithms satisfy the energy conservation through the interface not exactly but up to 2nd. order in time (even 3rd. order: see the recent paper [13]).

b. The space integration. Many authors have widely studied an important aspect of this question: the energy conservation in the effort transmitted between fluid (at momentum equation level) and structure, (see for instance [4]).

Under this light, the study of the energy budget for moving boundary problems, coupled or not, solved by means of ALE schemes can be carried out following two lines. On one hand, there is the work relationship between **coupled** fluid and structure: is the work *performed on* the fluid by the moving interface the same *done by* the solid through this interface? On the other hand, there is the work-energy relationship within the fluid, **regardless** the structure: is the work *performed on* the fluid by the moving interface properly translated in a *variation of the total energy* of the fluid?

The first line is considered in papers like [4] or [13]. Both systems are treated by their own equations, and a third one is included which governs the spatial grid movement. All three are coupled through the moving interface. In [4], the space discretization issue is analyzed, focusing on the load forces distribution at the interface, as seen either by the fluid or the solid. In [13], several ideas on the time integration scheme are proposed, based on a staggered formulation.

The present paper focuses on the second line. The action of the moving boundary on the fluid system produces a variation in its energy status. It is clear for the continuum problem, but the discrete form remains full of questions. The core of this problem remains in the flow equations discretization alone. How can an ALE scheme cope with this equivalence? How can the work performed on the system be properly defined? How does the fluid energy, defined as an extensive system property and related to the work, change? Are there fully energy preserving schemes that also verify the conservations and the Geometric Conservation Law? Once the questions are solved, their conclusions can be easily adapted to the coupled schemes.

The paper is organized as follows. First, the continuum problem is set: the Euler equations, that is to say the Navier-Stokes equations for compressible inviscid flow, are written down and the energy budget is presented. This is done in a fixed, Eulerian reference system but a moving domain $\Omega(t)$ is considered. Then, the flow equations are discretized in space following an ALE approach which deals with the moving boundary difficulty, and satisfies also the GCL. The variables approximation on the surface is carefully described. Next, the analysis on the energy conservation issue is conducted, resulting in a new ALE scheme. Finally, we conclude this paper in Section 5.

Notation and conventions The following notations and conventions are here adopted:
NC1: Variables notation:

- Scalars: ρ, p, E, etc.
- Vectors: $\boldsymbol{u}$, $\boldsymbol{U}$, $\boldsymbol{x}$, etc.
- Tensors: $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, etc. Space indicial notation for vectors and tensors is here scarcely preferred, only for neatness criteria.
- Extensive variables: $\mathcal{E}$, $\mathcal{V}$, etc., in upper case script style.
- Variables defined at nodes: ρ_i, $\boldsymbol{u}_i$, etc. Subindices *always* refer to nodal number, unless the contrary is explicitly said.
- Variables defined at cell boundaries: κ_{ij}, $\boldsymbol{\nu}_{ij}$, etc., labelled with two subindices.
- Variables defined at the domain boundaries: $\widehat{\boldsymbol{u}}$, $\widehat{p}$, $\widehat{\boldsymbol{\beta}_{ij}}$, etc., labelled or not.

Einstein's summation on repeated indices (index contraction) is never assumed, unless the contrary is explicitly said.

NC2: Considering integrals, the following notation is adopted, unless otherwise is explicitly said. For any field f,

$$\int_A f := \int_A f \, dA,$$

where A can be volume, surface or time interval, and dA labels the corresponding differential.

2 The physical problem: the Euler equations for compressible flow

The physical problem under study is the inviscid compressible laminar flow. Its physics is modelled by the Euler equations, basically the Navier - Stokes set without viscous terms. For compressible flows, this set of equations comprises transport equations for the mass, the momentum and the energy (see for instance [8]). Compressibility introduces a major change in flow behaviour: pressure p, density ρ and temperature T changes are now related through a state equation, which couples the energy budget with the mass and force balances.

Suppose a fluid is contained in a given spatial domain $\Omega \subset \mathbb{R}^n$, where n is the space dimension. All the variables that describe its behaviour are then functions of position $\boldsymbol{x}$ and time t, being $t \in [0, \infty)$. In conservative form, the unknowns of the Euler equations are respectively the density ρ, the momentum $\boldsymbol{U} = \rho \boldsymbol{u}$, and the total energy per unit volume

$E = \rho e$. The velocity vector is noted as $\boldsymbol{u}$ and $e = e^o + \frac{1}{2}u^2$ is the total energy per unit mass, being e^o the fluid's internal energy and $u^2 = \boldsymbol{u} \cdot \boldsymbol{u}$. Hence, the conservation form is

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(U_i) = 0, \\
&\frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i U_j + p\delta_{ij}\right) = 0, \\
&\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i(E+p)\right) = 0,
\end{aligned} \tag{1}$$

where the subindices note space components, i.e. $i, j = 1, ..., n$ and repeated indices contract. As viscous forces are absent, the stress tensor is $\sigma_{ij} = \ - \ p\delta_{ij}$. To close the equations, a state law is needed: the ideal gas state law $p = \rho RT$ for instance. Under this hypothesis, the internal energy is $e^o = C_{\mathrm{v}}T$. R is the universal gas constant and C_{v} and C_{P} are the specific heat at constant volume and at constant pressure respectively.

In addition to (1), a set of properly defined boundary (discussed later) and initial conditions are needed to tackle the solution of the problem. Initial conditions are of the kind

$$\begin{aligned}
&\boldsymbol{u}(\boldsymbol{x}, 0) := \boldsymbol{u}^0(\boldsymbol{x}), && \text{for all } \boldsymbol{x} \in \Omega \\
&\rho(\boldsymbol{x}, 0) := \rho^0(\boldsymbol{x}), && \text{for all } \boldsymbol{x} \in \Omega \\
&T(\boldsymbol{x}, 0) := T^0(\boldsymbol{x}), && \text{for all } \boldsymbol{x} \in \Omega
\end{aligned} \tag{2}$$

and the rest of the initial variables can be derived from them.

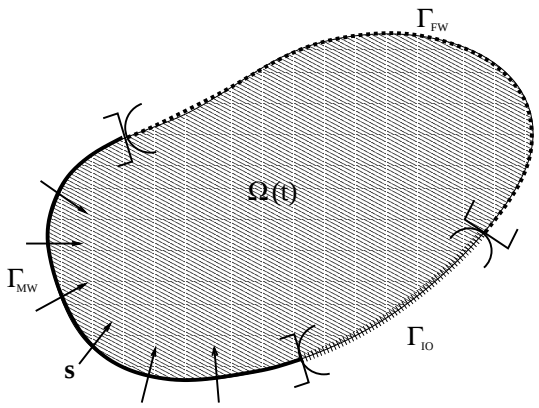

Figure 1: Sketch of the generic problem.

