Duality and Separation Theorems in Idempotent Semimodules

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Abstract: We consider subsemimodules and convex subsets of semimodules over semirings with an idempotent addition. We introduce a nonlinear projection on subsemimodules: the projection of a point is the maximal approximation from below of the point in the subsemimodule. We use this projection to separate a point from a convex set. We also show that the projection minimizes the analogue of Hilbert's projective metric. We develop more generally a theory of dual pairs for idempotent semimodules. We obtain as a corollary duality results between the row and column spaces of matrices with entries in idempotent semirings. We illustrate the results by showing polyhedra and half-spaces over the max-plus semiring.

Key-words: Max-plus semiring, semimodules, Hahn-Banach theorem, linear extension, duality, dual pairs, projection, residuation, Galois connection, generalized conjugacies, row space, column space.

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Dualité et Théorèmes de Séparation dans les Semimodules Idempotents

Résumé : Nous considérons les sous-semi-modules et sous-ensembles convexes des semi-modules sur des semi-anneaux dont l'addition est idempotente. Nous introduisons une projection non linéaire sur un sous-semi-module: la projection d'un point est le plus grand élément du sous-semi-module qui minore ce point. Nous utilisons le projeté pour séparer un point d'un convexe. Nous montrons aussi que le projeté minimise l'analogue de la métrique projective de Hilbert. Nous développons plus généralement une théorie des semi-modules en dualité. Nous obtenons comme corollaire des résultats de dualité entre l'espace des lignes et l'espace des colonnes de matrices à coefficients dans des semi-anneaux idempotents. Nous illustrons les résultats en montrant quelques polyèdres et demi-espaces sur le semi-anneau maxplus.

Mots-clés : Semianneau max-plus, semimodules, théorème de Hahn-Banach, prolongement linéaire, dualité, projection, résiduation, correspondance de Galois, conjugaisons généralisées, espace des lignes, espace des colonnes.

Contents

1. Introduction

In this paper, we study semimodules over semirings whose addition is idempotent, that we call *idempotent semimodules.*

The typical example of semiring with an idempotent addition that we shall consider is the *max-plus semiring*, $\mathbb{R}_{\max}$, which is the set $\mathbb{R} \cup \{-\infty\}$, equipped with the addition $(a, b) \mapsto \max(a, b)$ and the multiplication $(a, b) \mapsto a+b$. We shall also consider the completed max-plus semiring, $\overline{\mathbb{R}}_{\max}$, which is obtained by adjoining to $\mathbb{R}_{\max}$ a $+\infty$ element. The *Boolean semiring* $\mathbb{B}$ is a subsemiring of $\mathbb{R}_{\max}$ and $\overline{\mathbb{R}}_{\max}$ (obtained by keeping only the zero element, $-\infty$, and the unit element, 0).

Idempotent semimodules include a number of familiar examples. For instance, the set of convex functions defined on a vector space can be thought of as a semimodule over the max-plus semiring. Another familiar class of idempotent semimodules consists of sup-semilattices with a bottom element, which coincide with semimodules over the Boolean semiring. The interest for idempotent semimodules arose specially from the development of the max-plus algebraic approach to optimal control (Maslov 1973, Maslov and Samborskiĭ 1992, Kolokoltsov and Maslov 1997, Litvinov *et al.* 1998, Litvinov *et al.* 2001) and discrete event systems (Cohen *et al.* 1985, Baccelli *et al.* 1992, Cohen *et al.* 1996, Cohen *et al.* 1997, Gaubert and Plus 1997, Cohen *et al.* 1999). See also (Cuninghame-Green 1979, Zimmermann 1981, Kim 1982, Cao *et al.* 1984, Wagneur 1991*a*, Gunawardena 1998, Gondran and Minoux 2002) for more background.

