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## *2-microlocal Formalism*

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## 2-microlocal Formalism

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Thème 4 — Simulation et optimisation  
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**Abstract:** This paper is devoted to the study of a fine way to measure the local regularity of distributions. Starting from the 2-microlocal analysis introduced by J.M. Bony, we develop a *2-microlocal formalism*, much in the spirit of the multifractal formalism. This allows to define a new regularity function, that we call the *2-microlocal spectrum*. The 2-microlocal spectrum proves to be a powerful tool that we apply in three directions. First, it allows to recover all previously known results on local regularity exponents, as well as to discover new properties about them. Second, the 2-microlocal spectrum provides a deeper understanding of the 2-microlocal frontiers. It yields in particular a natural way of prescribing these frontiers on a countable dense set of points. Finally, we explore the close parallel between the multifractal and 2-microlocal formalisms. These applications are illustrated on examples such as the Weierstrass and the Riemann functions, as well as lacunary wavelet series.

**Key-words:** 2-microlocal Analysis, Wavelets, Regularity Exponents, Hölder Exponents, Local Regularity, Multifractal Formalism.

## Formalisme 2-microlocal

**Résumé :** Nous développons dans ce travail une nouvelle méthode pour mesurer finement la régularité locale de distributions. Nous utiliserons une fonction appelée *spectre 2-microlocal*, calculée à partir des coefficients d'ondelettes de distributions. En se fondant sur l'analyse 2-microlocale introduite par J.-M. Bony, nous démontrons un *formalisme 2-microlocal*, qui relie par une transformée de Legendre le spectre 2-microlocal et les frontières 2-microlocales. Ces frontières contiennent beaucoup d'informations sur la régularité locale de la distribution étudiée.

Le spectre 2-microlocal est un outil performant dont nous montrons quelques applications. Tout d'abord, il permet de retrouver et de généraliser les résultats déjà connus sur les exposants de régularité locaux. Il procure également un nouveau point de vue sur la régularité et les frontières 2-microlocales: en particulier, nous serons capables de prescrire des frontières sur des ensembles dénombrables denses. Enfin nous comparerons le *formalisme multifractal* et le formalisme 2-microlocal: cette étude sera instructive pour les deux domaines.

Nous calculerons en détails les spectres 2-microlocaux de fonctions "connues": la fonction de Weierstrass, la fonction de Riemann et les séries lacunaires d'ondelettes.

**Mots-clés :** Analyse 2-microlocale, Ondelettes, Exposants de Régularité, Exposants d'Hölder, Régularité locale, Formalisme multifractal.

## 1 Introduction and motivations

Local regularity analysis is useful in many fields, including PDE theory, fluid mechanics (turbulence), financial analysis, traffic analysis, signal and image processing. There are several motivations for looking at the local regularity. From a mathematical point of view, it is important to keep track of the singularities of, e.g., the solutions of a PDE [2]. In turbulence, the energy dissipation is directly linked with the singularity structure of the flow [6]. In traffic or financial data analysis, the local regularity gives an indication of the erraticity of the record around any given point, allowing better control [1], [24]. Finally, in the processing of irregular signals or images, the local regularity information is often more pertinent than, e.g., the amplitude, for purposes of segmentation or detection [23].

It is thus important to define relevant ways of measuring the local regularity of functions, and to be able to compare them, as well as to design numerical procedures for their estimation. In recent years, a multitude of regularity exponents have been proposed and studied in this view. See, among other references, [3], [11], [13], [16], [21], [25], [26]. An important task is to describe the structure of the regularity exponents, as well as their inter-relations. For instance, it is shown in [3] and [11] that the pointwise Hölder function, i.e. the function which associates to each point  $x_0$  its pointwise Hölder exponent  $\alpha_p(x_0)$ , is a liminf of a sequence of continuous functions. Likewise, the local Hölder function must be a lower semi-continuous function ([8], [26]). In addition, there are some mutual constraints between the exponents. For example the pointwise and the Hölder exponent functions must coincide on an uncountable set of points [26], and a similar relation holds between the pointwise Hölder exponent and the chirp exponent [13]. These second kinds of conditions are harder to analyze, as they are more deeply related to the internal structure of the functions. We recall in section 2 the definitions of some classical exponents.

2-microlocal analysis, introduced in [2] by J.M. Bony, generalizes the notion of exponents: It associates to each point  $x_0$  a curve in  $\mathbb{R}^2$ , called the 2-microlocal frontier, that describes very precisely the local regularity behaviour of a function or a distribution at  $x_0$ . Some steps towards an understanding of 2-microlocal frontiers were taken in [7] and [21]. For example these works give an answer to the problem of characterizing the most general 2-microlocal frontier at one point  $x_0$ . However, compatibility conditions between frontiers at different points are not well-understood. The present work makes some progress in this direction, by showing how to prescribe the frontiers simultaneously on any dense countable set. The basics of 2-microlocal analysis are recalled in subsection 2.3.

It was noticed in [8] that the 2-microlocal frontier  $\sigma$  of a distribution  $f$  at  $x_0$  may be obtained as the Legendre transform of a given function  $\chi$ , called the *2-microlocal spectrum*, of the wavelet coefficients of  $f$  in a neighbourhood of  $x_0$ . Exploring and refining this relation between  $\sigma$  and  $\chi$  allows us to define, in section 3, a *2-microlocal formalism*, which proves to be a powerful tool for a fine investigation of the local regularity. As an illustration, we propose in the sequel three applications of the 2-microlocal formalism and the 2-microlocal spectrum  $\chi$ .

In section 4 we explore the parallel between the 2-microlocal and multifractal formalisms. Although we restrict here to very preliminary studies, it seems clear that such a parallel sheds new light on both domains, which would be worth of further work. The second application is an extension of the works mentioned above that were dealing with the various exponents and their relations. We first show how to obtain most previously considered exponents from the 2-microlocal spectrum (subsection 3.5). The function  $\chi$  allows to obtain a deeper understanding of these exponents and their complex interactions. In particular, it leads naturally to the definition of a new exponent, the *neighbourhood* exponent. The study of this exponent yields necessary compatibility conditions between 2-microlocal frontiers: As a third application, we show how to prescribe the 2-microlocal frontiers on any countable set of points (Section 6).

The 2-microlocal spectrum contains more information than the 2-microlocal frontier. It thus gives an even more precise description of the local regularity of functions. We illustrate this on the case of the classical Riemann function and on lacunary wavelet series in section 7.

We start with a brief recall on the classical regularity exponents.

## 2 Regularity Exponents

### 2.1 Definitions of Classical Regularity Exponents

In most cases, one measures local regularity using the pointwise Hölder exponent  $\alpha_p$ . We recall here the definition of  $\alpha_p$ .

**Definition 1** *Let  $x_0 \in \mathbb{R}$ , and  $s$  be a positive real number with  $s \notin \mathbb{N}$ . A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  belongs to  $C_{x_0}^s$  if and only if there exist a constant  $C$  and a polynomial  $P$  of degree smaller than  $[s]$  such that*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s.$$

The *pointwise Hölder exponent* of  $f$  at  $x_0$ , denoted by  $\alpha_p(x_0)$ , is defined to be  $\sup\{s : f \in C_{x_0}^s\}$ .

It is well known that  $\alpha_p$  alone does not give sufficient information in some applications, and is even sometimes irrelevant. A simple example of this arises in signal processing: One often needs to act on a signal using a pseudo-differential operator, such as the Hilbert transform or a local fractional integration. The problem comes from the fact that  $\alpha_p$  is not stable under the action of such operators, i.e. it is not possible to predict what the pointwise Hölder exponent of the transformed signal will be. This is particularly crucial when one deals with very irregular signals, to which one would like to apply a multifractal analysis. In general, “multifractal functions”, such as attractors of Iterated Function Systems (IFS), have a wildly varying  $\alpha_p$ , and one would like to keep track of the evolution of the pointwise regularity after transformation.

There are several ways to supplement the information brought by  $\alpha_p$  in order to obtain a richer and more stable description. Five paths at least have been explored. In the first one, a second exponent  $\beta_c$ , called the chirp exponent, is defined as follows:

**Definition 2** [21] *Let  $f$  be a function in  $L_{loc}^\infty(\mathbb{R})$ , and denote by  $f^{(-l)}$  a primitive of  $f$  of order  $l$ .  $f$  is called a  $(h, \beta_c)$ -type chirp at  $x_0$  if*

$$\forall n \in \mathbb{N}, \quad f^{(-n)} \in C_{x_0}^{h+n(1+\beta_c)}.$$

This exponent thus characterizes the asymptotic behaviour of the function after a large number of integrations. In a very precise sense, it measures the oscillatory content of the function (see [21]). Another exponent is the *weak scaling* exponent  $\beta_w$  defined by

**Definition 3** [21] *Let  $f$  be a function in  $L_{loc}^\infty(\mathbb{R})$ , and denote by  $f^{(-l)}$  a primitive of  $f$  of order  $l$ . Then*

$$\beta_w(x_0) = \sup\{s : \exists n, \quad f^{(-n)} \in C_{x_0}^{s+n}\}.$$

A related exponent  $\beta_o$ , called the *oscillating exponent*, is obtained by considering what happens when on the contrary an infinitesimal integration is performed:

**Definition 4** [15] *Let  $f \in L_{loc}^\infty(\mathbb{R})$ . We denote by  $h_t(x_0)$  the pointwise Hölder exponent of a fractional primitive of order  $t$  of  $f$  at  $x_0$ . Then*

$$\beta_o(x_0) = \left( \frac{\partial}{\partial t} h_t(x_0) \right)_{t=0+} - 1.$$

A fourth possibility is to use the local Hölder exponent  $\alpha_l$ . To define  $\alpha_l$ , set first  $\alpha_l(f, x_0, \eta) = \sup\{\alpha : f \in C^\alpha(B(x_0, \eta))\}$ , where  $C^\alpha(E)$  is the usual global Hölder space on  $E$  and  $B(x_0, \eta)$  is the open ball centered at  $x_0$  of radius  $\eta$ . Clearly,  $\alpha_l(f, x_0, \eta)$  is non increasing as a function of  $\eta$ , and we set:

**Definition 5** [8] *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. The local Hölder exponent of  $f$  at  $x_0$  is the real number:*

$$\alpha_l(x_0) = \lim_{\eta \rightarrow 0} \alpha_l(f, x_0, \eta)$$

Contrarily to  $\alpha_p$ ,  $\alpha_l$  is stable under the action of pseudo-differential operators. A complete study of  $\alpha_l$  and its links with  $\alpha_p$  is proposed in [26].

Before moving to the fifth path for measuring local regularity, let us give some examples of functions to get a better understanding and to illustrate the role of the exponents defined above.



## 2.2 Examples of Exponents

In the case of an isolated singularity such as  $f(x) = |x|^\gamma$ ,  $\beta_c$ ,  $\beta_w$ ,  $\beta_o$  and  $\alpha_l$  do not yield any new information as compared to  $\alpha_p$ . The same holds for everywhere irregular functions with “regular” pointwise exponent, such as the Weierstrass function. In these cases,  $\beta_w = \alpha_l = \alpha_p$  and  $\beta_c = \beta_o = 0$  at all points.

The simplest function for which the additional exponents bring non trivial information is the so-called chirp, i.e. the function  $f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}$ ,  $\gamma > 0$ ,  $\beta \geq 0$ . In this case, one easily checks that  $\alpha_p = \gamma$ ,  $\beta_w = \infty$ ,  $\beta_c = \beta_o = \beta$  and  $\alpha_l = \frac{\gamma}{1+\beta}$ . In addition, in this simple situation,  $\beta_c$ ,  $\beta_o$  and  $\alpha_l$  allow to predict the evolution of  $\alpha_p$  under integro-differentiation of arbitrary order.

Let us now consider the attractor of an IFS (see for instance [3]). In this case, it is well-known that  $\alpha_p$  is everywhere discontinuous and ranges in an interval  $[\alpha_m, \alpha_M]$ . Furthermore, all the level sets of the function  $x \mapsto \alpha_p(x)$  are everywhere dense. Regarding the additional exponents, one has:  $\beta_w = \alpha_p$ ,  $\beta_c = \beta_o = 0$  and  $\alpha_l = \alpha_m$  at all points. In this case, neither  $\beta_c$ ,  $\beta_w$  nor  $\beta_o$  give a hint that there is a much more complex structure than in the case of, e.g., a Weierstrass function. In contrast, the local exponent, being different from  $\alpha_p$ , indicates that “something is going on”, i.e. that  $\alpha_p$  does not give exhaustive information. The explanation of this phenomenon is twofold. First, as Weierstrass functions, IFS are not oscillating functions, so that the chirp and weak scaling exponents are not of any help here. On the other hand, Weierstrass functions are “monofractal”, while IFS are “multifractal”. Now, the fact that  $\alpha_l$  allows to capture some of the greater complexity of IFS can be traced back to the property of IFS to be “multisingular” at each point, a notion we introduce in section 4. It thus seems that, at least for certain problems in signal processing and in multifractal analysis,  $\alpha_l$  is a relevant tool.

## 2.3 2-microlocal Analysis

The fifth path to characterize local regularity is by far the most complete, since it yields an infinity of exponents: It consists in performing a 2-microlocal analysis of the function. This notion was introduced by J.M. Bony in [2]. It relies on a Littlewood-Paley analysis and we recall it below.

### 2.3.1 Definition and Properties

Let  $\phi$  be a function that belongs to the Schwartz space  $\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall(\gamma, \delta) \in \mathbb{N}^2, \sup_x |x^\gamma \partial^\delta f(x)| < \infty\}$ , such that its Fourier transform satisfies

$$\begin{aligned} \hat{\phi}(\xi) &= 1 & \text{if } |\xi| \leq 1/2, \\ \hat{\phi}(\xi) &= 0 & \text{if } |\xi| \geq 1. \end{aligned}$$

One defines the dilated versions of  $\phi$ : For  $j \in \mathbb{N}$ ,  $\phi_j(x) = 2^j \phi(2^j x)$ , and  $\psi_j = \phi_{j+1} - \phi_j$ . Let  $f$  be a tempered distribution, i.e.  $f$  belongs to the space  $\mathcal{S}'(\mathbb{R})$  defined by

$$\mathcal{S}'(\mathbb{R}) = \{f : \exists C, \exists q \in \mathbb{N}, \forall g \in \mathcal{S}(\mathbb{R}), |\langle f, g \rangle| \leq C \pi_q(g)\},$$

where  $\pi_q(g) = \sup\{(1 + |x|)^q |\partial^\delta g(x)| : |\delta| \leq q, x \in \mathbb{R}\}$ . The Littlewood-Paley Analysis of  $f$  is the set of distributions

$$\begin{aligned} S_0 f &= \phi * f \\ \Delta_j f &= \psi_j * f. \end{aligned}$$

One has the fundamental decomposition (see [21] for details)

$$f = S_0 f + \sum_{j=0}^{+\infty} \Delta_j f.$$

We are now able to define the 2-microlocal spaces.

**Definition 6** Let  $x_0 \in \mathbb{R}$  and  $(s, s')$  two real numbers. A distribution  $f \in \mathcal{S}'(\mathbb{R})$  is said to belong to  $C_{x_0}^{s, s'}$  if there exists a constant  $C$  such that

$$\begin{aligned} |S_0 f(x)| &\leq C(1 + |x - x_0|)^{-s'}, \\ |\Delta_j f(x)| &\leq C 2^{-js} (1 + 2^j |x - x_0|)^{-s'}. \end{aligned}$$

In the following we focus on the local properties of functions. The above definition is not adapted to our study, because it takes into account the behaviour of  $f$  at infinity. We will thus use a local version of 2-microlocal spaces, defined as follows [21]:

**Definition 7** Let  $V$  be an open neighbourhood of  $x_0$  and  $f \in \mathcal{D}'(V)$  a distribution on  $V$ . We say that  $f$  belongs to  $C_{x_0}^{s, s'}$  locally if there exists a smaller neighbourhood  $V_0 \subset V$  of  $x_0$  and a distribution  $g \in C_{x_0}^{s, s'}$  (globally) such that  $f = g$  on  $V_0$ .

By convention, from now on,  $C_{x_0}^{s, s'}$  will denote local 2-microlocal spaces.

The most important property of 2-microlocal spaces, which motivates their definition, is their stability under integro-differentiation

**Proposition 1** Let  $n \in \mathbb{N}$ . For any  $f \in \mathcal{S}'(\mathbb{R})$ , for all  $(s, s') \in \mathbb{R}$ , for all  $x_0$ ,

$$f \in C_{x_0}^{s, s'} \iff f^{(n)} \in C_{x_0}^{s-n, s'}.$$

In fact more is true, as some pseudo-differential operators, such as fractional integro-differentiation, may be considered instead of plain integro-differentiation (see [21]).

A useful characterization of  $C_{x_0}^{s,s'}$  is given by the wavelet coefficients of  $f$  [10]: Let  $\psi$  be a function in the Schwartz class  $\mathcal{S}(\mathbb{R})$  such that  $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k)\}_{(j,k) \in \mathbb{Z}^2}$  forms an orthonormal basis of  $L^2(\mathbb{R})$ . Such a function exists, see for example [20]. The wavelet coefficients of  $f$  are defined by (note that we do not use an  $L^2$  normalization factor for the wavelet coefficients, and take  $2^j$  instead of  $2^{j/2}$ , for convenience in the following proofs)

$$d_{j,k} = 2^j \int f(x)\psi(2^jx - k)dx.$$

In addition, when the function  $\Psi$  is an admissible analyzing wavelet (i.e. satisfying the conditions developed, e.g., in [20]), the continuous wavelet transform of  $f$  is defined, for  $a > 0$  and  $b \in \mathbb{R}$ , by

$$C(a, b) = \frac{1}{a} \int f(x)\Psi\left(\frac{x-b}{a}\right) dx.$$

Then one has the following theorem [15]

**Theorem 1** *Let  $(r, N) \in \mathbb{N} \times \mathbb{N}^*$ . Assume  $\psi(x)$  and  $\Psi$  have at least  $N$  vanishing moments, and that their  $r$  first derivatives have fast decay.*

*Let  $s, s'$  be two real numbers such that  $r + s + \inf(s', 1) > 0$  and  $N > \sup(s, s + s')$ . Then, the three following conditions are equivalent:*

1.  $f \in C_{x_0}^{s,s'}$ ,
2.  $\forall j, k$  such that  $|x_0 - k2^{-j}| \leq 1$ ,  $|d_{j,k}| \leq C2^{-js}(1 + |k - 2^jx_0|)^{-s'}$ ,
3.  $\forall a > 0, |b - x_0| < 1$ ,  $|C(a, b)| \leq Ca^s(1 + \frac{|b-x_0|}{a})^{-s'}$ .

We shall use both characterizations **2.** and **3.** based on the discrete and continuous wavelet transforms. In the sequel, we always assume that  $\psi$  and  $\Psi$  have enough vanishing moments so that characterizations **2.** and **3.** hold.

Finally, there also exists a time domain characterization of 2-microlocal spaces. We recall the following theorem from [25] (see also [16]),

**Theorem 2** *Let  $x_0 \in \mathbb{R}$ , and  $s, s'$  be two real numbers satisfying  $s + s' > 0$ ,  $s + s' \notin \mathbb{N}$ , and  $s' < 0$  (and thus  $s \geq 0$ ).*

*Let  $m = [s + s']$ . A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  belongs to  $C_{x_0}^{s,s'}$  if and only if its  $m$ -th derivative exists around  $x_0$ , and if there exist  $0 < \delta < 1/4$ , a polynomial  $P$  of degree smaller than  $[s] - m$ , and a constant  $C$ , that verify*

$$\left| \frac{\partial^m f(x) - P(x)}{|x - x_0|^{[s]-m}} - \frac{\partial^m f(y) - P(y)}{|y - x_0|^{[s]-m}} \right| \leq C|x - y|^{s+s'-m}(|x - y| + |x - x_0|)^{-s'-[s]+m}$$

for all  $x, y$  such that  $0 < |x - x_0| < \delta$ ,  $0 < |y - x_0| < \delta$ .

The above inequality takes a particularly simple form when  $0 < s + s' < 1$ ,  $-1 \leq s' \leq 1$  and  $s \geq 0$ . This corresponds to the case of a continuous non-differentiable function. Then,  $f$  belongs to  $C_0^{s,s'}$  if there exist a positive real  $\delta$  and a constant  $C$  such that for all  $(x, y)$  with  $0 < |x - x_0| < \delta$ ,  $0 < |y - x_0| < \delta$ ,

$$|f(x) - f(y)| \leq C|x - y|^{s+s'}(|x - x_0| + |x - y|)^{-s'}.$$

Here are some basic properties of 2-microlocal spaces.

**Theorem 3**  $\forall x_0 \in \mathbb{R}$ ,

- $t \leq s$  and  $t + t' \leq s + s' \Rightarrow C_{x_0}^{s,s'} \subset C_{x_0}^{t,t'}$ .
- $\forall s > 0$ ,  $C_{x_0}^s \subset C_{x_0}^{s,-s}$ .
- $\forall (s, s')$  with  $s + s' > 0$ ,  $C_{x_0}^{s,s'} \subset C_{x_0}^s$ .

In the following, we will denote by  $\sigma$  the quantity  $\sigma = s + s'$ .

### 2.3.2 2-microlocal Frontier

2-microlocal analysis allows to give a rich geometrical interpretation of the behaviour of a function around a point. Let us introduce the 2-microlocal domain of  $f$  at  $x_0$ , i.e. the set  $E(f, x_0) = \{(s, s') : f \in C_{x_0}^{s,s'}\}$ . By Theorem 3,  $E(f, x_0)$  is a convex subset of the abstract plane  $(s, s')$ . It is completely determined by its frontier in the  $(s, s')$ -plane, a convex curve, called the 2-microlocal frontier  $\Gamma(f, x_0)$ . This curve possesses right and left derivatives at each point, and they take their value in the interval  $[-1, -\infty]$ . Formally, the frontier is defined by

$$\begin{aligned} \Gamma(f, x_0) : \quad \mathbb{R} &\rightarrow \mathbb{R} \\ s' &\rightarrow s(s') = \sup\{r : f \in C_{x_0}^{r,s'}\} \end{aligned}$$

There are several ways to parameterize the 2-microlocal frontier, e.g.  $s'(s)$ ,  $s'(\sigma)$ , ... (recall that  $\sigma = s + s'$ ). Using  $s$  as the free parameter does not allow to describe portions of the frontier where  $s$  is constant (i.e. where  $\frac{\partial s}{\partial s'}(s') = -\infty$ ). This may only happen on a half line  $-\infty \leq s' < s'_0$ . For an example, see the cusp function in Paragraph 2.5. Likewise, the use of  $\sigma$  as a free parameter is not possible on parts where  $\sigma = s + s'$  is constant (this can only happen on a half line  $s'_1 < s' < +\infty$ , see the Weierstrass function in Paragraph 2.5). Since no parts of the frontier may have  $s'$  constant, this problem does not occur if one takes  $s'$  as a free parameter. For reasons that will appear later, it is simpler to consider  $\sigma(s')$  than  $s(s')$ , and this is the parameterization we shall mainly use in the following.

A simple consequence of Theorem 3 is the following proposition:

**Proposition 2** *The 2-microlocal frontier of  $f$  at  $x_0$ , seen as a function  $s' \rightarrow \sigma(s')$ , verifies*

- $\sigma(s')$  is a concave, non-decreasing function,
- $\sigma(s')$  has left and right derivatives always between 0 and 1.

Remark that, if  $s' \rightarrow \sigma(s')$  is the 2-microlocal frontier of  $f$  at  $x_0$ , then  $\forall \varepsilon > 0$ , one has  $f \in C_{x_0}^{\sigma(s')-s', s'-\varepsilon}$ , but one cannot ensure in general that  $f \in C_{x_0}^{\sigma(s')-s', s'}$ .

## 2.4 Exponents vs 2-microlocal Frontier

As said above, performing a 2-microlocal analysis of a function  $f$  gives a far more complete description of the local regularity than the ones presented in Section 2. Indeed, to each point  $x$  is now associated a curve in  $\mathbb{R}^2$ , namely the set of couples  $(s', \sigma(s'))$ , where  $s' \in \mathbb{R}$ . In particular all the previous regularity exponents can be read from the 2-microlocal frontier:

**Proposition 3** *Let  $s' \rightarrow \sigma(s')$  denote the 2-microlocal frontier of a function  $f$  at  $x_0$ . Assume  $f \in C^\varepsilon(\mathbb{R})$  for some  $\varepsilon > 0$ . Then one has:*

- $\alpha_p(x_0) = -\inf\{s' : \sigma(s') \geq 0\}$ , with the convention that  $\alpha_p(x_0) = +\infty$  if  $\sigma(s') > 0$  for all  $s'$ ,
- $\alpha_l(x_0) = \sigma(0)$ ,
- $\beta_c(x_0) = \lim_{s' \rightarrow -\infty} \frac{s' - \sigma(s')}{\sigma(s')}$ ,
- $\beta_w(x_0) = \lim_{s' \rightarrow -\infty} (\sigma(s') - s')$ ,
- $\beta_o(x_0) = \lim_{s' \rightarrow -\alpha_p} \left( \frac{\sigma(s')}{s' + \alpha_p} \right)^{-1} - 1 = \left( \left( \frac{\partial \sigma}{\partial s'} \right)_- (-\alpha_p) \right)^{-1} - 1$ .

Thus the exponents have a simple interpretation in terms of 2-microlocal analysis: As soon as  $f$  has some minimal global regularity, e.g. when  $f$  belongs to  $C^\varepsilon$ , the pointwise Hölder exponent is simply the intercept between the frontier and the second bisector  $\sigma = 0$ , if it exists. If there is no intercept, then  $\alpha_p = +\infty$ . The exponent  $\beta_w$  is the supremum of the values  $\sigma(s') - s'$  for the couples  $(s, s')$  such that  $f$  belongs to a space  $C_{x_0}^{s, s'}$ . In other words, in the plane  $(s, s')$ ,  $\beta_w$  is the abscissa of the rightmost point in the 2-microlocal frontier.  $\beta_o$  is related to the right derivative of the frontier at the point  $(s, s') = (\alpha_p, -\alpha_p)$  in the  $(s, s')$ -plane (or to the left derivative of the frontier at  $(\sigma, s') = (0, -\alpha_p)$  in the  $(\sigma, s')$ -plane). Remark that these derivatives always exist, since the frontier is a concave function. The exponent  $\beta_c$  is related to the slope of the frontier when  $s' \rightarrow -\infty$ . Finally,  $\alpha_l$  is the intercept of the frontier with the  $s$ -axis, i.e. corresponding to  $s' = 0$ . This is illustrated on figure 1 in the  $(s, s')$  plane, and on figure 2 in the  $(s', \sigma)$  plane.

