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Abstract: A control approach for practical and asymptotic stabilization of driftless controllable control systems with perturbations is proposed. This type of systems naturally appears when addressing the trajectory stabilization problem of driftless control systems on Lie groups. The objective of the approach is to provide practical stability whatever the perturbation —e.g. practical stability of any trajectory in the state space—, and asymptotic stability or convergence to zero of the error variables when the perturbation term is zero or tends to zero. A general framework is presented in this paper and a control solution is proposed for the class of chained systems.

Key-words: practical stabilization, asymptotic stabilization, chained system, Lie group, transverse function

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Stabilisation pratique et asymptotique des systèmes chaînés via l'approche de commande par fonctions transverses

Résumé : Une approche de commande pour la stabilisation pratique et asymptotique des systèmes sans dérive commandables soumis à d'éventuelles perturbations est présentée. Ce type de système apparaît naturellement lorsque l'on s'intéresse à la stabilisation de trajectoires de systèmes sans dérive sur des groupes de Lie. L'objectif de commande est d'une part d'assurer la stabilisation pratique quelque soit la perturbation —dans le but, par exemple, de stabiliser toute trajectoire dans l'espace d'état, réalisable ou non—, et d'autre part la stabilisation asymptotique, ou la convergence vers zéro des variables d'erreur, lorsque le terme de perturbation est nul ou tend vers zéro. Un cadre général est présenté dans cet article, et une solution de commande particulière est proposée pour la classe des systèmes chaînés.

Mots-clés : stabilisation pratique, stabilisation asymptotique, système chaîné, groupe de Lie, fonction transverse

Contents

1 Introduction

The development of the transverse function (t.f.) approach [4] finds its original motivation in the problem of *practical* stabilization of the origin of a control system in the form

$$\mathcal{S}: \qquad \dot{x} = \sum_{i=1}^{m} u_i X_i(x) + P(x,t) \tag{1}$$

with $x \in \mathbb{R}^n$, $n > m$, $\{X_1, \ldots, X_m\}$ a set of smooth vector fields (v.f.) which satisfy the Lie Algebra Rank Condition (LARC) on an open ball centered at $x = 0$, and P an additive perturbation, continuous in x and t, but otherwise *arbitrary.* Note that such a perturbation may well forbid the existence of any equilibrium point for the controlled system. The t.f. approach provides a general solution to this problem. Up to now, and to our knowledge, this solution is unique in its class, eventhough several other methods and many control laws have been devised during the last decade to address the stabilization and path tracking problems when $P \equiv 0$. The importance of considering the perturbed case in association with the objective of practical stabilization is well illustrated when $\mathcal{S}$ is a system on a Lie group and the control objective consists of tracking a trajectory. Indeed, it is shown in [4] that the *error system* associated with this problem is in the same form as the original system, except for the apparition of a perturbation P. Moreover, when the trajectory is not a solution of the control system, asymptotic stabilization is not possible. Other reasons for considering practical stabilization as a reasonable control objective, in the case of nonlinear driftless systems, are also pointed out in [4]: lack of robustness of exponential (continuous/time-varying or discontinuous) stabilizers, non-existence of feedback controllers capable of stabilizing asymptotically every feasible trajectory [3],... and incapacity of most existing asymptotic stabilizers to ensure ε-ultimate boundedness of the closed-loop trajectories when a destabilizing perturbation P is present. However, it is important to realize that practical stabilization is by no means opposed to asymptotic stabilization. It is merely a weaker requirement, whose interest precisely resides in the fact that it is weaker and thus applicable to more numerous situations. Once practical stabilization is granted, it may still be possible, and desirable in some cases, to achieve asymptotic stabilization, or at least convergence to zero. When, for instance, P vanishes after some time. For the same reasons, feedback controllers derived with the t.f. approach should not be considered as antagonistic to other controllers proposed for nonlinear driftless systems —asymptotic stabilizers in particular. A more pertinent issue is the possibility of deriving a practical stabilizer which also ensures asymptotic stabilization when the perturbation P allows for it. For instance, can the t.f. approach be used to this purpose?

This question is addressed in the present paper, and a partial positive answer is brought to it. More precisely, an extension of the approach in [4] is proposed in order to achieve asymptotic stabilization of the origin of $\mathcal{S}$ when $P \equiv 0$, and asymptotic convergence to the origin when P tends to zero as time tends to infinity. The main ingredient of this extension is the concept of *generalized* transverse function introduced in Section 2. The principles of the t.f. approach and design of stabilizers are also exposed in this section. A solution to the problem of practical and asymptotic stabilization, for the popular class of chained systems, is proposed in Section 3, and illustrated by simulation results in Section 4. Finding a more general solution thus remains an open subject of research. In order to facilitate the reading of the paper, we have distributed the results' proofs along two sections: the cores are given in Section 5, whereas intermediate technical results of lesser conceptual significance are regrouped in the Appendix.

Since the t.f. approach finds its most natural exposition in the context of systems which are invariant on Lie groups, we have chosen to recast the systems and control problems evoked above in this framework. Let us recall the prominent role played by Lie groups in control theory [6, 2]. In particular, controllable driftless systems can always be approximated by controllable driftless homogeneous systems which are, after a possible dynamic extension, systems on Lie groups. The chained systems, which are more specifically addressed here, are systems on Lie groups.

The following notation is used throughout the paper. The tangent space of a manifold M at a point p is denoted as M_p. The differential of a smooth mapping f between manifolds, at a point p, is denoted as $df(p)$. We also use standard notation for Lie groups —see e.g [1] for more details on this topic. G denotes a Lie group of dimension n, with Lie algebra —of left-invariant v.f.— $\mathfrak{g}$. The identity element of G is denoted by e. Left and right translations are denoted by l and r respectively, i.e. $l_\sigma(\tau) = r_\tau(\sigma) = \sigma\tau$. As usual, if $X \in \mathfrak{g}$ and $p \in G$, $\exp tX$ is the solution at time t of $\dot{g} = X(g)$ with initial condition $g(0) = e$. The adjoint representation of G is Ad, i.e. for $\sigma \in G$, $\mathrm{Ad}(\sigma) = dI_\sigma(e)$ with $I_\sigma : G \longrightarrow G$ defined by $I_\sigma(g) = \sigma g \sigma^{-1}$. By extension we define $\mathrm{Ad}(\sigma)X(g) \triangleq dl_g(e)\mathrm{Ad}(\sigma)X(e)$. The differential of Ad is ad, defined by $(\mathrm{ad}X, Y) = [X, Y]$.

2 Control of perturbed driftless systems by the t.f. approach

Consider a control system

$$\mathcal{S}(g): \qquad \dot{g} = \sum_{i=1}^{m} u_i X_i(g) + P(g,t) \tag{2}$$

on a Lie group G, with $X_1, \ldots, X_m$ independent left-invariant smooth v.f. which satisfy the Lie Algebra Rank Condition (LARC). We assume that the drift term $P(g,t)$ is a continuous function of g and t, and that[1]

$$\forall (g,t) \in G \times \mathbb{R}, \qquad P(g,t) \in \mathrm{span}\{X_1(g), \ldots, X_m(g)\}^{\perp} \tag{3}$$

The definition of a *transverse function*, as originally given in [5] for v.f. on an arbitrary manifold —i.e. not necessarily on a Lie group—, is now recalled.

[1] Note that (3) can always be obtained after a possible preliminary feedback.

Definition 1 *Let $X_1, \dots, X_m$ denote smooth v.f. on a manifold M. A function $f \in \mathcal{C}^\infty(\mathbb{T}^p; M)$, with $\mathbb{T} \triangleq \mathbb{R}/2\pi\mathbb{Z}$ and $p \in \mathbb{N}$ is called a* transverse function *(for the v.f. $X_1, \dots, X_m$) if,*

$$\forall \sigma \in \mathbb{T}^p, \quad \text{span}\{X_1(f(\sigma)), \dots, X_m(f(\sigma)), \frac{\partial f}{\partial \sigma_1}(\sigma), \dots, \frac{\partial f}{\partial \sigma_p}(\sigma)\} = M_{f(\sigma)} \tag{4}$$

Note that by this definition, the image set $\text{Im}(f) = f(\mathbb{T}^p)$ is compact. The main contribution of [5] was to show that if a set of v.f. $X_1, \dots, X_m$ satisfies the LARC at some point $q \in M$, then for any neighborhood $\mathcal{U}$ of q, there exists a transverse function with values in $\mathcal{U}$.

In the context of stabilization, transverse functions allow to introduce $\dot{\sigma}$ as a new —virtual— control input vector. This leads us to introduce the following dynamic extension of $\mathcal{S}(g)$:

$$\mathcal{S}(g, \sigma): \qquad \begin{cases} \dot{g} = \sum_{i=1}^{m} u_i X_i(g) + P(g, t) \\ \dot{\sigma} = u_\sigma \end{cases} \tag{5}$$

where (u, u_σ) is viewed as an extended control vector. In the following subsection, the practical stabilization of $\mathcal{S}(g)$ with the t.f. approach is exposed —see [4] for more details.

2.1 Practical stabilization

Given a neighborhood $\mathcal{U}(e)$ of e, the problem consists in determining a (smooth) feedback control for $\mathcal{S}(g, \sigma)$ which asymptotically stabilizes a set contained in $\mathcal{U}(e) \times \mathbb{T}^p$.

Define $z \triangleq f(\sigma) g^{-1}$ with f any t.f. with values in $\mathcal{U}(e)$. All we have to do is find a feedback control which asymptotically stabilizes $z = e$. By differentiating the equality $f(\sigma) g^{-1} g = e$, one easily shows that along any smooth trajectory $(g, \sigma)(.)$ of $\mathcal{S}(g, \sigma)$,

$$\dot{z} = -dr_{g^{-1}}(f(\sigma)) \Big(\sum_{i=1}^{m} u_i X_i(f(\sigma)) - \frac{\partial f}{\partial \sigma}(\sigma) \dot{\sigma} + dl_z(g) P(g, t) \Big) \tag{6}$$

Now, for any v.f. Z on G, the property of transversality of f ensures that the equation

$$\sum_{i=1}^{m} u_i X_i(f(\sigma)) - \frac{\partial f}{\partial \sigma}(\sigma) u_\sigma = -dl_z(g) P(g, t) - dr_g(z) Z(z) \tag{7}$$

admits a feedback solution $(u, u_\sigma)(g, \sigma, t)$. Applying any[2] such feedback law to $\mathcal{S}(g, \sigma)$, and using the fact that $(dr_g(z))^{-1} = dr_{g^{-1}}(f)$, it follows from (6) that

$$\dot{z} = Z(z) \tag{8}$$

[2]The only (weak) requirement is that the solutions of $\mathcal{S}(g, \sigma)$ must be well defined for $t \in [0, \infty)$.

Therefore, provided that Z is chosen so as to asymptotically stabilize e for System (8), the feedback law (u, u_σ) defined by (7) makes the set $f(\mathbb{T}^p) \times \mathbb{T}^p \subset \mathcal{U}(e) \times \mathbb{T}^p$ asymptotically stable.

