Approximation of Normal Cycles

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Abstract: This report deals with approximations of geometric data defined on a hypersurface of the Euclidean space $\mathbb{E}^n$. Using geometric measure theory, we evaluate an upper bound on the flat norm of the difference of the normal cycle of a compact subset of $\mathbb{E}^n$ whose boundary is a smooth (closed oriented embedded) hypersurface, and the normal cycle of a compact geometric subset of $\mathbb{E}^n$ *"close to it"*. We deduce bounds between the difference of the curvature measures of the smooth hypersurface and the curvature measures of the geometric compact subset.

Key-words: Normal cycle, approximation, surface, curvature

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Approximation de cycles normaux

Résumé : Ce rapport étudie les approximations de quantités géométriques définies sur une hypersurface d'un espace euclidien $\mathbb{E}^n$. En utilisant la théorie de la mesure géométrique, nous évaluons une borne supérieure de la norme plate de la différence du cycle normal d'un compact de $\mathbb{E}^n$ dont le bord est une hypersurface (compacte orientée lisse plongée), et du cycle normal d'un compact géométrique qui lui est proche. Nous en déduisons une majoration de la différence des mesures de courbure de l'hypersurface et des mesures de courbure du compact géométrique.

Mots-clés : Cycle normal, approximation, surface, courbure

1 Introduction

This report deals with approximations of geometric data defined on a hypersurface of the Euclidean space $\mathbb{E}^n$. Basically, we answer to the following question: suppose that we want to evaluate different curvatures of a smooth hypersurface M_0, but that we can only access to the local geometry of a (smooth or discrete) object M_1, *"close" to it.* Can we deduce from the knowledge of the geometry of M_1 *"good approximations"* of the curvatures of M_0 ? In such a generality, the answer is obviously negative, but with simple suitable assumptions, much can be done. We deal with geometric measure theory, [4], and the use of currents. The main advantage lies in the fact that one can define curvature measures on non smooth objects, generalizing the Lipschitz-Killing curvatures of a submanifold, (which are smooth functions). This frame has been build with success by Wintgen and Zähle, [12], [13], [14],[15],[16], which defined the notion of normal cycle of singular objects, generalizing the unit normal bundle of a smooth submanifold, and by J. Fu, [6], [7], [8], [9], [10], who could characterize the normal cycles, and define the curvature measures of a large class of *"geometric objects"* admitting a normal cycle. He got an approximation result by comparing the curvature measures of a sequence of polyhedra converging to a smooth submanifold of an Euclidean space: if the polyhedra are *"closely inscribed in the submanifold"*, with a *"fatness"* bounded by below by a positive constant, and if the Hausdorff distance between the polyhedra and the submanifold tends to zero, then the curvature measures of the polyhedra tends to the curvature measures of the smooth submanifold, (see the definitions below). The proof of this theorem consists on checking that the masses of the sequence of the normal cycles of the polyhedra are bounded and using the compactness theorem of integral currents with bounded mass, [9].

Remark however that this result does not give a bound on the *"error"* between the curvature measures of a polyhedron *"close"* to the submanifold, and the curvature measures of the submanifold. This is the goal of this report: We evaluate an upper bound on the flat norm of the difference of the normal cycle of a compact subset of $\mathbb{E}^n$ whose boundary is a smooth (closed oriented embedded) hypersurface M^{n-1}, (basically, its unit normal bundle) and the normal cycle of a geometric compact subset $\mathcal{C}$ (in the sense of J. Fu, [8]) whose boundary $\mathcal{B}$ is *"close to it"*, in terms of the mass of the normal cycle of $\mathcal{C}$, the Hausdorff distance between M^{n-1} and $\mathcal{B}$, the maximum angle between the normals to M^{n-1} and the support of $N(\mathcal{C})$ over $\mathcal{B}$, and an *a priori* upperbound on the norm of the second fundamental form of M^{n-1}.

Recall that one can always build n differential forms $\omega_k, (0 \leq k \leq n-1)$ of degree $n-1$ on the tangent manifold $T\mathbb{E}^n$, which give rise to curvature measures. When one evaluates these forms on the unit normal bundle of a smooth hypersurface, one gets the integral of its Lipschitz Killing curvatures. By analogy, one can define the curvatures of any geometric object by evaluating these forms on their normal cycle. Our result will be applied at the end of the article to get explicit approximations of the curvatures of M^{n-1} by the curvatures of $\mathcal{C}$.