## 2.5 Examples of Frontiers

In order to obtain a more concrete understanding of the 2-microlocal frontier, we consider in this paragraph examples of simple functions.

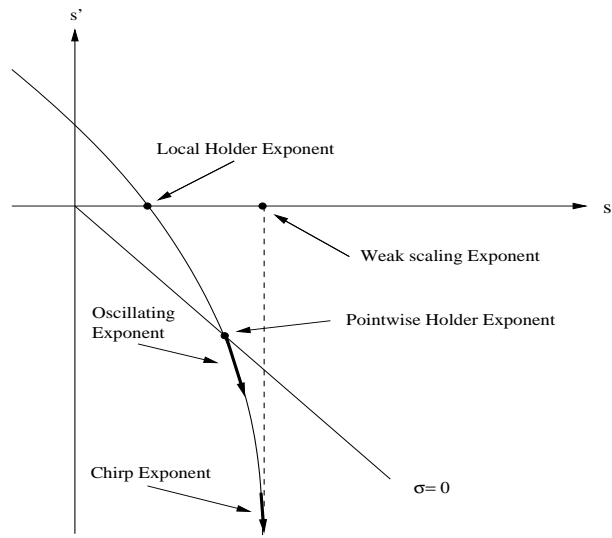


Figure 1: 2-microlocal Frontier and Exponents in the  $(s, s')$ -plane.

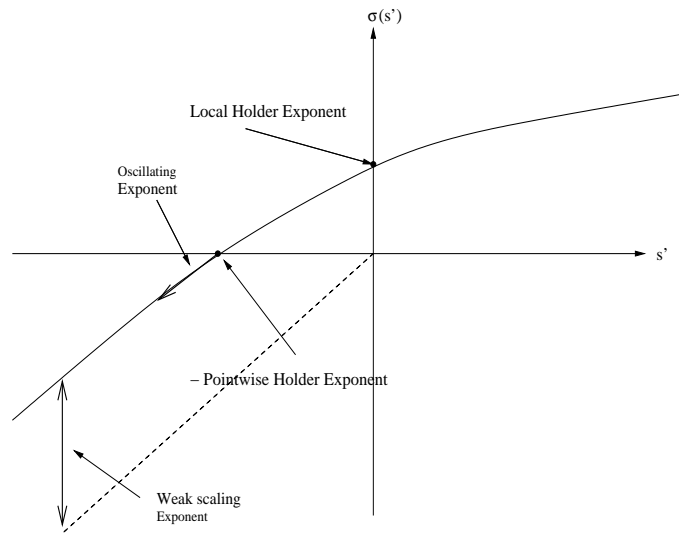


Figure 2: 2-microlocal Frontier and Exponents in the  $(s', \sigma)$ -plane.

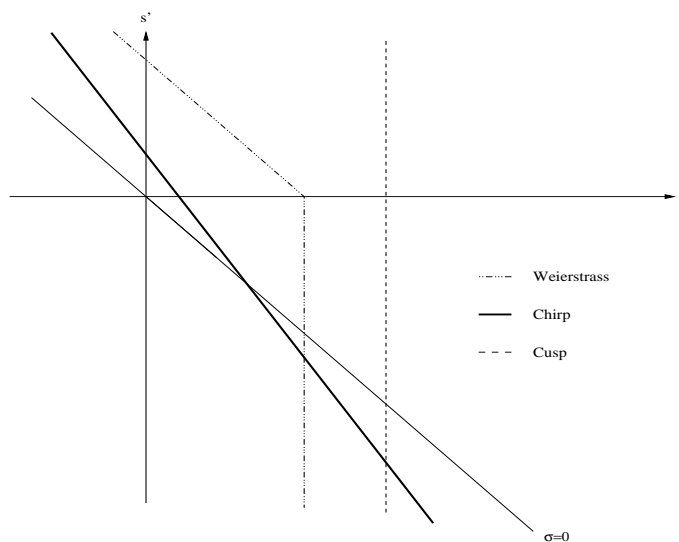


Figure 3: Examples of 2-microlocal Frontiers: Cusp function, chirp function, and a Weierstrass function.

In the case of a cusp function  $x \rightarrow |x|^\gamma$ , only the coefficients that are lying “above” 0 have an influence, since the other ones have a fast decay (i.e.  $C(a, b)$  decays faster than  $a^n$  for all  $n$ ). These large coefficients behave as  $2^{-j\gamma}$ . This leads to a 2-microlocal frontier at  $x = 0$  which is vertical and passes through  $(\gamma, 0)$  (see figure 3). Obviously, the cusp function belongs to all  $C_x^{s, s'}$  when  $x \neq 0$  (since it is  $C^\infty$  on  $\mathbb{R} \setminus \{0\}$ ).

In the case of the chirp function  $x \rightarrow |x|^\gamma \sin(\frac{1}{|x|^\beta})$ , the largest wavelet coefficients lie on the curve  $k2^{-j} = 2^{-j\frac{1}{1+\beta}}$ , and they behave like  $2^{-j\gamma}$ . This corresponds to a 2-microlocal frontier at 0 which is a straight line defined by  $\sigma(s') = \frac{1}{\beta+1}s' + \frac{\gamma}{\beta+1}$  (figure 3). The frontiers are trivial when  $x \neq 0$ .

The Weierstrass function  $W_s(x) = \sum_{n=1}^{+\infty} \lambda^{-ns} \sin(\lambda^n x)$ , where  $\lambda > 1$ ,  $0 < s < 1$ , is known to have  $\alpha_l(x) = \alpha_p(x) = s$  everywhere. As a consequence of Proposition 16, we will see that it implies that the 2-microlocal frontier of the Weierstrass function is the same at all points  $x$ : In the  $(s, s')$ -plane, the frontier is vertical with  $s = \alpha_p = \alpha_l$  for  $s' \leq 0$ , and it is parallel to the second bisector, i.e.  $s' = \alpha_p - s$ , for  $s' \geq 0$  (figure 3).

### 3 2-microlocal Formalism

In this section, we elaborate on a fact noticed in [8]: We show that  $\sigma(s')$  can be obtained as the Legendre transform of a certain function  $\chi$ . The function  $\chi(\rho)$  roughly measures the

exponential rate of decay of the wavelet coefficients on curves of the type  $|b| = a^\rho$ ,  $0 < \rho < 1$  in the time-frequency plane  $(b, a)$ . The relation between  $\chi$  and  $\sigma$  defines a *2-microlocal formalism*, much in the spirit of the multifractal formalism (see Section 4 for more details on this connection).

We start with an obvious remark. Remember that, by definition,  $f \in C_{x_0}^{s, s'}$  if and only if there exists  $b_0 > 0$  and a constant  $C_{s, s'}$  such that:

$$\forall a \in (0, 1], \forall b, |b - x_0| < b_0, |C(a, x_0 + b)| \leq C_{s, s'} a^\sigma (a + |b|)^{-s'},$$

where  $\sigma = s + s'$ . This yields the following simple expression for  $\sigma(s')$  (we take  $x_0 = 0$  to simplify the notations):

$$\sigma(s') = \liminf_{a \rightarrow 0} \inf_{|b| \leq b_0} \frac{s' \log(a + |b|) + \log |C(a, b)|}{\log a} \quad (1)$$

or, in a discrete setting:

$$\sigma(s') = \liminf_{j \rightarrow \infty} \inf_{|k|=0 \dots 2^j} \frac{s' \log(2^{-j} + |k2^{-j}|) + \log |d_{j, k}|}{-j} \quad (2)$$

### 3.1 The 2-microlocal Spectrum

Our aim is to obtain an expression for  $\sigma(s')$  more tractable than (1) or (2). We first need to set some definitions.

**Definition 8** Let  $f \in \mathcal{S}'(\mathbb{R})$ , and denote  $C(a, b)$  its wavelet transform using a wavelet of sufficient regularity. For a given  $x_0 \in \mathbb{R}$ , define:

- $\theta^0 : (0, 1) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$   
 $\theta^0(\varepsilon) = \sup\{\gamma : \exists b_o > 0, a^\varepsilon \leq |b - x_0| < b_o \Rightarrow |C(a, x_0 + b)| \leq K_\varepsilon a^\gamma\}$
- $\theta^1 : (0, 1) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$   
 $\theta^1(\varepsilon) = \sup\{\gamma : \exists b_o > 0, |b - x_0| < \min(b_o, a^\varepsilon) \Rightarrow |C(a, x_0 + b)| \leq K_\varepsilon a^\gamma\}$

Clearly,  $\theta^0$  is a non-increasing function of  $\varepsilon$ , and  $\theta^1$  is a non-decreasing function, so that we may define

$$\theta^0(0) = \lim_{\varepsilon \rightarrow 0^+} \theta^0(\varepsilon) = \sup\{\theta^0(\varepsilon) : \varepsilon \in (0, 1]\}$$

and

$$\theta^1(1) = \lim_{\varepsilon \rightarrow 1^-} \theta^1(\varepsilon) = \sup\{\theta^1(\varepsilon) : \varepsilon \in [0, 1)\},$$

with  $\theta^0(0)$  and  $\theta^1(1)$  in  $[0, +\infty]$ .

Loosely speaking,  $\theta^0(0)$  and  $\theta^1(1)$  characterize the behaviour of the wavelet coefficients respectively below all curves  $|b - x_0| \leq a^\rho$ ,  $\rho > 0$ , and in the neighbourhood of the cone of influence  $|b - x_0| \leq a$ .



**Definition 9** For  $x_0 \in \mathbb{R}$  and for  $\varepsilon > 0$ , define the function  $\chi^\varepsilon: (\varepsilon, 1 - \varepsilon) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$

$$\begin{aligned}\chi^\varepsilon(\rho) &= \sup\{\gamma : \forall a < b_0, \forall \beta \in [\rho - \varepsilon, \rho + \varepsilon], |C(a, x_0 \pm a^\beta)| \leq C_{\rho, \varepsilon} a^\gamma\} \\ &= \liminf_{a \rightarrow 0} \inf_{\rho - \varepsilon \leq \beta \leq \rho + \varepsilon} \frac{\log |C(a, x_0 \pm a^\beta)|}{\log a}\end{aligned}$$

Note that, for each fixed  $\rho, \varepsilon \rightarrow \chi^\varepsilon(\rho)$  is a non-increasing function, so that  $\lim_{\varepsilon \rightarrow 0^+} \chi^\varepsilon(\rho) = \sup\{\chi^\varepsilon(\rho) : \varepsilon > 0\}$  is well defined on  $[0, +\infty]$ .

**Definition 10** Define, for any given  $x_0, \chi: [0, 1] \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

- $\chi(0) = \theta^0(0)$
- $\rho \in (0, 1): \chi(\rho) = \lim_{\varepsilon \rightarrow 0^+} \chi^\varepsilon(\rho)$
- $\chi(1) = \theta^1(1)$

$\chi$  is called the **2-microlocal spectrum** of  $f$  at  $x_0$ . This denomination will be explained in Section 4, and the relation between  $\chi$  and  $\sigma(s')$  will be given by Theorem 1.

Roughly speaking,  $\chi(\rho)$  describes the behaviour of the largest wavelet coefficients that lie around the curve  $|b - x_0| = a^\rho$  (see section 3.6 for examples of computations of  $\chi$ ).

As a preparation, let us investigate some properties of  $\theta^0, \theta^1$  and  $\chi$ . Let  $\tilde{\chi}$  denote the convex envelop of  $\chi$ .

**Proposition 4** The following relations hold:

1.  $\forall \rho_0, \theta^0(\rho_0) \leq \inf_{\rho \in [0, \rho_0]} \chi(\rho)$  and  $\theta^1(\rho_0) \leq \inf_{\rho \in (\rho_0, 1]} \chi(\rho)$
2.  $\chi(0) \leq \liminf_{\rho \rightarrow 0^+} \chi(\rho)$  and  $\chi(1) \leq \liminf_{\rho \rightarrow 1^-} \chi(\rho)$
3.  $\tilde{\chi}(0) = \chi(0)$  and  $\tilde{\chi}(1) = \chi(1)$

**Proof:** To simplify notations, assume without loss of generality that  $x_0 = 0$ . We prove the three assertions for  $\chi(0)$ , as the other part of the proof (concerning  $\chi(1)$ ) follows the same lines.

(1) Fix  $\rho_0$ . For  $\rho \in (0, \rho_0)$  and  $\varepsilon < \min(\rho, \rho_0 - \rho)$ , choose  $\beta \in [\rho - \varepsilon, \rho + \varepsilon]$ . Set  $b = a^\beta$ .

First assume  $\theta^0(\rho_0) < +\infty$ . Then we have  $|C(a, a^\beta)| \leq K_{\rho_0, \eta} a^{\theta^0(\rho_0) - \eta}$  for all  $\eta > 0$ , by definition of  $\theta^0$ , since  $a^{\rho_0} \leq |b|$ . Thus,  $\chi(\rho) \geq \chi^\varepsilon(\rho) \geq \theta^0(\rho_0) - \eta, \forall \eta$ .

If  $\theta^0(\rho_0) = +\infty$ , for every  $N > 0$ ,  $|C(a, a^\beta)| \leq K_{\rho_0, \eta} a^N$ , and one concludes using the same argument that  $\chi(\rho) \geq \chi^\varepsilon(\rho) \geq N$ , for all  $N$ .

Let us now examine the case  $\rho = 0$ . By definition,  $\chi(0) = \theta^0(0)$  and, since  $\theta^0$  is non increasing,  $\theta^0(\rho_0) \leq \chi(0)$ .

(2) One simply uses the definition of  $\theta^0(0)$ :

If  $\theta^0(0) < +\infty$ ,  $\forall \varepsilon > 0$ ,  $\exists \rho_0$ ,  $\forall \rho \in [0, \rho_0]$ ,  $\theta^0(\rho) \geq \theta^0(0) - \varepsilon$ . Using (1), this yields

$$\forall \varepsilon > 0, \exists \rho_0 : \forall \rho \in [0, \rho_0], \chi(\rho) \geq \chi(0) - \varepsilon.$$

If  $\theta^0(0) = +\infty$ ,  $\forall N > 0$ ,  $\exists \rho_0$ ,  $\forall \rho \in [0, \rho_0]$ ,  $\theta^0(\rho) \geq N$ , and thus  $\chi(\rho) \geq N$  for all  $N > 0$ . This gives the result.

(3) This is a simple consequence of (2), the definition of  $\tilde{\chi}$  as the convex envelop of  $\tilde{\chi}$ , and the fact that 0 is an extremal point of the domain of definition of  $\chi$ . ■

**Remark:** The second function of Subsection 3.6 provides an example of a function such that, for an exponent  $\rho_0$ ,  $\theta(\rho_0) < \inf_{\rho \in [0, \rho_0]} \chi(\rho)$ .

**Corollary 1** *If  $\chi(0) < +\infty$ , then either there exists  $\varepsilon > 0$  such that  $\forall \rho \in (0, \varepsilon]$ ,  $\tilde{\chi}(\rho) = +\infty$ , or  $\chi(0) = \tilde{\chi}(0) = \lim_{\rho \rightarrow 0^+} \tilde{\chi}(\rho)$ .*

The following lemma shows that  $\chi$  is well-behaved.

**Lemma 1** *Let  $f \in \mathcal{S}'(\mathbb{R})$  and  $x_0 \in \mathbb{R}$ . Let  $I_{x_0}$  denote the interior of  $\{\rho : \chi(\rho) < +\infty\}$ . The function  $\rho \rightarrow \chi(\rho)$  is lower semi-continuous on  $I_{x_0}$ .*

**Proof:** Let  $\rho \in (0, 1)$ , with  $\rho \in I_{x_0}$ , and  $\varepsilon > 0$  small enough such that  $[\rho - \varepsilon, \rho + \varepsilon] \subset I_{x_0}$ . From the definition of  $\chi$  one gets

$$\chi^\varepsilon(\rho) \leq \inf_{\beta \in (\rho - \varepsilon, \rho + \varepsilon)} \chi(\beta). \quad (3)$$

Indeed the computation of  $\chi(\beta)$  for  $\beta \in (\rho - \varepsilon, \rho + \varepsilon)$  uses wavelet coefficients that are also taken into account when computing  $\chi^\varepsilon(\rho)$ . By definition,  $\lim_{\varepsilon \rightarrow 0^+} \chi^\varepsilon(\rho) = \chi(\rho)$ , thus  $\forall \eta > 0$ , there exists  $\varepsilon_\eta$  such that  $\varepsilon \leq \varepsilon_\eta$  implies  $\chi^\varepsilon(\rho) \geq \chi(\rho) - \eta$ . Using (3), one gets that,  $\forall \eta > 0$ , there exists  $\varepsilon_\eta$  such that  $\forall \varepsilon \leq \varepsilon_\eta$ ,

$$\chi(\rho) \leq \eta + \inf_{0 < \varepsilon \leq \varepsilon_\eta} \chi(\rho \pm \varepsilon).$$

Thus  $\chi(\rho) \leq \liminf_{\varepsilon \rightarrow 0^+} \chi(\rho \pm \varepsilon)$ .

The same arguments may be used to treat the cases  $\rho = 0$  and  $\rho = 1$ . ■

Combining Lemma 1 with Proposition 4 yields that the function  $\rho \rightarrow \chi(\rho)$  is also lower semi-continuous on the closure of  $I_{x_0}$ .

### 3.2 Main Result

For the analysis of the 2-microlocal frontier, it will be useful to define the following partition of  $D = (0, 1] \times [x - b_0, x + b_0]$  into three regions  $I_{\rho_0}$ ,  $II_{\rho_0}$  and  $III_{\rho_0}$  (where  $0 < \rho_0 < 1/2$ ):

$$\begin{aligned} I_{\rho_0} &= \{(a, b) \in D, |b - x_0| < a^{1-\rho_0}\} \\ II_{\rho_0} &= \{(a, b) \in D, a^{1-\rho_0} \leq |b - x_0| \leq a^{\rho_0}\} \\ III_{\rho_0} &= \{(a, b) \in D, a^{\rho_0} < |b - x_0|\}. \end{aligned}$$

We first give a result concerning  $\chi(0)$ .

**Proposition 5**

$$f \in C_x^{s, s'} \Rightarrow \sigma \leq \chi(0)$$

**Proof:** If  $\chi(0) = +\infty$ , the proposition is obvious.

If  $\chi(0) < +\infty$ , fix  $\eta > 0$ . By definition of  $\theta^0(\rho)$ , there exists a sequence  $(a_n, b_n)$  with  $(a_n)$  converging to 0 and  $a_n^\rho \leq |b_n| \leq b_0$  for all  $n$ , and such that:

$$\forall n, |C(a_n, b_n)| \geq K a_n^{\theta^0(\rho) + \eta}$$

for any fixed  $K$ . Indeed, otherwise, there would exist  $a_1 > 0$  and  $b_1 > 0$  such that, for all  $(a, b) \in (0, a_1) \times [-b_1, b_1]$ ,  $|C(a, b)| \leq K a^{\theta^0(\rho) + \eta}$ , contradicting the definition of  $\theta^0$  as a supremum.

Now if  $f \in C_x^{s, s'}$ , by item 3. of Theorem 1, one has, for all  $n$ :

$$K a_n^{\theta^0(\rho) + \eta} \leq |C(a_n, b_n)| \leq C_{\sigma, s'} a_n^\sigma (a_n + |b_n|)^{-s'}$$

This implies

$$K a_n^{\theta^0(\rho) + \eta - \sigma} \leq C_{\sigma, s'} (a_n + |b_n|)^{-s'} \quad (4)$$

Assume first  $s' \leq 0$ . Then, the right-hand side term of (4) remains bounded when  $n$  tends to infinity. Since  $a_n$  tends to 0 when  $n$  tends to infinity, this implies that  $\sigma \leq \theta^0(\rho) + \eta$ . Using  $\chi(0) = \sup_\rho \theta^0(\rho)$ , we get that, for all positive  $\eta$ ,  $\sigma \leq \chi(0) + \eta$ .

If now  $s' > 0$ , then (4) and  $a_n + |b_n| \geq a_n^\rho$  imply:

$$K a_n^{\theta^0(\rho) + \eta - \sigma} \leq C_{\sigma, s'} a_n^{-\rho s'}$$

And, by letting  $n$  go to infinity:

$$\theta^0(\rho) + \eta - \sigma + \rho s' \geq 0$$

Using again  $\chi(0) = \sup_\rho \theta^0(\rho)$ , we get  $\sigma \leq \chi(0) + \rho s' + \eta$  and the result follows by letting  $\rho$  go to 0. ■

We are now ready to state the 2-microlocal formalism. We shall denote by  $g^*$  the following Legendre transform of a function  $g$

$$g^*(y) = \inf_{x \in D_g} (xy - g(y)),$$

where  $D_g$  is the domain of definition of  $g$ .

**Theorem 4**<sup>1</sup> *Let  $f \in \mathcal{S}'(\mathbb{R})$ . The 2-microlocal frontier of  $f$  at any  $x_0$  is given by:*

$$\sigma(s') = (-\chi)^*(s') = \inf_{\rho \in [0,1]} (\rho s' + \chi(\rho)). \quad (5)$$

Recall that  $(-\chi)^*(s') = (-\tilde{\chi})^*(s') = \inf_{\rho \in [0,1]} (\rho s' + \tilde{\chi}(\rho))$ , since  $\tilde{\chi}$  is the convex envelop of  $\chi$ . More precisely, since  $\tilde{\chi}$  is positive, lower semi-continuous on its support and convex, one has  $(-\chi)^{**} = -\tilde{\chi}$ .

**Proof:**

Once again, we take  $x_0 = 0$ . Let us prove first that one has  $\sigma(s') \leq \inf_{\rho \in [0,1]} (\rho s' + \tilde{\chi}(\rho))$ .

By Theorem 1, if  $f \in C_0^{s,s'}$ , there exist  $b_0 > 0$  and  $C_{s,s'}$  such that:

$$\forall a \in (0, 1], \forall b, |b| < b_0, |C(a, b)| \leq C_{s,s'} a^\sigma (a + |b|)^{-s'}. \quad (6)$$

Applying (6) with  $b = a^\rho$  leads to

$$f \in C_0^{s,s'} \Rightarrow \forall \rho \in (0, 1], \forall a \in (0, b_0], |C(a, \pm a^\rho)| \leq C_{s,s'} a^\sigma (a + a^\rho)^{-s'}.$$

Since  $\forall a, \forall \rho, a^\rho < a + a^\rho \leq 2a^\rho$ , one gets:

$$f \in C_0^{s,s'} \Rightarrow \forall \rho \in (0, 1], \forall a \in (0, b_0], |C(a, \pm a^\rho)| \leq K_{s,s'} a^{\sigma - \rho s'},$$

where  $K_{s,s'} = C_{s,s'}$  if  $s' > 0$  and  $K_{s,s'} = 2^{-s'} C_{s,s'}$  if  $s' < 0$ .

Fix  $\rho \in (0, 1)$  and  $\varepsilon < \min(\rho, 1 - \rho)$ . For all  $\beta \in [\rho - \varepsilon, \rho + \varepsilon]$ , we have:

$$|C(a, \pm a^\beta)| \leq K_{s,s'} a^{\sigma - \beta s'} \leq K_{s,s'} a^{\sigma - (\rho \pm \varepsilon) s'},$$

where one chooses  $\rho + \varepsilon$  if  $s' > 0$  and  $\rho - \varepsilon$  if  $s' < 0$ . Thus, from the Definition (9) of  $\chi^\varepsilon$ ,

$$\chi^\varepsilon(\rho) \geq \sigma - (\rho \pm \varepsilon) s',$$

and letting  $\varepsilon$  go to 0,

$$\forall \rho \in (0, 1), \chi(\rho) \geq \sigma - \rho s'.$$

The case  $\rho = 1$  is treated similarly. Indeed, for  $\beta \in [1 - \varepsilon, 1]$ ,

$$|C(a, \pm a^\beta)| \leq K_{s,s'} a^{\sigma - \beta s'} \leq K_{s,s'} a^{\sigma - \max((1-\varepsilon)s', s')},$$

---

<sup>1</sup>A version of Theorem 1 was already proposed in [8]. However this version was in error, as the examples of Section 3.6 show.

and if  $|b| \leq a$ ,

$$|C(a, b)| \leq K_{s, s'} a^{\sigma - s'}.$$

Then, by definition of  $\theta^1$ ,  $\theta^1(1 - \varepsilon) \geq \sigma - \max((1 - \varepsilon)s', s')$ , and thus, letting  $\varepsilon$  go to 0,  $\chi(1) \geq \sigma - s'$ .

The case  $\rho = 0$  has already been taken care of by proposition 5. We end up with:

$$f \in C_0^{s, s'} \Rightarrow \sigma \leq \inf_{\rho \in [0, 1]} (\rho s' + \chi(\rho)).$$

Since  $-\tilde{\chi}$  is the concave envelop of  $-\chi$ , both functions have the same Legendre transform, and thus:

$$f \in C_0^{s, s'} \Rightarrow \sigma \leq \inf_{\rho \in [0, 1]} (\rho s' + \tilde{\chi}(\rho)).$$

This ends the first part of the proof.

Assume now that  $\sigma(s') \neq \inf_{\rho \in [0, 1]} (\rho s' + \chi(\rho))$ . Then there must exist  $s'_0$  and  $\eta > 0$  such that :

$$\sigma(s'_0) < \inf_{\rho \in [0, 1]} (\rho s'_0 + \chi(\rho)) - \eta. \quad (7)$$

Note that  $\sigma(s'_0)$  must verify :

$$\sigma(s'_0) \leq \chi(0). \quad (8)$$

By definition of the 2-microlocal frontier, with  $\eta$  defined in (7), for all  $K$ , there exists a sequence  $(a_n, b_n)_{n \in \mathbb{N}}$ , such that

$$\lim_{n \rightarrow \infty} a_n = 0 \text{ and } |C(a_n, b_n)| \geq K a_n^{\sigma(s'_0) + \eta} (a_n + |b_n|)^{-s'_0}. \quad (9)$$

Fix  $K$  and define:

$$\tilde{\rho} = \liminf_{n \rightarrow \infty} \frac{\log |b_n|}{\log a_n}.$$

There exists a infinite subsequence, still denoted  $(a_n, b_n)$  such that

$$\frac{\log |b_n|}{\log a_n} \rightarrow_{n \rightarrow +\infty} \tilde{\rho}. \quad (10)$$

We distinguish three cases, according to the values of  $\tilde{\rho}$ .