2.1 Moving domains

The core of the fluid-structure interaction problems is the boundary condition that couples both systems. By its motion, the solid *performs work* against the internal forces of the fluid,

producing a change in its energetic status. To focus on this issue, a generic problem is to be considered, sketched in Figure 1: suppose that $\Omega = \Omega(t)$, whose boundary is divided in three disjoint subsets: a *fixed wall* Γ_{FW} and a *moving wall* $\Gamma_{\text{MW}}(t)$, and an *input / output* Γ_{IO} boundary (additionally, let all boundaries be adiabatic ones):

$$\partial\Omega = \Gamma_{\text{FW}} + \Gamma_{\text{IO}} + \Gamma_{\text{MW}}(t). \tag{3}$$

Under these assumptions, the boundary conditions should be compatible with both the wall (moving or not) impermeability and adiabaticity, strongly imposed as Dirichlet's (on some velocity components) or weakly imposed as Neumann's (on the traction or the heat flow) Here, $\boldsymbol{s}$ will note **the velocity distribution for all the points belonging to the moving wall** measured in a fixed, Eulerian frame. For the sake of clarity, in the rest of the analysis fixed boundaries like Γ_{FW} and Γ_{IO} are to be noted as Γ and moving walls like $\Gamma_{\text{MW}}(t)$, as $\Gamma(t)$.

The conservation form of the flow equations is derived from a simple fact: some important properties are conserved as fluid gets in and out of a given volume. To evaluate this balance, all of the differential equations (1) can be integrated in space as follows. Let us write them generically as (recall **NC2** notation)

$$\frac{\partial v}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}(v) = 0,$$

where v and $\boldsymbol{F}$ represents the transported quantity and its flux respectively. Integrated in space, it is obtained

$$\int_{\Omega} \frac{\partial v}{\partial t} + \int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n} = 0, \tag{4}$$

after using the divergence Gauss' theorem. The vector $\boldsymbol{n}$ is the exterior normal to the boundary $\partial\Omega$. In this way, *extensive* flow quantities can be studied. When a locally defined variable like v is integrated over the volume Ω under study, we obtain an extensive quantity $\mathcal{V}$:

$$\mathcal{V} = \int_{\Omega} v.$$

The evolution of the extensive quantities is modeled by equations like (4). If in (4), the integration domain is changing in time, i.e. $\Omega = \Omega(t)$, the partial time derivative can be taken out of the integral by means of the so called Reynolds' formula:

$$\frac{d\mathcal{V}}{dt} = \frac{d}{dt}\left(\int_{\Omega(t)} v\right) = \int_{\Omega(t)} \frac{\partial v}{\partial t} + \int_{\partial\Omega(t)} v\ \boldsymbol{s} \cdot \boldsymbol{n}. \tag{5}$$

Again, $\boldsymbol{n}$ is the exterior normal and $\boldsymbol{s}$ is a the generic velocity of the moving boundaries. By using (5) in (4), it is obtained

$$\frac{d\mathcal{V}}{dt} + \int_{\partial\Omega(t)} \boldsymbol{F} \cdot \boldsymbol{n} - \int_{\partial\Omega(t)} v\ \boldsymbol{s} \cdot \boldsymbol{n} = 0, \tag{6}$$

which is the basis of any ALE formulation, a well-suited idea for solving flow problems with moving boundaries. The equation (6) is the key to understand the conservation principles under study in this paper: any ALE scheme should verify the energy conservation and the so-called geometric conservation. Both are introduced in the next two sections.

2.2 The energy budget

When applied to the total energy equation, as written in the set (1), by integrating over the **whole domain**, (5) gives the amount of energy introduced to (or extracted from) the fluid system by the solid via the mechanical action of the moving boundary. So let v in (5) be $v = e = \rho e^o + \rho\frac{u^2}{2}$, the total energy. Then, its extensive counterpart is

$$\mathcal{E} = \int_\Omega e,$$

which in turn verifies the Reynolds' formula (5)

$$\int_{\Omega(t)} \frac{\partial}{\partial t}\left(\rho e^o + \rho\frac{u^2}{2}\right) = \frac{d\mathcal{E}}{dt} - \int_{\partial\Omega(t)} \left(\rho e^o + \rho\frac{u^2}{2}\right)\ \boldsymbol{s}\cdot\boldsymbol{n}. \tag{7}$$

Therefore, the energy transport equation can be written as

$$\begin{aligned}\frac{d\mathcal{E}}{dt} = &- \int_{\partial\Omega(t)} \left(\rho e^o + \rho\frac{u^2}{2}\right)\ \boldsymbol{u}\cdot\boldsymbol{n} - \int_{\partial\Omega(t)} p\ \boldsymbol{u}\cdot\boldsymbol{n} \\ &+ \int_{\partial\Omega(t)} \left(\rho e^o + \rho\frac{u^2}{2}\right)\ \boldsymbol{s}\cdot\boldsymbol{n}.\end{aligned} \tag{8}$$

Since the impermeability condition

$$\begin{aligned}\boldsymbol{u}\Big|_{\Gamma_{\mathrm{MW}}}\cdot\boldsymbol{n} &= \boldsymbol{s}\cdot\boldsymbol{n} \\ \boldsymbol{u}\Big|_{\Gamma_{\mathrm{FW}}}\cdot\boldsymbol{n} &= 0\end{aligned} \tag{9}$$

is assumed both on the moving and fixed walls, the **normal flow velocity** $\boldsymbol{u}\cdot\boldsymbol{n}$ is there equal to the **normal solid wall velocity** $\boldsymbol{s}\cdot\boldsymbol{n}$. This additional condition is used to cancel some terms in (8) on the moving boundary, which now becomes

$$\frac{d\mathcal{E}}{dt} = - \int_{\Gamma_{\mathrm{IO}}} \left(\rho e^o + \rho\frac{u^2}{2}\right)\ \boldsymbol{u}\cdot\boldsymbol{n} - \int_{\Gamma_{\mathrm{IO}}} p\ \boldsymbol{u}\cdot\boldsymbol{n} - \int_{\Gamma_{\mathrm{MW}}} p\ \boldsymbol{s}\cdot\boldsymbol{n} \tag{10}$$

The energy variation comprises then three boundary terms, shown in Equation (10). The first one is the total energy convected in or out of the domain through the inflow and outflow. The second one is the work *performed by* the fluid against this in/outcoming energy. The third one represents the mechanical action of the solid on the fluid. It is the work *performed*

on the fluid by the moving wall. See that all three are co-linear with the velocity and it will be present regardless either viscosity or compressibility.