2 Background on currents

All the details can be found in [4].

2.1 General currents

Let M^n be a C^∞ n-dimensional manifold. We denote by $\mathcal{D}^m$ the $\mathbb{E}$-vector space of C^∞ *differential m-forms* with compact support on M^n. The norm of a m-differential form ϕ is the real number

$$||\phi|| = \sup_{p \in M^n} ||\phi_p||, \tag{1}$$

where, for each $p \in M^n$,

$$||\phi_p|| = \sup\{| < \phi_p, \zeta_p > |, \zeta_p \in \Lambda^m T_p M^n, |\zeta_p| = 1\}. \tag{2}$$

The dual of $\mathcal{D}^m$ is the $\mathbb{E}$-vector space $\mathcal{D}_m$ of *currents* on M^n. The subset of m-currents with compact support is denoted by $\mathcal{E}^m$. We endow $\mathcal{D}_m$ with the weak topology:

$$\lim_{j \to \infty} T_j = T \iff \lim_{j \to \infty} T_j(\phi) = T(\phi), \forall \phi \in \mathcal{D}^m. \tag{3}$$

2.2 Current representable by integration

We say that a current $T \in \mathcal{D}_m$ is *representable by integration* if there is a Borel regular measure $||T||$ on $\mathbb{E}^n$ finite on compact subsets and a unit m-vector fields $\overrightarrow{T}$ defined almost everywhere such that

$$T(\phi) = \int < \overrightarrow{T}, \phi > d||T||, \forall \phi \in \mathcal{D}^m. \tag{4}$$

2.3 Rectifiable and integral currents

In particular, we can associate a m-current to any oriented rectifiable subset S of dimension m of M^n: let $\vec{S}$ be the unit m-vector associated to almost every point x of S. For every $\phi \in \mathcal{D}^m(M)$, we define a current (still denoted by S) by

$$< S, \phi >= \int_S < \vec{S}, \phi >, \tag{5}$$

and more generally,

$$< \alpha S, \phi >= \alpha \int_S < \vec{S}, \phi >, \forall \alpha \in \mathbb{Z}. \tag{6}$$

If the support of S is compact, we say that S is *rectifiable.* We denote by $\mathcal{R}_m$ the space of rectifiable currents.

A current is said to be *integral* if it is rectifiable and if its boundary is rectifiable. The space of integral m-currents is denoted by I_m.

2.4 Mass and norms of currents

There are different interesting seminorms on the space of currents $\mathcal{D}_m$. We mention the main ones:

- The mass of a current $T \in \mathcal{D}_m$ is the real number

$$M(T) = \sup\{T(\phi), \text{ such that } \phi \in \mathcal{D}^m, ||\phi|| \leq 1.\} \tag{7}$$

  Using general results on representation of measure theory, it can be proved that if $M(T) < \infty$, T is representable by integration.

- The flat norm of a current $T \in \mathcal{D}_m$ is the real number

$$\mathcal{F}(T) = \inf\{M(A) + M(B) \text{ such that } T = A + \partial B, A \in \mathcal{R}_m, B \in \mathcal{R}_{m+1}\}. \tag{8}$$

Remark that the flat norm has another expression. One has:

$$\mathcal{F}(T) = \min\{M(A) + M(B) \text{ such that } T = A + \partial B, A \in \mathcal{E}_m, B \in \mathcal{E}_{m+1}\}, \tag{9}$$

or

$$\mathcal{F}(T) = \sup\{T(\phi), \text{ such that } \phi \in \mathcal{D}^m, ||\phi|| \leq 1, ||d\phi|| \leq 1\}. \tag{10}$$

2.5 The constancy theorem for integral currents

We shall use the following important result:

Theorem 1 *Let M^n be an oriented compact submanifold of $\mathbb{E}^N$, and T be an integral current whose support lies in M^n, and such that the support of ∂T lies in ∂M^n. Then, there exists an integer k such that $T = kM^n$.*

2.6 Normal cycle associated to a compact subset of $\mathbb{E}^n$

In this paragraph, we rephrase J. Fu. The goal is to define a generalisation of the unit normal bundle of a submanifold, to a very large compact subsets of $\mathbb{E}^n$.