**First case:**  $\tilde{\rho} \in (0, 1)$ . For  $n$  large enough, all the couples  $(a_n, b_n)$  belong to the region  $II_{\rho_0}$  for some  $\rho_0 < \tilde{\rho}$ . (10) implies

$$\forall \varepsilon > 0, \exists N, \forall n > N, a_n^{\tilde{\rho} + \varepsilon} \leq |b_n| \leq a_n^{\tilde{\rho} - \varepsilon},$$

$$\forall \varepsilon > 0, \exists N, \forall n > N, a_n^{\tilde{\rho} + \varepsilon} \leq a_n + |b_n| \leq 2a_n^{\tilde{\rho} - \varepsilon}.$$

Consider first the case  $s'_0 > 0$ , for which:

$$(a_n + |b_n|)^{-s'_0} \geq 2^{-s'_0} a_n^{-(\tilde{\rho} - \varepsilon)s'_0}.$$

(9) entails:

$$|C(a_n, b_n)| \geq K 2^{-s'_0} a_n^{\sigma(s'_0) + \eta - (\tilde{\rho} - \varepsilon)s'_0},$$

and

$$\frac{\log |C(a_n, b_n)|}{\log a_n} \leq \frac{\log K 2^{-s'_0}}{\log a_n} + \sigma(s'_0) + \eta - (\tilde{\rho} - \varepsilon)s'_0. \quad (11)$$

Now,  $a_n^{\tilde{\rho} + \varepsilon} \leq |b_n| \leq a_n^{\tilde{\rho} - \varepsilon}$  implies that:

$$\inf_{\beta \in [\tilde{\rho} - \varepsilon, \tilde{\rho} + \varepsilon]} |C(a_n, \pm a_n^\beta)| \leq |C(a_n, b_n)| \leq \sup_{\beta \in [\tilde{\rho} - \varepsilon, \tilde{\rho} + \varepsilon]} |C(a_n, \pm a_n^\beta)|,$$

and, for  $a_n < 1$ :

$$\frac{\log |C(a_n, b_n)|}{\log a_n} \geq \inf_{\beta \in [\tilde{\rho} - \varepsilon, \tilde{\rho} + \varepsilon]} \frac{\log |C(a_n, \pm a_n^\beta)|}{\log a_n},$$

Together with (11), this yields:  $\forall \varepsilon > 0, \exists N, \forall n > N$ ,

$$\inf_{\beta \in [\tilde{\rho} - \varepsilon, \tilde{\rho} + \varepsilon]} \frac{\log |C(a_n, \pm a_n^\beta)|}{\log a_n} \leq \frac{\log K 2^{-s'_0}}{\log a_n} + \sigma(s'_0) + \eta - (\tilde{\rho} - \varepsilon)s'_0.$$

Since  $a_n \rightarrow 0$  when  $n \rightarrow \infty$ , we get, by definition of  $\chi^\varepsilon$ :

$$\chi^\varepsilon(\tilde{\rho}) \leq \liminf_{n \rightarrow \infty} \inf_{\beta \in [\tilde{\rho} - \varepsilon, \tilde{\rho} + \varepsilon]} \frac{\log |C(a_n, \pm a_n^\beta)|}{\log a_n} \leq \sigma(s'_0) + \eta - (\tilde{\rho} - \varepsilon)s'_0.$$

Letting  $\varepsilon$  go to 0, one eventually finds

$$\sigma(s'_0) \geq \tilde{\rho}s'_0 + \chi(\tilde{\rho}) - \eta,$$

in contradiction with (7).

If now  $s'_0 < 0$ , using  $(a_n + |b_n|)^{-s'_0} \geq a_n^{-(\tilde{\rho} + \varepsilon)s'_0}$ , we get:

$$|C(a_n, b_n)| \geq K a_n^{\sigma(s'_0) + \eta - (\tilde{\rho} + \varepsilon)s'_0},$$

and

$$\frac{\log |C(a_n, b_n)|}{\log a_n} \leq \frac{\log K}{\log a_n} + \sigma(s'_0) + \eta - (\tilde{\rho} + \varepsilon)s'_0,$$

and we end up with the same contradiction.

**Second case:**  $\tilde{\rho} = 0$ . For  $n$  large enough, all the couples  $(a_n, b_n)$  belong to a region  $I_{\rho_0}$ , for  $\rho_0$  small enough. As a consequence, for  $n$  large enough, one has  $a_n^{\rho_0} \leq |b_n|$  and

$$|C(a_n, b_n)| \geq K a_n^{\sigma(s'_0) + \eta} (a_n + |b_n|)^{-s'_0}. \quad (12)$$

Now, if  $\chi(0) = +\infty$ , for all  $N > 0$ , there exist  $\rho_0$  and a constant  $K_{\rho_0, N}$  such that, for all  $(a_n, b_n)$  that belong to  $I_{\rho_0}$  (i.e. for  $n$  large enough)

$$|C(a_n, b_n)| \leq K_{\rho_0, N} a_n^N. \quad (13)$$

(12) and (13) imply

$$K a_n^{\sigma(s'_0) + \eta - N} \leq K_{\rho_0, N} (a_n + |b_n|)^{s'_0}.$$

If  $s'_0 \geq 0$ , the right hand side of (15) remains bounded when  $n$  tends to infinity, which implies that  $\sigma(s'_0) \geq N - \eta$  for all  $N$ .

If  $s'_0 < 0$ , we use  $a_n + |b_n| \geq |b_n| \geq a_n^{\rho_0}$  to write:

$$K a_n^{\sigma(s'_0) + \eta - N} \leq K_{\rho_0, N} a_n^{\rho_0 s'_0},$$

which leads to  $\sigma(s'_0) \geq N$  for all  $N$  (when  $\rho_0$  goes to 0). In both cases,  $\sigma(s'_0) = \chi(0) = +\infty$ , which is in contradiction with (7).

If  $\chi(0) < +\infty$ , by definition of  $\theta^0$ , for all  $\varepsilon > 0$  small enough, there exists  $K_{\rho_0, \varepsilon}$  such that, for all  $n$  large enough:

$$|C(a_n, b_n)| \leq K_{\rho_0, \varepsilon} a_n^{\theta^0(\rho_0) - \varepsilon}. \quad (14)$$

(12) and (14) entail:

$$K a_n^{\sigma(s'_0) + \eta - \theta^0(\rho_0) + \varepsilon} \leq K_{\rho_0, \varepsilon} (a_n + |b_n|)^{s'_0}. \quad (15)$$

The same arguments as before allow to conclude. Indeed, if  $s'_0 \geq 0$ , the right hand side of (15) remains bounded when  $n$  tends to infinity, which implies that:

$$\sigma(s'_0) \geq \theta^0(\rho_0) - \eta - \varepsilon.$$

Since this must be valid for all positive  $\varepsilon$  and  $\eta$  small enough, we get:

$$\sigma(s'_0) \geq \theta^0(\rho_0)$$

This inequality holds for all  $\rho_0 > 0$  small enough. Letting  $\rho_0$  tend to 0, we get

$$\sigma(s'_0) \geq \tilde{\chi}(0).$$

If  $s'_0 < 0$ , we use  $a_n + |b_n| \geq |b_n| \geq a_n^{\rho_0}$  to write:

$$K a_n^{\sigma(s'_0) + \eta - \theta^0(\rho_0) + \varepsilon} \leq K_{\rho_0, \varepsilon} a_n^{\rho_0 s'_0},$$

which entails:

$$\sigma(s'_0) \geq \theta^0(\rho_0) + \rho_0 s'_0 - \eta - \varepsilon,$$

and, again by letting  $\rho_0$  go to 0,

$$\sigma(s'_0) \geq \tilde{\chi}(0).$$

Comparing with (8), we get that, necessarily

$$\sigma(s'_0) = \tilde{\chi}(0).$$

Thus we get that

$$\sigma(s') = \inf_{\rho \in [0,1]} (\rho s' + \tilde{\chi}(\rho)),$$

except maybe for  $s' = s'_0$ , where  $s'_0$  is such that  $\sigma(s'_0) = \tilde{\chi}(0)$ , and for which we could have  $\sigma(s'_0) < \inf_{\rho \in [0,1]} (\rho s' + \tilde{\chi}(\rho))$ . However, this last inequality cannot occur since obviously  $\inf_{\rho \in [0,1]} (\rho s' + \tilde{\chi}(\rho)) \leq \tilde{\chi}(0)$ .

The case  $\tilde{\rho} = 1$  is treated as the case  $\tilde{\rho} = 0$ . This concludes the proof.  $\blacksquare$

**Remark:** Note that, since the Legendre transform is invertible for concave functions, Theorem 1 also allows to compute  $\tilde{\chi}$  from  $\sigma(s')$ :

$$-\tilde{\chi}(\rho) = \inf_{s' \in \mathbb{R}} \{\rho s' - \sigma(s')\}.$$

**Remark:** If one parameterizes the frontier using the function  $s'(\sigma)$ , then the result corresponding to Theorem 1 is the following: if the 2-microlocal frontier is nowhere parallel to the second bisector,

$$s'(\sigma) = \sup_{\rho \in [0,1]} \frac{\sigma - \tilde{\chi}(\rho)}{\rho} = \sup_{\rho \in [0,1]} \left( \frac{\sigma}{\rho} - u(1/\rho) \right) \quad (16)$$

where  $u(\beta) = \beta \tilde{\chi}\left(\frac{1}{\beta}\right)$ . It is easy to check that, if a function  $x \rightarrow g(x)$  is convex, then the function  $x \rightarrow xg(1/x)$  is also convex. Then (16) states that the (convex) function  $s'(\sigma)$  is the Legendre transform for convex functions of  $u$ .

The proof of this result follows the same lines as the ones of Theorem 1, and uses the following additional property, whose proof is omitted:

$$\forall \sigma \leq \chi(0), \quad \sup_{0 < \rho \leq 1} \frac{\sigma - \chi(\rho)}{\rho} \geq \sup_{0 \leq \rho \leq 1} \frac{\sigma - \tilde{\chi}(\rho)}{\rho}$$

### 3.3 Discrete Setting

It is useful to define the analog of  $\chi$  in the discrete setting. For every scale  $j$  and every  $\rho \in [0, 1]$ , denote by  $k_{j,\rho}$  the integer  $k_{j,\rho} = \lfloor 2^{j(1-\rho)} \rfloor$ .

**Definition 11** Let  $f \in \mathcal{S}'(\mathbb{R})$  and let  $(d_{j,k})$  denote its wavelet coefficients using a wavelet of sufficient regularity. For any given  $x_0$ , define



- $\theta^0 : (0, 1) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$   
 $\theta^0(\varepsilon) = \sup\{\gamma : \exists b_0 > 0, \forall \beta \leq \varepsilon, 2^{-j} \leq b_0 \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K_\varepsilon 2^{-j\gamma}\}$
- for any given  $\varepsilon > 0$ ,  $\chi^\varepsilon : (\varepsilon, 1 - \varepsilon) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ ,  
 $\chi^\varepsilon(\rho) = \sup\{\gamma : \exists b_0, \forall \beta \in [\rho - \varepsilon, \rho + \varepsilon], 2^{-j} \leq b_0 \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq C_{\rho,\varepsilon} 2^{-j\gamma}\}$   
 $= \liminf_{j \rightarrow +\infty} \inf_{\rho - \varepsilon \leq \beta \leq \rho + \varepsilon} \frac{\log |d_{j,[2^j x_0 \pm k_{j,\beta}]}|}{-j}$
- $\theta^1 : (0, 1) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$   
 $\theta^1(\varepsilon) = \sup\{\gamma : \exists b_0 > 0, \forall \beta \geq \varepsilon, 2^{-j} \leq b_0 \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K_\varepsilon 2^{-j\gamma}\}$

Clearly,  $\theta^0$  and  $\chi^\varepsilon$  are non-increasing functions, while  $\theta^1$  is non-decreasing. We thus define the analog of  $\chi$  (also denoted  $\chi$ ) in the discrete setting

**Definition 12** Define, for any given  $x_0$ :  $\chi : [0, 1] \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ :

- $\chi(0) = \lim_{\varepsilon \rightarrow 0^+} \theta^0(\varepsilon) = \sup_{\varepsilon > 0} \theta^0(\varepsilon)$ .
- $\rho \in (0, 1)$ :  $\chi(\rho) = \lim_{\varepsilon \rightarrow 0} \chi^\varepsilon(\rho)$
- $\chi(1) = \lim_{\varepsilon \rightarrow 1^-} \theta^1(\varepsilon) = \sup_{\varepsilon > 0} \theta^1(\varepsilon)$ .

Then the following theorem, analogous to Theorem 1 in the continuous case, holds:

**Theorem 4 (bis)** Let  $f$  be in  $\mathcal{S}'(\mathbb{R})$ . The 2-microlocal frontier of  $f$  at any  $x_0$  is given by:

$$\sigma(s') = (-\chi)^*(s') = \inf_{\rho \in [0,1]} (\rho s' + \chi(\rho)).$$

Theorem 4 bis is important since it allows to build functions with explicit formulas for their wavelet coefficients. The proof of Theorem 4 bis is an easy adaptation of the one of Theorem 1. It also uses Lemma 1. This is left to the reader.

### 3.4 Time Domain Version of $\chi$

In the spirit of [16] and [25], a “time domain” equivalent to Theorem 1 may be obtained as follows. As recalled in Theorem 2, when  $(\sigma, s') \in T_0 = \{(\sigma, s') : 0 < \sigma < 1, -1 \leq s' \leq 1, \sigma \geq s'\}$ ,  $f$  belongs to  $C_0^{s,s'}$  if and only if there exist a positive real  $\delta$  and a constant  $C$  such that  $\forall(x, y), 0 < |x - x_0| < \delta, 0 < |y - x_0| < \delta$ ,

$$|f(x) - f(y)| \leq C|x - y|^{s+s'}(|x - x_0| + |x - y|)^{-s'}.$$

This yields the following alternative equation for the 2-microlocal frontier:

$$\sigma(s') = \liminf_{x \rightarrow x_0} \inf_{y: |y-x_0| < |x-x_0|} \left( \frac{\log |f(x) - f(y)|}{\log |x - y|} + s' \frac{\log(|x - x_0| + |x - y|)}{\log |x - y|} \right)$$

This is formally analogous to (1) if we identify  $a$  with  $|x - y|$ ,  $b$  with  $|x - x_0|$ , and  $C(a, b)$  with  $|f(x) - f(y)|$ . Thus, we expect the following relation to hold:

$$(\sigma, s') \in T_0 \Rightarrow \sigma_{x_0}(s') = \inf_{\rho \in [0,1]} (\rho s' + \xi_{x_0}(\rho)) \quad (17)$$

where:

$$\begin{aligned} \xi_{x_0}(0) &= \lim_{\rho \rightarrow 0^+} \tilde{\theta}_{x_0}^0(\rho) \\ \xi_{x_0}(\rho) &= \lim_{\varepsilon \rightarrow 0^+} \tilde{\chi}_{x_0}^\varepsilon(\rho) \\ \xi_{x_0}(1) &= \lim_{\rho \rightarrow 1^-} \tilde{\theta}_{x_0}^1(\rho) \end{aligned}$$

and

$$\begin{aligned} \tilde{\theta}_{x_0}^0(\rho) &= \sup \left\{ \gamma : \begin{array}{l} \exists b_o > 0, \forall x, y \text{ with } |y - x_0| < |x - x_0|, \\ |x - y|^\rho \leq |x - x_0| < b_o \Rightarrow |f(x) - f(y)| \leq C|x - y|^\gamma \end{array} \right\}, \\ \tilde{\theta}_{x_0}^1(\rho) &= \sup \left\{ \gamma : \begin{array}{l} \exists b_o > 0, \forall x, y \text{ with } |y - x_0| < |x - x_0|, \\ |x - x_0| \leq |x - y|^\rho < b_o \Rightarrow |f(x) - f(y)| \leq C|x - y|^\gamma \end{array} \right\}, \\ \tilde{\chi}_{x_0}^\varepsilon(\rho) &= \sup \left\{ \gamma : \begin{array}{l} \exists b_o > 0, \forall x \text{ with } |x - x_0| \leq b_o, \forall \beta \in [\rho - \varepsilon, \rho + \varepsilon], \\ |f(x) - f(x - |x - x_0|^{\frac{1}{\beta}} \text{sgn}(x - x_0))| \leq C|x - x_0|^{\frac{\gamma}{\beta}} \end{array} \right\} \end{aligned}$$

( $\text{sgn}(x)$  denotes the sign of  $x$ ). These quantities should be compared to the ones in Definitions (8) and (9). Note that they need to be modified when  $(\sigma, s') \notin T_0$ . This is achieved in the same manner as in [25]. The details are left to the reader.

Let us put formula (17) to use in the simplest case of the function  $f(x) = |x|^\gamma$ ,  $0 < \gamma < 1$ . A straightforward computation yields  $\xi_0(\rho) = \rho(\gamma - 1) + 1$  and:

$$\begin{aligned} s' \leq 1 - \gamma & : \sigma_0(s') = s' + \gamma \\ s' \geq 1 - \gamma & : \sigma_0(s') = 1 \end{aligned}$$

This example shows that, as expected, (17) gives the right frontier inside  $T_0$ , but may yield wrong results outside  $T_0$ .

Remark finally that  $\chi_0$  and  $\xi_0$  differ: In general, there is no reason why the ‘‘time domain’’ and the ‘‘wavelet domain’’ 2-microlocal spectra should coincide.

### 3.5 Relations between the 2-microlocal Spectrum $\chi$ and the Regularity Exponents

To each point  $x_0$  is associated its 2-microlocal spectrum  $\chi_{x_0}$ . In the same way as the regularity exponents can be deduced from  $\sigma_{x_0}(s')$ , Theorem 1 allows to link them with  $\chi_{x_0}(\rho)$ . In particular, the following set of relations holds.

**Proposition 6** 1. If  $f \in C^\gamma(\mathbb{R})$  for some  $\gamma > 0$ ,  $\alpha_l(x_0) = \inf\{\chi_{x_0}(\rho) : \rho \in [0, 1]\}$

2. If  $f \in C^\gamma(\mathbb{R})$  for some  $\gamma > 0$ ,  $\alpha_p(x_0) = \inf\{\frac{\chi_{x_0}(\rho)}{\rho} : \rho \in (0, 1]\}$

3.  $\beta_w(x_0) = \chi_{x_0}(1) \in [0, +\infty]$

4. If  $\alpha_p(x_0) < +\infty$ , then  $\beta_0(x_0)$  is the smallest real number  $\beta$  that satisfies

$$\chi_{x_0}\left(\frac{1}{\beta+1}\right) = \frac{\alpha_p(x_0)}{\beta+1}.$$

5.  $\alpha_l(x_0) = \alpha_p(x_0) \Rightarrow \alpha_l(x_0) = \chi_{x_0}(1)$  and  $\beta_0(x_0) = 0$

6.  $\alpha_l(x_0) = \alpha_p(x_0) \Rightarrow \chi_{x_0}(\rho) \geq \alpha_p(x_0), \forall \rho$

**Proof:** We omit the subscript  $x_0$  in the proof to simplify the notations.

1. follows from Theorem 1. Indeed,  $\alpha_l$  corresponds to  $\sigma(0)$ , i.e. the intersection between the frontier and the  $s$ -axis. Then we use  $\sigma(0) = \inf_\rho(0s' + \chi(\rho)) = \inf_\rho \chi(\rho)$ .

$\alpha_p$  corresponds to the intersection between the frontier and the second bisector, thus to the  $s'$  (if it exists) such that  $\sigma(s') = 0 = \inf_{\rho \in [0, 1]}(\rho s' + \chi(\rho))$ . This leads to 2. If  $\alpha_p = +\infty$ , this intersection does not exist. By Legendre transform  $\chi(\rho) = +\infty$  if  $\rho \in (0, 1]$ . Thus  $\alpha_p = \inf\{\frac{\chi(\rho)}{\rho} : \rho \in (0, 1]\} = +\infty$ .

Relation 3. follows simply from:

$$\chi(1) = -\inf_{s'}(s' - \sigma(s')) = \sup_{s'}(\sigma(s') - s') = \lim_{s' \rightarrow -\infty} (\sigma(s') - s')$$

Proposition 3 gives  $\beta_0(x_0) = ((\frac{\partial \sigma}{\partial s'})_{\text{left}}(-\alpha_p))^{-1} - 1$ . To prove 4., remark first that if  $\alpha_p < +\infty$ , there exists at least one  $\rho > 0$  such that  $\chi(\rho) = \tilde{\chi}(\rho) = \rho \alpha_p$  (this is due to Lemma 1). Assume for simplicity that  $s' \rightarrow \sigma(s')$  is differentiable at all  $s'$ . Then the following parametric form holds for  $\tilde{\chi}$ :

$$\begin{cases} \rho & = \frac{d\sigma}{ds'}(s') \\ -\tilde{\chi}(\rho) & = s' \frac{d\sigma}{ds'}(s') - \sigma(s') \end{cases}$$

When  $s' = -\alpha_p$ ,  $\beta_0 = \rho^{-1} - 1$  and thus  $\beta_0$  is defined by

$$\begin{aligned} \tilde{\chi}(\rho) & = \tilde{\chi}\left(\frac{1}{\beta_0 + 1}\right) \\ & = \alpha_p b + \sigma(-\alpha_p) \\ & = \frac{\alpha_p}{\beta_0 + 1}. \end{aligned}$$

since  $\sigma(-\alpha_p) = 0$ .

Eventually, 5. and 6. are obvious. ■

**Remarks:**

- No remarkable relation seems to hold between  $\chi$  and  $\beta_c$ .
- The values of  $\chi(0)$  and  $\alpha_p$  are independent.
- $\chi(1)$  controls the shape of the asymptotic branch of the 2-microlocal frontier (when  $s' \rightarrow -\infty$ ). Heuristically, for a “multifractal” function for which all the levels sets of  $\alpha_p$  are dense, “most of the time”,  $\chi(1)$  is finite, and thus the 2-microlocal frontier, in the  $(s, s')$  plane, has a vertical asymptote  $s = \sup\{r : \exists s' \in \mathbb{R}, f \in C_{x_0}^{r, s'}\}$ .
- 5. shows that, when the local and the pointwise exponents coincide, their common value can be inferred using only the wavelet coefficients “above” the considered point. This case is favorable since it leads to simple estimation procedures.
- The uniform Hölder condition  $f \in C^\gamma(\mathbb{R})$  is necessary for 2. to hold in Proposition 6. See subsection 3.6.3 for an example of a function with  $\alpha_p(x_0) \neq \inf\{\frac{\chi_{x_0}(\rho)}{\rho} : \rho \in (0, 1]\}$ .

## 3.6 Examples

We provide now examples of computations of  $\sigma$  and  $\chi$  on various functions, in view of obtaining a more concrete understanding of their relation.

### 3.6.1 Simple Functions

Let us start with the simplest function, i.e. the cusp function  $x \rightarrow |x|^\gamma$ . The wavelet coefficients that are not “above” 0 have a fast decay. This implies that  $\chi_0(\rho) = +\infty$  for  $\rho \in [0, 1)$ . On the other hand, inside the cone, the largest wavelet coefficients behave like  $2^{-j\gamma}$ , thus  $\chi_0(1) = \gamma$ . One easily verifies that  $(-\chi_0)^*(s') = \sigma_0(s') = \gamma + s'$ .

Let us now apply Theorem 1 to the case of a chirp. Recall that, for  $f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}$ ,  $\gamma > 0$ ,  $\beta \geq 0$ , one has at 0:

$$s' = -\frac{\beta + 1}{\beta}s + \frac{\gamma}{\beta},$$

which is the same as

$$\sigma(s') = \frac{1}{\beta + 1}s' + \frac{\gamma}{\beta + 1}.$$

Applying the Legendre transform to  $\sigma(s')$  leads to

$$\begin{cases} \tilde{\chi}(\rho) = \infty & \rho \neq \frac{1}{\beta+1} \\ \tilde{\chi}(\frac{1}{\beta+1}) = \frac{\gamma}{\beta+1} \end{cases}$$

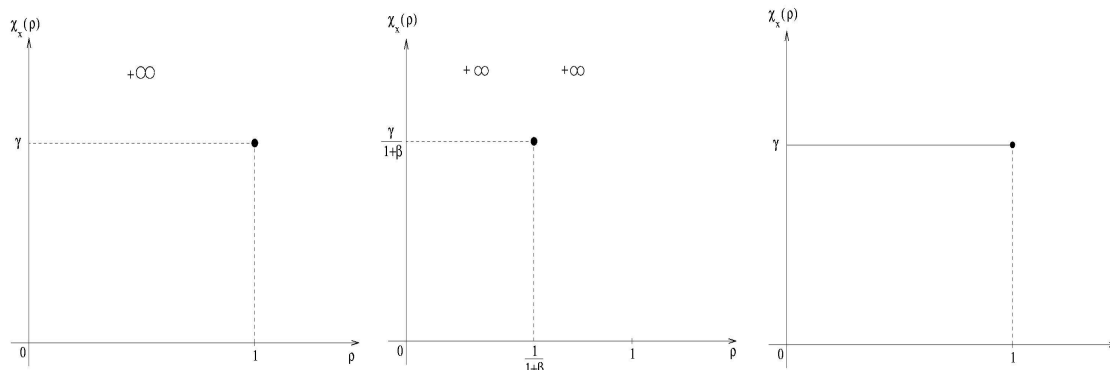


Figure 4: Typical 2-microlocal spectra: Cusp, Chirp and Weierstrass function.

and thus to  $\chi = \tilde{\chi}$ . We see that we recover the well-known fact that the wavelet coefficients of the chirp have fast decay everywhere except around the curve  $a = |b|^{\beta+1}$ , on which the largest coefficients verify  $C(a, \pm a^{\frac{1}{\beta+1}}) \sim a^{\frac{\gamma}{\beta+1}}$ .

Note that, while  $\sigma(s')$  is always concave,  $\chi$  does not have to be convex, and this is why  $\tilde{\chi}$  had to be introduced. (Take for instance the sum of two chirps  $f(x) = |x|^\gamma \sin \frac{1}{|x|^{\beta_1}} + |x|^\gamma \sin \frac{1}{|x|^{\beta_2}}$ , with  $\gamma > 0$ ,  $\beta_1 > 0$ ,  $\beta_2 > 0$ ,  $\beta_1 \neq \beta_2$ .)