Equation (10) has been calculated from the space integration of the total energy transport equation for the fluid. In turn, it can be integrated in time to get the energy variation $\Delta\mathcal{E}$

$$\Delta\mathcal{E}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \frac{d\mathcal{E}}{dt}, \tag{11}$$

between times t_1 and t_2 (being $t_1 < t_2$). It should verify

$$\Delta\mathcal{E}\Big|_{t_1}^{t_2} = (\Delta\mathcal{Q} + W)\Big|_{t_1}^{t_2} = W\Big|_{t_1}^{t_2} \tag{12}$$

where $\Delta\mathcal{Q}|_{t_1}^{t_2}$ holds for the thermal contribution to the gas from outside, and $W|_{t_1}^{t_2}$ for the work performed by the external forces on the fluid system between t_1 and t_2. We assume that $\Delta\mathcal{Q} = 0$. As highlighted in the introduction, Equality (12) is the key of the present work: trivial in the continuum, it should be analyzed when time and space discretizations are envisaged to solve the problem. $W|_{t_1}^{t_2}$ can be split as follows:

$$W\Big|_{t_1}^{t_2} = (W_{\text{IO}} + W_{\text{MW}})\Big|_{t_1}^{t_2}. \tag{13}$$

This includes the three contributions of Equation (10). Those coming from the input/output domain boundary, grouped in W_{IO}, also appear when all boundaries are fixed and therefore W_{IO} presents no particular difficulties in an ALE context. The differential fact is now the work performed by the moving wall on the fluid, noted as W_{MW}.

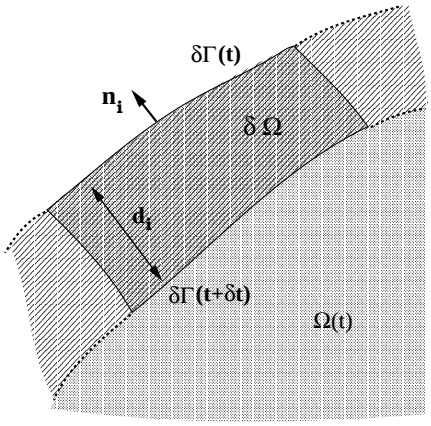

Figure 2: Work performed by the moving wall.

Alternatively to (11), calculated as said above as a domain integral of the energy transport equation, the right hand side of (12) can be evaluated from "outside" the fluid, considering the movement of $\Gamma(t)$, the work spent in this motion and the energy-work interchanges

in inputs and outputs. From now on, we will focus only on what happens with the moving part of the boundary, leaving aside W_{IO}. Let us suppose a small time increment δt for which the moving wall goes from $\Gamma(t)$ to $\Gamma(t+\delta t)$. Let f_i be the force per unit surface, δw its performed work and d_i the distance between $\delta\Gamma(t)$ and $\delta\Gamma(t+\delta t)$ which are very small portions of $\Gamma(t)$ and $\Gamma(t+\delta t)$ respectively (Figure 2). On these grounds, it can be defined

$$\begin{aligned} \boldsymbol{f} &= \boldsymbol{\sigma}\cdot\boldsymbol{n} \\ \boldsymbol{d} &= \boldsymbol{s}\ \delta t. \end{aligned}$$

where $\boldsymbol{\sigma}$ is the stress tensor defined above. Then,

$$\delta w = \boldsymbol{d}\cdot\boldsymbol{f} = \boldsymbol{d}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = (\delta t\ \boldsymbol{s})\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{n}) \tag{14}$$

which yields

$$\delta W = \delta\Gamma\ \delta t\ \boldsymbol{s}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}. \tag{15}$$

Integrated in time (between t_1 and t_2) and surface (over the whole $\partial\Omega(t)$, which corresponds to $\Gamma(t)$), it gives

$$W\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\partial\Omega(t)} \boldsymbol{s}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\ d\Gamma\ dt. \tag{16}$$

In few words: regardless the in/outlets contribution, the left hand side of (12) (LHS(12), from now on) is Equation (10) integrated in the given time interval, that is to say (11), and the right hand side (RHS(16)) is Equation (16). The LHS(12) is calculated in an Eulerian formulation for a moving domain, starting from total energy general conservation principles for fluid dynamics. It describes what happens "on the fluid side". The RHS(16) is calculated as the work introduced in the system via the moving boundaries, it is "on the solid side" (recall we have left aside W_{IO}). By the energy conservation principle, both must be strictly equal:

$$\Delta\mathcal{E}\Big|_{t_1}^{t_2} = W\Big|_{t_1}^{t_2}. \tag{17}$$

The LHS is based on the energy budget "within" the fluid domain. It requires that impermeability (10) is verified in $(\Omega(t), \delta t)$ and that both the velocity $\boldsymbol{u}$ and the stress tensor $\boldsymbol{\sigma}$ are properly defined there. Also, the moving boundary $\Gamma(t)$ must be entirely described by $\boldsymbol{n}$ and both $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ should be well defined on $\Gamma(t)$. On the other hand, the RHS(16) is calculated purely as the work introduced in the fluid system by the mechanical action of the moving boundary. In the continuum formulation both are the same because of the continuity of all the variables at the walls. But this fact is to be discussed for a discrete formulation and its solution scheme.

2.3 The Continuous Geometric Conservation Law

We analyze here the GCL idea. Basically, we will use the tools introduced in the preceding section, namely the Reynolds' formula and the extensive variables definition. Let us consider now a part of $\Omega(t)$ that we name $S(t)$. $\partial S(t)$ is its boundary (see Figure 3) which moves at speed $\boldsymbol{\beta}$. Typically, $S(t)$ can coincide with the support of a test (or characteristic) function for a given space discretization method, but for the moment it will be just a part of $\Omega(t)$. Suppose also that $\partial S(t) \cap \partial\Omega(t) \neq \emptyset$. While in this intersection the boundary velocity $\widehat{\boldsymbol{\beta}}$ is the same for both domains,

$$\widehat{\boldsymbol{\beta}} \equiv \boldsymbol{\beta}\Big|_{\partial S(t)\cap\partial\Omega(t)} = \boldsymbol{s}, \tag{18}$$

in the rest of $\partial S(t)$ the boundary velocity, $\boldsymbol{\beta}$, is determined from the fact that $S(t)$ can undergo deformations, which in principle, are not related to the fluid velocity.

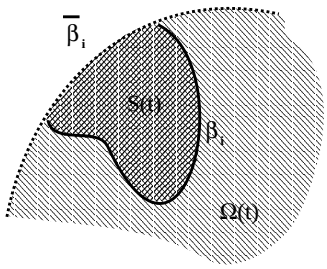

Figure 3: Subdomain $S(t)$ of $\Omega(t)$.