Let A be a compact subset of $\mathbb{E}^n$. Consider the function

$$i_A : ST\mathbb{E}^n \to \mathbb{E}, \tag{11}$$

defined by

$$i_A(x,\xi) = \lim_{r\to 0}\lim_{s\to 0}[\chi(A \cap B(x,r) \cap \{p \text{ such that } (p-x).\xi \leq t\})|_{t=-s}^{t=+s}]. \tag{12}$$

(Remark that i_A may have no sense if the Euler characteristic is taken for an object which has no finite homology...) However, if i_A exists, it is unique in a certain sense. More precisely, J. Fu proved the following, [9]:

Theorem 2 *There exists at most one closed compactly supported integral current $S_A \in I_{n-1}(ST\mathbb{E}^n)$ such that*

- *S_A is Legendrian,*
- *and for all smooth functions ϕ in $ST\mathbb{E}^n$,*

$$S_A(\phi(x,\xi)dv_{S^{n-1}}) = \int_{S^{n-1}} \sum_{x\in\mathbb{E}^n} \phi(x,\xi) i_A(x,\xi) dv_{S^{n-1}}.$$

Following [9], one gives the following:

Definition 1 *Let A be a compact subset of $\mathbb{E}^n$. If i_A and S_A exist, then A is said to be geometric.*

Examples: The main examples of geometric compact subsets of $T\mathbb{E}^n$ are the unit normal bundle of any smooth submanifold of $\mathbb{E}^n$, the generalized unit normal bundle of any convex subset of $\mathbb{E}^n$, the normal bundle of subsets with positive reach, the normal cycle of any polyhedron of $\mathbb{E}^n$.

3 An approximation result

In this paragraph, we shall evaluate an upper bound on the flat norm of the difference of the normal cycle of a compact subset K of $\mathbb{E}^n$ the boundary of which is a smooth (closed oriented embedded) hypersurface M^{n-1} and the normal cycle of a geometric compact subset $\mathcal{C}$ the boundary of which $\mathcal{B}$ is strongly close to M^{n-1}; (see the definitions below). We denote by $\delta(A, A')$ the Hausdorff distance between two subsets A and A' of $\mathbb{E}^n$.

3.1 The second fundamental form of a hypersurface of $\mathbb{E}^n$

A detailed background can be found in [1]. Let $x : (M, g) \hookrightarrow \mathbb{E}^n$ be a codimension one isometric immersion of a Riemannian submanifold M into $\mathbb{E}^n$. We will use the following notations: h denotes the second fundamental form of the immersion i, (that is the symmetric tensor with values in the normal bundle $T^\perp M$), A denotes the Weingarten endomorphism. One has, $\forall X, Y \in TM, \forall \xi \in T^\perp M$,

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{13}$$

$$\tilde{\nabla}_X \xi = \nabla^\perp_X \xi - A_\xi X. \tag{14}$$

3.1.1 The Gauss map associated to M^{n-1}

Let ξ be a unit normal vector field on the hypersurface M^{n-1}. We denote by G the Gauss map associated to the immersion of M^{n-1}:

$$G : M^{n-1} \hookrightarrow T\mathbb{E}^n \tag{15}$$

is defined by

$$G(m) = (m, \xi_m). \tag{16}$$

The differential of G

$$dG : TM^{n-1} \hookrightarrow TT\mathbb{E}^n$$

satisfies:

$$dG(x, X) = (x, \xi, X, \tilde{\nabla}_X \xi) = (x, \xi, X, -A_\xi X). \tag{17}$$

In particular,

$$||dG|| \leq \sup(1, ||h||). \tag{18}$$

3.1.2 The projection on a smooth hypersurface

Let M^{n-1} be a closed (oriented embedded) hypersurface of $\mathbb{E}^n$. Since M^{n-1} is smooth, there exists a tubular neighborhood U be of M^{n-1} on which the orthogonal projection $\mathrm{pr}_{|U}$ from U to $\mathbb{E}^n$ is well defined. Remark that if δ is the maximum radius of U, then

$$\delta < \frac{1}{||h||}. \tag{19}$$

The following result is classical, (see [5] for instance):