We end this subsection with an example where  $\chi(\rho)$  is finite for all  $\rho$ : consider the function  $f_\gamma$ , whose wavelet coefficients are  $d_{j,k} = 2^{-j\gamma}$  for all  $j, k$ , where  $\gamma \in (0, 1)$ . By definition,  $\chi(\rho) = \gamma$  for all  $x$  and  $\rho \in [0, 1]$ . Taking the Legendre transform of  $\chi(\rho)$ , we get  $\sigma(s') = \gamma$  for  $s' \geq 0$  and  $\sigma(s') = \gamma + s'$  for  $s' \leq \gamma$  (since all points  $x$  have the same 2-microlocal features, we drop the subscript  $x$ ). In particular,  $\alpha_p = \alpha_l = \gamma$ . In the  $(s, s')$  plane, the frontiers are parallel to the second bisector for  $s' > 0$  and vertical for  $s' \leq 0$ .  $f_\gamma$  is an example of what we will call in Section 4 a *multisingular* function. It is easy to see that the Weierstrass function  $\sum_{n=1}^{+\infty} \lambda^{-n\gamma} \sin(2\pi\lambda^n x)$  also has  $\chi_x(\rho) = \gamma$  for all  $\rho$  and all  $x$ .

### 3.6.2 More Elaborate Functions

In [8], a version of Theorem 1 was given using, in place of  $\chi(\rho)$ , the function  $\xi$  defined as follows

$$\begin{aligned} \xi(\rho) &= \sup\{\gamma : |C(a, x_0 \pm a^\rho)| \leq Ca^\gamma, \forall a < b_0\}, \\ \xi(1) &= \sup\{\gamma : |C(a, x_0 \pm b)| \leq Ca^\gamma, \forall 0 \leq b \leq a < 1\}. \end{aligned}$$

(note that  $\xi$  is not defined at 0).

In simple cases, it is true that the 2-microlocal frontier  $\sigma(s')$  is given by the Legendre transform of  $\xi$ <sup>2</sup>. However, this statement is wrong in general. The heuristic reason is that the function  $\xi$  does not consider “enough” wavelet coefficients, as shown by the two following examples.

**1** - Let us consider the function  $f_{\beta_1, \beta_2}$  (for  $0 < \beta_2 < \beta_1 < 1$ ) constructed in [25], Theorem 4.1, with  $f(x) = \beta_1$ ,  $g(x) = \beta_2$ . This specific function has a local Hölder exponent equal to  $\beta_2$  everywhere, while the pointwise Hölder exponent equals  $\beta_1$  everywhere except on a set of Hausdorff dimension 0.

Let us study the 2-microlocal frontier at 0 of  $f_{\beta_1, \beta_2}$ . With the help of Proposition 6.2 in [25], one easily computes that  $\xi(\rho) = \chi(\rho) = \beta_1$  for all  $\rho \in (0, 1]$ . The Legendre transform  $\tilde{\sigma}(s')$  of  $\xi$  reads

$$\begin{aligned}\tilde{\sigma}(s') &= \beta_1 + s' \text{ if } s' < 0 \\ \tilde{\sigma}(s') &= \beta_1 \text{ if } s' \geq 0\end{aligned}$$

In particular,  $\sigma = \tilde{\sigma}$  would imply that  $f_{\beta_1, \beta_2}$  belongs to  $C_0^{\beta_1 - \varepsilon, 0}$  for any  $\varepsilon > 0$ . This is not true: Indeed, since the local Hölder exponent of  $f_{\beta_1, \beta_2}$  at 0 is  $\beta_2$ ,  $f_{\beta_1, \beta_2}$  can not belong to any  $C_0^{s, 0}$  for  $s > \beta_2$ .

>From this first example one sees that considering only curves  $b = a^\rho$  for  $\rho > 0$  in the time-frequency plane is not enough. It is necessary to consider the case “ $\rho = 0$ ”, i.e. to define properly  $\chi(0)$ .

Indeed, the computation of  $\chi(0)$  for the function  $f_{\beta_1, \beta_2}$  yields  $\chi(0) = \beta_2$ , leading to

$$\begin{aligned}\sigma(s') &= \beta_2 + s' \text{ if } s' < (\beta_2 - \beta_1) \\ \sigma(s') &= \beta_2 \text{ if } s' \geq (\beta_2 - \beta_1),\end{aligned}$$

which is the correct 2-microlocal frontier of  $f_{\beta_1, \beta_2}$  at 0 (this is left to the reader). Note that  $\chi$  is not continuous at 0. ■

**2** - A second, and distinct, difficulty occurs when one considers  $\xi$  instead of  $\chi$ : The function  $\xi$  is “too focused” on the curves  $b = a^\rho$ , and this is the reason why a regularization procedure is necessary to properly define  $\chi$ .

Consider the function  $f$  defined by its wavelet coefficients  $d_{j,k}$  in an orthonormal wavelet basis  $\{\psi_{j,k}\}_{(j,k)}$ . The  $d_{j,k}$ 's are set as follows. For all  $n \in \mathbb{N}$ , define the integers  $j_n$  and  $k_n$  by

$$\begin{aligned}j_n &= 2^n, \\ k_n &= 2^{\frac{j_n}{2}} + 2n\end{aligned}$$

---

<sup>2</sup>In particular, this will occur when  $\varepsilon \rightarrow \chi^\varepsilon(\rho)$  is continuous at  $\varepsilon = 0^+$  and  $\rho \rightarrow \chi(\rho)$  is continuous at 0 and 1. This is because  $\xi(\rho) = \chi^{\varepsilon=0}(\rho)$  for  $\rho \in (0, 1)$ , and  $\chi$  and  $\tilde{\chi}$  coincide at 0 and 1

Now, we set, with  $\beta > \delta > 0$ ,

$$\begin{aligned} d_{j_n, k_n} &= 2^{-j_n \delta} \quad \forall n \in \mathbb{N}, \\ d_{j, k} &= 2^{-j\beta} \text{ for every other couple of indices } (j, k). \end{aligned}$$

This function  $f = \sum_{j, k} d_{j, k} \psi_{j, k}$  is clearly well-defined and continuous. Moreover,  $f \in C^\delta(\mathbb{R})$ , since  $\forall (j, k)$ ,  $|d_{j, k}| \leq 2^{-j\delta}$ . Let us compute the function  $\xi$  associated to this function  $f$  at point 0.

For all  $\rho > 0$ , we denote by  $k_{j, \rho}$  the integer  $[2^{j(1-\rho)}]$ .

- if  $\rho = 1/2$ , for all  $n \in \mathbb{N}$ ,  $k_n = k_{j_n, 1/2} + 2n$ , thus  $d_{j_n, k_{j_n, 1/2}} = 2^{-j_n \beta}$ . Thus, for all  $j$ ,  $d_{j, k_{j, 1/2}} = 2^{-j\beta}$ , and one deduces  $\xi(1/2) = \beta$ .
- if  $\rho \in ]0, 1[ \setminus \{1/2\}$ , then there exists  $N_\rho$  such that  $n \geq N_\rho$  implies  $|k_n - k_{j_n, \rho}| \geq 2$ . Indeed,  $k_n \sim 2^{j_n(1-1/2)}$  while  $k_{j_n, \rho} \sim 2^{j_n(1-\rho)}$  when  $n \rightarrow +\infty$ .  
Thus, for  $j \geq 2^{N_\rho}$ , one has  $d_{j, k_{j, \rho}} = 2^{-j\beta}$ . One thus concludes that  $\xi(\rho) = \beta$ .
- One eventually has  $\xi(1) = \xi(0) = \beta$ , since the “bad” coefficients (those equal to  $2^{-j\delta}$ ) are located around the curve  $k2^{-j} = 2^{-j/2}$ .

Thus  $\xi(\rho) = \beta$  for all  $\rho \in [0, 1]$ . Applying the Legendre transform to  $\xi$  yields

$$\begin{aligned} \sigma(s') &= \beta + s' \text{ if } s' < 0 \\ \sigma(s') &= \beta \text{ if } s' \geq 0 \end{aligned}$$

In particular this would imply that  $f$  belongs to  $C_0^{\beta-\varepsilon, 0}$  for all  $\varepsilon > 0$ . This means that for all  $(j, k)$  such that  $|k2^{-j}|$  is close enough to 0,  $|d_{j, k}| \leq C2^{-j(\beta-\varepsilon)}$ . This is obviously wrong, since by construction some of them are equal to  $2^{-j\delta}$ .

The problem here comes from the fact that the constants  $C_{\rho, \gamma}$  used in the definition of  $\xi(\rho) = \sup\{\gamma : |d_{j, k_{j, \rho}}| \leq C_{\rho, \gamma} 2^{-j\gamma}\}$  are not uniformly bounded in  $\rho$ . In particular, in this example, they tend to infinity, and this does not allow to obtain a global bound for the decay of the wavelet coefficients. This leads to a wrong 2-microlocal frontier. On the contrary, if one computes the values of  $\chi(\rho)$ , one finds

$$\begin{aligned} \chi(\rho) &= \beta \text{ if } \rho \neq 1/2, \\ \chi(1/2) &= \delta, \end{aligned}$$

which gives by Legendre transform the right 2-microlocal frontier (this is left to the reader). ■

### 3.6.3 A distribution and a function with no positive Hölder regularity

We end this section with two examples where the condition  $\alpha_l > 0$  is not verified.

**1** - Consider the function  $g : x \rightarrow \sqrt{|x|} \sin(2\pi \exp(1/|x|))$ ,  $g(0) = 0$ . Following the same lines of computation as in the case of the chirp (see [17]), it is easy to show that, in the neighbourhood of 0, the ratio  $|g(x) - g(y)|/|x - y|$  is large only around the sequences  $x_k = \frac{1}{\log((4k+1)/4)}$  and  $y_k = \frac{1}{\log((4k+3)/4)}$ . Now:

$$\log|x_k - y_k| \sim -\log k \text{ when } k \rightarrow +\infty,$$

and:

$$\log|g(x_k) - g(y_k)| \sim -1/2 \log \log(k).$$

As a consequence,  $\chi_0(\rho) = \infty$  for  $\rho \neq 0$ , and  $\chi_0(0) = \alpha_l = 0 < \alpha_p = 0.5$ . Theorem 1 yields that  $\sigma(s') = 0$  for all  $s'$ , i.e. the 2-microlocal frontier of  $g$  is the second bisector in the  $(s, s')$ -plane : In this case of a function with no positive uniform Hölder regularity,  $\alpha_p$  is not given by the intersection of the frontier with the  $s'$  axis. The formula  $\alpha_p(0) = \inf_{\rho \in (0,1]} \left( \frac{\chi_0(\rho)}{\rho} \right)$  does not apply.

Note that in this case the 2-microlocal frontier can be obtained without computations as follows. Any primitive  $g^{(-n)}$  of order  $n$  of  $g$  has infinitely fast oscillations around 0. The oscillating exponent of  $g^{(-n)}$  is thus  $+\infty$  for all  $n$ . The last item of Proposition 3 entails that the derivative of the function  $s' \rightarrow \sigma(s')$  is 0 at infinitely many points. As the frontier is convex, it has to be constant. The fact that  $\sigma(0) = 0$  allows to conclude that  $\sigma(s') = 0$  for all  $s'$ .

The same arguments clearly apply to any function of the type  $g_h : x \rightarrow |x|^h w(2\pi \exp(1/|x|))$ , where  $w$  is an oscillating function in the sense of [15].

**2** - On the other hand, the case of the Dirac distribution  $\delta_0$  at 0 shows that one may have  $\alpha_p(0) = \inf_{\rho \in (0,1]} \left( \frac{\chi_0(\rho)}{\rho} \right)$  even though  $\delta_0 \notin \cup_{\varepsilon > 0} C^\varepsilon(\mathbb{R})$ . Assume the wavelet  $\psi$  has compact support. Then the wavelet coefficients of  $\delta_0$  that are located outside the "cone" above 0 vanish. The largest wavelet coefficients inside the cone grow as  $2^j$  when  $j \rightarrow +\infty$ . As a consequence,  $\chi_0(1) = -1$  and  $\chi_0(\rho) = +\infty$  for  $\rho \in [0, 1)$ .

Theorem 1 entails that  $\sigma(s') = -1 + s'$  for all  $s'$ . If one could apply Proposition 6, one would get  $\alpha_l(0) = \alpha_p(0) = -1$ ,  $\beta_w(0) = -1$ ,  $\beta_o(0) = 0$ . Although the definitions we have set for  $\alpha_l$  and  $\alpha_p$  do not make sense for distributions, the values  $\alpha_l(0) = \alpha_p(0) = -1$  are perfectly meaningful:  $\delta_0$  is indeed not oscillatory at 0, with regularity exponents equal to  $-1$ , in the sense, e.g., that  $\delta_0(\lambda \cdot) = \lambda^{-1} \delta_0(\cdot)$ .

## 3.7 $d$ -dimensional case

The 2-microlocal spectrum, as well as the other exponents, can be defined in any dimension. The definitions are slightly modified as shown below, but the 2-microlocal formalism still holds.



**Definition 13** Let  $f \in \mathcal{S}'(\mathbb{R}^d)$ , and denote  $C(a, b)$  its wavelet transform using a wavelet of sufficient regularity. For a given  $x_0 \in \mathbb{R}^d$ , define:

- $\theta^0 : (0, 1) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$

$$\theta^0(\varepsilon) = \sup\{\gamma : \exists b_o > 0, a^\varepsilon \leq \|b - x_0\| < b_o \Rightarrow |C(a, x_0 + b)| \leq K_\varepsilon a^\gamma\}$$

- $\chi^\varepsilon : (\varepsilon, 1 - \varepsilon) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$

$$\chi^\varepsilon(\rho) = \sup\left\{\gamma : \begin{array}{l} \exists b_o > 0, \forall a < b_o, \forall b \text{ with } a^{\rho+\varepsilon} \leq \|b - x_0\| \leq a^{\rho-\varepsilon}, \\ |C(a, x_0 + b)| \leq C_{\rho, \varepsilon} a^\gamma \end{array}\right\}$$

- $\theta^1 : (0, 1) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$

$$\theta^1(\varepsilon) = \sup\{\gamma : \exists b_o > 0, \|b - x_0\| < \min(b_o, a^\varepsilon) \Rightarrow |C(a, x_0 + b)| \leq K_\varepsilon a^\gamma\}$$

The 2-microlocal spectrum  $\chi$  in the  $d$ -dimensional case is defined as in dimension 1:

**Definition 14** Define, for any given  $x_0$ ,  $\chi : [0, 1] \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  by

- $\chi(0) = \theta^0(0)$

- $\rho \in (0, 1): \chi(\rho) = \lim_{\varepsilon \rightarrow 0^+} \chi^\varepsilon(\rho)$

- $\chi(1) = \theta^1(1)$

The following analog to Theorem 1 holds:

**Theorem 4(ter)** Let  $f$  be a function in  $\mathcal{S}'(\mathbb{R}^d)$ . The 2-microlocal frontier of  $f$  at any  $x_0 \in \mathbb{R}^d$  is given by:

$$\sigma(s') = (-\chi)^*(s') = \inf_{\rho \in [0, 1]} (\rho s' + \chi(\rho))$$

## 4 2-microlocal Analysis, Large Deviations and Multifractal Analysis

### 4.1 Recalls on Multifractal Analysis

The 2-microlocal formalism stated in Theorem 1 allows to build a fruitful parallel between 2-microlocal analysis and the large deviation aspects of multifractal analysis. We do no more in this section than take a few steps in this direction. As we shall see, 2-microlocal analysis can be interpreted in a sense as “a multifractal analysis of a function at one point”.

Recall that the multifractal analysis of a function  $Z$  consists in first computing the pointwise Hölder function  $\alpha_p(t)$  of  $Z$ , and then associating to each level set  $E_\alpha = \{t :$

$\alpha_p(t) = \alpha$  its Hausdorff dimension  $f_h(\alpha) = \dim_H(E_\alpha)$ . The function  $f_h(\alpha)$  is called the *Hausdorff multifractal spectrum* of  $Z$ . In addition, one defines a *large deviation multifractal spectrum*  $f_g(\alpha)$  as follows:

$$f_g(\alpha) = \lim_{\varepsilon \rightarrow 0} f_g^\varepsilon(\alpha)$$

where:

$$f_g^\varepsilon(\alpha) = \liminf_{n \rightarrow \infty} \frac{\log N_n^\varepsilon(\alpha)}{n}$$

and:

$$N_n^\varepsilon(\alpha) = \#\{k : \alpha - \varepsilon \leq \alpha_n^k \leq \alpha + \varepsilon\}$$

In the formula above,  $\alpha_n^k$  is the coarse grained exponent corresponding to the interval  $I_n^k = [k2^{-n}, (k+1)2^{-n}]$ , i.e.:

$$\alpha_n^k = \frac{\log |Y_n^k|}{-n}$$

$Y_n^k$  is some quantity that measures the variation of the original function  $Z$  in the interval  $I_n^k$ . Usual choices are the increment  $Y_n^k = Z((k+1)2^{-n}) - Z(k2^{-n})$ , or the oscillation of  $Z$  inside  $I_n^k$ <sup>3</sup>.

When the so-called strong (resp. weak) multifractal formalism holds,  $f_h(\alpha)$  (resp.  $f_g(\alpha)$ ) can be obtained as the Legendre transform of the function  $\tau(q)$  defined as:

$$\tau(q) = \liminf_{n \rightarrow \infty} \frac{\log(S_n(q))}{-n}$$

where

$$S_n(q) = \sum_{k=0}^{2^n-1} |Y_n^k|^q$$

with the convention  $0^q = 0$  for all  $q$ .

Remark that  $\frac{N_n^\varepsilon(\alpha)}{2^n}$  may be interpreted as the probability of hitting an interval with coarse grained exponent roughly equal to  $\alpha$  when  $k$  is chosen randomly according to the uniform probability  $\mathcal{P}_n(k) = 2^{-n}$ ,  $k = 0, \dots, 2^n - 1$ . Thus, saying that the weak formalism holds amounts to stating that the sequence of random variables  $\frac{\log |Y_n^k|}{-n}$  satisfies a large deviation principle with rate function  $f_g = \tau^*$ . A common situation where this occurs is when  $\tau(q)$  exists as a limit rather than a liminf, and is differentiable. Then the Gärtner-Ellis Theorem ensures that  $\tau^* = f_g$  [5].

<sup>3</sup>In applications, one sometimes takes  $Y_n^k$  equal to the wavelet coefficient of  $Z$  at scale  $n$  and location  $k$ . This however does not lead to a well defined multifractal spectrum, since  $f_g$  would then depend on the choice of the wavelet.

## 4.2 Formal Correspondence between 2-microlocal and Multifractal Analysis

To build a parallel between the weak multifractal formalism and 2-microlocal analysis, we start by defining a sequence of measures indexed by scale. Each measure  $\mathcal{Q}_j$  is defined over the set  $\{0, \dots, 2^j - 1\}$ , or, alternatively, on the dyadic intervals  $I_j^k = [k2^{-j}, (k+1)2^{-j}]$ ,  $k = 0, \dots, 2^j - 1$  by:

$$\mathcal{Q}_j(k) = |d_{j,k}|$$

where, as usual,  $d_{j,k}$  is the wavelet coefficient of the function  $Z$  at scale  $j$  and location  $k$ <sup>4</sup>.

In contrast with  $\mathcal{P}_n(k)$ , which does not depend on  $Z$ ,  $\mathcal{Q}_j(k)$  picks up intervals according to the behaviour of  $Z$  at scale  $j$  and location  $k$ . Let us now make the following formal correspondences: we relate  $q$  to  $s'$  (or to  $s$ ),  $\tau(q)$  to  $\sigma(s')$  (or to  $s'(s)$ ), the Hölder exponent  $\alpha$  to the “curve width”  $\rho = \log |b| / \log a = \log((|k|+1)2^{-j}) / \log(2^{-j})$ , and finally the multifractal spectrum  $f_g$  to the 2-microlocal spectrum  $\chi$ . In both fields, i.e. multifractal analysis and 2-microlocal analysis, we have that  $\tau(q)$  (resp.  $\sigma(s')$ ) is the Legendre transform of  $f_g(\alpha)$  (resp.  $\chi(\rho)$ )<sup>5</sup>. When the weak multifractal formalism holds,  $f_g$  is given by the Legendre transform of  $\tau$ . This implies in particular that  $f_g$  is itself concave. By analogy, we shall say that the *weak 2-microlocal formalism* holds when  $\chi$  is convex, and is thus also equal to the Legendre transform of  $\sigma(s')$ <sup>6</sup>.

Again by analogy with multifractal analysis, one would then like to state an associated large deviation principle that would read:

$$\mathcal{Q}_j \left( \frac{\log k}{n} = 1 - \rho \right) \simeq 2^{-j\chi(\rho)}, j \rightarrow \infty \quad (18)$$

This would roughly mean that the measure (induced by the wavelet coefficients) of the interval  $[k2^{-j}, (k+1)2^{-j}]$  decays exponentially with rate  $\chi(\rho)$ , where  $k$  and  $\rho$  are related by  $k = \pm 2^{-j\rho}$ . Relation (18) is motivated by the fact that the definition of  $\chi^\varepsilon$  is the counterpart of the one of  $f_g^\varepsilon$ . Note that the “ $\varepsilon$ -tolerance” is in the (*size, increments*) domain for  $f_g^\varepsilon$ , while it is in the (*a, b*) domain for  $\chi^\varepsilon$ . This is in agreement with the fact that, in the multifractal frame, we are studying the probability that the increment of the signal in a dyadic interval  $I_n^k$  of size  $2^{-n}$  behaves roughly as  $2^{-n\alpha}$ ; In the 2-microlocal frame, the “probability” refers to a portion of the (*a, b*) domain having a behaviour of the form  $b \simeq a^\rho$  when this portion is chosen randomly with measure proportional to  $|C(a, b)|$ .

<sup>4</sup>Note that it is not desirable to normalize the  $\mathcal{Q}_j$ , and thus we are not dealing with a sequence a *probability* measures. Assume for instance that  $d_{j,k} = 2^{-j\alpha}$  for all  $j, k$ . Normalizing  $\mathcal{Q}_j$  would lead to  $\mathcal{Q}_j(k) = 2^{-j}$ : the regularity information, which is essential for our purpose, would be lost. A more precise interpretation of this fact is given below.

<sup>5</sup>The equality  $\tau = f_g^*$  is valid under mild conditions, see e.g. [18], [19].

<sup>6</sup>Since both  $\tau$  and  $\sigma$  are always concave, they carry less information than  $f_g$  and  $\chi$ . However, in contrast with multifractal analysis, where a non concave  $f_g$  has important consequences, it seems that, for the simple applications considered in this work, the information conveyed by  $\sigma$  will usually be sufficient.

### 4.3 Generalized 2-microlocal Spaces

The formal correspondence of the previous section is not exact, because in multifractal analysis, one considers *sums*, as in 2-microlocal analysis, one is dealing with *suprema*. More precisely, the exact 2-microlocal counterparts of the various quantities considered in multifractal analysis, i.e.  $S_n$ ,  $\tau$ ,  $N_n^\varepsilon$ ,  $f_g$ ,  $f_g^\varepsilon$  and  $\tau$  are defined as follows. The sequence of functions corresponding to  $S_n$  is:

$$\Sigma_j(s') = \sum_{k=0}^{2^j-1} |(k+1)2^{-j}|^{s'} |d_{j,k}|$$

$T$  will denote the function corresponding to  $\tau(q)$ , i.e.:

$$T(s') = \liminf_{j \rightarrow \infty} \frac{\log(\Sigma_j(s'))}{-j}$$

Finally, the counterparts of  $N_n^\varepsilon$ ,  $f_g^\varepsilon$ , and  $f_g$  are:

$$M_j^\varepsilon(\rho) = \sum_{k: |\rho_n^k - \rho| \leq \varepsilon} |d_{j,k}|$$

$$X^\varepsilon(\rho) = \liminf_{j \rightarrow \infty} \frac{\log M_j^\varepsilon(\rho)}{-j}$$

$$X(\rho) = \lim_{\varepsilon \rightarrow 0} X^\varepsilon(\rho)$$

(All these definitions assume that we are performing the analysis at  $x = 0$ .) When a weak formalism holds, the correct large deviation principle for  $Q_j$  reads:

$$Q_j \left( \frac{\log k}{n} = 1 - \rho \right) \simeq 2^{-jX(\rho)}, \quad j \rightarrow \infty$$

The time domain equivalents at 0 to the quantities above are defined as:

$$\tilde{\Sigma}_x(s') = \int_{|y| < x} |y|^{s'} |X(x) - X(y-x)| dy$$

$$\tilde{T}(s') = \liminf_{x \rightarrow 0} \frac{\log \tilde{\Sigma}_x(s')}{\log x}$$

$$\tilde{X}^\varepsilon(\rho) = \liminf_{x \rightarrow 0} \frac{\log \int_{x^{\rho-\varepsilon}}^{x^{\rho+\varepsilon}} |X(x) - X(y-x)| dy}{\log x}$$

Of course, we need to explore the links between these new “2-microlocal functions” and the ones used above:

**Proposition 7** *For any distribution:*

$$\begin{aligned} \forall s', \quad T(s') &\leq \sigma(s') \leq T(s' - 1) + 1 \\ \forall \rho, \quad X^{**}(\rho) &\leq \chi^{**}(\rho) \leq X^{**}(\rho) + \rho + 1 \end{aligned}$$

**Proof:** Define  $\rho_k = 1 - \frac{\log(k+1)}{j}$ , so that  $(k+1)2^{-j} = 2^{-j\rho_k}$ . When  $k$  ranges in  $\{0, \dots, 2^j - 1\}$ ,  $\rho_k$  ranges in  $[0, 1]$ . We have:

$$\Sigma_j(s') = \sum_{k=0}^{2^j-1} 2^{-js'\rho_k} |d_{j,k}|$$

Fix  $\varepsilon > 0$  and let  $\beta_i = i\varepsilon, i = 0, \dots, K := \lceil 1/\varepsilon \rceil$ , where  $\lceil u \rceil$  denotes the integer part of  $u$ . Then:

$$\Sigma_j(s') \leq \sum_{i=0}^K \sum_{k: |\rho_k - \beta_i| \leq \varepsilon} 2^{-js'\rho_k} |d_{j,k}|$$

Denote  $\beta = \operatorname{argmax}(\sum_{k: |\rho_k - \beta_i| \leq \varepsilon} 2^{-js'\rho_k} |d_{j,k}|)$  (if there are several such  $\beta$ , take for instance the smallest one). This yields:

$$\begin{aligned} \Sigma_j(s') &\leq (K+1) \sum_{k: |\rho_k - \beta| \leq \varepsilon} 2^{-js'\rho_k} |d_{j,k}| \\ &\leq (K+1) 2^{-js'(\beta \pm \varepsilon)} \sum_{k: |\rho_k - \beta| \leq \varepsilon} |d_{j,k}| \end{aligned}$$

(choose  $\beta + \varepsilon$  is  $s' \leq 0$ ,  $\beta - \varepsilon$  otherwise). Fix  $\eta > 0$ . By definition of  $\chi^\varepsilon$ , for  $j$  large enough,

$$\sup\{|d_{j,k}| : |\rho_k - \beta| \leq \varepsilon\} \leq C 2^{-j(\chi^\varepsilon(\beta) - \eta)}$$

On the other hand,  $\#\{k : |\rho_k - \beta| \leq \varepsilon\} \leq 2^{j(1 - \beta + \varepsilon)}$ , so that:

$$\Sigma_j(s') \leq (K+1) 2^{-j(s'(\beta \pm \varepsilon) + \chi^\varepsilon(\beta) - \eta + \beta - 1 - \varepsilon)}$$

Letting  $j$  go to infinity, and then  $\varepsilon$  and  $\eta$  go to zero, we finally get:

$$T(s') \geq \beta(s' + 1) + \chi(\beta) - 1$$

and thus

$$T(s') \geq \sigma(s' + 1) - 1$$

To find an upper bound to  $T(s')$ , one simply writes:

$$\Sigma_j(s') \geq \sup_{k=0 \dots 2^j - 1} ((k+1)2^{-j})^{s'} |d_{j,k}|$$

This yields:

$$\frac{\log \Sigma_j(s')}{-j} \leq \inf_{k=0 \dots 2^j - 1} \frac{s' \log((k+1)2^{-j}) + \log |d_{j,k}|}{-j}$$

and thus:

$$T(s') \leq \sigma(s')$$

The inequalities involving  $X$  and  $\chi$  follow by Legendre transform. ■

The bounds in proposition 7 are optimal, as the following examples show:

*Lower bound:* Simply take the function  $x \rightarrow |x|^\gamma$ . It is straightforward to check that, in this case,  $T = \sigma$ .