The meaning of Equation (6) is then the following: $\mathcal{V}$, which is now defined in $S(t)$, varies in time due to: (a) the flux $\boldsymbol{F}$ going in and out across $\partial S(t)$ and (b) the amount of v "captured" by the moving part of the boundary as it moves. As firstly pointed out in [14], and described in papers as [12, 9, 11], a very important thing can be deduced by simply inspecting the terms of (6): the ALE formulation must verify the so-called *Geometric Conservation Law (GCL)* which is independent of the chosen time and space discretization method.

This law is deduced as follows. Integrating in time (6), the difference $\Delta\mathcal{V}$ between times $t_1 < t_2$ is

$$\Delta\mathcal{V}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \left(- \int_{\partial S(t)} \boldsymbol{F}\cdot\boldsymbol{n} + \int_{\partial S(t)} v\,\boldsymbol{\beta}\cdot\boldsymbol{n} \right) dt. \tag{19}$$

Suppose now that v is a constant field in space and time. Then, its flux F_i is zero and v can be taken out of all the integrals and simplified, yielding the GCL:

$$\Delta A\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \left(\int_{\partial S(t)} \boldsymbol{\beta}\cdot\boldsymbol{n} \right) dt, \tag{20}$$

where $A(t)$ is the area (or volume in 3-D) of $S(t)$.

Again, in the continuum the GCL is automatically verified. But, as in the energy budget issue, the same cannot be said once the problem is discretized. Suppose now that $S(t)$ is indeed an element of the partition of $\Omega(t)$ (a finite volume control cell, for instance), a partition obtained by means of a spatial grid. In principle, the grid updating (i.e., the $S(t)$ deformation) is completely independent of the flow state. It should be conformed to the moving boundary evolution, while adapting the position of the interior nodes to the new situation in order to maintain the grid quality. This is done according to a certain grid dynamic equation. What the GCL (20) says is that the areas (volumes) difference between t_1 and t_2 for $S(t)$ must be equal to the area (volume) swept by $\partial S(t)$ in the same interval. Like in the energy conservation issue discussed above, the elements in contact with the moving walls are also particularly sensible to this conservation problem. In order to carry on with the analysis, a discretized spatial formulation of the flow equations must be chosen. This is addressed in the next section.

3 Discrete standpoint

The spatial discretization of the flow equations is based on a finite volume formulation on tetrahedral, non-structured grids, extended to an ALE formulation [12].

3.1 Spatial discretization of the Euler equations

Equations (1) can be written in a compact form. For each component,

$$\frac{\partial v}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}(v) = 0, \tag{21}$$

taking

$$\begin{aligned} v &= (\rho\,,\, \rho u_1\,,\, \rho u_2\,,\, \rho u_3\,,\, \rho e)^T, \\ F_k(v) &= (U_k\,,\, u_k U_1 + p\delta_{k1}\,,\, u_k U_2 + p\delta_{k2}\,,\, u_k U_3 + p\delta_{k3}\,,\, u_k(\rho e + p))^T \end{aligned} \tag{22}$$

for a tridimensional problem, the subindices running through the three space coordinates.

The weak form of (21) is discretized using a nodal centered finite volume method. The basis function space is then made of compact support functions, defined by pieces upon a nonstructured partition in control cells of the discretized domain Ω_{h}. In order to gain insight on the problem, consider from now on a bi-dimensional space. The finite volumes basis functions, the *characteristic functions* are defined with the aid of *control cells*. The cells construction is performed using the medians of the triangles that have a^i as a constitutive node, as sketched in Figure 4. The characteristic function can be constructed as a constant-by-pieces one:

$$\phi = \chi^i(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in C^i, \\ 0 & \text{otherwise.} \end{cases}$$

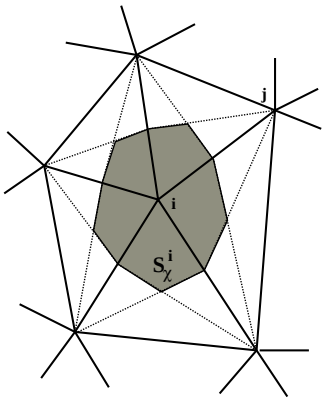

Figure 4: FV support C^i corresponding to node a^i: the "control cell" around a^i.

In this way, the control cells cover completely Ω_h:

$$\Omega_h = \bigcup_{i=1}^{n_c} C^i. \tag{23}$$

Then, the weak form of (21) is integrated in the partitioned Ω_h and using χ^i following these ideas. For each node, we have

$$\int_{C^i} \frac{\partial v_i}{\partial t} \quad + \quad \int_{\partial C^i} \boldsymbol{n}_i \boldsymbol{F}_i(v) = 0 \tag{24}$$

3.2 Discretization of moving domains: ALE method

Going back to the original problem: let $\Omega = \Omega(t)$, according to the generic problem sketched in Figure 1. It follows that in this case, Equation (24) can be treated as Equation (4) above, which has lead to (6). Now, suppose that the discretized domain is partitioned as follows:

$$\Omega_h(t) = \bigcup_{i=1}^{n_c} C^i(t).$$

It is assumed that the space discretizations are on meshes of constant topology but showing a smooth deformation in time. Then,

$$\frac{d\mathcal{V}^i}{dt} \quad + \quad \int_{\partial C^i(t)} \boldsymbol{F}_i(v) \cdot \boldsymbol{n}_i \quad - \quad \int_{\partial C^i(t)} v_i \, \boldsymbol{\beta}_i \cdot \boldsymbol{n}_i = 0, \tag{25}$$

where,

$$\frac{d\mathcal{V}^i}{dt} = \frac{d}{dt} \left(\int_{C^i(t)} v \right)$$

and $\boldsymbol{\beta}_i$ is the speed of the boundary cell $\partial C^i(t)$, that is to say, the **mesh velocity**.