In general, the solution (u, u_σ) of (7) is not unique. It is shown in [4], however, that one can always find[3] transverse functions $f \in \mathcal{C}^\infty(\mathbb{T}^{n-m}; G)$, i.e. $p = n - m$ with the notation of Definition 1. It is clear from the transversality condition (4) that this value of p is minimal, and that the solution (u, u_σ) of (7) is unique in this case. Allowing the transverse function f to depend on a larger number of variables provides complementary control inputs which can be used to guarantee complementary control objectives. The asymptotic stabilization problem will be addressed in this way.

2.2 A framework for asymptotic stabilization

Let us introduce, in the framework of Lie groups, the following specific class of transverse functions. From now on, variables in $\mathbb{T}^{n-m}$ will be indexed starting from $m+1$, i.e. if $\theta \in \mathbb{T}^{n-m}$, $\theta = (\theta_{m+1}, \ldots, \theta_n)$.

Definition 2 *A function $f \in \mathcal{C}^\infty(\mathbb{T}^{n-m} \times \mathbb{T}^{n-m}; G)$ is called a* generalized *transverse function for the v.f $X_1, \ldots, X_m$ on the Lie group G if, for any $\sigma = (\theta, \beta) \in \mathbb{T}^{n-m} \times \mathbb{T}^{n-m}$,*

$$\operatorname{span}\{X_1(f(\sigma)), \ldots, X_m(f(\sigma)), \frac{\partial f}{\partial \theta_{m+1}}(\sigma), \ldots, \frac{\partial f}{\partial \theta_n}(\sigma)\} = G_{f(\sigma)} \tag{9}$$

and for any $\beta \in \mathbb{T}^{n-m}$,

$$f(0, \beta) = e \tag{10}$$

It is clear that any generalized transverse function is a transverse function. It is also quite simple to build a generalized transverse function $f \in \mathcal{C}^\infty(\mathbb{T}^{n-m} \times \mathbb{T}^{n-m}; G)$ from a transverse function $\bar{f} \in \mathcal{C}^\infty(\mathbb{T}^{n-m}; G)$. For example, define

$$\forall(\theta, \beta) \in \mathbb{T}^{n-m} \times \mathbb{T}^{n-m}, \quad f(\theta, \beta) = (\bar{f}(\beta))^{-1} \bar{f}(\theta + \beta)$$

Let us now consider any generalized transverse function. We let

$$\dot{\theta} = v, \qquad \dot{\beta} = w \tag{11}$$

so that $\dot{\sigma} = u_\sigma = (v, w)$. With this notation, Equation (7) is equivalent to

$$\sum_{i=1}^{m} u_i X_i(f(\sigma)) - \frac{\partial f}{\partial \theta}(\sigma) v = \frac{\partial f}{\partial \beta}(\sigma) w - dl_z(g) P(g, t) - dr_g(z) Z(z) \tag{12}$$

From (9), this equation has a unique feedback solution $(u, v)(g, \sigma, t)$ for any function w. The v.f. Z is again chosen so as to make $z = e$ asymptotically stable. Now the objective is to

[3] Expressions of such functions are given in that paper —see also the next subsection.

define w in order to make θ tend to zero. Indeed, this latter property implies, in view of (10), that f tends to e so that, from the fact that $z = f(\sigma)g^{-1}$ tends to e, the asymptotic convergence of g to e follows. Note that such a convergence cannot be obtained without the drift term P satisfying some extra conditions. For instance, if $P(e,t)$ is different from zero, then it follows from (3) that e cannot be an equilibrium for System (2), whatever the control u. Therefore, convergence of $P(g,t)$ to zero when g tends to e and t tends to infinity is an absolute requirement to the convergence of the system's solutions to e.

The feedback law (u,v) defined by (12) ensures the convergence of z to e independently of w. Hence, the asymptotic behavior of $\theta(t)$ and $\beta(t)$, for the controlled system, is described by the *zero-dynamics* obtained by setting $z = e$ in (12), i.e.

$$\sum_{i=1}^{m} u_i(g,\sigma,t)X_i(f(\sigma)) - \frac{\partial f}{\partial \theta}(\sigma)v(g,\sigma,t) = \frac{\partial f}{\partial \beta}(\sigma)w - P(f(\sigma),t) \tag{13}$$

From the initial assumption that $X_1,\dots,X_m$ are independent, there exist v.f. $X_{m+1},\dots,X_n$ such that $\text{span}\{X_1,\dots,X_n\} = \mathfrak{g}$. For any such set of v.f., there exist smooth functions $a_{i,j}$ and $b_{i,j}$ such that

$$\forall j = m+1,\dots,n\,, \qquad \frac{\partial f}{\partial \theta_j}(\sigma) = \sum_{i=1}^{n} a_{i,j}(\sigma)X_i(f(\sigma))\,, \quad \frac{\partial f}{\partial \beta_j}(\sigma) = \sum_{i=1}^{n} b_{i,j}(\sigma)X_i(f(\sigma)) \tag{14}$$

With d_i $(i = m+1,\dots,n)$ denoting the one-forms defined by $\langle d_i, X_k\rangle = \delta_{i,k}$, the application of d_i to each side of (13) yields, since $\dot{\theta} = v$,

$$A(\sigma)\dot{\theta} = -B(\sigma)w + \sum_{i=m+1}^{n} \langle d_i(f(\sigma)), P(f(\sigma),t)\rangle e_i \tag{15}$$

with

$$A(\sigma) \triangleq (a_{i,j}(\sigma))_{i,j=m+1,\dots,n}\,, \text{ and } B(\sigma) \triangleq (b_{i,j}(\sigma))_{i,j=m+1,\dots,n} \tag{16}$$

and e_i the $(i-m)$-th unit vector in $\mathbb{R}^{n-m}$. Note that the transversality condition (9) is equivalent to the matrix $A(\sigma)$ being invertible for any σ.

Equation (15) is important because it explicitly relates the control w (the time-derivative of β) to the variation of θ. In particular, the simplification obtained when $P \equiv 0$, i.e.

$$\dot{\theta} = -A^{-1}(\sigma)B(\sigma)w \tag{17}$$

suggests various ways of choosing w to make $|\theta(t)|$ non-increasing on the zero-dynamics. However, a difficulty arising at this stage, to ensure the convergence of $\theta(t)$ to zero, comes from the fact that $B(\sigma)$ tends to the null matrix when θ tends to zero —since $f(0,\beta) = e,\ \forall\beta \Rightarrow \frac{\partial f}{\partial \beta}(0,\beta) = 0,\ \forall\beta$. This difficulty is itself related to the well-known impossibility of ensuring *exponential* stabilization of e by means of a *smooth* feedback. The matter would still be easily settled if $B(\sigma)$ were invertible everywhere except at $\theta = 0$. Unfortunately, this is not true in general and further inspection of this matrix, in relation to the way the structure of f combines with the structure of Lie algebra $\mathfrak{g}$, is required. Although we do not know whether a solution always exists, we were able to use the specific structure of the Lie algebra associated with chained systems and derive a solution in this case. Prior to reporting it in the next section, we propose below a formulation of the problem which, whereas it is restricted to the zero-dynamics (17), prepares the ground to a solution for the complete system.

Problem 1 Given a neighborhood $\mathcal{U}(e)$ of e, determine a triplet (f, w, V) consisting of a generalized t.f. $f \in \mathcal{C}^\infty(\mathbb{T}^{n-m} \times \mathbb{T}^{n-m}; \mathcal{U}(e))$, a function $w \in \mathcal{C}^1(\mathcal{U}(0) \subset \mathbb{T}^{n-m}; \mathbb{R}^{n-m})$, and a positive definite[4] function $V \in \mathcal{C}^1(\mathcal{U}(0) \subset \mathbb{T}^{n-m}; \mathbb{R})$ with bounded first-order partial derivatives, such that, along the trajectories of the zero-dynamics (17),

$$V(\theta) < V_{\max} \Longrightarrow \dot{V}(\theta) \leq -W(\theta) \tag{18}$$

for some positive definite function W and some $V_{\max} > 0$ such that $V^{-1}([0, V_{\max})) \subset \mathcal{U}(0)$.

Note that (18) clearly implies that $\theta = 0$ is locally asymptotically stable for the system (17). Once the above problem is solved, it is not difficult to infer a solution to the problem of asymptotic stabilization of e for System $\mathcal{S}(g, \sigma)$. Such a solution is pointed out in the following proposition.

Proposition 1 *Assume that Problem 1 is solved by a triplet $(f, w^\star, V)$, and let Z denote a smooth v.f. which asymptotically stabilizes e for the system $\dot{z} = Z(z)$. Consider for $\mathcal{S}(g, \sigma)$ the feedback control (u, v, w) with (u, v) defined by (12) and w defined by*

$$w(\theta) = k\left(\frac{1}{V_{\max} - V(\theta)}\right) w^\star(\theta) \tag{19}$$

with k denoting any $\mathcal{K}_\infty$-function[5]. Then,

1. *the set $f(\mathbb{T}^{n-m}) \times \mathbb{T}^{n-m} \times \mathbb{T}^{n-m}$ is asymptotically stable for $\mathcal{S}(g, \sigma)$ whatever P.*
2. *the set $\{e\} \times \{0\} \times \mathbb{T}^{n-m}$ is asymptotically stable for $\mathcal{S}(g, \sigma)$, if $P = 0$.*
3. *$(g, \theta)(t) \longrightarrow (e, 0)$ as $t \longrightarrow +\infty$, if $P(g, t)$ tends to zero as $t \longrightarrow +\infty$ —uniformly w.r.t. g in compact sets.*

The domain of attraction of the controller contains —in the coordinates (z, θ, β)— the set $\mathcal{D}_z \times V^{-1}([0, V_{\max})) \times \mathbb{T}^{n-m}$, with $\mathcal{D}_z$ the domain of attraction of the vector field Z. Note also that, when P and Z are differentiable, the stabilizing feedback control (u, v, w) so obtained is differentiable on the set $G \times \mathcal{U}(0) \times \mathbb{T}^{n-m}$. In particular, if $P = 0$, this rules out a convergence rate as fast as exponential, as in the case of a time-periodic Lipschitz-continuous asymptotic stabilizer of $\mathcal{S}(g)$. However, while the frequency of a time-periodic stabilizer is constant, the time-derivatives of θ and β, which may be interpreted as self-adapting frequencies in the case of a stabilizer derived with the t.f. approach, asymptotically tend to zero.

[4] i.e. $\forall \theta, V(\theta) \geq 0$ and $V(\theta) = 0 \Longleftrightarrow \theta = 0$.

[5] i.e. $k \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$, $k(0) = 0$, k is strictly increasing, and $k(s) \longrightarrow +\infty$ as $s \longrightarrow +\infty$.

2.3 A class of generalized transverse functions

In this section, we introduce a class of generalized transverse functions which is instrumental in solving Problem 1 for the class of chained system. First, we need to recall the definition of a graded basis of $\mathfrak{g}$ ([4]).