In this paper, we give Hahn-Banach type theorems for complete idempotent semimodules (the notion of completeness is defined in terms of the natural order of the semimodule). We show that a universal separation result holds (Theorem 8 below), without any additional assumptions on the semimodule or on the semiring, if one takes as a nonlinear dual space an opposite semimodule. To recover a separation theorem involving a linear dual space, we study more generally dual pairs, similar to the ones that arise classically in the theory of topological vector spaces: a ***predual pair*** consists of two complete semimodules X, Y, equipped with a bilinear continuous pairing $\langle \cdot \mid \cdot \rangle$, and a ***dual pair*** is a predual pair which separates points (see §4). We introduce a Galois connection $X \to Y, x \mapsto {}^{-}x$, $Y \to X, y \mapsto y^{-}$, which yields anti-isomorphisms between the lattices of the elements of X and Y which are closed for this correspondence. For instance, when $X = \overline{\mathbb{R}}_{\max}^{n\times 1}$ is the semimodule of n-dimensional column vectors over the completed max-plus semiring $\overline{\mathbb{R}}_{\max}$, $Y = \overline{\mathbb{R}}_{\max}^{1\times n}$, and $\langle y \mid x\rangle = \max_{1\leq i\leq n}(y_i + x_i)$, all elements of X and Y are closed, and the conjugation operation is simply ${}^{-}x = (-x)^{\top}$ and $y^{-} = (-y)^{\top}$ where $\top$ denotes the transposition. For a class of idempotent semirings that we call reflexive, we show that dual pairs satisfy a more familiar, linear, geometric Hahn-Banach theorem, which has the following form (see Theorem 34 below): if V is a complete subsemimodule of X, if $x \in X$ but $x \notin V$, then there exist elements $y, z \in Y$ such that

$$\langle y \mid v\rangle = \langle z \mid v\rangle, \forall v \in V, \quad \text{and} \quad \langle y \mid x\rangle \neq \langle z \mid x\rangle \ . \tag{1}$$

The separating pair (y, z) is nothing but the pair of conjugates $({}^{-}x, {}^{-}P_V(x))$, where $P_V(x)$

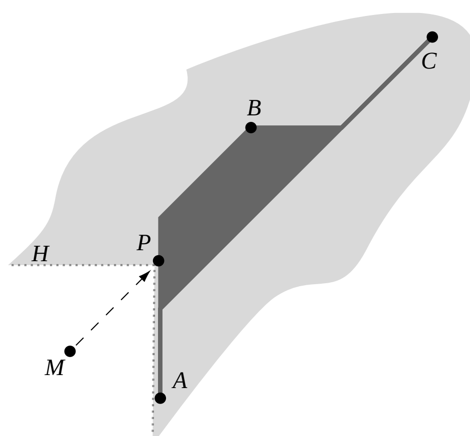

FIGURE 1. Separation of the convex ABC and the point M by the half-space H.

is the best approximation from below of x by an element of V. Since $P_V(x)$ minimizes an analogue of Hilbert's projective metric, (1) is similar to the separation property in Euclidean spaces, where P_V is the orthogonal projector on V and the vector $(x, P_V(x))$ gives the direction orthogonal to a separating hyperplane. The key discrepancy, by comparison with vector spaces, is that one needs *pairs* of linear forms to separate a point from a subspace, or more generally, from a convex set. The affine form of the separation theorem is illustrated in Fig. 1, which shows a max-plus polyhedron generated by three extremal points, A, B, C, a point M which does not belong to the polyhedron, together with a half space H (in light gray) which contains the polyhedron, but not the point M. The half-space is obtained from the projection P of M. See §3.4 for details.