The integrals above are computed numerically following the adopted finite volume scheme. Together with the appropriate boundary conditions, a numerical solution will be obtained. Consider the space domain partition as set in (23). For each **interior** cell C^i, with its corresponding set $V(i)$ of neighboring cells,

$$\partial C^i = \bigcup_{j \in V(i)} \partial C^{ij}, \tag{26}$$

where ∂C^{ij} represents the segment (faces in 3-D) shared by cells C^i and C^j. Then, (25) can be written as

$$\frac{d\mathcal{V}^i}{dt} + \sum_{j \in V(i)} \int_{\partial C^{ij}(t)} \boldsymbol{F}_i(v) \cdot \boldsymbol{n}_{ij} - \sum_{j \in V(i)} \int_{\partial C^{ij}(t)} v_i \, \boldsymbol{\beta}_i \cdot \boldsymbol{n}_{ij} = 0. \tag{27}$$

In order to evaluate this integral, following [3], let us define

$$\boldsymbol{\nu}_{ij}(t) = \frac{1}{|\partial C_{ij}(t)|} \int_{\partial C_{ij}(t)} \boldsymbol{n}_{ij}(t) \, ds \tag{28}$$

and

$$\kappa_{ij}(t) = \frac{1}{|\partial C_{ij}(t)|} \int_{\partial C_{ij}(t)} \boldsymbol{\beta}(t) \cdot \boldsymbol{n}_{ij}(t) \, ds \, . \tag{29}$$

$\boldsymbol{\nu}_{ij}(t)$ is the **mean normal** corresponding to $\partial C_{ij}(t)$ and $\kappa_{ij}(t)$, the **mean normal mesh velocity** projection for the same boundary cell segment (the full meaning of this "mean" will be grasped below, when it becomes also a temporal one). We get then an integral ALE semi-discretization of the conservation law:

$$\frac{d\mathcal{V}^i}{dt} + \sum_{j \in V(i)} |\partial C_{ij}(t)| \; \Phi(v_i, v_j, \boldsymbol{\nu}_{ij}(t), \kappa_{ij}(t)) = 0, \tag{30}$$

where Φ is a numerical flux function, typically an approximate Riemann solver, with mesh velocity normal component $\kappa_{ij}(t)$ and with mean value of unknown v over cell i denoted by v_i. In particular it satisfies the following consistency condition:

$$\Phi(v, v, \boldsymbol{\nu}, \kappa) = \boldsymbol{F}(v) \cdot \boldsymbol{\nu} - \kappa v.$$

3.3 GCL for discretized problem

Consider now a time discretization of the above formula. Up to first order, v_i can be taken as constant within each cell. Then, if the volume of the partition's cell i is $|C_i(t)|$,

$$\mathcal{V}_i(t) = |C_i(t)| \; v_i(t). \tag{31}$$

A time discretization yields

$$|C_i^{n+1}|v_i^{n+1} \;=\; |C_i^n|v_i^n \;-$$

$$\Delta t\theta \sum_{j\in V(i)} |\partial\bar{C}_{ij}|\Phi\left(v_i^{n+1}, v_j^{n+1}, \overline{\boldsymbol{\nu}}_{ij}, \overline{\kappa}_{ij}\right) \;-\; \Delta t(1-\theta) \sum_{j\in V(i)} |\partial\bar{C}_{ij}|\Phi\left(v_i^{n}, v_j^{n}, \overline{\boldsymbol{\nu}}_{ij}, \overline{\kappa}_{ij}\right)$$

where the overlines mean that time averaged values are taken.

Assume the above system able to reproduce the constant solution $v^n = v^{n+1} = v^*$, it should satisfy:

$$|C_i^{n+1}|v_i^{*} \;=\; |C_i^n|v_i^{*} \;-$$

$$\Delta t\theta \sum_{j\in V(i)} |\partial\bar{C}_{ij}|\Phi\left(v_i^{*}, v_j^{*}, \overline{\boldsymbol{\nu}}_{ij}, \overline{\kappa}_{ij}\right) \;-\; \Delta t(1-\theta) \sum_{j\in V(i)} |\partial\bar{C}_{ij}|\Phi\left(v_i^{*}, v_j^{*}, \overline{\boldsymbol{\nu}}_{ij}, \overline{\kappa}_{ij}\right)$$

Invoking the consistency condition for Φ and the fact that cells remain closed during the motion, which writes:

$$\sum_{j\in V(i)} |\partial\bar{C}_{ij}|\overline{\boldsymbol{\nu}}_{ij} \;=\; 0,$$

we see that the above condition simplifies to:

$$|C_i^{n+1}|v_i^{*} \;=\; |C_i^n|v_i^{*} \;-\; \Delta t \sum_{j\in V(i)} |\partial\bar{C}_{ij}|\overline{\kappa}_{ij}.$$

As stated in papers like [12], [2] or [1], the Discrete GCL (DGCL) becomes a design condition to impose for the time averaged values $|\partial\bar{C}_{ij}|$, $\overline{\boldsymbol{\nu}}_{ij}$, and $\overline{\kappa}_{ij}$. Taken over $\partial C^{ij}(t)$, they should be carefully computed. In [12], both the cell's normals and the grid's velocity mean values determine the geometrical parameters which enforces the GCL. In [2], this is attained by means of a proper node's position updating and grid's velocity evaluation, showing also an equivalence with the former paper ideas. On the other hand, in [1], a scheme verifying GCL is proposed by tuning how the cell volume is evaluated. We also refer to [3] for examples of averagings satisfying the DGCL for the above time advancing scheme.

However, from an energy conservation point of view, this is not enough. While certainly it is for **interior** cells, additional care should be taken for **boundary** ones in order to keep the work performed on the fluid equal to the energy gain/loss of the fluid. While the GCL condition provides a way to calculate the mean normal vector and the mean normal *mesh velocity* $\boldsymbol{\beta}$ which does not introduce spurious effects due to the mesh movement, it says nothing about the *flow velocity* $\boldsymbol{u}$. In the interior nodes, both can be completely independent. But at the moving boundaries, that move at speed $\widehat{\boldsymbol{\beta}}$, some questions arise about the relationship between them. This analysis is completed in the next section.

3.4 The CGL integration on a facet

As stated above in (25), the discretized space volume can be divided in control cells C^i, each of them limited by a faceted frontier ∂C^i. For a 3D problem, the control cells are polyhedra whose constitutive "facets" are triangles, like Σ^i_{jk+} or Σ^i_{kj+}. This is shown in Figure 5, where C is the center of each of the segments connecting two nodes, B represents the center of each of the tetrahedra's faces and D, the center of the tetrahedra themselves. The facet notation is the following: Σ^i_{jk+} means "the facet corresponding to node i, relating it with node j and with one of its sides lying on node k's median". By inspection it can be seen that there are only two possibilities for indices (i, j, k) in this strict order. Therefore, labels $+$ or $-$ distinguish between both facets.