*Upper bound:* We consider this time the function defined by the wavelet coefficients  $d_{j,k} = 2^{-j\gamma}$  for all  $j, k$ . One has:

$$\Sigma_j(s') = 2^{j(s'-\gamma)} \sum_k (k+1)^{s'}$$

Using that  $\frac{1}{j} \log \sum_k (k+1)^{s'} \rightarrow -s' - 1$  for  $s' \leq -1$  and  $\frac{1}{j} \log \sum_k (k+1)^{s'} \rightarrow 0$  for  $s' \geq -1$ , one verifies that  $\sigma(s') = T(s' - 1) + 1$ .

We remark in passing that, denoting  $Z(t) = t\mathbf{1}_{\mathbb{Q}}(t)$ , it is easy to check that the 2-microlocal analysis of  $Z$  at 0 is exactly the mutual multifractal analysis of  $Z$  with respect to the sequence of measures  $\mathcal{Q}_j$  (in fact, any function  $Z$  such that  $Y_n^k = (k+1)2^{-n}$ , where  $Y_n^k$  is the oscillation of  $Z$  in the interval  $I_n^k$ , would do). See [19] for more on mutual multifractal analysis.

>From a different point of view, the definition of  $\sigma(s')$  and  $T(s')$  suggest the introduction of the following functional spaces, as a natural generalization of 2-microlocal spaces:

**Definition 15** *A distribution belongs to the space  $C_0^{p,\sigma,s'}$  if its wavelet coefficients satisfy:*

$$\sup_{j \geq 0} 2^{j\sigma} \left( \sum_k |(k+1)2^{-j}|^{ps'} |d_{j,k}|^p \right)^{1/p} < \infty$$

(the definition of  $C_x^{p,\sigma,s'}$  follows by translation). Note that the  $C_x^{p,\sigma,s'}$  spaces are stable through pseudo-differentiation, e.g.  $X$  belongs to  $C_x^{p,\sigma,s'}$  iff  $X'$  belongs to  $C_x^{p,\sigma-1,s'}$ . Of course,  $C_x^{s,s'} = C_x^{\infty,\sigma-s',s'}$ , and thus  $\sigma_x(s') = \sup\{t : X \in C_x^{\infty,t,s'}\}$ , while  $T_x(s')$  is related to  $C_x^{1,\sigma,s'}$  through:

$$T_x(s') = \sup\{t : X \in C_x^{1,t,s'}\} \tag{19}$$

The spaces  $C_x^{p,\sigma,s'}$  have a flavour somewhat reminiscent of “weighted” Besov spaces  $B_{p,\infty}^s$ . Recall that the Besov space  $B_{p,\infty}^s, p > 0$  is defined as the set of functions  $X$  with wavelet coefficients verifying:

$$\sup_{j \geq 0} 2^{js} \left( \sum_k |d_{j,k}|^p \right)^{1/p} < \infty$$

See [4], [20] for details on Besov spaces. The well-known statement corresponding to (19) in the multifractal frame is: For all  $p > 0$ ,  $\tau(p) = \sup\{s : X \in B_{p,\infty}^{\frac{s}{p}}\}$ .

In fact, if one defines the mapping from  $\mathcal{S}(\mathbb{R})$  to itself which associates to  $X$  the distribution  $\hat{X}_{s'}$  defined through its wavelet coefficients  $\hat{d}_{j,k} = ((k+1)2^{-j})^{s'} d_{j,k}$ , where the  $d_{j,k}$  are the wavelet coefficients of  $X$ , we have the obvious equivalence:

$$X \in C_0^{p,\sigma,s'} \iff \hat{X}_{s'} \in B_{p,\infty}^\sigma$$

Note that  $\hat{X}_{s'}$  is essentially equal to  $|x|^{s'} X$ , so that, roughly speaking,  $X$  is in  $C_{x_0}^{p,\sigma,s'}$  when  $|x - x_0|^{s'} X$  is in  $B_{p,\infty}^\sigma$ .

**Remark:** In the time domain and for  $0 < s < 1$ , the Besov space  $B_{p,\infty}^s$  is characterized by the condition  $\sup_{y>0} y^{-s} \sup_{0<h<y} \int |X(x+h) - X(x)|^p dx < \infty$ , while  $C_x^{p,\sigma,s'}$  is characterized by  $\sup_{t>0} t^{-s} \int |h|^{s'} |X(x+h) - X(x)| dh < \infty$ .

#### 4.4 Multisingular Functions

In multifractal analysis, one says that  $Z$  is monofractal if its spectrum is reduced to a point, and/or if the function  $\tau$  is a straight line. Conversely, a function with a non-degenerate spectrum (usually one requires that  $f_g$  is positive on an interval) and/or a non linear  $\tau$  is called multifractal. Following the correspondences set in Section 4.2, we shall call *monosingular* (at a given point  $x$ ) a function such that  $\sigma(s')$  is linear and/or  $\chi(\rho)$  is finite for a single value of  $\rho$ . The function  $|x|^\alpha$  as well as chirps are thus monosingular (and also monofractal). *Multisingular* functions have finite  $\chi$  for  $\rho$  ranging in an interval and non linear  $\sigma(s')$ . Thus, the notion of multisingularity is concerned with the behaviour of a function around a given point. The notion of multifractality, on the other hand, deals with the variation of regularity along time.

We have already encountered an example of a multisingular function in Section 3.6.2: The function  $f_\gamma$ , whose wavelet coefficients are  $d_{j,k} = 2^{-j\gamma}$  for all  $j, k$ , where  $\gamma \in (0, 1)$ , is such that  $\chi_x(\rho) = \gamma$  for all  $x$  and  $\rho \in [0, 1]$ . In terms of the 2-microlocal frontier, this means  $\sigma(s') = \gamma$  for  $s' \geq 0$ , and  $\sigma(s') = \gamma + s'$  for  $s' \leq 0$ . In addition,  $\alpha_p = \alpha_l = \gamma$ .  $f_\gamma$  is an example where each point  $x$  has a seemingly simple behaviour, with same pointwise and local exponents, but is in fact multisingular because  $\chi(\rho)$  is finite for all  $\rho$ . More general examples of functions that are multisingular at all points include attractors of IFS or the Riemann function, as we shall see below. The 2-microlocal frontier at any given point  $x$  of an IFS is again the union of two half lines, but the frontier varies with  $x$ . IFS serve as a paradigm in multifractal analysis, because they are probably the simplest multifractal functions. We believe that they may also be the simplest multisingular functions.

>From a practical point of view, multifractal functions are richer than monofractal ones because their regularity vary along time. In the same way, a multisingular function displays a whole range of different decays for the wavelet coefficients along different curves  $b = \pm a^\rho$ .

In the time domain, this translates into different behaviours for  $|f(x_0 + x) - f(x_0 + y)|$  when  $x$  and  $y$  tend to  $x_0$  at different rates.

In the thermodynamical interpretation of the multifractal formalism,  $q$  corresponds to temperature,  $\alpha$  to energy,  $\tau(q)$  to free energy, and  $f_g(\alpha)$  to entropy. To get an intuitive understanding of this, note for instance that, when  $q \rightarrow \infty$ , the value of  $\tau(q)$  will be related to those regions for which the local increments are the largest; this corresponds to small values of  $\alpha$ . Conversely,  $q \rightarrow -\infty$  is in relation with points having large  $\alpha$  and small increments. Thus, the “temperature” parameter  $q$  allows to pick different regions in the support of  $X$  according to their regularity.

For a multifractal function, the behaviour of the moments of a given order  $q$ , i.e.  $\sum_k |Y_n^k|^q$ , cannot be deduced from the one of the moment of order  $q_0$ , since  $\tau(q)$  is not linear (remember that  $\sum_k |Y_n^k|^q \sim 2^{-n\tau(q)}$ ): Different “temperatures”  $q$  lead to different scalings of the moments.

In the 2-microlocal frame, we shall interpret the “temperature parameter”  $s$  as a degree of integro-differentiation (we consider here for convenience the parameterization  $s'(s)$  of the frontier): This is best understood if one recalls that changing  $s$  to  $s + \varepsilon$  amounts to translating by  $\varepsilon$  the frontier along the  $s$  axis, and thus to performing an integro-differentiation of order  $\varepsilon$ . We measure the influence of a change of the temperature  $s$  on our function by the variation of  $\alpha_p$  that it induces. A linear  $s'(s)$  means that the behaviour of  $X$  (and most notably the variation of its pointwise Hölder exponent) through integro-differentiation of any order  $\varepsilon$  may be deduced once we know its variation for a single  $\varepsilon_0$ . In contrast, for a multisingular  $X$ , the evolution of  $\alpha_p$  under integration of order  $\varepsilon_1$  cannot be predicted from the one under integration of order  $\varepsilon_0$ , because  $s'$  is not linear. Multisingularity means that knowledge of the whole frontier is necessary in order to predict the evolution of  $\alpha_p$  under arbitrary orders of integration. For instance, in the simplest case of the function  $f_\gamma$  above, one observes a “phase transition” at  $s' = 0$ .

It is of interest to explore the links between multisingularity and multifractality. How does the multisingular formalism relate with the multifractal one? Is there a simple condition on the evolution in time of the  $\chi$  functions that implies the multifractal formalism? What is in general the relation between the level sets  $E_\alpha$  of  $\alpha_p$  and the classes  $C_{\chi_0} = \{x : \chi_x = \chi_0\}$  (of course each class  $C_\chi$  is included in a single  $E_\alpha$ )? A few preliminary results in this direction are given in Section 5.



Multifractal Analysis	2-microlocal Analysis
$q$	$s'$ or $s$
$\tau(q)$	$\sigma(s')$ or $s'(s)$
$\alpha$	$\rho$
$f_g(\alpha)$	$\chi(\rho)$
$\dim_B(\text{Supp}(\mu))$	$-\alpha_l$
$D_q = \frac{\tau(q)}{q-1}$	$\Delta(s') = \frac{\sigma(s')}{s'+\alpha_p}$
Information Dimension $\alpha^* = D_1$	Information Exponent $\rho^* = \sigma'(-\alpha_p)$

**Table1:** Formal correspondences between 2-microlocal and multifractal analysis.

Table 1 recapitulates the formal correspondences between quantities in multifractal analysis and in 2-microlocal analysis. >From this table, one sees that the counterpart of the box dimension of the support of the analyzed measure  $\mu$  is the opposite of the local exponent (because  $\tau(0) = -\dim_B$  while  $\sigma(0) = \alpha_l$ ). This is why it is not desirable in general to normalize the sequence of measures  $\mathcal{Q}_j$  defined in 4.2: Normalizing  $\mathcal{Q}_j$  amounts to setting  $\alpha_l = 1$  (see footnote 3). It would correspond to ignoring the dimension of the support of the measure in multifractal analysis.

The quantities  $\Delta(s') = \frac{\sigma(s')}{s'+\alpha_p}$ ,  $s' \neq -\alpha_p$ , correspond to the well-known generalized dimensions  $D_q = \frac{\tau(q)}{q-1}$ ,  $q \neq 1$ , defined in multifractal analysis. The denominator  $q - 1$  in the definition of  $D_q$  has been replaced by  $s' + \alpha_p$  since  $\sigma(s')$  crosses the  $s'$  axis at  $s' = -\alpha_p$  while  $\tau(q)$  crosses the  $q$  axis at  $q = 1$ . The multifractal spectrum of a measure is always below the first bisector with at least an exponent  $\alpha^*$  such that  $\alpha^* = f(\alpha^*) = D_1$ .  $D_1$  is called the information dimension. Similarly, in 2-microlocal analysis,  $\chi(\rho)$  is always larger than or equal to  $\alpha_p \rho$  (because  $\alpha_p = \inf_\rho \frac{\chi(\rho)}{\rho}$ ). When  $f \in C^\gamma$  for some  $\gamma > 0$ , the smallest  $\rho$  such that  $\chi(\rho) = \alpha_p \rho$ , denoted  $\rho^*$ , may rightfully be called the *information exponent*. It is equal to  $\Delta(-\alpha_p)$ , where the definition of  $\Delta(-\alpha_p)$  parallels the one of the information dimension in multifractal analysis:

$$\rho^* = \Delta(-\alpha_p) = \lim_{s' \rightarrow -\alpha_p} \frac{\sigma(s')}{s' + \alpha_p} = \lim_{s' \rightarrow -\alpha_p} \frac{\sigma(s') - \sigma(-\alpha_p)}{s' + \alpha_p} = \sigma'(-\alpha_p)$$

(if the derivative does not exist, take the left derivative, which always exists). We see that  $\rho^*$  is equal to  $\frac{1}{1+\beta_0}$ . The intuitive meaning of the information exponent  $\rho^*$  is the following one: The rate of increase of  $\alpha_p$  through infinitesimal integration is governed by the wavelet coefficients lying on the curve  $|b - x_0| = a^{\rho^*}$ . Thus, for any  $f$  and any  $x_0$ , the situation w.r.t. infinitesimal integration can be reduced to that of a chirp: *There always exists a single curve  $|b - x_0| = a^{\rho^*}$ , with  $\rho^* = \sigma'(-\alpha_p)$ , that dictates the rate of increase of  $\alpha_p$ , which equals  $\frac{1}{\rho^*}$ .* Likewise, in the “time domain”, the behaviour of  $\alpha_p$  through infinitesimal integration is controlled by differences of the form  $f(|x|^{\rho^*}) - f(|x|^{\rho^*} - |x|)$  for  $x$  tending to  $x_0 = 0$ .

## 5 The Neighbourhood Exponent $\chi_x(0)$

The value of  $\chi_x(0)$  provides a new regularity exponent, which gives information complementary to  $\alpha_l$  and  $\alpha_p$ . Since  $\chi_x(0)$  is concerned only with what happens in the neighbourhood of  $x$ , we shall call it the *neighbourhood exponent*. It appears that  $\chi_x(0)$  as well as the function  $x \rightarrow \chi_x(0)$  hold essential information both from a multifractal and a 2-microlocal point of view. In this section, we describe some of the properties of the neighbourhood exponent.

### 5.1 Relations with other exponents

The following proposition is an obvious consequence of the shape of the frontier:

**Proposition 8** *Assume  $\alpha_l(x) > 0$ . Then  $\alpha_l(x) \leq \min(\alpha_p(x), \chi_x(0))$ .*

Note that there is no general relation between  $\alpha_p(x)$  and  $\chi_x(0)$ , as the following examples show (such a relation is not expected, since, for instance, the wavelet coefficients that contribute to  $\alpha_p(x)$  and  $\chi_x(0)$  may belong to different regions of the  $(a, b)$  plane).

Consider first the function  $x \rightarrow |x|^\gamma$ . In this case, the exponents at 0 are  $\alpha_p(0) = \alpha_l(0) = \gamma < \chi_0(0) = +\infty$ .

For the reverse inequality  $\chi_x(0) < \alpha_p(x)$ , recall subsection 3.5.2 and the function  $f_{\beta_1, \beta_2}$ . We have proved that in this case,  $\chi_0(0) = \beta_2$  while  $\alpha_p(x) = \beta_1 > \beta_2$ . For a more explicit example, consider the function  $f$  given by  $f(x) = \exp(-1/|x|) \sin(2\pi \exp(1/|x|))$ ,  $f(0) = 0$ . Following the same lines of computation as in the case of the chirp [17], one finds that  $|f(x) - f(y)|/|x - y|$  is large only around the sequences  $x_k = \frac{1}{\log((4k+1)/4)}$ ,  $y_k = \frac{1}{\log((4k+3)/4)}$ . Since

$$\log(x_k - y_k) \sim -\log(k), k \rightarrow \infty$$

and

$$\log |f(x_k) - f(y_k)| \sim -\log(k), k \rightarrow \infty,$$

one gets that  $\chi_0(\alpha) = +\infty$  for  $\alpha \neq 0$  and  $\chi_0(0) = 1$ . In particular,  $\alpha_l(0) = \chi_0(0) = 1 < \alpha_p(0) = +\infty$ . Note that such a function, which has uniform regularity 1, has for frontier the straight line  $\sigma(s') = 1$ , which never crosses the second bisector. This is consistent with the fact that  $\alpha_p = +\infty$  and Proposition 2.

**Proposition 9** *Let  $f \in \mathcal{S}'(\mathbb{R})$ .*

$$\chi_x(0) = \sup_{s' \in \mathbb{R}} \sigma(s') = \lim_{s' \rightarrow +\infty} \sigma(s')$$

The proof is obvious.

The exponent  $\chi_x(0)$  shares an important property with  $\alpha_l$ . Indeed, the stability of  $C_x^{s, s'}$  spaces with respect to fractional integro-differentiation obviously entails the one of  $\chi_x(0)$  (remember that this is not the case for  $\alpha_p$ ). Thus, for instance, for any  $f$  and  $x$ ,  $\chi_x^{f'}(0) = \chi_x^f(0) - 1$ .

**Proposition 10** *The exponent  $\chi_x(0)$  is stable under the action of fractional integro-differentiation.*

## 5.2 Properties of $x \rightarrow \chi_x(0)$ , Links with Multifractal Analysis and Compatibility Conditions between 2-microlocal Frontiers

The first two propositions show that, not surprisingly,  $\chi_x(0)$  is intimately related to the regularity of the points lying in the neighbourhood of  $x$ .

**Proposition 11** *If  $\chi_x(0) < +\infty$ , then  $\forall \delta > 0$ , there exists a neighbourhood  $V_x^\delta$  of  $x$  such that, for all  $y \in V_x^\delta$ , for all  $\rho \in [0, 1]$ ,  $\chi_y(\rho) \geq \chi_x(0) - \delta$ .*

*Moreover, for every  $y \in V_x^\delta$ , one has  $\sigma_y(s') \geq \sigma^{\chi(0), \delta}(s')$ , where  $\sigma^{\chi(0), \delta}(s')$  is defined by*

$$\begin{aligned}\sigma^{\chi(0), \delta}(s') &= (\chi_x(0) - \delta) + s' \text{ if } s' < 0 \\ \sigma^{\chi(0), \delta}(s') &= (\chi_x(0) - \delta) \text{ if } s' \geq 0\end{aligned}$$

**Proof:** Let  $x$  be such that  $\chi_x(0) < +\infty$ . By definition, for all  $\delta > 0$ , there exists  $\rho_\delta > 0$  such that  $\theta_x^0(\rho_\delta) \geq \chi_x(0) - \delta/2$ , where we recall that

$$\theta_x^0(\rho) = \sup \left\{ \gamma : \begin{array}{l} \exists b, C_{\rho, \gamma} \text{ such that } \forall (j, k) \text{ with} \\ 2^{-j\rho} \leq |k2^{-j} - x| \leq b, \text{ one has } |d_{j, k}| \leq C_{\rho, \gamma} 2^{-j\gamma} \end{array} \right\}$$

Thus there exists  $b_\delta$  and a constant  $C$  such that,  $\forall (j, k)$  with  $2^{-j\rho_\delta} \leq |k2^{-j} - x| \leq b_\delta$ , one has  $|d_{j, k}| \leq C2^{-j(\chi_x(0) - \delta)}$ . Let us denote by  $\Gamma_{x, \delta}$  this set of coefficients.

Let now  $y$  be in  $(x - b_\delta, x + b_\delta)$ , and consider  $\eta_y > 0$  such that  $[y - \eta_y, y + \eta_y] \subset ((x - b_\delta, x) \cup (x, x + b_\delta))$ .

Let  $\rho \in [0, 1]$ ,  $\varepsilon > 0$ , and let us compute  $\chi_y^\varepsilon(\rho)$ . One knows that, for all  $(j, k)$  such that  $|k2^{-j} - y| \leq \eta_y$  and  $2^{-j\rho_\delta} \leq |k2^{-j} - x|$ ,

$$|d_{j, k}| \leq C2^{-j(\chi_x(0) - \delta)}. \quad (20)$$

These wavelet coefficients are the ones located around  $y$ , with a scale  $2^{-j}$  small enough so that  $d_{j, k} \in \Gamma_{x, \delta}$ .

Hence, for  $j$  large enough, all the coefficients used for the computations of  $\chi_y^\varepsilon(\rho)$  verify (20). One deduces that  $\chi_y^\varepsilon(\rho) \geq \chi_x(0) - \delta$ , for all  $\varepsilon > 0$ .

Letting  $\varepsilon \rightarrow 0$  leads to the result, i.e. for all  $y \in (x - b_\delta, x + b_\delta)$ ,  $\forall \rho \in [0, 1]$ ,  $\chi_y(\rho) \geq \chi_x(0) - \delta$ .

The second part of Proposition 11 simply follows from the first part, since  $\chi_y(\rho) \geq \chi_x(0) - \delta \forall \rho \in [0, 1]$  implies, by applying the Legendre Transform, that  $\sigma_y(s') \geq \sigma^{\chi(0), \delta}(s') \forall s' \in \mathbb{R}$ . ■

The next proposition, somehow complementary to the previous one, shows that if  $\chi_x(0) = +\infty$ , then  $f$  must be very regular in the (excluded) neighbourhood of  $x$ .

**Proposition 12** *If  $\chi_x(0) = +\infty$ , then for all  $n \in \mathbb{N}$ , there exists a neighbourhood  $V_x^n$  of  $x$  such that  $f \in C^n(V_x^n \setminus \{x\})$ .*

**Proof:** The proof is similar to the one of Proposition 11. Let  $x$  be such that  $\chi_x(0) = +\infty$ . For all  $n > 0$ , there exists  $\rho_n > 0$  such that  $\theta_x^0(\rho_n) \geq n$ .

Thus there exists  $b_n$  and a constant  $C$  such that,  $\forall (j, k)$  such that  $2^{-j\rho_n} \leq |k2^{-j} - x| \leq b_n$ , one has  $|d_{j,k}| \leq C2^{-jn}$ . Denote by  $\Gamma_{x,n}$  this set of coefficients.

Let now  $y$  be in  $(x - b_n, x + b_n)$ , and consider  $\eta_y$  such that  $(y - \eta_y, y + \eta_y) \subset ((x - b_n, x) \cup (x, x + b_n))$ . Let  $\rho \in [0, 1]$ ,  $\varepsilon > 0$ , and let us compute  $\chi_y^\varepsilon(\rho)$ . One knows that, for all  $(j, k)$  such that  $|k2^{-j} - y| \leq \eta_y$  and  $d_{j,k} \in \Gamma_{x,n}$ , one has

$$|d_{j,k}| \leq C2^{-jn}.$$

Hence, using the same arguments as in Proposition 11,  $\chi_y^\varepsilon(\rho) \geq n$ , for all  $\varepsilon > 0$ .

This leads to the following property: for all  $y \in (x - b_n, x + b_n)$ ,  $\forall \rho \in [0, 1]$ ,  $\chi_y(\rho) \geq n$ .

In particular, for all  $y \in (x - b_n, x + b_n)$ , the local Hölder exponent at  $y$ , which is equal to  $\inf_\rho(\chi_y(\rho))$ , is larger than  $n$ . This concludes the proof. ■

Rephrasing Proposition 12 as follows yields a link between 2-microlocal and multifractal analysis. Recall first that, for a function  $f$ , the set  $E_\alpha$  is defined by

$$E_\alpha = \{x : \alpha_p(x) = \alpha\},$$

and the Hausdorff spectrum  $f_h$  of  $f$  is

$$f_h(\alpha) = d_H(E_\alpha),$$

where  $d_H$  is the Hausdorff dimension.

**Corollary 2** *Assume  $f \in C^\gamma$  for some  $\gamma > 0$ . Then:*

$$\sup_{x \in \mathbb{R}} \chi_x(0) = +\infty \Rightarrow \text{Supp}(f_h) \text{ unbounded.}$$

**Proof:** The result simply follows from the inequality  $\alpha_p(x) \geq \alpha_l(x)$  and Proposition 12. ■

In the same spirit, one has

**Proposition 13** *Let  $f \in C^\gamma(\mathbb{R})$  with  $\gamma > 0$ . Then  $E_\infty = \{x : \alpha_p(x) = +\infty\}$  is included in  $\{x : f \text{ is monosingular at } x\}$ .*

**Proof:** If  $\alpha_p(x) = +\infty$ , then the 2-microlocal frontier of  $f$  at  $x$  (in the  $(s, s')$ -plane) never crosses the second bisector. Using that  $\alpha_p(x) = \inf_\rho(\frac{\chi_x(\rho)}{\rho})$ , this means that  $\chi_x(\rho) = +\infty$  whenever  $\rho \in (0, 1]$ . Only  $\chi_x(0)$  may be finite, thus  $f$  is monosingular at  $x$ . ■

**Proposition 14** *Let  $f_h$  denote the Hausdorff spectrum of a function  $f \in C^\gamma$  for some  $\gamma > 0$ . Then:*

$$\text{supp}(f_h) \text{ bounded} \Rightarrow f \text{ is everywhere multisingular.}$$

**Proof:** Assume there exists a point  $x$  where  $f$  is monosingular. The 2-microlocal frontier of  $f$  at  $x$  is by assumption a straight line, which has a slope less or equal than  $-1$  (in the  $(s, s')$ -plane).