Proposition 1 *The map*

$$pr_{|U} : U \rightarrow M^{n-1}$$

is differentiable; moreover, at each point $p \in U$, $Dpr_{|U}$ is given by the following matrix, in a frame of (unit) principal vectors of M^n completed by the unit normal vector:

$$\mathbf{Dpr_{|U}} = \begin{pmatrix} \frac{1}{1+\delta\epsilon\lambda_1} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{1}{1+\delta\epsilon\lambda_{n-1}} & 0 \end{pmatrix}$$

where $\lambda_1, ..., \lambda_{n-1}$ are the principal curvatures of M^{n-1}, and $\epsilon = \pm 1$. In particular,

$$||Dpr_{|U}|| \leq \frac{1}{1 - \delta||h||}. \tag{20}$$

In the following of this section, we shall say that a compact subset $\mathcal{B}$ lying in U is *close to* M^{n-1}. Moreover, if the orthogonal projection onto M^{n-1} induces a bijection between $\mathcal{B}$ and M^{n-1} we shall say that $\mathcal{B}$ is *strongly close to* M^{n-1}. If $\mathcal{B}$ is strongly close to M^{n-1}, then it is homeomorphic to it. In the following, we shall assume that M^{n-1} is the boundary of a compact subset K, and that $\mathcal{B}$ is the boundary of a geometric compact subset $\mathcal{C}$, (that is, admitting a normal cycle).

3.2 Comparing the normals

Let $\mathcal{C}$ be a geometric compact subset of $\mathbb{E}^n$ such that $\mathcal{B} = \partial\mathcal{C}$ is strongly close to M^{n-1}. Let B be a Borel subset of $\mathbb{E}^n$ included in $\mathcal{B}$. For every point m in M^{n-1}, one can compare the normal ξ_m of M^{n-1} with

$$\{(p,\nu) \in ST\mathbb{E}^n \text{such that } \mathrm{pr}(p) = m \text{and } (p,\nu) \in \text{spt } N(\mathcal{C})_{|B\times\mathbb{E}^n}\}.$$

For every (Borel) subset B of $\mathcal{C}$, we put

$$\alpha_B = \sup_{m\in\mathrm{pr}(B)} \sup_{\nu\in\text{spt } (N(\mathcal{C})_{|B\times\mathbb{E}^n})\text{such that } \mathrm{pr}(p)=\nu} |\angle(\xi_m,\nu)|. \tag{21}$$

3.3 A homotopy between normal cycles

With the previous notations, consider the map f defined by the following diagram:

$$\begin{array}{ccc} TU \simeq U\times\mathbb{E}^n & \xrightarrow{f} & \text{spt } N(K) \subset TK \simeq K\times\mathbb{E}^n \\ p_1\downarrow & & \uparrow G \\ U & \xrightarrow{pr} & M^{n-1} \end{array}$$

Let h be the affine homotopy between f and the identity, [4]:

$$h : (U\times\mathbb{E}^n)\times[0,1] \to \text{spt } N(K),$$

given by

$$h(x,X,t) = tf(x,X) + (1-t)(x,X).$$

Let B be a Borel subset included in $\mathcal{B}$. To simplify the notations, we define the $(n-1)$-current D by $D = N(\mathcal{C})_{|(B\times\mathbb{E}^n)}$. We define also the $(n-1)$-current E by $E = N(K)_{|(pr(B)\times\mathbb{E}^n)}$ and the n-current $C = h_\sharp(D\times[0,1])$. For technical reasons, we shall assume that B is regular enough to be sure that D is an integral current, (or at least representable by integration). We shall say that such a Borel subset is *regular*.

The homotopy formula for currents (cf. [4]), gives immediately

$$\partial C = f_\sharp(D) - D - h_\sharp(\partial D\times[0,1]).$$

Proposition 2 *1. One has:*

$$f_\sharp(D) = E.$$

2. Moreover,

$$\mathcal{F}(D-E) \leq (M(D)+M(\partial D)) \sup_{spt\ D} |f - Id| \sup_{spt\ D} (||Df||^{n-2}, ||Df||^{n-1}, 1).$$

Proof of Proposition 2:

1. We apply the constancy theorem, [4], 4.1.31: the support of the image by f of D (resp. ∂D) is included in the support of E, (resp. ∂E). Consequently, there exists an integer c such that $f_\sharp(D) = cE$. By evaluating $f_\sharp(D)$ and E on particular differential forms, as the volume form ω_0, and using the fact that the restriction og f to B is one-one, we see that $c = 1$; (see below for a precise definition of ω_0).