Definition 3 *Let $X_1, \ldots, X_m \in \mathfrak{g}$ denote independent v.f. such that* $\mathrm{Lie}(X_1, \ldots, X_m) = \mathfrak{g}$. *Define inductively $\mathfrak{u}^k = \mathfrak{u}^{k-1} + [\mathfrak{u}, \mathfrak{u}^{k-1}]$ with $\mathfrak{u} = \mathrm{span}(X_1, \ldots, X_m)$, and let $K = \min\{k : \mathfrak{u}^k = \mathfrak{g}\}$. A* graded basis *of $\mathfrak{g}$ associated with $X_1, \ldots, X_m$ is an ordered basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ associated with two mappings $\lambda, \rho : \{m+1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ such that:*

1. For any $k = 1, \ldots, K$, $\mathfrak{u}^k = \mathrm{span}\{X_1, X_2, \ldots, X_{\dim \mathfrak{u}^k}\}$.

2. For $k \geq 2$ and $\dim u^{k-1} < i \leq \dim u^k$, $X_i = [X_{\lambda(i)}, X_{\rho(i)}]$ with $X_{\lambda(i)} \in \mathfrak{u}^a$, $X_{\rho(i)} \in \mathfrak{u}^b$, and $a + b = k$.

With any graded basis of $\mathfrak{g}$, one can associate a *weight-vector* $(r_1, \ldots, r_n)$ defined by

$$r_i = k \Longleftrightarrow X_i \in \mathfrak{u}^k \setminus \mathfrak{u}^{k-1} \Longleftrightarrow \dim u^{k-1} < i \leq \dim u^k$$

Note that, $1 = r_1 \leq r_2 \leq \cdots \leq r_n = K$ and, from Definition 3, $\forall i > m,\ r_i = r_{\lambda(i)} + r_{\rho(i)}$.

With $\{X_1, \ldots, X_n\}$ any graded basis of $\mathfrak{g}$, let us define $f \in \mathcal{C}^\infty(\mathbb{T}^{n-m} \times \mathbb{T}^{n-m}; G)$ by

$$\forall \sigma = (\theta, \beta) \in \mathbb{T}^{n-m} \times \mathbb{T}^{n-m}, \quad f(\sigma) = f_n(\sigma_n) \cdots f_{m+1}(\sigma_{m+1}) \tag{20}$$

with $f_j : \mathbb{T} \times \mathbb{T} \longrightarrow G$ defined by

$$\forall \sigma_j = (\theta_j, \beta_j), \quad f_j(\sigma_j) = \exp\left(\alpha_j(\sigma_j) X_j\right) \exp\left(\alpha_{j,\lambda}(\sigma_j) X_{\lambda(j)} + \alpha_{j,\rho}(\sigma_j) X_{\rho(j)}\right) \tag{21}$$

where

$$\begin{gathered} \alpha_{j,\lambda}(\sigma_j) = \varepsilon_j^{r_{\lambda(j)}} \left(\sin(\theta_j + \beta_j) - \sin \beta_j\right), \ \alpha_{j,\rho}(\sigma_j) = \varepsilon_j^{r_{\rho(j)}} \left(\cos(\theta_j + \beta_j) - \cos \beta_j\right) \\ \alpha_j(\sigma_j) = \frac{\varepsilon_j^{r_j}}{2} \sin \theta_j \end{gathered} \tag{22}$$

and the ε_j's are positive real numbers. This function obviously satisfies (10). As for the transversality condition (9), we have:

Proposition 2 *Let $X_1, \ldots, X_m$ denote independent v.f. on a Lie group G of dimension n. Assume that* $\mathrm{Lie}(X_1, \ldots, X_m) = \mathfrak{g}$. *Let $f \in \mathcal{C}^\infty(\mathbb{T}^{n-m} \times \mathbb{T}^{n-m}; G)$ be defined by (20)-(21)-(22) with $X_1, \ldots, X_n$ a graded basis of $\mathfrak{g}$. Then, there exist real numbers $\eta_{m+1}, \ldots, \eta_n, \varepsilon_0 > 0$ such that, for $(\varepsilon_{m+1}, \ldots, \varepsilon_n) = \varepsilon(\eta_{m+1}, \ldots, \eta_n)$ with $\varepsilon \in (0, \varepsilon_0)$, f satisfies (9). More precisely, the η_k's can be defined recursively by choosing any $\eta_{m+1} > 0$ and, for $k = m+2, \ldots, n$, by choosing η_k large enough w.r.t. $\eta_{m+1}, \ldots, \eta_{k-1}$.*

Remark 1 The same result can be obtained when f_j in (21) is defined by

$$f_j(\sigma_j) = \exp\left((\alpha_j + \frac{1}{2}\alpha_{j,\lambda}\alpha_{j,\rho})(\sigma_j) X_j\right) \exp\left(\alpha_{j,\rho}(\sigma_j) X_{\rho(j)}\right) \exp\left(\alpha_{j,\lambda}(\sigma_j) X_{\lambda(j)}\right)$$

Details about this point can be obtained from the authors.

3 Asymptotic stabilization of chained systems

A solution to Problem 1 is provided in the case where $G = \mathbb{R}^n$, $m = 2$ and the control v.f. X_1, X_2 are defined by

$$X_1(x) = (1, 0, x_2, \ldots, x_{n-1})', \quad X_2 = (0, 1, 0, \ldots, 0)' \tag{23}$$

with $g = x = (x_1, \ldots, x_n)$ and $e = 0$. The practical relevance of this case comes from the widespread use of chained systems to model the kinematic equations of various mechanical systems subjected to nonholonomic constraints (unicycle and car-like mobile robots, for instance) and also the possibility of using them as homogeneous approximations of dynamics involved in several other physical systems (ships, induction motors,...).

The v.f. X_1 and X_2 defined by (23) are left-invariant w.r.t. the group operation

$$(xy)_i = \begin{cases} x_i + y_i & \text{if } i = 1, 2 \\ x_i + y_i + \sum_{j=2}^{i-1} \frac{y_1^{i-j}}{(i-j)!} x_j & \text{otherwise} \end{cases}$$

Furthermore, $\text{Lie}(X_1, X_2) = \mathfrak{g}$, so that chained systems (with $P \equiv 0$) are controllable, and the v.f.

$$X_1, X_2, X_k \triangleq [X_1, X_{k-1}] \quad (k = 3, \ldots, n) \tag{24}$$

define a graded basis. The associated weight-vector r is given by

$$r_1 = r_2 = 1, r_k = k - 1 \quad (k = 3, \ldots, n) \tag{25}$$

Since the underlying Lie group G is $\mathbb{R}^n$, a simple example of v.f. which globally exponentially stabilizes the origin of $\dot{z} = Z(z)$ on $\mathbb{R}^n$ is defined by $Z(z) = Kz$ with K denoting any $n \times n$ Hurwitz-stable matrix.

The following notation for $\theta_i \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$|\theta_i| = \begin{cases} \theta_i & \text{if } \theta_i \in [0, \pi) \\ -\theta_i & \text{if } \theta_i \in [-\pi, 0) \end{cases}$$

is used in the main result stated next.

Theorem 1 *When $m = 2$ and the v.f. X_1, X_2 are given by (23), a solution to Problem 1 is the triplet (f, w, V) consisting of*

1. *the generalized t.f. defined by (20)-(21)-(22) with $(\varepsilon_{m+1}, \ldots, \varepsilon_n) = \varepsilon(\eta_{m+1}, \ldots, \eta_n)$, ε beeing any strictly positive value such that f ranges in $\mathcal{U}(e)$, and the η_k's beeing chosen as indicated in Proposition 2.*

2. *the function $w \in \mathcal{C}^1((-\pi, \pi)^{n-2}; \mathbb{R}^{n-2})$ defined by*

$$w_i(\theta_i) = \frac{1}{\eta_i^{i-2}} |\theta_i|^{(i-3)} \theta_i \qquad (i = 3, \ldots, n) \tag{26}$$

3. the function $V \in \mathcal{C}^1((-\pi,\pi)^{n-2};\mathbb{R})$ *defined by*

$$V(\theta) \triangleq \sum_{i=3}^{n} \eta_i^{i-3/2}|\theta_i|^{n+2-i} \quad \text{with} \quad V_{\max} = \min_{i=3,\dots,n}\{\eta_i^{i-3/2}\pi^{n+2-i}\}$$

Remark 2 The solution to Problem 1 given in Theorem 1 applies also to a unicycle-like mobile robot without having to transform its kinematic equations into the chain form —the only restriction is that ε must be smaller than some finite upper bound $\varepsilon_0 > 0$ whatever $\mathcal{U}(e)$ whereas, in the case of chained system, $\varepsilon_0 = +\infty$. One only has to check that the proof of Theorem 1 works as well in this case with $n = 3$, $G = \mathbb{R}^2 \times S^1$, $g = (x, y, \alpha)$, and the system's control v.f. defined by

$$X_1(g) = (\cos\alpha, \sin\alpha, 0)', \quad X_2(g) = (0, 0, 1)' \tag{27}$$

These v.f. are left-invariant w.r.t. the group operation

$$g_1 g_2 = \begin{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + R(\alpha_1)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \alpha_1 + \alpha_2 \end{pmatrix}$$

with $g_i = (x_i, y_i, \alpha_i)'$ and $R(\alpha_1)$ the rotation matrix of angle α_1. Also, $\text{Lie}(X_1, X_2) = \mathfrak{g}$ and $\{X_1, X_2, X_3 = [X_1, X_2]\}$ constitutes a graded basis of $\mathfrak{g}$ with weight vector $(r_1 = 1, r_2 = 1, r_3 = 2)$.

Let us comment on the rate of convergence provided by a feedback control derived according to Proposition 1 and Theorem 1, when $P \equiv 0$. This will be the starting point of a more general discussion about what the t.f. approach can offer in comparison with other control design methods, its limitations and assets. Assuming that the v.f. Z used in the expression of (u, v) is chosen so as stabilize the origin of $\dot{z} = Z(z)$ exponentially, the rate of convergence of $g(t)$ to e coincides with the slower rate of convergence of $\theta(t)$ to zero on the zero-dynamics. This latter rate is itself given by the rate of convergence of $V(\theta(t))$ to zero, and thus related to the lowerbound W of $|\dot{V}|$ as pointed out in (18). In the proof of Theorem 1, $W(\theta)$ is proportional to $|\theta|^{n+1}$ while $k_1|\theta|^{n-1} \leq V(\theta) \leq k_2|\theta|^2$ in the neighborhood of $\theta = 0$. One deduces that $V(\theta(t))$ tends to zero at least as quickly as $t^{-\frac{2}{n-1}}$. In fact, a complementary analysis would show that $V(\theta(t))$ cannot tend to zero faster. Therefore $|\theta(t)|$ tends to zero like $t^{-\frac{2}{(n-1)^2}}$, and so does $|g(t)|$ towards e. This polynomial rate of convergence is similar to the one which can be obtained by applying a smooth time-periodic stabilizer to $\mathcal{S}(g)$. Therefore, one can conclude that, as far as asymptotic stabilization is concerned, no clear advantage results from designing a stabilizer with the t.f. approach. In the authors' opinion this conclusion is correct, but it conveys only a partial picture of the properties granted by the approach. Indeed, the primary feature of such a controller, which has motivated the development of the t.f. approach in the first place, is the capacity of ensuring practical stabilization, with easily tunable arbitrary small ultimate bound of the state error, independently of the "perturbation" P acting on the system. As shown in [4], this allows for example to track *any* trajectory in the state space (it does not have to be a solution to the system's equations) with arbitrary good precision, in the sense that tracking errors are ultimately bounded by a pre-specified (non-zero, but otherwise as small as desired) threshold. To our knowledge, no other controller proposed so far in the literature has this capacity. Our motivation, for the present paper, was to show that such a controller can also be endowed with the extra property of ensuring asymptotic point-stabilization, when such a feature is desirable. This is achieved via the concept of a generalized t.f. depending upon two sets of variables whose time-derivatives are used as extra control inputs. Transversality is maintained with respect to the first set θ, while the second set β is used to enforce some type of "phase-tuning" which allows to reduce the size of the t.f. when the perturbation P vanishes.