Assume the slope is strictly less than  $-1$ . Then, using that  $\chi_x(0) = \lim_{s' \rightarrow +\infty} \sigma(s')$ , we get that  $\chi_x(0) = +\infty$ . Corollary 2 then entails that  $\text{Supp}(f_h)$  is unbounded, hence a contradiction.

Thus the 2-microlocal frontier of  $f$  at  $x$  must be parallel to the second bisector, which implies  $\chi_x(\rho) = +\infty$  for all  $\rho \in (0, 1]$ , and thus  $\alpha_p(x) = +\infty$ . This leads to the same contradiction. ■

The following proposition shows that the neighbourhood exponent function satisfies the same regularity constraint as the local Hölder function.

**Proposition 15** *The function  $x \rightarrow \chi_x(0)$  (which maps  $\mathbb{R}$  to  $[0, +\infty]$ ) is a lower semi-continuous function.*

**Proof:** This is a consequence of Propositions 11 and 12. Indeed, if  $\chi_x(0) < +\infty$ , for all  $\delta > 0$ , there exists a neighbourhood  $V_\delta$  of  $x$  such that for all  $y \in V_\delta$ ,  $\chi_y(0) \geq \chi_x(0) - \delta$ .

If  $\chi_x(0) = +\infty$ , for all  $n > 0$ , there exists a neighbourhood  $V_n$  of  $x$  such that for all  $y \in V_n$ ,  $\chi_y(0) \geq n$ .

These two facts entail that  $x \rightarrow \chi_x(0)$  is a lower semi-continuous function. ■

A straightforward consequence of Proposition 15 is given below

**Proposition 16** *If  $\alpha_l(x) = \chi_x(0)$ , then the 2-microlocal frontier at  $x$  verifies:  $\forall s' > 0$ ,  $\sigma(s') = \alpha_l(x)$ . This means that, in the  $(s, s')$  plane, the 2-microlocal frontier is a half line parallel to the second bisector for  $s' > 0$ , passing through the point  $(\alpha_l(x), 0)$ .*

The proof is obvious and is left to the reader. The last results imply some compatibility conditions on the 2-microlocal frontiers of a function  $f$  at neighbouring points.

**Proposition 17** *Let  $f \in C^\gamma$  for some  $\gamma > 0$ .*

$$\chi_x(0) \leq \liminf_{y \rightarrow x} \left( \inf_{\rho \in [0, 1]} (\chi_y(\rho)) \right) = \liminf_{y \rightarrow x} \alpha_l(y)$$

**Proof:** This is a simple consequence of Propositions 11 and 12.

Indeed, Proposition 11 shows that if  $\chi_x(0) < +\infty$ ,  $\forall \delta > 0$ , there exists a neighbourhood  $V_\delta$  of  $x$  such that for every  $y \in V_\delta$ ,  $\inf_{\rho \in [0, 1]} \chi_y(\rho) \geq \chi_x(0) - \delta$ .

If  $\chi_x(0) = +\infty$ , there exists a neighbourhood  $V_n$  of  $x$  such that for every  $y \in V_n$ ,  $\inf_{\rho \in [0,1]} \chi_y(\rho) \geq n$ . Hence the required result, i.e.  $\chi_x(0) \leq \liminf_{y \rightarrow x} (\inf_{\rho \in [0,1]} (\chi_y(\rho)))$ .  
Eventually, Proposition 6 gives the relation with the local Hölder exponent  $\alpha_l$ . ■

Combining Proposition 8 and Proposition 17, one obtains

**Proposition 18** *For any function  $f$  in  $C^\gamma$ , one has*

$$\alpha_l(x) \leq \chi_x(0) \leq \liminf_{y \rightarrow x} \alpha_l(y)$$

Using the last Proposition, one recovers a well-known result [8], [26]

**Corollary 3** *For any function  $f \in C^\gamma(\mathbb{R})$  for some  $\gamma > 0$ , for all  $x$ , one has*

$$\alpha_l(x) \leq \liminf_{y \rightarrow x} \alpha_l(y),$$

thus  $x \rightarrow \alpha_l(x)$  is a lower semi-continuous (lsc) function.

We indicate here, just as a parenthesis, that one can also recover the fact that  $x \rightarrow \alpha_p(x)$  is a lim inf of a sequence of continuous functions ([3], [13]). In fact, one can prove even more:

**Proposition 19** *Let  $f \in C^\gamma(\mathbb{R})$  for some  $\gamma > 0$ . Then*

- *Let  $\rho \in (0, 1]$ . The function  $x \rightarrow \chi_x(\rho)$  is a lim inf of a sequence of continuous functions.*
- *The function  $x \rightarrow \alpha_p(x)$  is a lim inf of a sequence of continuous functions.*

**Proof:**

$\forall n \in \mathbb{N}$ , define the function  $g_n$  by

$$g_n : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}^+$$

$$(\rho, x) \rightarrow \inf_{2^{-(n+1)} \leq a < 2^{-n}} \left( \inf_{\rho - 2^{-n} \leq \beta \leq \rho + 2^{-n}} \left( \frac{\log |C(a, x \pm a^\beta)|}{\log a} \right) \right)$$

When  $n$  is fixed,  $(x, \rho) \rightarrow g_n(\rho, x)$  is continuous in the variables  $(x, \rho)$ , since the wavelet transform  $(x, a, b) \rightarrow C(a, x + b)$  is a continuous function in  $(x, a, b)$ .

Let  $\rho \in (0, 1)$ . The function  $x \rightarrow g_n(\rho, x)$  is thus obviously continuous in  $x$ . We let the reader check that, for all  $x$ , one has  $\chi_x(\rho) = \liminf_{n \rightarrow +\infty} g_n(\rho, x)$ .

If  $\rho = 1$ , one uses the functions  $g_n(1, \cdot)$

$$g_n(1, \cdot) : \mathbb{R} \rightarrow \mathbb{R}^+$$

$$x \rightarrow \inf_{2^{-(n+1)} \leq a < 2^{-n}} \left( \inf_{-a^{1-2^{-n}} \leq b \leq a^{1-2^{-n}}} \left( \frac{\log |C(a, x + b)|}{\log a} \right) \right),$$

and one checks that  $\chi_x(1) = \liminf_{n \rightarrow +\infty} g_n(1, x)$ . This concludes the proof of the first item.

To prove the second item, one uses the continuity of  $g_n$  with respect to  $(x, \rho)$ . Indeed, this continuity implies that,  $\forall n \in \mathbb{N}$ , the function  $h_n$  defined by

$$h_n : \mathbb{R} \rightarrow \mathbb{R}^+ \\ x \rightarrow \inf_{\rho \in [2^{-n}, 1]} \left( \frac{g_n^\rho(x)}{\rho} \right)$$

is continuous. It is now easily verified that

$$\liminf_{n \rightarrow +\infty} h_n(x) = \inf_{\rho \in (0, 1]} \frac{\chi_x(\rho)}{\rho} = \alpha_p(x).$$

■

This leads to a new result on the constraints on the regularity exponents of a function

**Corollary 4** *The weak scaling exponent function  $x \rightarrow \beta_w(x)$  is a lim inf of a sequence of continuous functions.*

**Proof:** This is a direct by-product of the last Proposition: Indeed, by Proposition 6,  $\forall x$ ,  $\chi_x(1) = \beta_w(x)$  and one has proved that  $x \rightarrow \chi_x(1)$  is a lim inf of a sequence of continuous functions. ■

Proposition 16 and 18 have in particular a consequence in multifractal analysis. Indeed, for multifractal functions, the local Hölder function  $x \rightarrow \alpha_l(x)$  is often a continuous function (for IFS or Weierstrass functions,  $x \rightarrow \alpha_l(x)$  is a constant). This implies that all the frontiers have their upper-part (corresponding to  $s' \geq 0$ ) parallel to the second bisector.

More generally, one has

**Proposition 20** *For any  $f$  defined on an interval  $I$ , there exists a residual set<sup>7</sup> of  $I$  such that, for all  $x \in I$ ,  $\sigma(s') = \alpha_l(x)$  for all  $s' > 0$ .*

**Proof:** This is a simple consequence of the following fact: If  $g$  is a lower semi-continuous function, then  $g$  is continuous on a residual set.

> From Proposition 18,  $\alpha_l(x) = \chi_x(0)$  at all points  $x$  where  $\alpha_l$  is continuous. Combining this with the fact that  $x \rightarrow \alpha_l(x)$  is lower semi-continuous gives the result. ■

**Remark:** A consequence of Proposition 20 is that no function  $f$  with  $\alpha_l(x) < +\infty \forall x$  can behave as a “pure” cusp (i.e. have a vertical frontier) at all points.

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<sup>7</sup>Recall that  $R$  is a residual set of  $I$  if  $R = \bigcap_{n \in \mathbb{N}} \Omega_n$ , where  $\{\Omega_n\}_{n \in \mathbb{N}}$  is a sequence of open sets, such that for all  $n$ ,  $\Omega_n$  is dense in  $I$ .

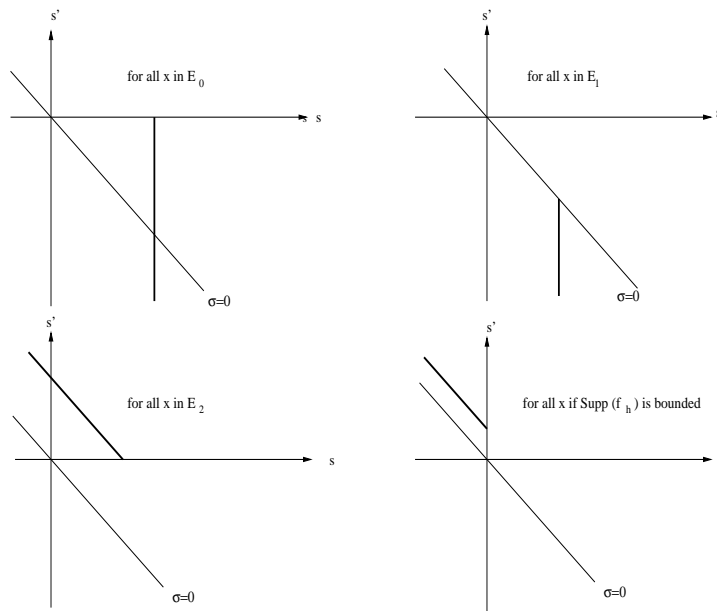


Figure 5: Typical parts of 2-microlocal frontiers.



In view of the propositions in this section, one has that the generic 2-microlocal frontier has an upper-part parallel to the second bisector, and an vertical asymptote when  $s' \rightarrow -\infty$ . Figure 5 illustrates this: The upper-left graph displays the frontier on a set of Hausdorff dimension 0  $E_0$  (this comes from [26]), the upper-right graph shows the frontier on a set  $E_1$  of measure 0 (where  $\beta_c = 0$ , see [13]), the lower-left graph shows the frontier of points that belong to a residual set  $E_2$  (Proposition 20), and the last graph illustrates Proposition 14.

## 6 2-microlocal Frontier Prescription

The problem of prescribing the 2-microlocal frontier of a distribution at one point has been solved in [7] and [21]. We propose here another way of doing so, using the 2-microlocal spectrum.

The advantage of such a method is that it can be extended to any countable dense set of points. More precisely we will be able in Section 6.2 to exhibit a function whose 2-microlocal frontiers are prescribed on a countable dense set of points.

We shall use the parameterization  $s' \rightarrow \sigma(s')$  for the 2-microlocal frontier.

### 6.1 Prescription at one point

**Theorem 5** *Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a concave, non-decreasing function, with slope between 0 and 1. Assume that  $g(0) > 0$ . There exists a function  $f$  such that the 2-microlocal frontier of  $f$  at 0 is  $\sigma_0(s') = g(s')$ .*

**Proof:**

The Legendre transform of  $g$ ,  $g^*(\rho) = \inf_{s' \in \mathbb{R}}(\rho s' - g(s'))$ , is continuous on its support, and ranges in  $[-\infty, -g(0)]$ .

Let us first define the functions  $\chi_0^j(\rho) = \min(j, -g^*(\rho))$ . We are going to build a function  $f$ , defined on  $[0, 1]$ , by its wavelet coefficients  $\{d_{j,k}\}_{j \geq 1, k}$  on an orthonormal wavelet basis  $\{\psi_{j,k}\}_{j,k}$ . We will do so in a way such that  $\chi_0(\rho) = -g^*(\rho)$  for all  $\rho \in [0, 1]$ .

Let  $(j, k)$  be a couple of indices, such that  $|k2^{-j}| \leq 1$  and  $j \geq 1$ . We denote by  $E_{j,k}$  the set of exponents  $\rho$  such that  $0 \leq \rho \leq 1$  and  $k = [2^{j(1-\rho)}]$ , and by  $\beta_{j,k}$  the minimum of  $\chi_0^j$  on  $E_{j,k}$ . Then we set, for all  $(j, k)$ ,  $d_{j,k} = 2^{-j\beta_{j,k}}$ .

By construction,  $-g^*(\rho) \geq g(0)$  for all  $\rho$ . The function  $f = \sum_{j,k} d_{j,k} \psi_{j,k}$  belongs to  $C^{g(0)}$  around 0.

For every couple  $(j, k)$  and for every exponent  $\rho \in [0, 1]$ , we denote by  $k_{j,\rho}$  the integer  $[2^{j(1-\rho)}]$ . Remark that  $\lim_{j \rightarrow +\infty} \beta_{j,k_{j,\rho}} = -g^*(\rho)$ .

Let us now compute the function  $\rho \rightarrow \chi_0(\rho)$  for this function  $f$  at 0.

When  $0 < \rho < 1$ , one must distinguish two cases:

- $-g^*(\rho) < +\infty$ :

Let  $\varepsilon > 0$ . Since  $-g^*$  is convex, there exists  $\eta > 0$ , such that  $|\gamma - \rho| \leq \eta$  implies  $-g^*(\gamma) \geq -g^*(\rho) - \varepsilon$ . Thus, by construction,  $|\gamma - \rho| \leq \eta$  also implies  $\chi_0^j(\gamma) \geq \chi_0^j(\rho) - \varepsilon$ . In particular,  $\forall \gamma \in [\rho - \eta/2, \rho + \eta/2]$ , and for  $j$  such that  $2^{-j} \leq \eta/2$ , one has  $\beta_{j,k_j,\gamma} \geq \chi_0^j(\gamma)$ . Thus,

$$\begin{aligned} |d_{j,k_j,\gamma}| &\leq 2^{-j\beta_{j,k_j,\gamma}} \\ &\leq 2^{-j\chi_0^j(\gamma)} \\ &\leq 2^{-j(\chi_0^j(\rho) - \varepsilon)}. \end{aligned}$$

Using that  $\lim_j \chi_0^j(\rho) = -g^*(\rho)$ , one concludes that  $\chi_0^\eta(\rho) \geq -g^*(\rho) - \varepsilon$  (remember that  $\chi_0^\eta$  takes into account the wavelet coefficients  $d_{j,k}$  such that  $k \in [[2^{j(1-(\rho+\eta))}], [2^{j(1-(\rho-\eta))}]]$ ).

Reciprocally, let us have a look at the coefficients  $d_{j,k_j,\rho}$ . For all  $j$ , one has  $d_{j,k_j,\rho} = 2^{-j\beta_{j,k_j,\rho}}$ , and  $\lim_{j \rightarrow +\infty} \beta_{j,k_j,\rho} = -g^*(\rho)$ . Thus  $\exists J, j \geq J \Rightarrow \beta_{j,k_j,\rho} \leq -g^*(\rho) + \varepsilon$ . This implies  $\chi_0^\eta(\rho) \leq -g^*(\rho) + \varepsilon$ . Finally, for all  $\varepsilon > 0$ , there exists  $\eta$  such that

$$-g^*(\rho) - \varepsilon \leq \chi_0^\eta(\rho) \leq -g^*(\rho) + \varepsilon.$$

Letting  $\varepsilon$  go to 0 gives  $\chi_0(\rho) = -g^*(\rho)$ .

- $-g^*(\rho) = +\infty$ :

by construction of  $g^*$ , for all  $N$ , there exists  $\eta$  such that  $|\gamma - \rho| \leq \eta$  implies  $-g^*(\gamma) \geq N$ . Thus  $|\gamma - \rho| \leq \eta$  also implies that  $\chi_0^j(\gamma) \geq N$  for all  $j > N$ . In particular,  $\forall \gamma \in [\rho - \eta, \rho + \eta]$ , for all  $j > N$ , one has

$$\begin{aligned} |d_{j,k_j,\gamma}| &\leq 2^{-j\beta_{j,k_j,\gamma}} \\ &\leq 2^{-jN}. \end{aligned}$$

One concludes that  $\chi_0^\eta(\rho) \geq N$ . This can be done for all  $N > 0$ , thus  $\chi_0(\rho) = +\infty = -g^*(\rho)$ .

Let us compute now  $\chi_0(0)$ . If  $-g^*(0) < +\infty$ , then, for all  $\varepsilon > 0$ , there exists  $\eta > 0$  such that  $\forall \gamma \in [0, \eta]$ ,  $-g^*(\gamma) \geq -g^*(0) - \varepsilon$ , and thus  $\chi_0^j(\gamma) \geq \chi_0^j(0) - \varepsilon$ . This means that, for all coefficients  $d_{j,k}$  such that  $k \geq [2^{j(1-\eta)}]$ , one has

$$\begin{aligned} |d_{j,k}| &\leq 2^{-j\beta_{j,k}} \\ &\leq 2^{-j(\chi_0^j(0) - \varepsilon)}. \end{aligned}$$

Thus  $\chi_0^\eta(0) \geq -g^*(0) - \varepsilon$ . One concludes that  $\chi_0(0) \geq -g^*(0)$ .

On the other hand,  $d_{j,k_j,0} = 2^{-j\beta_{j,0}}$ , and  $\lim_{j \rightarrow +\infty} \beta_{j,0} = -g^*(0)$ . Thus for all  $\eta > 0$ ,  $\chi_0^\eta(0) \leq -g^*(0)$ . This leads to  $\chi_0(0) \leq -g^*(0)$ , and finally  $\chi_0(0) = -g^*(0)$ .

The case  $-g^*(0) = +\infty$  is treated similarly to the case  $-g^*(\rho) = +\infty$  for  $0 < \rho < 1$ . Eventually, one computes  $\chi_0(1)$  using the same method, and one obtains

$$\forall \rho \in [0, 1], \quad \chi_0(\rho) = -g^*(\rho). \quad (21)$$

■

Notice that this prescription method chooses the “smoothest” function  $\chi_0$  among the ones that lead to the same frontier, because it forces  $\chi_0 = \tilde{\chi}_0$ .

## 6.2 Prescription on a set

The main advantage of the prescription method used in Theorem 5 is that it can be generalized to a countable dense set of points.

**Theorem 6** *Let  $h : (x, \rho) \rightarrow h(x, \rho)$  be a function from  $[0, 1] \times [0, 1]$  to  $[\gamma, +\infty]$  (with  $\gamma > 0$ ), such that*

- $x \rightarrow h(x, 0)$  is an lsc function.
- $x \rightarrow \inf_{\rho \in [0, 1]}(h(x, \rho))$  is an lsc function.
- For all  $x$ ,  $h(x, 0) \leq \liminf_{y \rightarrow x}(\inf_{\rho \in [0, 1]}(h(y, \rho)))$ .

Let  $\{x_n\}_n$  be a countable set of points in  $[0, 1]$ . For each  $x_n$ , denote by  $\tilde{h}_n$  the convex envelop of  $\rho \rightarrow h(x_n, \rho)$ .

There exists a function  $f$  such that the 2-microlocal spectrum  $\rho \rightarrow \chi_{x_n}(\rho)$  of  $f$  at each point  $x_n$  is exactly  $\rho \rightarrow \tilde{h}_n(\rho)$ .

**Remark:** To each continuous function  $f$  can be associated a two-variables function  $(x, \rho) \rightarrow \chi_x(\rho)$ . In view of Propositions 15 and 18, the conditions imposed on the function  $h : (x, \rho) \rightarrow h(x, \rho)$  make it an admissible candidate to be equal to  $\chi_x$ .

**Proof:** For each  $x_n$ , let us denote by  $g_n$  the Legendre transform of  $\rho \rightarrow -h(x_n, \rho)$ :  $g_n(s') = \inf_{\rho \in [0, 1]}(\rho s' + h(x_n, \rho))$ .

Theorem 5 explains how to force the 2-microlocal frontier to be equal to  $g_n$  at one  $x_n$ , or equivalently to force  $\chi_{x_n}$  to be equal to  $\tilde{h}_n$ . Here we are going to adapt this construction in order to do it simultaneously on all the  $x_n$ 's.

The construction is iterative: Let  $\Gamma$  denote the set of the wavelet coefficients  $\{d_{j,k}\}_{j,k}$  of the function  $f$  we are going to build:  $\Gamma = \{d_{j,k} : j \geq 1, k \in \{0, \dots, 2^j - 1\}\}$ . We only consider the  $d_{j,k}$ 's such that  $j \geq 1$ , since they include all the ones that contribute to the local regularity. Then, for any  $n$ , we define the "cone"  $\Gamma_{x_n} = \{d_{j,k} : |k2^{-j} - x_n| \leq 2^{-j/\log 2}\}$ . One now proceeds to the following iterative construction:

- one first imposes  $d_{j,k} = 2^{-j^2}$ , for all  $d_{j,k} \in \Gamma$ .
- at step 1, one modifies the wavelet coefficients that lie inside the cone  $S_1 = \Gamma_{x_1}$ , and one prescribes them according to  $\tilde{h}_1$ , the exponent function expected for  $x_1$ , following the construction used in Theorem 5 when prescribing the 2-microlocal frontier at one point.
- at step 2, one modifies the wavelet coefficients that lie inside the set  $S_2 = \Gamma_{x_2} \setminus \Gamma_{x_1}$ , and one prescribes them according to  $\tilde{h}_2$ .
- at step  $n$ , one modifies the wavelet coefficients that lie inside the set  $S_n = \Gamma_{x_n} \setminus \{\bigcup_{i=1, \dots, n-1} \Gamma_{x_i}\}$ , and one prescribes them according to  $\tilde{h}_n$ .

**Lemma 2**  $\Gamma = \bigcup_n S_n$ . Moreover, for all  $n > 0$ ,  $S_n$  is non-empty, and there exists  $j_n$  such that, if  $j \geq j_n$  and  $|k2^{-j} - x_n| \leq 2^{-j/\log j}$ , then  $d_{j,k} \in S_n$ .

**Proof:** Let  $d_{j,k} \in \Gamma$ . Since  $\{x_n\}$  is dense in  $[0, 1]$ , the set  $\{i : d_{j,k} \in \Gamma_{x_i}\}$  is non-empty, and it has a smallest element  $i_{\min}$ . It is easy to verify that  $d_{j,k} \in S_{i_{\min}}$ .

Let  $n$  be an integer greater than 2. For all  $i$  such that  $1 \leq i \leq n-1$ , one has

$$\lim_{j \rightarrow +\infty} |(x_i \pm 2^{-j/\log j}) - (x_n \pm 2^{-j/\log j})| = |x_i - x_n|.$$

Thus there exists  $\eta_n \leq \frac{1}{2} \min_i(|x_n - x_i|)$ , and  $j_n$ , such that  $\forall 1 \leq i \leq n-1, j \geq j_n$  implies

$$|(x_i \pm 2^{-j/\log j}) - (x_n \pm 2^{-j/\log j})| \geq \eta_n$$

This equivalently means that, for all  $(j, k)$  such that  $j \geq j_n$  and  $x_n - 2^{-j/\log j} \leq k2^{-j} \leq x_n + 2^{-j/\log j}$ ,  $d_{j,k} \notin S_i$ , for  $i \leq n-1$ . This concludes the proof. ■

Lemma 2 shows that the above construction allows to build a function  $F$  by prescribing all its wavelet coefficients. It is obvious that, for any  $(j, k)$ ,  $|d_{j,k}| \leq 2^{-j^\gamma}$ , thus  $F$  is well defined and belongs to  $C^\gamma([0, 1])$ .

The crucial point is the following lemma:

**Lemma 3** For any point  $x_n$  such that  $\lim_{\rho \rightarrow 0^+} \tilde{h}_n(\rho) < +\infty$ , one has  $\chi_{x_n}(\rho) = \tilde{h}_n(\rho)$  for every  $\rho \in [0, 1]$ .

Lemma 3 says that it is sufficient to prescribe the wavelet coefficients inside the cone  $\Gamma_{x_n}$  to prescribe the whole 2-microlocal frontier. We are going to apply this to effectively compute the regularity of the function.

**Proof:** (of Lemma 3) Using Lemma 2, one knows that there exists a scale  $j_n$ , such that, for all  $(j, k)$  such that  $j \geq j_n$  and  $|k2^{-j} - x_n| \leq 2^{-j/\log j}$ , the wavelet coefficients  $d_{j,k}$  are

chosen as in Theorem 5, i.e. in such a way that  $\chi_{x_n}(\rho) = \tilde{h}_n(\rho)$  for every  $\rho \in [0, 1]$ . The problems come from the fact that we have only prescribed *a priori* the coefficients that are inside the cone  $\Gamma_{x_n}$ , and Theorem 5 does not exactly apply here.

As noticed in Sections 3 and 5, and also in Corollary 1,  $\lim_{\rho \rightarrow 0^+} \tilde{h}_n(\rho) < +\infty$  implies that

$$\lim_{\rho \rightarrow 0^+} \tilde{h}_n(\rho) = \tilde{h}_n(0) < +\infty. \quad (22)$$

The only problem is in fact the computation of  $\chi_{x_n}(0)$ . Indeed, if  $\rho > 0$ , then the computation of  $\chi_{x_n}(\rho)$  uses only the coefficients that are lying inside the cone  $\Gamma_{x_n} = \{d_{j,k} : |k2^{-j} - x_n| \leq 2^{-j/\log j}\}$ , i.e. those which are correctly scaled.

Let us compute  $\theta_{x_n}^0(\eta)$ , for  $\eta > 0$  small. Two kinds of coefficients need to be taken into account: those that are inside the cone  $\Gamma_{x_n}$ , and those that are outside.