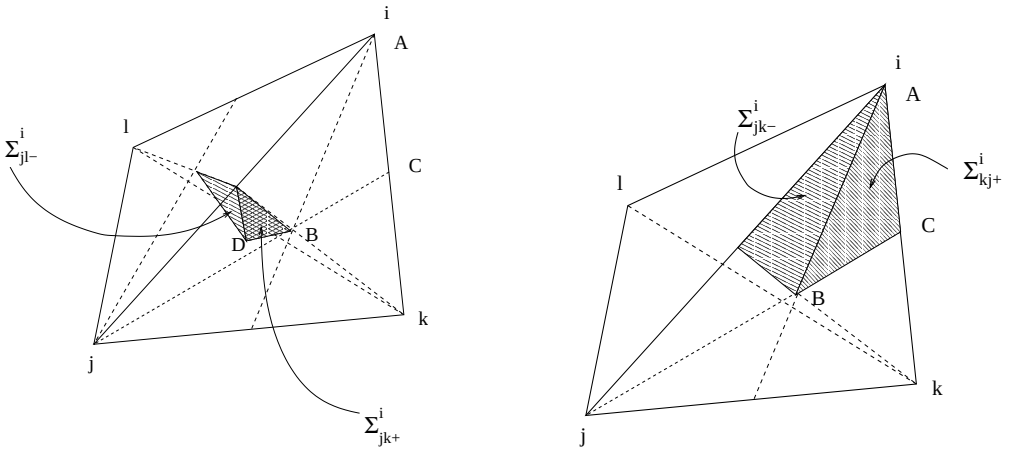

Figure 5: Tetrahedra and facets. Left, interior facets Σ^i_{jl-} and Σ^i_{jk+}. Right, boundary facets Σ^i_{jk-} and Σ^i_{kj+}.

Therefore, Equation (26) can be pushed further:

$$\partial C^i = \bigcup_{j \in V(i)} \partial C^{ij} = \bigcup_{j \in V(i)} \left(\bigcup_{k \in V(i)} (\Sigma^i_{jk+} + \Sigma^i_{kj-}) \right). \tag{32}$$

The (D)GCL can then be set for an arbitrary triangular facet. As said above, by enforcing this law, the time averaged values of the cell faceted surface $|\partial \bar{C}_{ij}|$, its normals $\overline{\boldsymbol{\nu}}_{ij}$, and its speeds $\overline{\kappa}_{ij}$ can be properly evaluated. In [12], this analysis is carried out for an internal node. Here, using the same strategy, we complete it for a boundary one. Let us consider one of these facets, which is a moving triangle embedded in a fluid, as in Figure 6. It is shown in [12] that, for a given triangle $\Sigma^{\mathrm{M}}_{\mathrm{NP}}$, the mean normal $\overline{\boldsymbol{\nu}}^{\Sigma}$ and the mean normal velocity

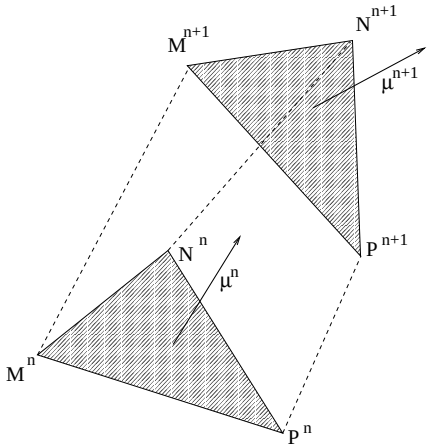

Figure 6: Triangle (representing a facet) $\Sigma^{\rm M}_{\rm NP}$ with its normal μ.

$\overline{\kappa}^{\Sigma}$, defined above in (28) and (29) are

$$\overline{\boldsymbol{\nu}}^{\Sigma} = \frac{1}{3}(\boldsymbol{\mu^n} + \boldsymbol{\mu^{n+1}} + \boldsymbol{\mu^*}) \tag{33}$$

$$\overline{\kappa}^{\Sigma} = \frac{\overline{\boldsymbol{\nu}}^{\Sigma}}{||\overline{\boldsymbol{\nu}}^{\Sigma}||} \cdot \boldsymbol{\beta}^{\Sigma}, \tag{34}$$

where $\boldsymbol{\mu}$ is the non-normalized exterior normal to the facet Σ, and the auxiliar variable $\boldsymbol{\mu^*}$ and the **facet velocity** $\boldsymbol{\beta}^{\Sigma}$ are defined as

$$\boldsymbol{\mu^*} = \frac{(\boldsymbol{x}^n_{\rm P} - \boldsymbol{x}^n_{\rm M}) \wedge (\boldsymbol{x}^{n+1}_{\rm N} - \boldsymbol{x}^{n+1}_{\rm M})}{4} + \frac{(\boldsymbol{x}^{n+1}_{\rm P} - \boldsymbol{x}^{n+1}_{\rm M}) \wedge (\boldsymbol{x}^n_{\rm N} - \boldsymbol{x}^n_{\rm M})}{4}$$

$$\boldsymbol{\beta}^{\Sigma} = \frac{\boldsymbol{\beta}^n_{\rm M} + \boldsymbol{\beta}^n_{\rm N} + \boldsymbol{\beta}^n_{\rm P}}{3}.$$

For a given facet, the vertex velocities $\beta_{\rm M,N,P}$ are linearly interpolated from those at the corresponding tetrahedra nodes, because the related vertex M, N, or P can be a node of the mesh, a mid-point segment, a face center or a tetrahedron center, noted respectively as A, C, B and D in Figure 5. For instance, for an interior facet like Σ^i_{jl-} in Figure 5 (left), the facet velocity is (see [12])

$$\boldsymbol{\beta}^{\Sigma,\rm INT} = \frac{1}{2}\left(\frac{13}{36}(\boldsymbol{\beta}_i + \boldsymbol{\beta}_j) + \frac{5}{36}(\boldsymbol{\beta}_k + \boldsymbol{\beta}_l)\right). \tag{35}$$

On the other hand, for a boundary facet like Σ^i_{jk-}, Figure 5 (right),

$$\boldsymbol{\beta}^{\Sigma,\rm BOU} = \frac{1}{2}\left(\frac{22}{36}\boldsymbol{\beta}_i + \frac{7}{36}(\boldsymbol{\beta}_j + \boldsymbol{\beta}_k)\right). \tag{36}$$

This formula is applied to all boundary facets, including in/output ones. In general, those latter boundaries are on the Euler part of the ALE domain, where $\boldsymbol{\beta}$, is supposedly zero.

3.5 The energy budget for the discretized problem

Once the discretization method is set, the energy budget problem can be revisited, starting by considering the application of (25) to the energy flow equation. Taking $v := \rho e^o + \rho\frac{u^2}{2} = \rho e = E$ in (25), we obtain for an Euler problem

$$\frac{d\mathcal{E}_i}{dt} \quad + \quad \int_{\partial C_i(t)} \boldsymbol{n}_i \cdot \Big((E+p)_i \boldsymbol{u}_i - E_i \boldsymbol{\beta}_i \Big) = 0. \tag{37}$$

Let us consider the control cells C_i which share part of their facet boundaries with the moving wall boundary (a two-dimensional representation is given by Figure 7). For such cells we have

$$\partial C_i(t) \cap \partial\Omega_h(t) = \partial\widehat{C}_i(t) \neq \emptyset. \tag{38}$$

It can be expected that in order to maintain the impermeability condition, the normal boundary velocity at the intersection with the moving wall boundary should be equal to the flow velocity.