4 Simulation results

The control law proposed in the previous section has been tested by simulation on a four-dimensional chained system. The following parameters for the definition of the transverse function have been used: $\varepsilon = 0.2$, $\eta_3 = 1; \eta_4 = 8$. The v.f. $Z(z)$ in (12) has been chosen as $Z(z) = -0.3z$. Finally the $\mathcal{K}_\infty$-function k in (19) has been defined by $k(s) = 10V_{\max}s$ with $V_{\max}$ defined according to Theorem 1. The initial condition for the simulation was $x(0) = (0,0,0,10)'$, and $\sigma(0) = 0$. Figure 1 displays the state variables versus time. As discussed in the previous section, the convergence rate to zero is slow. For comparison, Figure 2 displays the same variables when no attempt is made to achieve convergence to zero, i.e. with $w = 0$ and $\beta = 0$ in the control law defined by (12). In this case $\theta(t)$ exponentially converges to some $\theta_{\lim} \in \mathbb{T}^{n-m}$, and $x(t)$ exponentially converges to $f(\theta_{\lim}, 0)$. Note that the solution to Problem 1 given by Theorem 1 is only one of its kind and that much room is left for improving the proposed stabilization method.

5 Proofs

The following notation is used in the proofs. With v denoting a smooth function of the variables x and y —possibly vector-valued—, we write $v = \mathrm{o}(x^k)$ (resp. $v = \mathrm{O}(x^k)$) if $\frac{|v(x,y)|}{|x|^k} \longrightarrow 0$ as $|x| \longrightarrow 0$ (resp. if $\frac{|v(x,y)|}{|x|^k} \leq K < \infty$ in some neighborhood of $x = 0$) uniformly w.r.t. y belonging to compact set. Finally, for indexed variables x_i with $i = k, \dots, n$ we define the set of indexed vectors $\{\bar{x}_p\}_{p \in \{k,\dots,n\}}$ by setting $\bar{x}_p = (x_k, \dots, x_p)$.

5.1 Proof of Proposition 1

Let us first recall, as shown in Section 2.2, that the control $(u,v)(g,\sigma,t)$ defined by (12) yields

$$\dot{z} = Z(z) \tag{28}$$

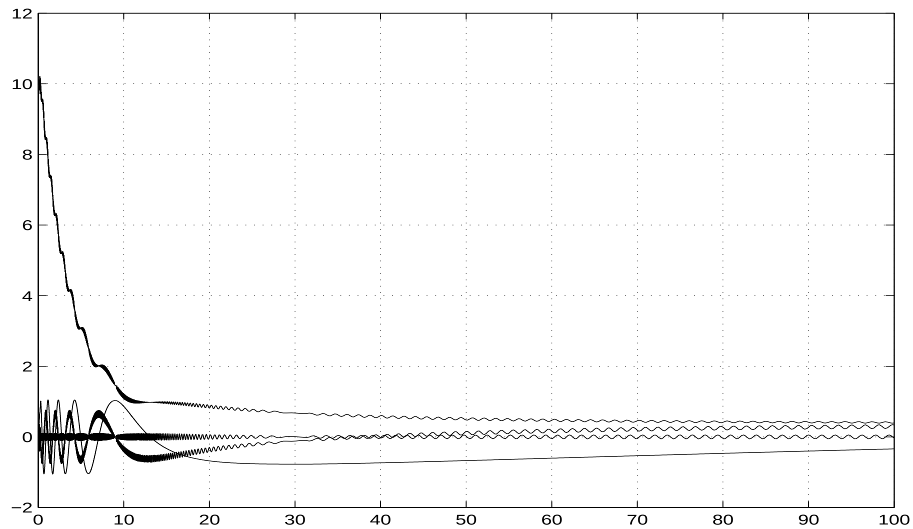

Figure 1: State variables for the 4-d chained system, asymptotic stabilization.

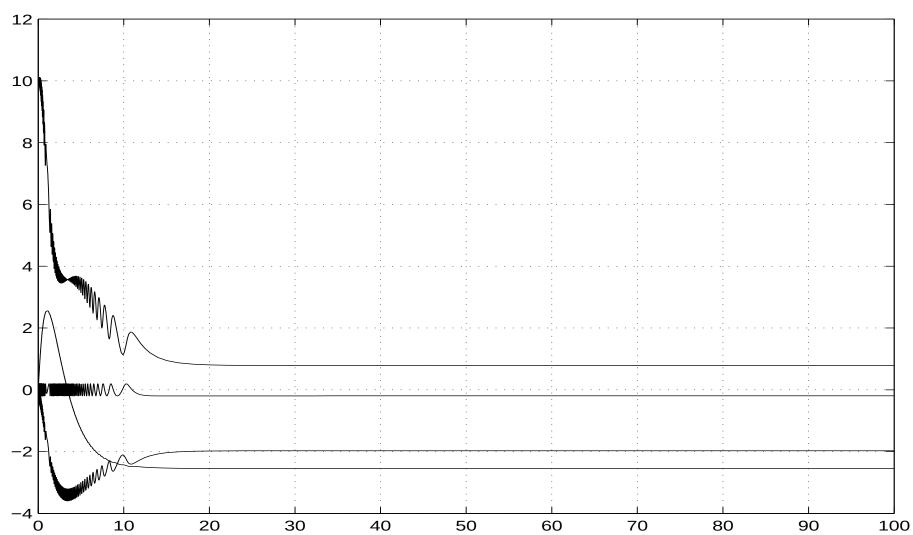

Figure 2: State variables for the 4-d chained system, practical stabilization.

with $z = fg^{-1}$ and Z chosen so as to ensure asymptotic stability of e for the above system. Now, the projection of both members of the equality (12) onto $(\text{span}\{X_1, \ldots, X_m\})^{\perp}$ yields the following equality —compare with (15)—

$$\dot{\theta} = -A^{-1}(\sigma)B(\sigma)w + A^{-1}(\sigma) \sum_{i=m+1}^{n} \langle d_i(f), dl_z(g)P(g,t) + dr_g(z)Z(z)\rangle e_i \tag{29}$$

where we have used the same notation as in Section 2.2. Using the fact that $z = fg^{-1}$, we rewrite this equation as

$$\dot{\theta} = -A^{-1}(\sigma)B(\sigma)w + A^{-1}(\sigma) \sum_{i=m+1}^{n} \langle d_i(f), dl_z(z^{-1}f)P(z^{-1}f,t) + dr_{z^{-1}f}(z)Z(z)\rangle e_i \tag{30}$$

Since $w(\theta)$, as defined by (19), tends to infinity when $V(\theta)$ tends to $V_{\max}$, boundedness of w along a trajectory of the closed-loop system is ensured provided that $\theta(0)$ is chosen so that $V(\theta(0)) < V_{\max}$ and

$$V(\theta(0)) < V_{\max} \Longrightarrow \exists \delta > 0 : \forall t, \quad V(\theta(t)) < V_{\max} - \delta \tag{31}$$

To show that the above proposition is true, we proceed by contradiction. Assume that (31) is not true for some trajectory, then there exists a sequence $\theta_n \triangleq \theta(t_n))_{n\in\mathbb{N}}$ such that

$$V(\theta_n) \longrightarrow V_{\max} \text{ and } \dot{V}(\theta_n) \geq 0 \tag{32}$$

On the other hand, by differentiating V along this trajectory and by using (18), (19), and (30), we obtain

$$\begin{aligned}\dot{V}(\theta) \leq & -k(\frac{1}{V_{\max} - V(\theta)})W(\theta) \\ & + \frac{\partial V}{\partial \theta}(\theta)A^{-1}(\sigma) \sum_{i=m+1}^{n} \langle d_i(f), dl_z(z^{-1}f)P(z^{-1}f,t) + dr_{z^{-1}f}(z)Z(z)\rangle e_i\end{aligned} \tag{33}$$

In view of (32), this implies that

$$k(\frac{1}{V_{\max} - V(\theta_n)})W(\theta_n) \leq \frac{\partial V}{\partial \theta}(\theta_n)A^{-1}(\sigma) \sum_{i=m+1}^{n} \langle d_i(f), dl_z(z^{-1}f)P(z^{-1}f,t) - dr_{z^{-1}f}(z)Z(z)\rangle e_i$$

As n tends to infinity, the term on the left-hand side tends to infinity because $V(\theta_n)$ tends to $V_{\max} > 0$, whereas the term on the right hand-side is bounded because, i) by assumption $\frac{\partial V}{\partial \theta}$ is bounded on $\mathcal{U}(0) \supset V^{-1}([0, V_{\max}))$, ii) z, and subsequently z^{-1}, are bounded due to the asymptotic stability of e for the system (28), and iii) $P(z^{-1}f, t)$ is bounded because P is continuous and $P(g,t)$ tends, by assumption, to zero as t tends to infinity uniformly w.r.t g

in compact sets. The resulting contradiction implies that (31) is true. This in turn implies that the control law is well defined and bounded along any trajectory of the closed-loop system and that the trajectories are complete.
From here the asymptotic stability of $(g,\theta)=(e,0)$ when $P\equiv 0$, and the convergence of $(g,\theta)(t)$ to $(e,0)$ when P tends to zero, follow immediately from the asymptotic stability of e for the system (28), and from (33).

5.2 Proof of Proposition 2

We proceed in three steps summarized in the form of three lemmas which are proved in the appendix. The first two lemmas also point out specific properties of the partial derivative of $f(\theta,\beta)$ with respect to β that are used in the proof of Theorem 1. For $j=m+1$, the convention $\mathrm{O}(\bar\varepsilon_{j-1})\triangleq 0$ is used in Lemma 2.