By construction,  $\tilde{h}_n(\rho) \leq \tilde{h}_n(0) - \varepsilon$  if  $\eta$  is small enough. Thus, if  $d_{j,k} \in \Gamma_{x_n}$ , if  $|k2^{-j} - x_n| \leq 2^{-j\eta}$  and if  $|k2^{-j} - x_n| \leq \eta_1 = 2^{-j_n/\log j_n}$ , then

$$d_{j,k} \leq C2^{-j(\tilde{h}_n(0) - \varepsilon)}. \quad (23)$$

Assume for the moment that  $\forall \varepsilon > 0$ , there exists  $\eta_2 > 0$  such that for all the  $d_{j,k}$ 's with  $(j, k)$  satisfying  $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta_2$ , one has

$$|d_{j,k}| \leq 2^{-j(\tilde{h}_n(0) - \varepsilon)} \quad (24)$$

(this concerns all the wavelet coefficients, not only those that belong to  $\Gamma_{x_0}$ ). Then one sees that the coefficients which are outside the cone  $\Gamma_{x_n}$  also satisfy (23) if  $|k2^{-j} - x_n| \leq \eta_2$ .

This means that, for  $\eta \leq \min(\eta_1, \eta_2)$ , if  $|k2^{-j} - x_n| \leq \eta$  and  $|k2^{-j} - x_n| \leq 2^{-j\eta}$ , then  $|d_{j,k}| \leq C2^{-j(\tilde{h}_n(0) - \varepsilon)}$ . This implies that  $\theta_{x_n}^0(\eta) \geq \tilde{h}_n(0) - \varepsilon$  for every  $\eta \leq \min(\eta_1, \eta_2)$ . This entails  $\chi_{x_n}(0) \geq \theta_{x_n}^0(\eta) \geq \tilde{h}_n(0) - \varepsilon$ , for all  $\varepsilon > 0$ . Thus  $\chi_{x_n}(0) \geq \tilde{h}_n(0)$ .

On the other hand, one always has  $\chi_{x_n}(0) \leq \lim_{\rho \rightarrow 0^+} \chi_{x_n}(\rho)$  by Proposition 4. But, due to (22),  $\lim_{\rho \rightarrow 0^+} \chi_{x_n}(\rho) = \lim_{\rho \rightarrow 0^+} \tilde{h}_n(\rho) = \tilde{h}_n(0) < +\infty$ .

This proves  $\chi_{x_n}(0) \leq \tilde{h}_n(0)$ .

The only thing that remains to prove to ensure that one has prescribed the 2-microlocal frontier at each  $x_n$  is inequality (24). This will be done now in Lemma 4, and will be the consequence of the second condition imposed on  $h$  in Theorem 6.

**Lemma 4** *If  $\tilde{h}_n(0) < +\infty$ , then  $\forall \varepsilon > 0$ , there exists  $\eta > 0$  such that, for all the  $(j, k)$ 's that satisfy  $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta$ , one has*

$$|d_{j,k}| \leq 2^{-j(\tilde{h}_n(0) - \varepsilon)} \quad (25)$$

**Proof:** (of Lemma 4) Let  $\varepsilon > 0$ . Since  $x \rightarrow \inf_{\rho \in [0,1]}(h(x, \rho))$  is an lsc function, there exists  $\eta_2 > 0$  such that  $|y - x_n| \leq \eta_2$  implies

$$\inf_{\rho \in [0,1]}(h(y, \rho)) \geq \inf_{\rho \in [0,1]}(h(x_n, \rho)) - \varepsilon. \quad (26)$$

This also means that for every point  $x_i$  such that  $|x_n - x_i| \leq \eta_2$ , one has  $\inf_{\rho \in [0,1]}(h(x_i, \rho)) \geq \inf_{\rho \in [0,1]}(h(x_n, \rho)) - \varepsilon$ , or even better,

$$\alpha_l(x_i) = \inf_{\rho \in [0,1]} \tilde{h}_i(\rho) \geq \inf_{\rho \in [0,1]} \tilde{h}_n(\rho) - \varepsilon.$$

Let us focus on the wavelet coefficients  $d_{j,k}$  such that  $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta_2$ . Such a wavelet coefficient  $d_{j,k}$  belong to some  $S_i$  (defined during the construction process), it has been modified by the process. One can locate the corresponding  $x_i$ . Indeed, one obviously has  $|x_i - k2^{-j}| \leq 2^{-j/\log j}$ , thus  $x_i \in [k2^{-j} - 2^{-j/\log j}, k2^{-j} + 2^{-j/\log j}]$ . Using that  $|x_n - k2^{-j}| \leq \eta$ , one concludes that

$$x_n - 2^{-j/\log j} - \eta \leq x_i \leq x_n + 2^{-j/\log j} + \eta.$$

We now fix  $j_n$  and  $\eta$  such that  $2^{-j_n/\log j_n} + \eta \leq \min(\eta_2, 2^{-J/\log J})$ .

Let us sum up our findings: if  $(j, k)$  verifies  $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta$ , then  $d_{j,k}$  belongs to one  $S_i$ , whose associated point  $x_i$  satisfies  $|x_i - x_n| \leq \eta \leq \eta_2$ . Using (26), one knows that the value of  $d_{j,k}$  after modification by the process is smaller than  $2^{-j \inf_{\rho \in [0,1]}(h(y, \rho))}$ , i.e. smaller than  $2^{-j(\inf_{\rho \in [0,1]}(h(x_n, \rho)) - \varepsilon)}$ .

In particular, this shows that, if  $(j, k)$  verifies  $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta$ , then  $|d_{j,k}| \leq 2^{-j(h(x_n, 0) - \varepsilon)} = 2^{-j(\tilde{h}_n(0) - \varepsilon)}$ . This is the result claimed in (25). ■

One can thus conclude that for all  $n \in \mathbb{N}$  such that  $\lim_{\rho \rightarrow 0^+} \tilde{h}_n(\rho) = +\infty$ , one has  $\chi_{x_n} = \tilde{h}_n$ . This ends the proof of Lemma 3. ■

The next lemma, whose proof is omitted, is an easy adaptation of Lemma 3 to the case  $\lim_{\rho \rightarrow 0^+} \tilde{h}_n(\rho) = h_n(0) = +\infty$ .

**Lemma 5** *For any point  $x_n$  such that  $\lim_{\rho \rightarrow 0^+} \tilde{h}_n(\rho) = +\infty$ , one has  $\chi_{x_n}(\rho) = \tilde{h}_n(\rho)$  for every  $\rho \in [0, 1]$ .*

Eventually, one has exactly prescribed the 2-microlocal frontier simultaneously at all  $x_n$ . ■

One shall remark that a “miracle” is happening here: It is enough to prescribe only a part of the wavelet coefficients that influence the 2-microlocal frontier of a function at some (countable dense) points  $\{x_n\}_n$  to recover the whole 2-microlocal frontier at each of these

points  $x_n$ . The lack of information is compensated by the regularity imposed on the function  $x \rightarrow \chi_x(0)$ .

The loss of information incurred when prescribing the convex envelop  $\tilde{h}_n$  of  $\rho \rightarrow h(x_n, \rho)$  instead of  $\rho \rightarrow h(x_n, \rho)$  itself, explains why, with the above method, one can not expect to prescribe more than what Theorem 6 allows to. Formally the loss of information could be compared to the one in multifractal analysis when considering the Legendre spectrum instead of the Hausdorff or large deviation spectrum.

A natural question is to enquire about the behaviour of the points that do not belong to the  $\{x_n\}_n$ . Recall that

- the local Hölder function satisfies  $\forall n, \alpha_l(x_n) = \inf_{\rho \in [0,1]} \tilde{h}_n(\rho)$  and  $\alpha_l(x_n) \leq \liminf_{x_i \rightarrow x_n} \alpha_l(x_i)$ .
- $\forall n, \chi_{x_n}(0) \leq \liminf_{x_i \rightarrow x_n} \chi_{x_i}(0)$ .

In addition, recall that, since both functions  $x \rightarrow h(x, 0)$  and  $x \rightarrow \inf_{\rho \in [0,1]} (h(x, \rho))$  are lsc functions, they are completely characterized by their values on a dense set of points (see [18]), respectively  $\{y_n\}_n$  and  $\{z_n\}_n$ . Thus if one prescribes the 2-microlocal frontiers on the three (countable) sets of points  $\{x_n\}$ ,  $\{y_n\}_n$  and  $\{z_n\}_n$ , one obtains the following proposition

**Proposition 21** *If  $h$  satisfies the properties of Theorem 6, and if one prescribes the 2-microlocal frontiers on the three sets of points  $\{x_n\}$ ,  $\{y_n\}_n$  and  $\{z_n\}_n$ , then one obtains a function  $f$  such that, in addition to the properties of Theorem 6, one has for all  $x$ ,*

$$\begin{aligned} \alpha_l(x) &\leq \inf_{\rho \in [0,1]} (h(x, \rho)), \\ \chi_x(0) &\leq h(x, 0). \end{aligned}$$

**Proof:** We use the following lemma, whose proof is omitted.

**Lemma 6** *Let  $v$  be an lsc function, and  $D = \{t_n\}_n$  be a sequence of points in  $[0, 1]$  that characterizes  $v$  in  $[0, 1]$  (see [18]). Then if  $w$  is an lsc function such that  $\forall n, w(t_n) = v(t_n)$ , then  $w \leq v$  on  $[0, 1]$ .*

Now, by construction,  $x \rightarrow \alpha_l(x)$  and  $x \rightarrow h_{min}(x) = \inf_{\rho \in [0,1]} (h(x, \rho))$  coincide on a dense set of points, namely  $\{z_n\}_n$ , which entirely characterizes  $x \rightarrow \alpha_l(x)$ . Hence, since they both are lsc, using Lemma 6, for all  $x$ , one has

$$h_{min}(x) \leq \alpha_l(x).$$

The same argument applies to  $x \rightarrow \chi_x(0)$ . ■

## 7 Riemann's function and lacunary wavelet series from the $\chi$ -point of view

We conclude this work with the explicit computation of  $\chi$  and  $\sigma$  for the celebrated one- and two-dimensional Riemann's function, as well as for lacunary wavelet series.

### 7.1 One-dimensional Riemann function

The function

$$\mathcal{R}(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2} \sin(\pi n^2 x),$$

was introduced by Riemann and was originally thought to be a continuous but nowhere differentiable function. Since then, deep studies have shown that it is in fact differentiable at rational points  $\frac{2p+1}{2q+1}$ ,  $p, q \in \mathbb{Z}^2$  (more precisely  $\mathcal{R}$  is  $C^{\frac{3}{2}}$  at these points), and that  $\mathcal{R}$  is a multifractal function whose spectrum has been calculated in [12].

Our aim here is to compute the 2-microlocal frontier of  $\mathcal{R}$  at all points. It is known ([9], [12]) that the behaviour of the continuous wavelet transform of  $\mathcal{R}(x)$  can be reduced to the one of the function  $T(a, b)$  defined by

$$T(a, b) = a(\operatorname{Im}(\Theta(b + ia)) - 1), \quad (27)$$

where  $\Theta$  is the Jacobi theta function defined by

$$\Theta(z) = \sum_{n=1}^{+\infty} e^{i\pi n^2 z}, \text{ for } \operatorname{Im}(z) > 0.$$

This is achieved when using the analyzing wavelet  $\psi(x) = \frac{1}{(i+x)^2}$ , which has only one vanishing moment. One can not thus *a priori* reach exponents greater than 1. However, taking for reconstructing wavelet any function  $\tilde{\psi} \in \mathcal{S}(\mathbb{R})$  supported by  $[-\varepsilon, \varepsilon]$  such that  $\int_{-\varepsilon}^{\varepsilon} \tilde{\psi}(x) dx = 0$ , one can verify that

$$f(x) = \int_0^{+\infty} \int_{-\infty}^{+\infty} \tilde{\psi}\left(\frac{x-b}{a}\right) \theta(b+ia) db \frac{da}{a} = \sum_{n=1}^{+\infty} \frac{e^{i\pi n^2 x}}{n^2},$$

whose imaginary part is exactly  $\mathcal{R}(x)$ . Thus there is no loss of information when using this specific analyzing wavelet.

The function  $\Theta$  has some invariance properties

$$\Theta(z) = \Theta(z+2) \text{ and } \Theta(z) = \sqrt{\frac{i}{z}} \Theta\left(-\frac{1}{z}\right).$$



Thus the rational points can be split into two parts: those which belong to the orbit of 0 under the two transforms above, and those which belong to the orbit of 1. They correspond respectively to the rational points that can be written  $\frac{2p}{2q+1}$  or  $\frac{2p+1}{2q}$  ( $p, q \in \mathbb{Z} \times \mathbb{Z}^*$ ) and to those that can be written  $\frac{2p+1}{2q+1}$  ( $p, q \in \mathbb{Z}^2$ ).

To compute the 2-microlocal spectrum  $\chi_x^1(\rho)$  of  $\mathcal{R}$  (the 1 in  $\chi_x^1$  stands for dimension one), we use the following estimation of the Jacobi theta function found in [9]

**Proposition 22 1.** *In a neighbourhood of any point  $x$  in the orbit of 0, one has*

$$T(a, x + b) = C_x \operatorname{Im} \left( a \sqrt{\frac{i}{b + ia}} + a\phi(b + ia) \right), \quad (28)$$

where  $\operatorname{Im}$  denotes the imaginary part, and  $\phi$  satisfies

$$\begin{aligned} \phi(b + ia) &= \mathcal{O}((b^2 + a^2)^{-\frac{1}{4}} e^{-\frac{a}{b^2 + a^2}}) \text{ if } \frac{a}{b^2 + a^2} > 1, \\ &= \mathcal{O}(a^{-\frac{1}{2}}(b^2 + a^2)^{\frac{1}{4}}) \text{ uniformly.} \end{aligned}$$

**2.** *In a neighbourhood of any point  $x$  in the orbit of 1, one has*

$$T(a, x + b) = C_x \operatorname{Im} (a\phi(b + ia)), \quad (29)$$

with  $\phi$  satisfying the same properties as before.

Let us define the following sets:

$$\begin{aligned} R_{1/2} &= \{x \in \mathbb{Q} : x \text{ is in the orbit of 0}\} \\ R_{3/2} &= \{x \in \mathbb{Q} : x \text{ is in the orbit of 1}\} \\ \forall \tau \geq 2, S_\tau &= \{x \in \mathbb{R} \setminus \mathbb{Q} : \eta(x) = \tau\} \end{aligned}$$

- **If  $x \in R_{1/2}$ :**

The first term in (28) is easily estimated. We set  $b = \pm a^\rho$  and see that

$$\begin{aligned} \left| \operatorname{Im} \left( a \sqrt{\frac{i}{\pm a^\rho + ia}} \right) \right| &= \frac{1}{\sqrt{2}} a^{1-\rho/2} \left| \operatorname{Im}((1 \pm ia^{1-\rho})^{-\frac{1}{2}}) \right| \\ &= \frac{1}{\sqrt{2}} a^{1-\rho/2} \left| \operatorname{Im}(1 \mp \frac{i}{2} a^{1-\rho} + \mathcal{O}(a^{2(1-\rho)})) \right| \\ &= \frac{1}{2\sqrt{2}} a^{2-\frac{3}{2}\rho} + \mathcal{O}(a^{\beta+2-\frac{5}{2}\rho}). \end{aligned}$$

It is clear then that the 2-microlocal spectrum corresponding to the first term of (28) is  $\chi_{x,1}(\rho) = 2 - \frac{3}{2}\rho$ .

The second term of (28) is more delicate: Set  $b = a^\rho$ . If  $\frac{1}{2} < \rho < 1$ ,  $\lim_{a \rightarrow 0} \frac{a}{b^2 + a^2} = +\infty$ , thus we can use the first bound of  $\phi$ , i.e.

$$\begin{aligned}\phi(a^\rho + ia) &= \mathcal{O}((a^{2\rho} + a^2)^{-\frac{1}{4}} e^{-\frac{a}{a^{2\rho} + a^2}}) \\ &= \mathcal{O}(a^{-\frac{1}{2}\rho} e^{-a^{1-2\rho}}),\end{aligned}$$

which gives a decay with  $a$  faster than any  $a^N$ ,  $N \in \mathbb{N}$ . If  $0 < \rho < \frac{1}{2}$ ,  $\lim_{a \rightarrow 0} \frac{a}{b^2 + a^2} = \lim_{a \rightarrow 0} \frac{a}{(a^\rho)^2 + a^2} = 0$  and we can only use the second bound of  $\phi$ , which leads to

$$\begin{aligned}\phi(b + ia) &= \mathcal{O}(a^{-\frac{1}{2}}(a^{2\rho} + a^2)^{\frac{1}{4}}) \\ &= \mathcal{O}(a^{\frac{1}{2}\rho - \frac{1}{2}})\end{aligned}$$

Using (27), it comes that  $\chi_{x,2}(\rho) = \frac{1}{2} + \frac{1}{2}\rho$  if  $0 < \rho \leq \frac{1}{2}$ , and  $\chi_{x,2}(\rho) = +\infty$  otherwise. It is also obvious that  $\chi_{x,2}(\frac{1}{2}) = \frac{3}{4}$ .

The computations of  $\chi_{x,2}(0)$  and  $\chi_{x,2}(1)$  easily follow from the above computation:  $\chi_{x,2}(0) = \chi_{x,2}(1) = 1/2$ .

Eventually, if  $x$  is in the orbit of 0, one has

$$\chi_x^1(\rho) = \min(\chi_{x,1}(\rho), \chi_{x,2}(\rho)).$$

$\chi_{x,1}(\rho) \geq \chi_{x,2}(\rho)$  if  $0 \leq \rho \leq 1/2$ , thus an explicit formula for  $\chi_x^1$  is

$$\begin{aligned}\chi_x^1(\rho) &= \frac{1}{2} + \frac{1}{2}\rho \text{ if } 0 \leq \rho \leq 1/2 \\ &= 2 - \frac{3}{2}\rho \text{ if } 1/2 < \rho \leq 1\end{aligned}$$

It is interesting to remark that  $\chi_x^1(0) = \chi_x^1(1) = 1/2$ . Thus  $\tilde{\chi}_x^1(\rho) = 1/2$  for all  $\rho \in [0, 1]$  and the 2-microlocal frontier of  $\mathcal{R}$  at  $x$  is

$$\begin{aligned}\sigma(s') &= 1/2 \text{ if } s' \geq 0, \\ \sigma(s') &= 1 + s' \text{ if } s' < 0,\end{aligned}$$

which corresponds to a cusp at each point  $x$  in the orbit of 0. Moreover, one recovers that  $\alpha_l(x) = \inf_\rho \chi_x^1(\rho) = 1/2$  and  $\alpha_p(x) = \inf_\rho \frac{\chi_x^1(\rho)}{\rho} = 1/2$ . These points are cusps with regularity 1/2, but one shall remark that the function  $\chi_x^1$  at these points contains more information than the single 2-microlocal frontier.

- **If  $x \in S_\tau$ :** The case of the irrational points has been studied in [12]: if  $x_0$  is an irrational point, and  $\frac{p_n}{q_n}$  its sequence of approximations by continued fractions, then there exists an infinity numbers of integers  $n$  such that

$$C(a_n, x_0) \sim C a_n^{1/2 + \frac{1}{2\tau n}},$$

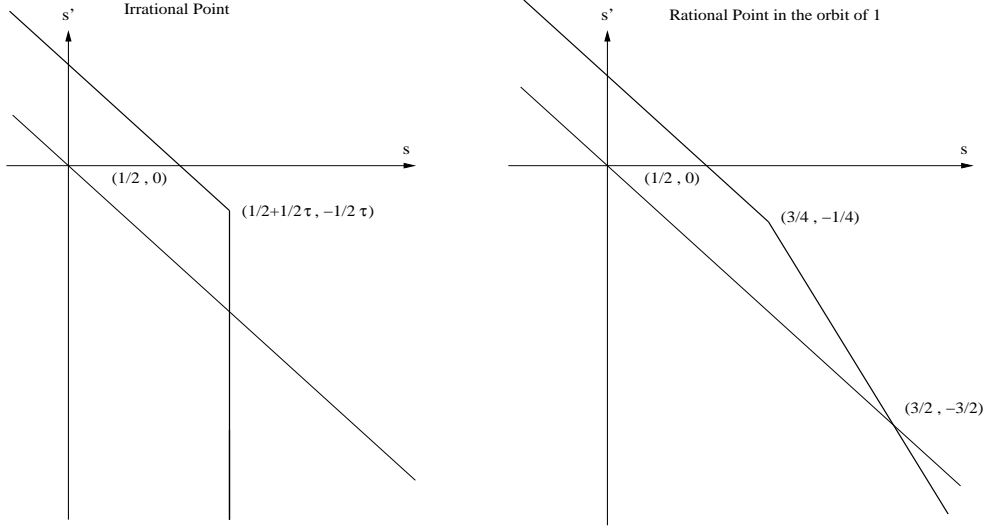


Figure 6: 2-microlocal Frontiers of the Riemann Function on  $R_{1/2}$  (left),  $R_{3/2}$  (right).

where  $|x_0 - \frac{p_n}{q_n}| = \frac{1}{q_n^2} = a_n$ , and at the same time, if  $\eta(x_0) = \limsup_n \tau_n$ , then  $R \in C_{x_0}^{1/2 + \frac{1}{2\eta(x_0)} - \varepsilon}$ , for every  $\varepsilon > 0$ . The first point implies that  $\chi_{x_0}^1(1) \leq 1/2 + \frac{1}{2\eta(x_0)}$ . Together with  $\chi_{x_0}^1(0) = 1/2$  (which is in fact true for all  $x \in \mathbb{R}$ ), one obtains  $\tilde{\chi}_{x_0}^1(\rho) = 1/2 + \rho \frac{1}{2\eta(x_0)}$  for all  $\rho \in [0, 1]$ . In fact, more can be done, i.e. one can show that

$$\tilde{\chi}_{x_0}^1(\rho) = 1/2 + \frac{\rho}{2\eta(x_0)} \text{ for } \rho \in [0, 1]. \quad (30)$$

These points have a 2-microlocal frontier equal to  $\sigma(s') = 1/2$  if  $s' \geq -\frac{1}{2\eta(x_0)}$ , and  $\sigma(s') = 1/2 + \frac{1}{2\eta(x_0)} + s'$  if  $s' < -\frac{1}{2\eta(x_0)}$ , which looks like a “cusp” frontier.

**Remark:** If  $x \in S_2$ , the wavelet coefficients  $T(a, x + b)$  effectively behave like indicated by the corresponding 2-microlocal spectra, i.e.  $T(a, x + a^\rho) \sim a^{\chi_{x_0}^1(\rho)}$ .

- If  $x \in R_{3/2}$ :

If  $x$  is in the orbit of 1,  $\chi_x^1(\rho) = \chi_{x,2}(\rho)$ , where  $\chi_{x,2}$  has been computed above. Indeed, only the second term needs to be estimated. Thus

$$\begin{aligned} \tilde{\chi}_x^1(\rho) &= \frac{1}{2} + \frac{1}{2}\rho \text{ if } 0 \leq \rho \leq 1/2 \\ &= +\infty \text{ if } 1/2 < \rho \leq 1, \end{aligned}$$

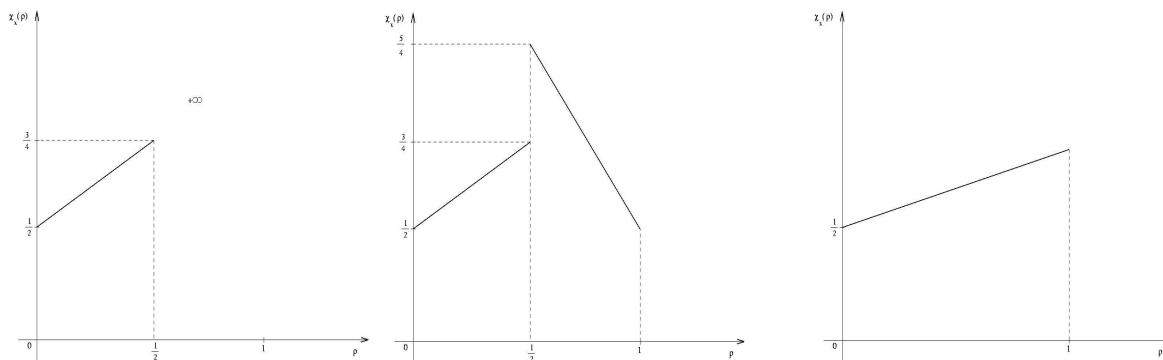


Figure 7: 2-microlocal spectra of the Riemann Function on  $R_{3/2}$  (left),  $R_{1/2}$  (middle) and  $S_\tau$  (right).

which gives a 2-microlocal frontier equal to

$$\begin{aligned} \sigma(s') &= \beta - 1/2 \text{ if } s' \geq -1/4, \\ \sigma(s') &= \beta - 1/4 + \frac{1}{2}s' \text{ if } s' < -1/4. \end{aligned}$$

One recovers that, at such points,  $\alpha_l(x) = \inf_\rho \chi_x^1(\rho) = 1/2$  and  $\alpha_p(x) = \inf_\rho \frac{\chi_x^1(\rho)}{\rho} = \frac{3/4}{1/2} = 3/2$ .

The complementary information provided by this analysis is that these points are chirps  $(3/2, 1)$ , i.e. with  $\alpha_p = 3/2$ , and  $\beta_c = \beta_o = 1$ .

**Remark:** Let us insist on the fact that the wavelet coefficients  $T(a, x + b)$  effectively behave around the rational points like indicated by the corresponding 2-microlocal spectra, i.e.  $T(a, x + a^\rho) \sim a^{\chi_x(\rho)}$ . This will be of great importance to treat the two-dimensional case.