2. In order to evaluate the flat norm of $D - E$, we decompose $D - E$ in a sum of a $(n-1)$-current and the boundary of a n current, by writing:

$$D - E = \partial C - h_\sharp([0,1] \times \partial D). \tag{22}$$

   By definition of the flat norm, we deduce immediately that the flat norm of $(D - E)$ satisfies

$$\mathcal{F}(D-E) \leq M(C) + M(h_\sharp([0,1] \times \partial D)). \tag{23}$$

   On the other hand, since D is representable by integration, we have, ([4] 4.1.9.):

$$M(C) = M(h_\sharp(D \times [0,1])) \leq M(D) \sup_{\text{spt } D} |f - Id| \sup_{\text{spt } D} (||Df||^{n-1}, ||Id||^{n-1}), \tag{24}$$

   and

$$M(h_\sharp(\partial D \times [0,1])) \leq M(\partial D)) \sup_{\text{spt } D} |f - Id| \sup_{\text{spt } D} (||Df||^{n-2}, ||Id||^{n-2}), \tag{25}$$

   from which we deduce Proposition 2.

Proposition 3 *Let B be a regular Borel subset included in $\mathcal{B}$. Then*

1. $\sup_{spt\ D} |f - Id| \leq \max(\delta_B, \alpha_B)$, *and*

2. $\forall k \geq 1, \sup_{spt\ D} ||Df|| \leq \frac{\sup(1, ||h_B||)}{1 - \delta_B ||h_B||}$,

where $\delta_B = \delta(B, pr(B))$ is the Hausdorff distance between B and $pr(B)$ and $||h_B||$ is the maximum of the norm of the second fundamental form of M^{n-1} restricted to $pr(B)$.

Proof of Proposition 3:

1. The first item is trivial;

2. For the second item, we remark that

$$f = G \circ \mathrm{pr} \circ \mathrm{p}_1 \text{ and } Df = DG \circ D\mathrm{pr} \circ D\mathrm{p}_1. \tag{26}$$

   On the other hand, one has, almost everywhere,

$$\sup_{\mathrm{pr}(B)} ||DG|| \leq \sup_{\mathrm{pr}B}(1, ||h_B||), ||D\mathrm{pr}_B|| \leq \frac{1}{1-\delta_B||h_B||}, ||D\mathrm{p}_1|| = 1. \tag{27}$$

   The conclusion follows.

3.4 Approximation of curvature measures

From Propositions 2 and 3, we shall deduce the following result:

Theorem 3 *Let M^{n-1} be a closed (oriented) hypersurface of $\mathbb{E}^n$ bounding a compact subset K and $\mathcal{C}$ be a geometric compact subset of $\mathbb{E}^n$ the boundary of which $\mathcal{B}$ is strongly closed to M^{n-1}. Let B be any regular Borel subset of $\mathbb{E}^n$ included in $\mathcal{B}$. Then,*

$$\mathcal{F}(N(\mathcal{C})_{|(B\times\mathbb{E}^n)} - N(K)_{|(pr(B)\times\mathbb{E}^n)}) \leq$$

$$\max(\delta_B, \alpha_B)(\frac{\sup_B(1,||h_B||)}{1-\delta_B||h_B||})^{n-1}(M(N(\mathcal{C})_{|(B\times\mathbb{E}^n)}) + M(\partial N(\mathcal{C})_{|(B\times\mathbb{E}^n)})),$$

where $\delta_B = \delta(B, pr(B))$, $||h_B||$ denotes the maximum of the norm of the second fundamental form h of M^{n-1} restricted to $pr(B)$.

On the tangent bundle $T\mathbb{E}^n$ of $\mathbb{E}^n$, one can define $(n-1)$-forms which give rise to curvature measures when they are evaluated on the normal cycle of geometric compact subsets of M^n. These forms are invariant by rigid motions of $\mathbb{E}^n$ and by rotation on each fiber when they are restricted to the unit tangent bundle $ST\mathbb{E}^n$. We recall now their construction: we identify $T\mathbb{E}^n$ with $\mathbb{E}^n \times \mathbb{E}^n$ itself identified with $\mathbb{C}^n$, endowed with its canonical complex structure J. At any point (m, ξ) of $ST\mathbb{E}^n$, consider an orthonormal frame $(e_1, ..., e_{n-1})$ of $\xi^\perp$, and $(\epsilon_1 = Je_1, ..., \epsilon_{n-1} = Je_n)$ its image by J. On $ST\mathbb{E}^n$, we can build the $(n-1)$-differential form

$$(e_1^* + t\epsilon_1^*) \wedge ... \wedge (e_{n-1}^* + t\epsilon_{n-1}^*),$$

(the e_i^* and ϵ_i^* denote the dual frame). Consider this expression as a polynomial in the variable t, and remark that the coefficient of every t^i is a differential form ω_i which does not depend on the orthonormal frame (e_i). Each $(n-1)$-form ω_i is invariant under the action of the orthogonal group. That is why we call these n-forms the standard invariant $(n-1)$-forms on $ST\mathbb{E}^n$.