The present idempotent Hahn-Banach theorem has a long history. The first result of this kind seems to have been proved by Zimmermann (1977), for closed convex subsets of $\mathcal{K}^n$, where $\mathcal{K}$ is a semiring with an idempotent addition, satisfying some axioms which hold when $\mathcal{K} = \mathbb{R}_{\max}$. A similar result was proved by Samborskiĭ and Shpiz (1992) under more general assumptions on the semiring, but assuming that the point to separate has invertible coordinates. The present Hahn-Banach theorem holds under more general assumptions, and yields direct explicit formulæ for separating hyperplanes. This generality is possible because we work in *complete* ordered structures. In the case of the max-plus semiring, this means that the coefficients of the separating half-spaces that we build can take the $+\infty$ value, so that these half-spaces need not be closed for the usual topology. Hence, our results apply even to some convex subsets which are not closed, see the example at the end of Remark 16 below. In (Cohen *et al.* 2002), we apply the present results to convex functions over the max-plus semiring, and recover in particular a separation theorem à la Zimmerman for closed convex sets. The spirit of the present work is also very close to that of the theory developed by Litvinov, Maslov, and Shpiz (2001), who establish idempotent analogues of several classical theorems of functional analysis. The representation theorem for linear forms (Corollary 39 below) and the related analytic form the Hahn-Banach theorem (Corollary 40 below) are extensions of the corresponding results of (Litvinov *et al.* 2001). V. Kolokoltsov (1999) suggested to the second author the interest of revisiting max-plus residuation theory with a Galois connection point of view, and the present work illustrates the fruitful character of this idea, which is also applied to different problems in (Akian *et al.* 2002). We thank him, and we thank also M. Akian, P. Lotito, E. Mancinelli, I. Singer and E. Wagneur, for useful discussions. Finally, we note that a preliminary version of the present results appeared in (Cohen *et al.* 2000).

2. Preliminaries

2.1. Complete Ordered Sets and Residuated Maps.

We first recall some classical notions about ordered sets and residuated maps. We say that a partially ordered set $(S, \leq)$ is *complete* if any subset $X \subset S$ has a least upper bound (denoted by $\bigvee X$). In particular, S has both a minimal (bottom) element $\bot S$ denoted β_S equal to $\bigvee \varnothing$, and a maximal (top) element $\top S$ denoted τ_S which is equal to $\bigvee S$. Since the greatest lower bound of a set X can be defined by $\bigwedge X = \bigvee\{y \in S \mid y \leq x, \ \forall x \in X\}$, S is a complete lattice. See (Birkhoff 1940, Dubreil-Jacotin *et al.* 1953, Blyth and Janowitz 1972) for more details on ordered algebraic structures and residuation.

If $(S, \leq)$ and $(T, \leq)$ are ordered sets, we say that a map $f : S \to T$ is *residuated* if there exists a map $f^\sharp : T \to S$ such that

$$f(s) \leq t \iff s \leq f^\sharp(t) \ , \tag{2}$$

which means that for all $t \in T$, the set of subsolutions, $\{s \in S \mid f(s) \leq t\}$ has a maximal element, $f^\sharp(t)$. It is not difficult to see that f is residuated if, and only if, it is monotone (i.e. $s \leq s' \implies f(s) \leq f(s')$), and

$$f \circ f^\sharp \leq I_T, \qquad f^\sharp \circ f \geq I_S \ , \tag{3}$$

where I_X denotes the identity map on a set X. If $(X, \leq)$ is an ordered set, we denote by $(X^{\mathrm{op}}, \overset{\mathrm{op}}{\leq})$ the *opposite* ordered set, for which $x \overset{\mathrm{op}}{\leq} y \iff x \geq y$. Due to the symmetry of the defining property (2), it is clear that $f : S \to T$ is residuated if, and only if, $f^\sharp : T^{\mathrm{op}} \to S^{\mathrm{op}}$ is residuated. When S, T are complete ordered sets, there is a simple characterization of residuated maps. We say that a monotone map $f : S \to T$ is *continuous* if for all $U \subset S$, $f(\bigvee U) = \bigvee f(U)$, where $f(U) = \{f(x) \mid x \in U\}$. In particular, when $U = \varnothing$, we get $f(\beta_S) = \beta_T$.