Lemma 1 *There exist analytic functions $v_{i,j}$ and $w_{i,j}$ ($i\in\{1,\dots,n\},\ j\in\{m+1,\dots,n\}$) such that:*

$$\frac{\partial f_j}{\partial\theta_j}(\sigma_j)=\sum_{i=1}^n v_{i,j}(\sigma_j)X_i(f_j(\sigma_j))\,,\quad \frac{\partial f_j}{\partial\beta_j}(\sigma_j)=\sum_{i=1}^n w_{i,j}(\sigma_j)X_i(f_j(\sigma_j)) \tag{34}$$

with

$$v_{i,j}=\begin{cases}\mathrm{O}(\varepsilon_j^{r_i}) & \forall i\\ \mathrm{o}(\varepsilon_j^{r_i}) & \text{if } i<j\ \&\ r_i=r_j\\ \dfrac{\varepsilon_j^{r_j}}{2}+\mathrm{o}(\varepsilon_j^{r_j}) & \text{if } i=j\end{cases} \tag{35}$$

and

$$w_{i,j}=\begin{cases}\mathrm{O}(\varepsilon_j^{r_i})\mathrm{O}(\theta_j) & \forall i\\ \varepsilon_j^{r_j}(1-\cos\theta_j)+\mathrm{o}(\varepsilon_j^{r_j})\mathrm{o}(\theta_j^2) & \text{if } i=j\end{cases} \tag{36}$$

Lemma 2 *There exist analytic functions $a_{i,j}$ and $b_{i,j}$ ($i\in\{1,\dots,n\},\ j\in\{m+1,\dots,n\}$) such that*

$$\frac{\partial f}{\partial\theta_j}(\sigma)=\sum_{i=1}^n a_{i,j}(\sigma)X_i(f(\sigma))\,,\quad \frac{\partial f}{\partial\beta_j}(\sigma)=\sum_{i=1}^n b_{i,j}(\sigma)X_i(f(\sigma)) \tag{37}$$

with

$$a_{i,j}=\begin{cases}\mathrm{O}(\bar\varepsilon_j^{r_i}) & \forall i\\ \mathrm{O}(\bar\varepsilon_{j-1})\mathrm{O}(\bar\varepsilon_j^{r_i-1})+\mathrm{o}(\bar\varepsilon_j^{r_i}) & \text{if } i<j\ \&\ r_i=r_j\\ \dfrac{\varepsilon_j^{r_j}}{2}+\mathrm{O}(\bar\varepsilon_{j-1})\mathrm{O}(\bar\varepsilon_j^{r_j-1})+\mathrm{o}(\bar\varepsilon_j^{r_j}) & \text{if } i=j\end{cases} \tag{38}$$

and

$$b_{i,j} = \begin{cases} O(\bar{\varepsilon}_j^{r_i})O(\bar{\theta}_j) & \forall i \\ \varepsilon_j^{r_j}(1-\cos\theta_j) + O(\bar{\varepsilon}_{j-1})O(\bar{\varepsilon}_j^{r_j-1})O(\theta_j)O(\bar{\theta}_{j-1}) + o(\bar{\varepsilon}_j^{r_j})o(\bar{\theta}_j^2) & \text{if } i=j \end{cases} \tag{39}$$

The second lemma would be a direct consequence of Lemma 1 if the partial derivative of f and f_j with respect to θ_j and β_j were equal. Note also that, if all O and o terms in the above expressions were equal to zero, then the transversality property would follow from (37-38) and from the fact that $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$. Although this is not the case, one can show that these terms can be neglected provided that the ε_j's are adequately chosen.

Lemma 3 *There exist $n-m$ numbers $\eta_{m+1}, \ldots, \eta_n$, and $\varepsilon_0 > 0$ such that choosing*

$$(\varepsilon_{m+1}, \ldots, \varepsilon_n) = \varepsilon(\eta_{m+1}, \ldots, \eta_n)$$

with $\varepsilon \in (0, \varepsilon_0]$ yields

$$\forall \sigma \in \mathbb{T}^{n-m} \times \mathbb{T}^{n-m}, \quad \text{Det}\, A(\sigma) \neq 0 \quad (A(\sigma) = (a_{i,j}(\sigma))_{i,j=m+1,\ldots,n}) \tag{40}$$

5.3 Proof of Theorem 1

The proof relies on the following lemma, proved in the appendix, which points out complementary properties of the functions $a_{i,j}$ and $b_{i,j}$ in Lemma 2 in the case of chained systems.

Lemma 4 *In the case of chained systems, the functions $a_{i,j}$ and $b_{i,j}$ ($i = 1, \ldots, n,\ j = 3, \ldots, n$) are homogeneous polynomials of degree r_i in $\varepsilon_3, \ldots, \varepsilon_j$. Furthermore,*

$$a_{i,j} = O(\bar{\theta}_j^{r_i - r_j}), \quad b_{i,j} = O(\bar{\theta}_j^{\max(1, r_i - r_j + 2)}) \tag{41}$$

Let

$$A_p(\sigma) \triangleq (a_{i,j}(\sigma))_{i,j=3,\ldots,p}, \quad B_p(\sigma) \triangleq (b_{i,j}(\sigma))_{i,j=3,\ldots,p} \tag{42}$$

We note that $A_n = A$ and $B_n = B$, with A and B defined by (16). We also remark that A_p and B_p only depend on $\bar{\sigma}_p = (\sigma_3, \ldots, \sigma_p)$, and $\bar{\varepsilon}_p$ —this is easily verified from (85). The dependence w.r.t the latter vector of parameters is omitted in the notation of these matrices for the sake of lightening the notation.

Theorem 1 is obtained as a particular case, corresponding to $p = n$, of the following proposition.

Proposition 3 *For any $p = 3, \ldots, n$, there exists a set of positive numbers $\{\eta_3, \ldots, \eta_p\}$ such that setting $(\varepsilon_3, \ldots, \varepsilon_p) = \varepsilon(\eta_3, \ldots, \eta_p)$ with $\varepsilon > 0$ implies that*

i) the matrix $A_p(\sigma)$ is invertible for any σ, and

$$\forall i, j = 3, \ldots, p, \quad (A_p^{-1}(\sigma))_{i,j} = O(\bar{\theta}_p^{r_i - r_j}) \tag{43}$$

ii) the origin of

$$\dot{\bar\theta}_p = -A_p^{-1}(\sigma)B_p(\sigma)\bar{w}_p \tag{44}$$

is locally asymptotically stable, and along the trajectories of (44),

$$V_p(\bar\theta_p) < V_{p,max} \Longrightarrow \dot{V}_p(\bar\theta_p) \leq -\alpha_p|\bar\theta_p|^{n+1} \quad (\alpha_p > 0) \tag{45}$$

with

$$V_p(\bar\theta_p) \triangleq \sum_{i=3}^{p} \eta_i^{i-3/2}|\theta_i|^{n+2-i}, \qquad V_{p,max} = \min_{i=3,\dots,p}\{\eta_i^{i-3/2}\pi^{n+2-i}\} \tag{46}$$

The proof of the proposition proceeds by induction. For $p = 3$, it follows from (25) and Lemmas 2 and 4 that

$$a_{3,3}(\sigma) = \frac{\varepsilon_3^{r_3}}{2} = \frac{\varepsilon_3^2}{2}, \quad b_{3,3}(\sigma) = \varepsilon_3^{r_3}(1-\cos\theta_3) = \varepsilon_3^2(1-\cos\theta_3) \tag{47}$$

Therefore, $a_{3,3}(\sigma) > 0$ for any $\varepsilon_3 > 0$, and Property i) is satisfied.

From their definitions, w_3 and V_3 are of class $\mathcal{C}^1$ on $\mathcal{U} \triangleq V_3^{-1}([0, V_{3,\max})) = \mathbb{T}\setminus\{\pi\}$, and V_3 has a bounded first-order derivative on this set. Let $\eta_3 = 1$. From (26) and (47), we can rewrite (44) as

$$\dot\theta_3 = -a_{3,3}^{-1}(\sigma)\left(\varepsilon_3^{r_3}(1-\cos\theta_3)\right)\theta_3 = -2(1-\cos\theta_3)\theta_3 \tag{48}$$

From (46), one easily verifies that, along the trajectories of (48),

$$\theta_3 \neq \pi \Longrightarrow \qquad \dot{V}_3(\theta_3) = -2(n-1)(1-\cos\theta_3)|\theta_3|^{n-1} \tag{49}$$

We deduce from (49) that

$$\theta_3 \neq \pi \Longrightarrow \qquad \dot{V}_3(\theta_3) \leq -\alpha_3|\theta_3|^{n+1} \quad (\alpha_3 > 0)$$

This concludes the proof of Proposition 3 for $p = 3$.

Let us now assume that i) and ii) hold true up to some $p < n$, with $\bar\varepsilon_p = \bar\eta_p$, and show that they are also true for $p+1$, with $\bar\varepsilon_{p+1} = \bar\eta_{p+1}$. Then i) and ii) are still true when $\bar\varepsilon_{p+1} = \varepsilon\bar\eta_{p+1}$ with $\varepsilon > 0$, thanks to the homogeneity properties of the $a_{i,j}$'s and $b_{i,j}$'s —see Lemma 4. Indeed, when $\bar\eta_{p+1}$ is multiplied by ε, then A_{p+1} and B_{p+1} are just pre-multiplied by the diagonal matrix $\text{Diag}(\varepsilon^{r_3}, \dots, \varepsilon^{r_{p+1}})$, thus leaving the equation (44) and the subsequent analysis unchanged.