Proposition 4 *One has*

- $\forall k, 0 \leq k \leq n-1, ||\omega_k|| = C_1(n,k)$;
- $\forall k, 0 \leq k \leq n-1, ||d\omega_k|| = C_2(n,k)$;

where $C_1(n,k), C_2(n,k)$ *are constant depending only on* n *and* k*, satisfying* $C_1(n,k) leq C_2(n,k)$.

The proof is obvious and let to the reader. The exact values of C_1 and C_2 are $C_1(n,k) = C^k_{n-1}, C_2(n,k) = (k+1)(n-1-k) + k(n-k))C^k_{n-1}$.

The standard forms $(\omega_0, ..., \omega_{n-1})$ can be used to define curvature measures on $\mathbb{E}^n$, associated to any geometric compact subset $\mathcal{B}$.

Definition 2 *Let* $\mathcal{C}$ *be a geometric compact subset of* $\mathbb{E}^n$. *Let* B *be a Borel subset of* $\mathbb{E}^n$.
The k*-th curvature measure of* B *is the real number*

$$\mathcal{M}^{\mathcal{C}}_k(B) = < N(\mathcal{C})_{|B \times \mathbb{E}^n}, \omega_k > .$$

Recall that when M^{n-1} is a smooth hypersurface, $\mathcal{M}^{\mathcal{K}}_k(B)$ are nothing but the classical Lipschitz-Killing curvature of $B \cap M^{n-1}$, [5].

Let ξ denote a unit normal vector field defined on M^{n-1}, and h denote the second fundamental form of M^{n-1}. We denote by $\lambda^{\xi}_1, ..., \lambda^{\xi}_{n-1}$ the principal curvatures of M^{n-1}, (in the direction ξ).

Definition 3 *The* k^{-th}*-elementary symmetric function* $\Xi_k(\xi) = \{\lambda^{\xi}_{i_1}, ..., \lambda^{\xi}_{i_k}\}$ *of the principal curvatures of* M^{n-1} *in the direction* ξ *is called the* k^{-th}*-mean curvature of* M^n *in the direction* ξ.

Remark that

$$\det\,(I + tA_{\xi}) = \sum_{k=0}^{k=n-1} \Xi_k(\xi) t^k, \tag{28}$$

with $\Xi_k(\xi) = \sum_{I_k} \Delta_{I_k}(\xi)$, where $\Delta_{I_k}(\xi)$ is the sum of all k-minors of the matrix A_{ξ}, (I_k denotes the class of subsets of $\{1, ..., (N-1)\}$ with k elements).

The following result is well known, (see [1] for instance):

Proposition 5 $\mathcal{M}^K_k(B) = \int_{M^{n-1} \cap B} \Xi_k(\xi) dv$, *where* dv *denotes the volume form of* M^{n-1}.

Using Theorem 3 and Proposition 4, we get immediately the following

Corollary 1 *Let* K *be a compact subset of* $\mathbb{E}^n$ *whose boundary* M^{n-1} *is a smooth (closed oriented embedded) hypersurface of* $\mathbb{E}^n$. *Let* $\mathcal{C}$ *be a geometric compact subset of* $\mathbb{E}^n$ *whose* *boundary $\mathcal{B}$ is strongly close to M^{n-1}. Let B be any regular Borel subset of $\mathcal{B}$. Then, for every $k, 0 \leq k \leq n-1$,*

$$|\mathcal{M}_k^{\mathcal{C}}(B) - \mathcal{M}_k^K(pr(B))| \leq$$

$$C_2(n,k) \max(\delta_B, \alpha_B)((\frac{\sup_B(1, ||h_B||)}{1-\delta_B||h_B||})^{n-1}(M(N(\mathcal{C})_{|(B\times\mathbb{E}^n)}) + M(\partial(N(\mathcal{C})_{|B\times\mathbb{E}^n}))),$$

where $\delta_B = \delta(B, pr(B))$ is the Hausdorff distance between B and $pr(B)$ and $||h_B||$ is the maximum of the norm of the second fundamental form of M^{n-1} restricted to $pr(B)$.

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