Lemma 1. *If $(S, \leq)$ and $(T, \leq)$ are complete ordered sets, then, a map $f : S \to T$ is residuated if, and only if, it is continuous.* □

By symmetry, $f^\sharp : T^{\mathrm{op}} \to S^{\mathrm{op}}$ is continuous, which means that:

$$f^\sharp(\bigwedge U) = \bigwedge f^\sharp(U) \ , \forall U \subset T \ . \tag{4}$$

Using the monotonicity of f and $f^\sharp$, together with (3), we easily get that

$$f \circ f^\sharp \circ f = f \ , \tag{5a}$$

$$f^\sharp \circ f \circ f^\sharp = f^\sharp \ , \tag{5b}$$

$$f^\sharp \circ g^\sharp = (g \circ f)^\sharp \ , \tag{5c}$$

where $g : T \to W$ is another residuated map. It is not difficult to check that

$$f \text{ is injective} \Leftrightarrow f^\sharp \circ f = I_S \Leftrightarrow f^\sharp \text{ is surjective,} \tag{6a}$$

$$f \text{ is surjective} \Leftrightarrow f \circ f^\sharp = I_T \Leftrightarrow f^\sharp \text{ is injective.} \tag{6b}$$

Moreover, if $\{f_i\}_{i \in I}$ is an arbitrary family of residuated maps $S \to T$,

$$(\bigvee_{i \in I} f_i)^\sharp = \bigwedge_{i \in I} f_i^\sharp \ . \tag{7}$$

2.2. **Semimodules over Idempotent Semirings.** In the sequel, $(\mathcal{K}, \oplus, \otimes, \varepsilon, e)$ denotes a semiring whose addition is idempotent (i.e. $a \oplus a = a$), and ε and e are the neutral elements for $\oplus$ and $\otimes$, respectively. We shall adopt the usual conventions, and write for instance ab instead of $a \otimes b$. An idempotent monoid $(S, \oplus, \varepsilon)$ can be equipped with the *natural* order relation, $a \leq b \Leftrightarrow a \oplus b = b$, for which $a \oplus b = \bigvee\{a, b\}$ and $\varepsilon = \beta_{\mathcal{K}}$. We say that the semiring $\mathcal{K}$ is *complete* if it is complete as a naturally ordered set, and if the left and right multiplications, $L_a^{\mathcal{K}}, R_a^{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$, $L_a^{\mathcal{K}}(x) = ax$, $R_a^{\mathcal{K}}(x) = xa$, are continuous.

A (right) $\mathcal{K}$-*semimodule* X is a commutative monoid $(X,\oplus,\varepsilon)$, equipped with a map $X\times\mathcal{K}\to X$, $(x,\lambda)\to x\lambda$ (right action), that satisfies

$$x(\lambda\mu)=(x\lambda)\mu \ , \tag{8a}$$

$$(x\oplus y)\lambda = x\lambda\oplus y\lambda\ ,\quad x(\lambda\oplus\mu)=x\lambda\oplus x\mu \ , \tag{8b}$$

$$x\varepsilon=\varepsilon\ ,\quad \varepsilon\lambda=\varepsilon\ ,\quad xe=x \ , \tag{8c}$$

for all $x,y\in X$, $\lambda,\mu\in\mathcal{K}$. Since $(\mathcal{K},\oplus)$ is idempotent, $(X,\oplus)$ is idempotent (indeed, it follows from (8b) and (8c) that $x=xe=x(e\oplus e)=xe\oplus xe=x\oplus x$). The notion of left $\mathcal{K}$-semimodule is defined dually. Throughout the paper, all the semimodules that we shall consider will be over idempotent semirings. We shall also consider $\mathcal{K}$-*bisemimodules*: a bisemimodule is a set equipped with two, right and left, $\mathcal{K}$-semimodule structures, such that the right and left actions commute. In particular, an idempotent semiring $\mathcal{K}$ is a $\mathcal{K}$-bisemimodule if one take as left and right actions the semiring product $(a,b)\mapsto a\otimes b$.

We say that a right $\mathcal{K}$-semimodule X is *complete* if it is complete as a naturally ordered set, and if, for all $x\in X$ and $\lambda\in\mathcal{K}$, the left and right multiplications, $R^X_\lambda:\ X\to X$, $x\mapsto x\lambda$ and $L^X_v:\ \mathcal{K}\to X$, $\mu\mapsto v\mu$, are both continuous. Complete left $\mathcal{K}$-semimodules and complete $\mathcal{K}$-bisemimodules are defined in a similar way. In the sequel, all semimodules will be right semimodules, unless otherwise specified.