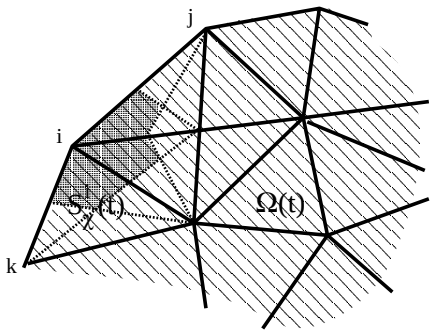

Figure 7: Two-dimensional representation of a cell on the moving wall boundary

The previous boundary cells C_i have both an "internal" and a "moving wall" parts, $\partial C_i = \partial C_{\rm O}^i + \partial C_{\rm MW}^i$, where

$$\partial C_{\rm MW}^i = \partial C_i(t) \quad \cap \quad (\bigcup \Sigma_{\rm MW})$$

$$\partial C_{\rm O}^i = \partial C_i(t) \quad \setminus \quad \partial C_{\rm MW}^i.$$

where $\Sigma_{\rm MW}$ denote the facets which belong to the moving part of $\partial\Omega_h(t)$, and the union is done over all of this facets. It should be remarked that in this case, (26) which expresses

that the cell boundary is the union of interfaces with neighboring cells is not true because it is associated now to a *boundary* node. Instead, (37) can be written as

$$\begin{aligned} \frac{d\mathcal{E}_i}{dt} &+ \int_{\partial C_{\mathrm{O}}^i} \boldsymbol{n}_i \cdot \Big(E\ \boldsymbol{w} + p\ \boldsymbol{u}\Big)_i \\ &+ \int_{\partial C_{\mathrm{MW}}^i} \boldsymbol{n}_i \cdot \Big(\widehat{E}\ \widehat{\boldsymbol{w}} + \widehat{p}\ \widehat{\boldsymbol{u}}\Big)_i = 0, \end{aligned} \tag{39}$$

where

$$\boldsymbol{w}_i = \boldsymbol{u}_i\ -\ \boldsymbol{\beta}_i. \tag{40}$$

This is the velocity difference between the mesh and the flow field at node i. As said before, in a general ALE formulation, mesh speed and flow velocities are not related, **except** at the moving boundaries. After (30), Equation (39) becomes

$$\begin{aligned} \frac{d\mathcal{E}_i}{dt} &+ \sum_{j \in V(i)} |\partial C_{ij}(t)|\ \ \Phi_{\mathrm{O}}(W_i, W_j, \boldsymbol{\nu}_{ij}(t), \kappa_{ij}(t)) \\ &+ \sum_{\Sigma, i \in \Sigma} |\partial \widehat{C}_i(t) \cap \Sigma|\ \ \Phi_{\mathrm{MW}}(\widehat{W}^{\Sigma}, \widehat{\boldsymbol{w}}^{\Sigma}, \widehat{\boldsymbol{\nu}}^{\Sigma}(t), \widehat{\kappa}^{\Sigma}(t)) \end{aligned} \tag{41}$$

where W is the flow solution vector and Σ is a triangular facet which belongs to the moving wall boundary.

Equation (41) is the core of the problem. For the interior cells, the second summation vanishes and the first summation over the "internal" segments completes the cell boundaries. But for the boundary cells, the second summation introduces a substancial difference. How is evaluated $\widehat{\boldsymbol{w}}$ on a moving facet? Once integrated in time and summed up to consider the extensive energy $\mathcal{E}$ of the whole domain, Equations (39) and (41) will give the amount of energy gained or loss by the discretized system (11), that could be in turn compared with the work performed on it (16). Equation (11) states that equation (39) is integrated as follows:

$$\begin{aligned} \Delta\mathcal{E}\Big|_{t_1}^{t_2} &= \int_{t_1}^{t_2} \frac{d\mathcal{E}}{dt} = \int_{t_1}^{t_2} \left(\sum_i^{n_c} \frac{d\mathcal{E}_i}{dt} \right) \\ &= -\ \int_{t_1}^{t_2} \left(\sum_i^{n_c} \left(\int_{\partial C_{\mathrm{MW}}^i} \boldsymbol{n}_i \cdot \Big(\widehat{E}\ \widehat{\boldsymbol{w}} + \widehat{p}\ \widehat{\boldsymbol{u}}\Big)_i \right) \right) \end{aligned} \tag{42}$$

where the summation runs over all the cells. The terms corresponding to the energy interchange between interior cells cancel each other and the remaining terms are those corresponding to the boundary contribution.

On the other hand, the work performed on the fluid by the moving boundary is

$$W\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\partial\Omega(t)} \widehat{p}\ \ \widehat{\boldsymbol{\beta}} \cdot \boldsymbol{n}$$

$$= \quad - \int_{t_1}^{t_2} \left(\sum_i^{n_c} \left(\int_{\partial C^i_{\mathrm{MW}}} \boldsymbol{n}_i \cdot \left(\widehat{p}\, \widehat{\boldsymbol{\beta}} \right)_i \right) \right). \tag{43}$$

From Equations (42) and (43), as in the GCL case, several questions on how are approximated the boundary values and fluxes must be considered.

4 Global conservation

The objective of the rest of the paper is to sketch a discretized ALE scheme that satisfies both of the conservation laws described above. It has been shown that the question of the fluid energy conservation is particularly important in boundary cells and regarding the boundary flows approximation. In order to fix the ideas, recall that:

- The **solid** moving boundary $\partial\Omega(t)$ moves at speed $\boldsymbol{s}$ defined at each of its vertices.
- The **flow** moves at velocity $\boldsymbol{u}$ wherever, and particularly at velocity $\widehat{\boldsymbol{u}}$ defined at any vertex of $\partial\Omega(t)$.
- The **mesh** moves at speed $\boldsymbol{\beta}$ wherever, and particularly at speed $\widehat{\boldsymbol{\beta}}$ defined at any vertex of $\partial\Omega(t)$.
- The **difference** between mesh speed and velocity flow is $\boldsymbol{w}$, and particularly $\widehat{\boldsymbol{w}}$ defined at any vertex of $\partial\Omega(t)$.