## 7.2 Two-dimensional Riemann function

A generalized version of the Riemann function, for  $x = (x_1, x_2) \in \mathbb{R}^2$ , is (see [22])

$$\mathcal{R}^2(x) = \sum_{(m_1, m_2) \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{e^{i\pi(m_1^2 x_1 + m_2^2 x_2)}}{(m_1^2 + m_2^2)^2}.$$

As in the one-dimensional case, the study of the wavelet transform of  $\mathcal{R}^2$  can be reduced to the study of the function

$$C(a, b) = a^\beta \Theta(x_1 + b_1) \Theta(x_2 + b_2), \tag{31}$$

where  $b = (b_1, b_2)$ . If  $x = (x_1, x_2)$  is fixed, the 2-microlocal spectrum of  $\mathcal{R}^2$  at  $x$   $\chi_x^2(\rho)$  is related to both  $\chi_{x_1}^1(\rho)$  and  $\chi_{x_2}^1(\rho)$ , where  $\chi_x^1$  is the 2-microlocal spectrum of the 1-D Riemann function  $\mathcal{R}$ . More precisely,

**Proposition 23** *Let  $x = (x_1, x_2) \in \mathbb{R}^2$ . Then, for all  $\rho \in (0, 1]$ ,*

$$\chi_x^2(\rho) \geq \min \left( \chi_{x_1}^1(\rho) + \inf_{\rho' \in [\rho, 1]} \chi_{x_2}^1(\rho'), \inf_{\rho' \in [\rho, 1]} \chi_{x_1}^1(\rho') + \chi_{x_2}^1(\rho) \right) \quad (32)$$

We will use the following lemma

**Lemma 7** *Let  $f \in C^\gamma$  for  $\gamma > 0$ ,  $x \in \mathbb{R}$  and  $\rho_0 \in (0, 1]$ . We denote by  $C_f(a, b)$  the continuous wavelet transform of  $f$ , and  $\chi_x$  the 2-microlocal spectrum of  $f$  at  $x$ . For all  $\eta > 0$  small enough, there exists  $a_0$  such that, if  $a \leq a_0$ , for all  $b$  with  $|x - b| \leq a^{\rho_0}$ ,*

$$|C_f(a, b)| \leq a^{\inf_{\rho \in [\rho_0, 1]} \chi_x(\rho) - \eta}. \quad (33)$$

**Proof:** (of Lemma 7)

For every  $\rho \in [\rho_0, 1]$ , by definition of  $\chi_x(\rho)$ , there exists an interval  $I_\rho = [\rho - \xi_\rho, \rho + \xi_\rho]$  and a constant  $a_\rho$  such that if  $a \leq a_\rho$ , for all  $\rho' \in I_\rho$ ,  $|C_f(a, a^\rho)| \leq a^{\chi_x(\rho) - \eta}$ . The interval  $[\rho_0, 1]$  is compact and can be covered by a finite number of such intervals  $(I_\rho)_{\rho \in \{\rho_1, \dots, \rho_n\}}$ . Now, if  $a \leq \min_{i=\{1, \dots, n\}} a_i$ , for all  $b$  such that  $|x - b| \leq \rho_0$ , one has

$$|C_f(a, b)| \leq a^{\min_{i=\{1, \dots, n\}} \chi_x(\rho_i) - \eta}.$$

In particular, this implies (33). ■

**Proof:** (of Proposition 23)

We shall work with the  $\|\cdot\|_\infty$  norm in  $\mathbb{R}^2$ . Let  $\rho \in (0, 1]$ , and  $\varepsilon > 0$ . We need to estimate  $\chi_x^{2, \varepsilon}(\rho)$ , which can be written

$$\begin{aligned} \chi_x^{2, \varepsilon}(\rho) &= \liminf_{a \rightarrow 0} \inf_{(a, b) \in \Gamma_{x, \rho, \varepsilon}} \frac{\log |a^2 \Theta(x_1 + b_1) \Theta(x_2 + b_2)|}{\log a} \\ &= \liminf_{a \rightarrow 0} \inf_{(a, b) \in \Gamma_{x, \rho, \varepsilon}} \frac{\log |a \Theta(x_1 + b_1)|}{\log a} + \frac{\log |a \Theta(x_2 + b_2)|}{\log a} \end{aligned}$$

where  $\Gamma_{x, \rho, \varepsilon} = \{(a, b) : a^{\rho + \varepsilon} \leq \|x - b\|_\infty \leq a^{\rho - \varepsilon}\}$ . If  $(a, b) \in \Gamma_{x, \rho, \varepsilon}$ , then  $|x_1 - b_1| \leq a^{\rho - \varepsilon}$ ,  $|x_2 - b_2| \leq a^{\rho - \varepsilon}$  and  $\max(|x_1 - b_1|, |x_2 - b_2|) \in [a^{\rho + \varepsilon}, a^{\rho - \varepsilon}]$ .

Let  $\eta > 0$ . Assume without loss of generality that  $|x_1 - b_1| \in [a^{\rho + \varepsilon}, a^{\rho - \varepsilon}]$ . For  $\varepsilon$  small enough,  $\chi_{x_1}^{1, \varepsilon}(\rho) \geq \chi_{x_1}^1(\rho) - \eta/2$ , and there exists  $a_1$  such that, for all  $a \leq a_1$ , for all  $b_1$  such that  $|x_1 - b_1| \in [a^{\rho + \varepsilon}, a^{\rho - \varepsilon}]$ ,

$$\frac{\log |a \Theta(x_1 + b_1)|}{\log a} \geq \chi_{x_1}^{1, \varepsilon}(\rho) - \eta/2 \geq \chi_{x_1}^1(\rho) - \eta. \quad (34)$$

At the same time, Lemma 7 applied to  $\mathcal{R}$  holds for  $x_2$  and yields that, if  $a \leq a_2$  and  $|x_2 - b_2| \leq a^{\rho-\varepsilon}$

$$\frac{\log |a\Theta(x_2 + b_2)|}{\log a} \geq \inf_{\rho' \in [\rho-\varepsilon, 1]} \chi_{x_2}^1(\rho') - \eta. \quad (35)$$

Both (34) and (35) are true if  $a \leq \min(a_1, a_2)$ , and together they imply

$$\frac{\log |C(a, b)|}{\log a} \geq \chi_{x_1}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon, 1]} \chi_{x_2}^1(\rho') - 2\eta. \quad (36)$$

The symmetric case  $|x_2 - b_2| \in [a^{\rho+\varepsilon}, a^{\rho-\varepsilon}]$  is treated similarly, and yields for  $a$  small enough,

$$\frac{\log |C(a, b)|}{\log a} \geq \chi_{x_2}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon, 1]} \chi_{x_1}^1(\rho') - 2\eta. \quad (37)$$

Since at least one of (36) and (37) holds, one has

$$\chi_x^{2,\varepsilon}(\rho) \geq \min \left( \chi_{x_1}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon, 1]} \chi_{x_2}^1(\rho'), \chi_{x_2}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon, 1]} \chi_{x_1}^1(\rho') \right) - 2\eta.$$

Now, letting  $\eta$  go to zero, and using that  $\varepsilon \rightarrow \chi_x^{2,\varepsilon}(\rho)$  is a non-decreasing function yields the announced result.  $\blacksquare$

An important remark is that if for example for  $x_1$ , the wavelet transform around  $x_1$  effectively behaves like indicated by the 2-microlocal spectrum  $\chi_x^1$  (i.e. if for all  $a \leq a_0$ , for all  $\rho > 0$ ,  $|C(a, a^\rho)| = \mathcal{O}(a^{\chi_x^1(\rho)})$ ), then one has equality in (32).

This is fundamental in our case, since, as noticed in Remarks 1 and 2, this is the case for  $\mathcal{R}$  around the rational points and the set  $S_2$ : This will allow to compute exactly the 2-microlocal spectrum of  $\mathcal{R}^2$  at some points  $x = (x_1, x_2)$ .

**Theorem 7** *Let  $x = (x_1, x_2) \in \mathbb{R}^2$ .*

1. *if  $(x_1, x_2) \in R_{3/2} \times R_{3/2}$ , the 2-microlocal spectrum of  $\mathcal{R}^2$  at  $x$  is  $\chi_x^2(\rho) = 1 + \rho$  if  $\rho \in [0, 1/2]$ , and  $+\infty$  if  $\rho \in (1/2, 1]$ .*
2. *if  $(x_1, x_2) \in R_{1/2} \times R_{3/2} \cup R_{3/2} \times R_{1/2}$ , the 2-microlocal spectrum of  $\mathcal{R}^2$  at  $x$  is  $\chi_x^2(\rho) = 1 + \rho/2$  if  $\rho \in [0, 1/2]$ , and  $+\infty$  if  $\rho \in (1/2, 1]$ .*
3. *if  $(x_1, x_2) \in R_{1/2} \times R_{1/2}$ , the 2-microlocal spectrum of  $\mathcal{R}^2$  at  $x$  is  $\chi_x^2(\rho) = 1 + \rho/2$  if  $\rho \in [0, 1/2]$ , and  $\chi_x^2(\rho) = 4 - 3\rho$  if  $\rho \in (1/2, 1]$ .*
4. *if  $(x_1, x_2) \in R_{3/2} \times S_\tau \cup S_\tau \times R_{3/2}$ , the 2-microlocal spectrum of  $\mathcal{R}^2$  at  $x$  is  $\chi_x^2(\rho) = 1 + \frac{\rho}{2}(1 + 1/\tau)$  if  $\rho \in [0, 1/2]$ , and  $+\infty$  if  $\rho \in (1/2, 1]$ .*
5. *if  $(x_1, x_2) \in R_{1/2} \times S_\tau \cup S_\tau \times R_{1/2}$ , the 2-microlocal spectrum of  $\mathcal{R}^2$  at  $x$  is  $\chi_x^2(\rho) = 1 + \frac{\rho}{2\tau}$  if  $\rho \in [0, 1/(1 + 1/3\tau)]$ , and  $\chi_x^2(\rho) = 5/2 - \frac{3}{2}\rho$  if  $\rho \in (1/(1 + 1/3\tau), 1]$ .*

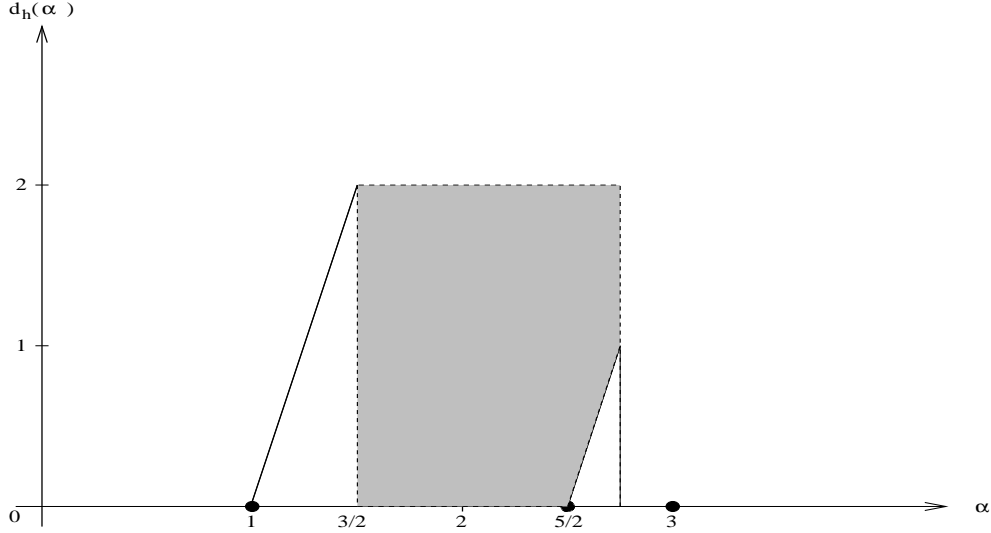


Figure 8: Multifractal spectrum of the 2-D Riemann Function.

6. if  $(x_1, x_2) \in S_\tau \times S_{\tau'} \cup S_{\tau'} \times S_\tau$ , the 2-microlocal spectrum of  $\mathcal{R}^2$  at  $x$  is  $\chi_x^2(\rho) \geq 1 + \frac{\rho}{2}(1/\tau + 1/\tau')$ . There is equality if either  $\tau$  or  $\tau'$  equals 2.

It is easy to compute the Hausdorff dimensions of the above sets, and the corresponding pointwise Hölder exponents of their points:

1.  $d_H(R_{3/2} \times R_{3/2}) = 0$ , they correspond to chirps  $(3, 1)$ .
2.  $d_H(R_{1/2} \times R_{1/2} \cup R_{3/2} \times R_{1/2}) = 0$ , they correspond to chirps  $(5/2, 1)$ .
3.  $d_H(R_{1/2} \times R_{1/2}) = 0$ , they correspond to cusps  $(1, 0)$ .
4.  $d_H(R_{3/2} \times S_\tau \cup S_\tau \times R_{3/2}) = 2/\tau$ , they correspond to chirps  $(5/2 + 1/2\tau, 1)$ .
5.  $d_H(R_{3/2} \times S_\tau \cup S_\tau \times R_{3/2}) = 2/\tau$ , they correspond to cusps  $(1 + 1/2\tau, 0)$ .
6.  $d_H(S_2 \times S_\tau \cup S_\tau \times S_2) = 1 + 2/\tau$  and they correspond to cusps  $(5/4 + 1/2 \max(\tau, 2), 0)$ .

The other cases, i.e.  $x \in S_\tau \times S_{\tau'}$  with  $\tau$  and  $\tau'$  strictly greater than 2, are hard to handle since the pointwise Hölder exponent at  $x$  can sometimes be higher than simply  $1 + 1/2\tau + 1/2\tau'$ . Nevertheless one knows that for every  $x \in S_\tau \times S_{\tau'}$ ,  $\alpha_p(x) \leq 11/4$ .

Parts of the spectrum are drawn on Figure 8. The shaded zone corresponds to areas where the spectrum is not known exactly, but where some bounds hold.

### 7.3 Lacunary wavelet series

Fix  $0 < \eta < 1$ , and  $\alpha > 0$ . A lacunary wavelet series [14] is a random process  $F$  defined through its wavelet coefficients as follows : Independently at each scale  $j \geq 0$ , one picks randomly  $[2^{nj}]$  coefficients among the  $2^j$  ones, according to the uniform probability distribution. These coefficients are attributed the value  $2^{-\alpha j}$ . The remaining ones are set to 0.

Let us compute  $\chi_x(\rho)$  for a sample path of  $F$  on  $[0, 1]$ . Remark first that it is obvious that  $\forall x, \forall \rho \in [0, 1], \chi_x(\rho) \in \{\alpha, +\infty\}$ .

Define  $G_j = \{k : d_{j,k} = 2^{-j\alpha}\}$ . For all  $j$ ,  $\text{card}(G_j) = [2^{nj}]$ .

**Proposition 24**  $\{x : \chi_x(0) = \alpha\} = [0, 1]$  almost surely.

**Proof:** Define the set  $F_j$  by

$$F_j = \{x : \exists k \in G_j, 2^{-j\frac{1}{\log j}} \leq |k2^{-j} - x| \leq 2^{-j(\frac{1}{\log j})^2}\}.$$

The following lemma is left to the reader

**Lemma 8** If  $x \in \bigcap_{J \in \mathbb{N}} \bigcup_{j \geq J} F_j$ , then  $\chi_x(0) = \alpha$ .

For any  $j$ , let  $k_i^j, i \in \{0, \dots, [2^{nj}] - 1\}$  be the integers such that  $d_{j,k_i^j} = 2^{-j\alpha}$  (there are for  $[2^{nj}]$  of them). Let  $n_j = \sum_{i=1}^j [2^{ni}]$ . For  $n \in [n_{j-1}, n_j]$ , one sets, for  $t_n = k_{n-n_{j-1}}^j 2^{-j}$ ,  $I_n^l = [t_n - 2^{-j(\frac{1}{\log j})^2}, t_n - 2^{-j\frac{1}{\log j}}]$ , and  $I_n^r = [t_n + 2^{-j\frac{1}{\log j}}, t_n + 2^{-j(\frac{1}{\log j})^2}]$ .

$F_j$  is the union of the  $2[2^{nj}]$  intervals  $I_n^l$  and  $I_n^r$  for  $n \in [n_{j-1}, n_j]$  of size  $2^{-j(\frac{1}{\log j})^2} - 2^{-j\frac{1}{\log j}}$ .

As explained in [14], one can consider that the  $t_n$  are chosen randomly and uniformly in  $[0, 1]$ . Thus the intervals  $I_n^l$  (respectively  $I_n^r$ ) are chosen randomly and uniformly in  $[0, 1]$ , so that one may apply the following Lemma of [14] to  $I_n^l$  (and  $I_n^r$ ):

**Lemma 9** If  $\limsup_{n \rightarrow +\infty} (\sum_{j=1}^n |I_n^l| - \log n) = +\infty$ , then  $\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} I_n^l = [0, 1]$  almost surely.

Lemma 9 applies to our intervals  $I_n^l$  and  $I_n^r$ . Combining this with Lemma 8 shows that  $\bigcap_{J \in \mathbb{N}} \bigcup_{j \geq J} F_j = [0, 1]$  a.s., and Proposition 24 is proved.  $\blacksquare$

**Proposition 25**  $\{x : \chi_x(\eta) = \alpha\} = [0, 1]$  almost surely.

**Proof:** The same arguments as before apply to the sets  $F_j^\eta$  defined by

$$F_j^\eta = \{x : \exists k \in G_j, 2^{-j(\eta + \frac{1}{\log j})} \leq |k2^{-j} - x| \leq 2^{-j(\eta - \frac{1}{\log j})}\}.$$



Indeed,  $F_j^\eta$  is the union of  $2[2^{2^j}]$  intervals of size  $2^{-j\eta}(2^{j\frac{1}{1-\delta}} - 2^{-j\frac{1}{1-\delta}})$ . Combining this with the fact that if  $x \in \bigcap_{J \in \mathbb{N}} \bigcup_{j \geq J} F_j^\eta$ ,  $\chi_x(\eta) = \alpha$ , the result follows at once. ■

Since  $\chi_x(\rho) \in \{\alpha, +\infty\}$  for every  $\rho$  and every  $x$ , one easily derives the form of the 2-microlocal spectrum at every point  $x$

**Proposition 26**  $\{x : \forall \rho \in [0, \eta], \tilde{\chi}_x(\rho) = \alpha\} = [0, 1]$  almost surely.

**Proposition 27** For almost every sample path of the process  $F$ , the Hausdorff dimension of the sets  $R_\delta = \{x : \chi_x(\eta + \delta) = \alpha\}$ , for  $\delta \in (0, 1 - \eta]$ , is  $\frac{\eta}{\eta + \delta}$ .

The 2-microlocal frontier of  $F$  at an arbitrary point of  $R_\delta$ , is drawn on Figure 9.

**Proof:** Let us consider the open balls  $B_{j,k}^\gamma = (k2^{-j} - 2^{-j\gamma}, k2^{-j} + 2^{-j\gamma})$ , for all couples  $(j, k)$  such that  $d_{j,k} \neq 0$ . This set of balls satisfies the conditions of Theorem 2 of [14]. This implies that the Hausdorff dimension of  $E_\gamma$  defined by

$$E_\gamma = \limsup_{j \rightarrow +\infty} B_{j,k}^\gamma = \bigcap_{n \in \mathbb{N}} \bigcup_{j \geq n} \bigcup_k B_{j,k}^\gamma \quad (38)$$

is  $\frac{\eta}{\gamma}$ . If  $x \in E_\gamma$ , then  $x$  belongs to an infinite number of balls  $B_{j,k}^\gamma$ . The following lemmas are obvious consequences of the definition of  $E_\gamma$

**Lemma 10** If  $x \notin E_\gamma$ ,  $\chi_x(\rho) = +\infty$  for  $\rho > \gamma$ .

**Lemma 11** If  $x \in E_\gamma \setminus E_{\gamma+\varepsilon}$ , there exists  $\rho \in [\gamma, \gamma + \varepsilon]$  such that  $\chi_x(\rho) = \alpha$ .

Using both Lemma 10 and 11 yields that if  $x \in \bigcap_{n \in \mathbb{N}} E_{\eta+\delta-1/n}$  but if  $x$  does not belong to  $\bigcup_{\varepsilon > \eta+\delta} E_\varepsilon$ , then  $\chi_x(\eta + \delta) = \alpha$ , and  $\chi_x(\rho) = +\infty$  if  $\rho > \eta + \delta$ .

Now, we apply Theorem 2 of [14]: Almost surely the Hausdorff dimension of the set  $E_{\eta+\delta-1/n} \setminus \bigcup_{\varepsilon > \eta+\delta} E_\varepsilon$  is exactly  $\frac{\eta}{\eta+\delta}$ . This concludes the proof. ■

The following proposition recapitulates the results obtained above. It sheds interesting light on the interplay between the 2-microlocal and multifractal analysis of lacunary wavelet series

**Proposition 28** For almost every sample path of the process  $F$ , for all  $x$ , there exists  $\delta \in [0, 1 - \eta]$  such that  $\tilde{\chi}_x(\rho) = \alpha$  if  $0 \leq \rho \leq \eta + \delta_x$ , and  $\tilde{\chi}_x(\rho) = +\infty$  elsewhere. The pointwise exponent at  $x$  is  $\frac{\alpha}{\eta+\delta_x}$  and the chirp exponent at  $x$  is  $\frac{1}{\eta+\delta_x} - 1$ . For a fixed  $\delta$  in  $[0, 1 - \eta]$ , the Hausdorff dimension of the points satisfying the above conditions is  $\frac{\eta}{\eta+\delta}$ .

Proposition 28 allows to recover the fact (proved in [14]) that the corresponding multifractal spectrum is  $d(h) = h\frac{\eta}{\alpha}$ , for  $h \in [\alpha, \frac{\alpha}{\eta}]$ .

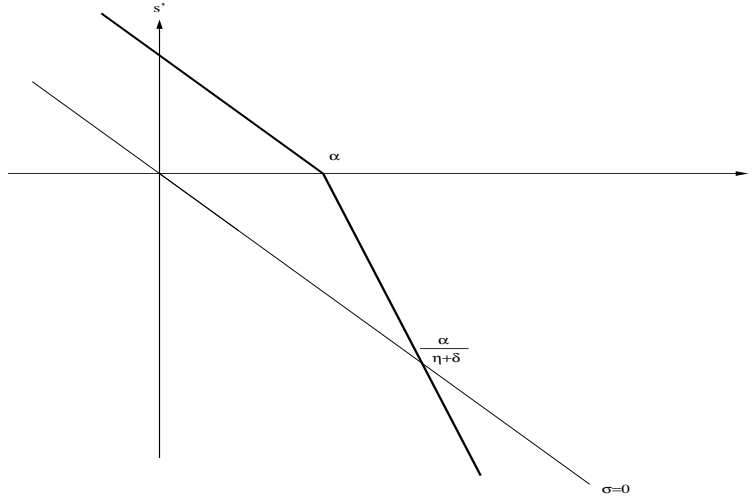


Figure 9: 2-microlocal frontier for a lacunary wavelet series at  $x \in R_\delta$ .

An interesting generalization of the construction above is to set the "remaining" coefficients to the value  $2^{-\beta j}$  instead of 0. Let  $F_\beta$  denote this new process. Under the condition that  $\beta > \alpha$ , the above computations are still valid with one modification: One has now  $\chi_x(\rho) = \beta$  instead of  $\chi_x(\rho) = +\infty$  when no coefficients equal to  $2^{-\alpha j}$  appears in the computation of  $\chi_x(\rho)$ . The proof of the following proposition is left to the reader.

**Proposition 29** *If  $\beta > \frac{\alpha}{\eta}$ , for almost every sample path of the process  $F_\beta$ , the multifractal spectrum is the same as before, i.e.  $d(h) = h\frac{\eta}{\alpha}$  if  $h \in [\alpha, \frac{\alpha}{\eta}]$ . Moreover, one still has that if  $\alpha_p(x) = h$ , then the chirp exponent at  $x$  is  $\frac{h}{\alpha} - 1$ .*

*If  $\beta < \frac{\alpha}{\eta}$ , for almost every sample path of the process  $F_\beta$ , the multifractal spectrum is  $d(h) = h\frac{\eta}{\alpha}$  if  $h \in [\alpha, \beta]$ ,  $d(\beta) = 1$ , and  $d(h) = -\infty$  elsewhere. Moreover, if  $\alpha_p(x) = h < \beta$ , then the chirp exponent at  $x$  is  $\frac{h}{\alpha} - 1$ , but if  $\alpha_p(x) = \beta$ , then the chirp exponent at  $x$  is 0 (there are no "oscillations" around  $x$ ).*

*If  $\beta = \frac{\alpha}{\eta}$ , for almost every sample path of the process  $F_\beta$ , the multifractal spectrum is the same as the one of the usual lacunary series, except that if  $\alpha_p(x) = \beta$ , then the chirp exponent at  $x$  is 0 (there are no more "oscillations" around  $x$ ).*

See Figure 10 for examples of 2-microlocal frontiers of  $F_\beta$ . It is interesting to notice that in the second case ( $\beta < \frac{\alpha}{\eta}$ ), one observes a jump in the multifractal spectrum at the critical value  $\beta$ .

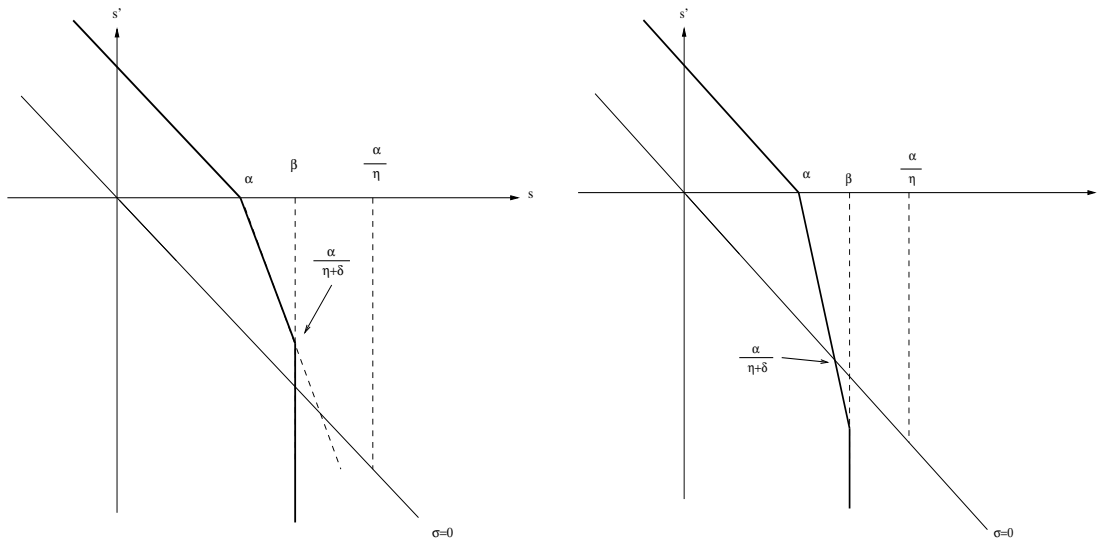


Figure 10: 2-microlocal frontiers depending on the value of  $\frac{\alpha}{\eta+\delta}$  for the modified lacunary wavelet series  $F_\beta$ .

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