From (42), A_{p+1} and B_{p+1} can be written as

$$A_{p+1} = \begin{pmatrix} A_p & a_{*,p+1} \\ a_{p+1,*} & a_{p+1,p+1} \end{pmatrix}, \quad B_{p+1} = \begin{pmatrix} B_p & b_{*,p+1} \\ b_{p+1,*} & b_{p+1,p+1} \end{pmatrix} \tag{50}$$

Let us recall (see e.g. [8, Ch. 2]) that if A_{11} and A_{22} are square matrices with A_{11} nonsingular, the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is invertible if and only if the Schur complement of A_{11} in A: $S \triangleq A_{22} - A_{21}A_{11}^{-1}A_{12}$ is invertible. Then

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{pmatrix} \tag{51}$$

From (50), the Schur complement of A_p in A_{p+1} is $S = a_{p+1,p+1} - a_{p+1,*}A_p^{-1}a_{*,p+1}$ and, in view of (25) and Lemmas 2 and 4,

$$S = \frac{\varepsilon_{p+1}^p}{2} + q^{p-1}(\varepsilon_{p+1}) \tag{52}$$

with $q^{p-1}(\varepsilon_{p+1})$ a polynomial of degree $p-1$ in ε_{p+1} —note that the term $a_{p+1,*}A_p^{-1}a_{*,p+1}$ depends on ε_{p+1} only through $a_{*,p+1}$. This implies that S, and thus A_{p+1}, are invertible for ε_{p+1} large enough. In order to prove i), there remains to show that (43) holds true for $p+1$. Since (43) is true for p, this follows directly from the fact that for any $p = 3, \dots, n-1$ and $\varepsilon_3, \dots, \varepsilon_{p+1}$ such that A_p and A_{p+1} are invertible,

$$A_{p+1}^{-1} = \begin{pmatrix} A_p^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} + \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & 0 \end{pmatrix} \triangleq \begin{pmatrix} A_p^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} + R \tag{53}$$

with

$$\varepsilon_{p+1} \geq 1 \Longrightarrow \begin{cases} |R_{i,j}| \leq C\varepsilon_{p+1}^{-1}|\bar{\theta}_{p+1}|^{r_i - r_j} & \text{for } i \leq p \\ |R_{i,j}| \leq C\varepsilon_{p+1}^{-p}|\bar{\theta}_{p+1}|^{r_i - r_j} & \text{for } i = p+1 \end{cases} \tag{54}$$

for some constant C. (53) is just a rewriting of (51) with

$$\begin{aligned} R_{11} &= S^{-1}A_p^{-1}a_{*,p+1}a_{p+1,*}A_p^{-1} \\ R_{12} &= -S^{-1}A_p^{-1}a_{*,p+1} \\ R_{21} &= -S^{-1}a_{p+1,*}A_p^{-1} \end{aligned}$$

and from here, (54) is easily obtained from Lemma 4, (43), and (52), using the fact that neither A_p nor $a_{p+1,*}$ depend on ε_{p+1}, and the fact that each $a_{i,j}$ satisfies

$$|a_{i,j}| \leq C|\bar{\varepsilon}_j|^{r_i}|\bar{\theta}_j|^{r_i - r_j} \tag{55}$$

for some constant C since, by Lemma 4, $a_{i,j}$ is a polynomial in $\varepsilon_3, \dots, \varepsilon_j$ of degree r_i and satisfies (41). Note that a relation similar to (55) holds for $b_{i,j}$, i.e.

$$|b_{i,j}| \leq C|\bar{\varepsilon}_j|^{r_i}|\bar{\theta}_j|^{\max(1, r_i - r_j + 2)} \tag{56}$$

This relation will be used later on.

Let us now examine the case of ii). Throughout the rest of the proof, we assume that $\varepsilon_{p+1} \geq 1$. From (50) and (53)

$$A_{p+1}^{-1}B_{p+1}\bar{w}_{p+1} = \begin{pmatrix} A_p^{-1}B_p\bar{w}_p \\ S^{-1}b_{p+1,p+1}w_{p+1} \end{pmatrix} + F_2 \tag{57}$$

with

$$F_2 = RB_{p+1}\bar{w}_{p+1} + \begin{pmatrix} A_p^{-1}b_{*,p+1}w_{p+1} \\ S^{-1}b_{p+1,*}\bar{w}_p \end{pmatrix} \tag{58}$$

From (39) and Lemma 4, it is not difficult to deduce that

$$b_{p+1,p+1} = \varepsilon_{p+1}^p(1-\cos\theta_{p+1}) + R_b \tag{59}$$

with

$$|R_b| \leq C\varepsilon_{p+1}^{p-1}|\bar{\theta}_{p+1}|^2 \tag{60}$$

for some constant C —recall that $\varepsilon_{p+1} \geq 1$. From (57) and (59), System (44) for $p+1$ can be written as

$$\dot{\bar{\theta}}_{p+1} = \underbrace{\begin{pmatrix} -A_p^{-1}B_p\bar{w}_p \\ -S^{-1}\varepsilon_{p+1}^p(1-\cos\theta_{p+1})w_{p+1} \end{pmatrix}}_{F_0} - \underbrace{\begin{pmatrix} 0 \\ S^{-1}R_bw_{p+1} \end{pmatrix}}_{F_1} - F_2 \tag{61}$$

We claim that the Lie derivative —denoted here as $F_0(V_{p+1})$— of V_{p+1} along F_0 defined by (61) satisfies

$$F_0(V_{p+1}) \leq -\alpha_p|\bar{\theta}_p|^{n+1} - \alpha_1\varepsilon_{p+1}^{1/2}|\theta_{p+1}|^{n+1} \quad (\alpha_p, \alpha_1 > 0) \tag{62}$$

Indeed, by (46), $V_{p+1} = V_p + \varepsilon_{p+1}^{p-1/2}|\theta_{p+1}|^{n-p+1}$ —recall that $\bar{\varepsilon}_{p+1} = \bar{\eta}_{p+1}$— and it follows from (61) that

$$F_0(V_{p+1}) = -A_p^{-1}B_p\bar{w}_p(V_p) - (n-p+1)S^{-1}\varepsilon_{p+1}^{2p-1/2}(1-\cos\theta_{p+1})w_{p+1}\theta_{p+1}^{\{n-p\}} \tag{63}$$

with the notation $x^{\{n\}} = |x|^{n-1}x$, also used in subsequent relations. From (45),

$$-A_p^{-1}B_p\bar{w}_p(V_p) = \dot{V}_p \leq -\alpha_p|\bar{\theta}_p|^{n+1} \tag{64}$$

and, proceeding as for $a_{3,3}$, it is simple to verify, by using (26), (52), and the fact that $\varepsilon_{p+1} = \eta_{p+1} \geq 1$, that

$$-(n-p+1)S^{-1}\varepsilon_{p+1}^{2p-1/2}(1-\cos\theta_{p+1})w_{p+1}\theta_{p+1}^{\{n-p\}} \leq -\alpha_1\varepsilon_{p+1}^{1/2}|\theta_{p+1}|^{n+1} \tag{65}$$

Then, (62) follows from (63), (64), and (65).

From (26), (52), (60) and (61), it is straightforward to verify —using again the condition $\varepsilon_{p+1} \geq 1$— that

$$|F_1(V_{p+1})| \leq \alpha_2\varepsilon_{p+1}^{-1/2}|\bar{\theta}_p|^{n+1} + \alpha_2|\theta_{p+1}|^{n+1} \tag{66}$$

Finally, we claim that

$$|F_2(V_{p+1})| \leq \left(\frac{\alpha_p}{2} + \alpha_3\varepsilon_{p+1}^{-1/2}\right)|\bar{\theta}_p|^{n+1} + \alpha_4|\theta_{p+1}|^{n+1} \tag{67}$$

Indeed, from (50), (53), and (58),

$$F_2 = \begin{pmatrix} (R_{11}B_p + R_{12}b_{p+1,*})\bar{w}_p + (R_{11}b_{*,p+1} + R_{12}b_{p+1,p+1})w_{p+1} + A_p^{-1}b_{*,p+1}w_{p+1} \\ R_{21}B_p\bar{w}_p + R_{21}b_{*,p+1}w_{p+1} + S^{-1}b_{p+1,*}\bar{w}_p \end{pmatrix}$$

Using (25), (26), (43), (54), (55), and (56), it is tedious but not difficult to show that

$$\begin{cases} |(F_2)_i| & \leq & C\varepsilon_{p+1}^{-1}|\bar{\theta}_{p+1}|^i + C|\bar{\theta}_{p+1}|^{i-p+1}|\theta_{p+1}|^{p-1} \quad \text{for } i = 3, \dots, p \\ |(F_2)_{p+1}| & \leq & C\varepsilon_{p+1}^{-p}|\bar{\theta}_{p+1}|^{p+1} \end{cases} \tag{68}$$

We infer from (46) and (68) that

$$|F_2(V_p)| \leq \alpha_5\varepsilon_{p+1}^{-1}|\bar{\theta}_{p+1}|^{n+1} + \alpha_6|\bar{\theta}_{p+1}|^{n-p+2}|\theta_{p+1}|^{p-1} \tag{69}$$

Using Young's inequality one can obtain

$$\begin{aligned} \alpha_6|\bar{\theta}_{p+1}|^{n-p+2}|\theta_{p+1}|^{p-1} &\leq \frac{\alpha_p}{2}|\bar{\theta}_{p+1}|^{n+1} + \alpha_7|\theta_{p+1}|^{n+1} \\ &\leq \frac{\alpha_p}{2}|\bar{\theta}_p|^{n+1} + \alpha_8|\theta_{p+1}|^{n+1} \end{aligned} \tag{70}$$

for other constants α_7, α_8. We deduce from (69) and (70) that

$$|F_2(V_p)| \leq \left(\frac{\alpha_p}{2} + \alpha_9\varepsilon_{p+1}^{-1}\right)|\bar{\theta}_p|^{n+1} + \alpha_{10}|\theta_{p+1}|^{n+1} \tag{71}$$

We also deduce from (68) that

$$|F_2(V_{p+1} - V_p)| \leq \alpha_{11}\varepsilon_{p+1}^{-1/2}|\bar{\theta}_{p+1}|^{n+1} \tag{72}$$

and (67) then follows from (71), (72), and the condition $\varepsilon_{p+1} \geq 1$.

Let us now use (62), (66), and (67) to determine an upperbound for $\dot{V}_{p+1}$. We obtain:

$$\begin{aligned} \dot{V}_{p+1} &= F_0(V_{p+1}) - F_1(V_{p+1}) - F_2(V_{p+1}) \\ &\leq -\left(\frac{\alpha_p}{2} - \alpha_{12}\varepsilon_{p+1}^{-1/2}\right)|\bar{\theta}_p|^{n+1} - \left(\alpha_1\varepsilon_{p+1}^{1/2} - \alpha_{13}\right)|\theta_{p+1}|^{n+1} \end{aligned}$$

Since by (62), α_p and α_1 are strictly positive, for ε_{p+1} large enough,

$$\dot{V}_{p+1} \leq -\alpha_{p+1}|\bar{\theta}_{p+1}|^{n+1} \quad (\alpha_{p+1} > 0)$$

This concludes the proofs of Proposition 3 and Theorem 1.

Appendix: proofs of Lemmas 1-4

The proofs of these lemmas rely on the following two properties.

Claim 1 *Let Y and Z denote two time-dependent left-invariant v.f. on G, and σ, τ solutions of $\dot{\sigma} = Y(\sigma,t)$ and $\dot{\tau} = Z(\tau,t)$ respectively. Then $\nu \triangleq \sigma\tau$ is a solution of $\dot{\nu} = \mathrm{Ad}(\tau^{-1})\, Y(\nu,t) + Z(\nu,t)$.*

This is well known and simple to verify.

Claim 2 *Let $X_1,\ldots,X_n$ denote a graded basis of the Lie algebra $\mathfrak{g}$ of a Lie group G. Let $\lambda,\rho,q \in \{1,\ldots,n\}$, $\alpha_p \in \mathbb{R}$, and $s \in \mathbb{N}$. Then, there exist analytic functions $g_1,\ldots,g_n$ such that, for any $\alpha_\lambda,\alpha_\rho \in \mathbb{R}$,*

$$\sum_{j=s}^{\infty} \frac{1}{j!}(\mathrm{ad}^j(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho), X_q) = \sum_{k=1}^{n} g_k(\alpha_\lambda,\alpha_\rho) X_k$$

Furthermore, if $\alpha_\lambda,\alpha_\rho$ are analytic function of x and y such that $\alpha_\lambda = \mathrm{O}(x^{r_\lambda})$ and $\alpha_\rho = \mathrm{O}(x^{r_\rho})$, then $g_k(\alpha_\lambda,\alpha_\rho)$ is an analytic function of x and y and

$$g_k(\alpha_\lambda,\alpha_\rho) = \mathrm{O}(x^{\max\{s\ \min\{r_\lambda,r_\rho\},r_k - r_q\}})$$

The proof can be viewed as a direct adaptation of [7, Section 2]. See also [4] where a similar result is proved.