Having this in mind, together with both geometric and energy conservation as set by GCL and Equations (42) - (43), it is observed that for any boundary cell, there exists a great deal of approximations for computing the boundary flows. Nevertheless, not all of them are properly conservative. On one hand, the DGCL has to be accomplished. Equations (35) and (36) tell us how to compute the facet velocity *once* we know the mesh velocities of each interior or boundary node i, noted $\boldsymbol{\beta_i}$ and $\widehat{\boldsymbol{\beta_i}}$ respectively. In the interior nodes, $\boldsymbol{\beta_i}$ can be arbitrarily assigned. As said above, the in/output boundaries belong to the purely Eulerian part of the domain, where $\widehat{\boldsymbol{\beta_i}} = 0$. However, for those lying at the moving boundaries, $\widehat{\boldsymbol{\beta_i}}$ **cannot** be arbitrarily set. Then (34) can be used to compute the mean values. Independently of the use of (36) in (34), the second conservation principle involved here must be verified.

We summarize this as follows. Let us consider the "moving wall" boundary $\partial C^i_{\mathrm{MW}}$ made of boundary facets (see Fig.5, right). Then, we can write the following lemmas:

Lemma 1
A **GCL-compatible** *integration of the energy flux through the "moving wall" boundary $\partial C^i_{\mathrm{MW}}$ can be computed so that Equation (42) is approximated as follows:*

$$\Delta\mathcal{E} \;=\; \sum_{i=1}^{n_c} \mathcal{E}_i \Big|_{t_1}^{t_2} \quad ; \quad \mathcal{E}_i \Big|_{t_1}^{t_2} = \; - \int_{t_1}^{t_2} \int_{\partial C^i_{\mathrm{MW}}} \left(\widehat{E}_i \; \widehat{\boldsymbol{w}}_i + \widehat{p}_i \; \widehat{\boldsymbol{u}}_i \right) \cdot \boldsymbol{n}_i$$

where, for each boundary facet Σ which intersects with $\partial C^i_{\rm MW}$, the time-space integration of the flow and mesh normal velocities is computed using (34) and (36), and where facet mean values $\widehat{p}_i$ and $\widehat{E}_i$ of the pressure and the total energy are used.

Lemma 2
A time-space integration scheme is **energy conservative** *if the energy loss (or gain) given by the discretized counterpart of Equation (42) is exactely the work produced (or received), defined by the discretized counterpart of Equation (43):*

$$\Delta\mathcal{E}\Big|_{t_1}^{t_2} = W\Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2}\sum_{i=1}^{n_c}\int_{\partial C^i_{\rm MW}} \widehat{p}_i \;\; \widehat{\boldsymbol{\beta}}_i\cdot\boldsymbol{n}_i$$

where all the above quantities have to be seen as time and space discretized ones. In particular, for a GCL-compatible sheme, the previous integral over each facet of $\partial C^i_{\rm MW}$ is computed with a time-space integration based on (34) *and* (36)*, and a facet mean value $\widehat{p}_i$ of the pressure.*

From the above two lemmas, we can build a simple conservative scheme in the sense it verifies both the GCL and the fluid energy conservation. Such a scheme, which satisfies Lemma 1 and Lemma 2, is built as follows:

a) It is a GCL-compatible integration scheme which leads to an approximation of Equation (42) as given by Lemma 1.

b) In Lemma 2, the mean moving facet pressure is evaluated as in Lemma 1. Furthermore, in Lemma 1, for each moving facet $\Sigma_{\rm MW}$ the mean flow normal velocity $\widehat{\boldsymbol{u}}_i\cdot\boldsymbol{n}_i$ is taken equal to the mean mesh normal velocity $\widehat{\boldsymbol{\beta}}_i\cdot\boldsymbol{n}_i$, so that the energy conservation stated by Lemma 2 is satisfied:

$$\begin{aligned}
\Delta\mathcal{E}\Big|_{t_1}^{t_2} &= -\sum_{i=1}^{n_c}\int_{t_1}^{t_2}\int_{\partial C^i_{\rm MW}}\left(\widehat{E}_i\;\widehat{\boldsymbol{w}}_i + \widehat{p}_i\;\widehat{\boldsymbol{u}}_i\right)\cdot\boldsymbol{n}_i \\
&= -\sum_{i=1}^{n_c}\int_{t_1}^{t_2}\int_{\partial C^i_{\rm MW}}\widehat{p}_i\;\widehat{\boldsymbol{u}}_i\cdot\boldsymbol{n}_i \\
&= -\sum_{i=1}^{n_c}\int_{t_1}^{t_2}\int_{\partial C^i_{\rm MW}}\widehat{p}_i\;\widehat{\boldsymbol{\beta}}_i\cdot\boldsymbol{n}_i \\
&= W\Big|_{t_1}^{t_2}
\end{aligned}$$

The previous condition on the flow and mesh normal velocities can be achieved by requiring that $\widehat{\boldsymbol{w}}_i \cdot \boldsymbol{n}_i = (\widehat{\boldsymbol{u}}_i - \widehat{\boldsymbol{\beta}}_i) \cdot \boldsymbol{n}_i = 0$ for each boundary node. However, both lemmas refer to **mean facet** values, not to nodal values. Therefore small nodal differences are allowed, although this can ultimately violate locally the impermeability condition if a solid is considered, because in this case we have at the boundaries $\widehat{\boldsymbol{\beta}}.\boldsymbol{n} = \boldsymbol{s}.\boldsymbol{n}$.

The impact of the second conservation principle involved here can be the following: if the fluid model is coupled with a structural one, then any transmission of the fluid pressure consistent with the above integration will allow a perfect energy budget, that is, any Joule lost by the fluid will be gained by the structure and vice versa. In practice, this situation is not obtained with time advancing schemes weakly coupling the two materials. In that latter case, energy conservation can be satisfied up to a higher order error ([13]). Exact budget are attainable with the so-called strongly coupled monolithic time advancing schemes.

5 Conclusion

Three important classes of conservation relations must be satisfied by numerical models when applied to nonlinear/unsteady fluid-structure interactions: the usual conservations (mass, moment, energy) in *each* medium, the Geometric Conservation Law, in the fluid, and the conservation of energy.

It is indeed important that work is conserved when passed from fluid to structure. But it is as important, in the case of a compressible fluid, that the work yielded/received by the fluid is exactly its total energy loss/gain.

We have shown that this can be satisfied by some spatially discretized numerical models, when some attention is paid to the approximation of some boundary fluxes, and that this can be obtained without losing the other conservation properties.

Under these assumptions, the energy budget will show a good transmission of the fluid energy, from the total fluid energy variable to the total structure energy, as in the physical differential model.

A numerical scheme which satisfies this property will produce more reliable results for violent transient problems where energy transfers of high local (in space or time) strength occur. It can also be very useful to evaluate more accurately energy budgets when several small energy losses are in competition (radiation,...).

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