Proof of Lemma 1

In order to simplify the notation, let

$$X_\lambda = X_{\lambda(j)}\ ,\ X_\rho = X_{\rho(j)}\ ,\ \ \alpha_\lambda = \alpha_{j,\lambda}\ ,\ \alpha_\rho = \alpha_{j,\rho} \tag{73}$$

With this notation, it follows from (21) that $f_j = \exp(\alpha_j X_j)\exp(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho)$. From Claim 1,

$$df_j = d(\exp(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho))(f_j) + d\alpha_j \mathrm{Ad}(\exp(-\alpha_\lambda X_\lambda - \alpha_\rho X_\rho))X_j(f_j) \tag{74}$$

Using the fact (see e.g. [1]) that $\mathrm{Ad}(\exp Y)Z = (\exp \mathrm{ad}Y, Z)$, and

$$\frac{d}{ds}\exp(X+sY)_{|s=0} = (\phi(\mathrm{ad}X),Y)(\exp X)\,, \quad \phi(z) \triangleq \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}z^k$$

we infer from (74) that

$$\begin{aligned} df_j &= (\phi(\mathrm{ad}(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho)), d\alpha_\lambda X_\lambda + d\alpha_\rho X_\rho)(f_j) + d\alpha_j(\exp \mathrm{ad}(-\alpha_\lambda X_\lambda - \alpha_\rho X_\rho), X_j)(f_j) \\ &= d\alpha_\lambda X_\lambda(f_j) + d\alpha_\rho X_\rho(f_j) - \tfrac{1}{2}[\alpha_\lambda X_\lambda + \alpha_\rho X_\rho, d\alpha_\lambda X_\lambda + d\alpha_\rho X_\rho](f_j) \\ &\quad + \sum_{k=2}^{\infty} \frac{(-1)^k}{(k+1)!}(\mathrm{ad}^k(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho), d\alpha_\lambda X_\lambda + d\alpha_\rho X_\rho)(f_j) \\ &\quad + d\alpha_j X_j(f_j) + d\alpha_j \sum_{k=1}^{\infty} \frac{1}{k!}(\mathrm{ad}^k(-\alpha_\lambda X_\lambda - \alpha_\rho X_\rho), X_j)(f_j) \end{aligned} \tag{75}$$

Since $X_j = [X_\lambda, X_\rho]$ —by Definition 3—, it comes from (75) that

$$\begin{aligned} df_j &= d\alpha_\lambda X_\lambda(f_j) + d\alpha_\rho X_\rho(f_j) + \left(d\alpha_j - \tfrac{1}{2}(\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda)\right) X_j(f_j) \\ &\quad + (\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+2)!}(\mathrm{ad}^k(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho), X_j)(f_j) \\ &\quad + d\alpha_j \sum_{k=1}^{\infty} \frac{1}{k!}(\mathrm{ad}^k(-\alpha_\lambda X_\lambda - \alpha_\rho X_\rho), X_j)(f_j) \end{aligned} \tag{76}$$

It follows from (22) that

$$\alpha_\lambda,\ d\alpha_\lambda = \mathrm{O}(\varepsilon_j^{r_\lambda}); \quad \alpha_\rho,\ d\alpha_\rho = \mathrm{O}(\varepsilon_j^{r_\rho}); \quad \alpha_j,\ d\alpha_j = \mathrm{O}(\varepsilon_j^{r_j}) \tag{77}$$

Therefore, by application of Claim 2 —with $x = \varepsilon_j$ and $y = \theta_j$—,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+2)!}(\mathrm{ad}^k(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho), X_j)(f_j) = \sum_{k=1}^{n} g_k(\alpha_\lambda, \alpha_\rho) X_k(f_j) \tag{78}$$

for some analytic functions $g_1, \dots, g_n$ which verify:

$$g_k(\alpha_\lambda, \alpha_\rho) = \mathrm{O}(\varepsilon_j^{\max\{1, r_k - r_j\}}) \tag{79}$$

Similarly, by applying Claim 2 again,

$$\sum_{k=1}^{\infty} \frac{1}{k!}(\mathrm{ad}^k(-\alpha_\lambda X_\lambda - \alpha_\rho X_\rho), X_j)(f_j) = \sum_{k=1}^{n} h_k(\alpha_\lambda, \alpha_\rho) X_k \tag{80}$$

with

$$h_k(\alpha_\lambda, \alpha_\rho) = \mathrm{O}(\varepsilon_j^{\max\{1, r_k - r_j\}}) \tag{81}$$

From (76), (78), and (80), we get

$$\begin{aligned} df_j = & \left(d\alpha_\lambda + (\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda)\, g_\lambda(\alpha_\lambda, \alpha_\rho) + d\alpha_j\, h_\lambda(\alpha_\lambda, \alpha_\rho)\right) X_\lambda(f_j) \\ & + \left(d\alpha_\rho + (\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda)\, g_\rho(\alpha_\lambda, \alpha_\rho) + d\alpha_j\, h_\rho(\alpha_\lambda, \alpha_\rho)\right) X_\rho(f_j) \\ & + \left(d\alpha_j - \frac{1}{2}(\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda) + (\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda)\, g_j(\alpha_\lambda, \alpha_\rho) + d\alpha_j\, h_j(\alpha_\lambda, \alpha_\rho)\right) X_j(f_j) \\ & + \sum_{k \notin \{\lambda, \rho, j\}} \left((\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda)\, g_k(\alpha_\lambda, \alpha_\rho) + d\alpha_j\, h_k(\alpha_\lambda, \alpha_\rho)\right) X_k(f_j) \end{aligned} \tag{82}$$

From (22), it is simple to show that

$$d\alpha_j - \frac{1}{2}(\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda) = \frac{\varepsilon_j^{r_j}}{2} d\theta_j + \varepsilon_j^{r_j}(1 - \cos\theta_j) d\beta_j \tag{83}$$

and

$$\alpha_\lambda,\ \frac{\partial \alpha_\lambda}{\partial \beta_j},\ \alpha_\rho,\ \frac{\partial \alpha_\rho}{\partial \beta_j},\ \alpha_j = \mathrm{O}(\theta_j); \quad \frac{\partial \alpha_j}{\partial \beta_j} = 0 \tag{84}$$

If f is an analytic function of ε and θ such that $f = \mathrm{O}(|\varepsilon|^p)$ and $f = \mathrm{O}(|\theta|^q)$, then $f = \mathrm{O}(|\varepsilon|^p)\mathrm{O}(|\theta|^q)$. Therefore, by using (77), (79), (81), (83), and (84), in (82), it is tedious but simple to recover all relations in Lemma 1 —for the last relation of (36), note that g_j and h_j are $\mathrm{O}(\theta_j)$ because they are functions of α_λ and α_ρ which vanish when $\alpha_\lambda = \alpha_\rho = 0$.

Proof of Lemma 2

From Claim 1, and relations (20) and (34),

$$\frac{\partial f}{\partial \theta_j} = \sum_{k=1}^{n} v_{k,j} \mathrm{Ad}(f_{m+1}^{-1} \cdots f_{j-1}^{-1}) X_k(f)\,, \quad \frac{\partial f}{\partial \beta_j} = \sum_{k=1}^{n} w_{k,j} \mathrm{Ad}(f_{m+1}^{-1} \cdots f_{j-1}^{-1}) X_k(f) \tag{85}$$

From the fact that $\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1)\mathrm{Ad}(g_2)$ and (21),

$$\begin{aligned} \mathrm{Ad}(f_{m+1}^{-1} \cdots f_{j-1}^{-1}) &= \prod_{p=m+1}^{j-1} \mathrm{Ad}(f_p^{-1}) \\ &= \prod_{p=m+1}^{j-1} \mathrm{Ad}(\exp(-\alpha_{p,\lambda} X_{\lambda(p)} - \alpha_{p,\rho} X_{\rho(p)}) \mathrm{Ad}(\exp -\alpha_p X_p) \end{aligned} \tag{86}$$

By application of Claim 2, for any $p, q, k = 1, \ldots, n$ and $(\alpha_p, \alpha_q) \in \mathbb{R}^2$

$$\mathrm{Ad}(\exp -\alpha_p X_p - \alpha_q X_q) X_k = X_k + \sum_{i=1}^{n} h_{p,q}^i(\alpha_p, \alpha_q) X_i$$

for some analytic functions $h^i_{p,q}$. Moreover, if $\alpha_p = \mathrm{O}(\varepsilon^{r_p})$ and $\alpha_q = \mathrm{O}(\varepsilon^{r_q})$ are analytic functions, then $h^i_{p,q}(\alpha_p, \alpha_q) = \mathrm{O}(\varepsilon^{\max(1, r_i - r_k)})$. This is used to infer from (77) and (86) that

$$\mathrm{Ad}(f_{m+1}^{-1} \cdots f_{j-1}^{-1}) X_k = X_k + \sum_i g^i_{j,k} X_i \quad \text{with } g^i_{j,k} = \mathrm{O}(\bar{\varepsilon}_{j-1}^{\max(1, r_i - r_k)}) \tag{87}$$

From (87),

$$\sum_{k=1}^{n} v_{k,j} \mathrm{Ad}(f_{m+1}^{-1} \cdots f_{j-1}^{-1}) X_k(f) = \sum_{i=1}^{n} \left(v_{i,j} + \sum_{k=1}^{n} v_{k,j} g^i_{j,k} \right) X_i(f)$$

and a similar expression holds when replacing v by w. Therefore, in view of (85), (37) holds with

$$a_{i,j} \triangleq v_{i,j} + \sum_{k=1}^{n} v_{k,j}\, g^i_{j,k} = A + B + C\,, \quad A = \sum_{r_k \le r_i} v_{k,j}\, g^i_{j,k}\,,\ B = v_{i,j}\,,\ C = \sum_{r_k > r_i} v_{k,j}\, g^i_{j,k} \tag{88}$$

and

$$b_{i,j} \triangleq w_{i,j} + \sum_{k=1}^{n} w_{k,j}\, g^i_{j,k} = D + E + F\,, \quad D = \sum_{r_k \le r_i} w_{k,j}\, g^i_{j,k}\,,\ E = w_{i,j}\,,\ F = \sum_{r_k > r_i} w_{k,j}\, g^i_{j,k} \tag{89}$$

Lemma 2 follows from this decomposition. Let us first show how (38) is obtained. From (35) and (87), A, B,and C in (88) are $\mathrm{O}(\bar{\varepsilon}_j^{r_i})$. This gives the first relation of (38).

For $i < j$ and $r_i = r_j$, A vanishes at $\bar{\varepsilon}_{j-1} = 0$ because of (87), and in view of (35), $B = \mathrm{o}(\bar{\varepsilon}_j^{r_i})$ and $C = \mathrm{O}(\bar{\varepsilon}_j^{r_k}) = \mathrm{o}(\bar{\varepsilon}_j^{r_i})$. This gives the second relation of (38).

For $i = j$, the only difference with the previous case comes from the B term which, in view of (35), is equal to $\varepsilon_j^{r_j}/2 + \mathrm{o}(\varepsilon_j^{r_j})$. This gives the third relation of (38).

Let us now show how (39) is obtained. From (87),

$$g^i_{j,k} = \mathrm{O}(\bar{\theta}_{j-1}) \tag{90}$$

because, by (21) and (22),

$$\bar{\theta}_{j-1} = 0 \Longrightarrow f_{m+1} = \cdots = f_{j-1} = e \Longrightarrow \mathrm{Ad}(f_{m+1}^{-1} \cdots f_{j-1}^{-1}) X_k = X_k$$

The first relation of (39) is then simply obtained from (36), (87), (89), and (90).
For $i = j$, E in (89) accounts for the term $\varepsilon_j^{r_j}(1 - \cos\theta_j)$ —up to higher order terms— in the second relation of (39), whereas D and F account for the remaining term by inspection of (36), (87), and (90).

Proof of Lemma 3

We refer to the proof of [4, Lemma 3]. More precisely, the property stated in Lemma 3 depends only on the properties of the functions $a_{i,j}$ in Lemma 2, and these properties have been established in the proof of [4, Lemma 3].

Proof of Lemma 4

Let us first show how Lemma 4 —relation (41), in particular— is obtained from the following two claims.

Claim 3 *For any i, j,*

$$\begin{cases} v_{i,j} = \varepsilon_j^{r_i}\tilde{v}_{i,j} & \text{with } \tilde{v}_{i,j} = \mathrm{O}(\theta_j^{r_i - r_j}) \\ w_{i,j} = \varepsilon_j^{r_i}\tilde{w}_{i,j} & \text{with } \tilde{w}_{i,j} = \mathrm{O}(\theta_j^{\max(1, r_i - r_j + 2)}) \end{cases} \tag{91}$$

where the functions $\tilde{v}_{i,j}$ and $\tilde{w}_{i,j}$ do not depend on the ε_k's.

Claim 4 *Each function $g_{j,k}^i$ in (87) is a polynomial in $\varepsilon_3, \ldots, \varepsilon_{j-1}$ homogeneous of degree $r_i - r_k$. Furthermore,*

$$g_{j,k}^i = \begin{cases} \mathrm{O}(\bar{\theta}_{j-1}^{r_i - r_k}) & \text{if } r_j \le r_k < r_i \\ \mathrm{O}(\bar{\theta}_{j-1}^{r_i - r_j + 1}) & \text{if } r_k < r_j < r_i \end{cases} \tag{92}$$

By using this last claim, and from (88) and (89), it is straightforward to show that $a_{i,j}$ and $b_{i,j}$ are polynomials homogeneous of degree r_i in $\varepsilon_3, \ldots, \varepsilon_j$. Then, by (91), E and F in (89) are $\mathrm{O}(\bar{\theta}_j^{\max(1, r_i - r_j + 2)})$. As for the term D, it can be decomposed as

$$D = \sum_{r_k < r_i} w_{k,j}\, g_{j,k}^i + \sum_{r_k = r_i} w_{k,j}\, g_{j,k}^i \tag{93}$$

From (91) and (92), the first sum in (93) is a $\mathrm{O}(\theta_j^{\max(1, r_k - r_j + 2)})\mathrm{O}(\bar{\theta}_{j-1}^{r_i - r_k})$ if $r_j \le r_k < r_i$, and a $\mathrm{O}(\theta_j)\mathrm{O}(\bar{\theta}_{j-1}^{r_i - r_j + 1})$ if $r_k < r_j < r_i$. Therefore, in both cases, it is a $\mathrm{O}(\bar{\theta}_j^{\max(1, r_i - r_j + 2)})$. As for the second sum in (93), it follows from (91) that it is a $\mathrm{O}(\bar{\theta}_j^{\max(1, r_i - r_j + 2)})$. This proves (41) for the term $b_{i,j}$. The proof for $a_{i,j}$ is similar and left as an exercise.

There remains to prove Claims 3 and 4.

In the case of a chained system, each element X_j of the graded basis, for $j = 3, \ldots, n$, is equal to $[X_{\lambda(j)}, X_{\rho(j)}]$ with $\lambda(j) = 1$ and $\rho(j) = j - 1$. It is also a constant v.f.. With the notation used in the proof of Lemma 1 these two facts imply that

$$(\mathrm{ad}(\alpha_\lambda X_\lambda + \alpha_\rho X_\rho), X_j) = \begin{cases} \alpha_\lambda X_{j+1} & \text{if } j < n \\ 0 & \text{if } j = n \end{cases}$$

Relation (76) in Lemma 1 then becomes

$$\begin{aligned} df_j &= d\alpha_\lambda X_\lambda(f_j) + d\alpha_\rho X_\rho(f_j) + \left(d\alpha_j - \tfrac{1}{2}(\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda)\right) X_j(f_j) \\ &\quad + (\alpha_\lambda d\alpha_\rho - \alpha_\rho d\alpha_\lambda) \sum_{k=1}^{n-j} \frac{(-1)^{k+1}}{(k+2)!} \alpha_\lambda^k X_{j+k}(f_j) + d\alpha_j \sum_{k=1}^{n-j} \frac{(-\alpha_\lambda)^k}{k!} X_{j+k}(f_j) \end{aligned}$$

Claim 3 is easily obtained by identifying this equality with (34), and by using (22) and (25).

Let us now prove Claim 4 by showing how relation (92) is obtained. The first step involves the evaluation of $\mathrm{Ad}(f_p^{-1})X_k$, for $p \in \{3, \dots, n-1\}$ and $k \in \{1, \dots, n\}$. We distinguish two cases.

Case 1: $k \neq 1$. From the definition (24) of $X_1, \dots, X_n$ and (21),

$$\begin{aligned}\mathrm{Ad}(f_p^{-1})X_k &= \mathrm{Ad}(\exp(-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}))\mathrm{Ad}(\exp -\alpha_p X_p)X_k \\ &= \mathrm{Ad}(\exp(-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}))X_k \\ &= X_k + \sum_{i=1}^{n-k} \frac{(-\alpha_{p,\lambda})^j}{j!} X_{k+j} \qquad (94) \\ &= X_k + \sum_{j=1}^{n-k} \varepsilon_p^j \, h_{p,k}^{k+j} X_{k+j} \quad \text{with } h_{p,k}^{k+j} = \mathrm{O}(\theta_p^j)\end{aligned}$$

where the last equality comes from (22) and (25), and $h_{p,k}^{k+j}$ is a function which does not depend on ε_p. From (25), $r_{k+j} = r_k + j$ for $k > 1$ and $0 \leq j \leq n-k$. Therefore, from (22) and (94)

$$\mathrm{Ad}(f_p^{-1})X_k = X_k + \sum_{i>k} \varepsilon_p^{r_i - r_k} h_{p,k}^i X_i \quad \text{with } h_{p,k}^i = \mathrm{O}(\theta_p^{r_i - r_k}) \qquad (95)$$

By applying (95) recursively, it follows that, for any $k \neq 1$,

$$\mathrm{Ad}(f_3^{-1} \cdots f_{j-1}^{-1})X_k = X_k + \sum_{i>k} g_{j,k}^i X_i \quad \text{with } g_{j,k}^i = \mathrm{O}(\bar{\theta}_{j-1}^{r_i - r_k}) \qquad (96)$$

where each $g_{j,k}^i$ is a polynomial homogeneous of degree $r_i - r_k$ in $\varepsilon_3, \dots, \varepsilon_{j-1}$. This yields (92) for $r_j \leq r_k \leq r_i$, and also for $r_k < r_j < r_i$ (and $k \neq 1$) after noticing that, in this case, $r_i - r_k \geq r_i - r_j + 1$.

Case 2: $k = 1$. We have

$$
\begin{aligned}
\mathrm{Ad}(f_p^{-1})X_1 &= \mathrm{Ad}(\exp(-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}))\mathrm{Ad}(\exp -\alpha_p X_p)X_1 \\
&= \mathrm{Ad}(\exp(-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}))(X_1 + \alpha_p X_{p+1}) \\
&= X_1 + \alpha_p X_{p+1} + [-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}, X_1 + \alpha_p X_{p+1}] \\
&\quad + \sum_{k=2}^{\infty} \frac{1}{k!}(\mathrm{ad}^{k-1}(-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}), [-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}, X_1 + \alpha_p X_{p+1}]) \\
&= X_1 + \alpha_p X_{p+1} - \alpha_{p,\lambda}\alpha_p X_{p+2} + \alpha_{p,\rho}X_p \\
&\quad + \sum_{k=2}^{\infty} \frac{1}{k!}(\mathrm{ad}^{k-1}(-\alpha_{p,\lambda}X_1 - \alpha_{p,\rho}X_{p-1}), -\alpha_{p,\lambda}\alpha_p X_{p+2} + \alpha_{p,\rho}X_p) \\
&= X_1 + \alpha_p X_{p+1} - \alpha_{p,\lambda}\alpha_p X_{p+2} + \alpha_{p,\rho}X_p \\
&\quad - \alpha_{p,\lambda}\alpha_p \sum_{k=2}^{\infty} \frac{(-\alpha_{p,\lambda})^{k-1}}{k!} X_{p+2+k-1} + \alpha_{p,\rho} \sum_{k=2}^{\infty} \frac{(-\alpha_{p,\lambda})^{k-1}}{k!} X_{p+k-1}
\end{aligned}
\tag{97}
$$

It follows from (22) and (97) that

$$
\mathrm{Ad}(f_p^{-1})X_1 = X_1 + \sum_{i>1} \varepsilon_p^{r_i - r_1} h_p^i X_i \quad \text{with } h_p^i = \mathrm{O}(\theta_p^{r_i - r_p}) \tag{98}
$$

and h_p^i does not depend on ε_p. By applying (98) recursively, and by using (96), it follows that,

$$
\mathrm{Ad}(f_3^{-1} \cdots f_{j-1}^{-1})X_1 = X_1 + \sum_{i>1} g_{j,1}^i X_i \quad \text{with } g_{j,1}^i = \mathrm{O}(\bar{\theta}_{j-1}^{r_i - r_{j-1}}) = \mathrm{O}(\bar{\theta}_{j-1}^{r_i - r_j + 1})
$$

where each $g_{j,1}^i$ is a polynomial homogeneous of degree $r_i - r_1$ in $\varepsilon_3, \ldots, \varepsilon_{j-1}$. This concludes the proof of Claim 4.

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