Example 2 (Free Complete Semimodules and Semimodules of Functions)**.** Let $\mathcal{K}$ denote a complete idempotent semiring. A *free complete right $\mathcal{K}$-semimodule* is of the form $\mathcal{K}^I$ for some arbitrary set I: the elements of $\mathcal{K}^I$ are functions $I\to\mathcal{K}$, and $\mathcal{K}^I$ is equipped with the addition $(a,b)\mapsto a\oplus b$, $(a\oplus b)(i)=a(i)\oplus b(i)$, and the action $(a,\lambda)\mapsto a\lambda$, $(a\lambda)(i)=a(i)\lambda$, for all $a,b\in\mathcal{K}^I,\lambda\in\mathcal{K}$. By considering the action $(a,\lambda)\mapsto\lambda a$, $(\lambda a)(i)=\lambda a(i)$, one can see $\mathcal{K}^I$ as a left semimodule.

The semimodule $\overline{\mathbb{R}}_{\max}^{n\times 1}$, evoked in the introduction, is an example of a free complete right $\overline{\mathbb{R}}_{\max}$-semimodule. Another example in the same category, to which we will return from time to time in this paper, is the set $\overline{\mathbb{R}}_{\max}^{\mathcal{U}}$ of functions from a set $\mathcal{U}$ to $\overline{\mathbb{R}}_{\max}$, with the pointwise supremum as $\oplus$ operation and the conventional addition of a real constant as (left of right) action. This semimodule (that we refer as $\mathcal{F}$ for short in the sequel) is complete.

In a complete semimodule X, we define, for all $x,y\in X$ and $\lambda\in\mathcal{K}$,

$$x\backslash y \stackrel{\text{def}}{=} (L^X_x)^\sharp(y)=\top\{\lambda\in\mathcal{K}\mid x\lambda\leq y\} \ , \tag{9a}$$

$$x/\lambda \stackrel{\text{def}}{=} (R^X_\lambda)^\sharp(x)=\top\{y\in X\mid y\lambda\leq x\} \ . \tag{9b}$$

Paraphrasing the definition of residuated maps,

$$x\lambda\leq y \iff \lambda\leq x\backslash y \iff x\leq y/\lambda \ . \tag{10}$$

The residuation formulæ (3), (5) and (7) yield

(11a) $x(x\backslash y) \leq y$, $(x/\lambda)\lambda \leq x$,

(11b) $(x\backslash y)\lambda \leq x\backslash(y\lambda)$, $x(\lambda/\mu) \leq (x\lambda)/\mu$,

(11c) $x\backslash(x\lambda) \geq \lambda$, $(x\lambda)/\lambda \geq x$,

(11d) $x\backslash(\bigwedge \Lambda) = \bigwedge(x\backslash\Lambda)$, $(\bigwedge U)/\lambda = \bigwedge(U/\lambda)$,

(11e) $x(x\backslash(xy)) = xy$, $((x\lambda)/\lambda)\lambda = x\lambda$,

(11f) $x\backslash(x(x\backslash y)) = x\backslash y$, $((x/\lambda)\lambda)/\lambda = x/\lambda$,

(11g) $\lambda\backslash(x\backslash z) = (x\lambda)\backslash z$, $(x/\mu)/\lambda = x/(\lambda\mu)$,

(11h) $(\bigvee U)\backslash y = \bigwedge(U\backslash y)$, $x/(\bigvee \Lambda) = \bigwedge(x/\Lambda)$,

for all $x, y, z \in X$, $U \subset X$, $\Lambda \subset \mathcal{K}$, where $(U\backslash y) = \{u\backslash y \mid u \in U\}$, and $x/\Lambda = \{x/\lambda \mid \lambda \in \Lambda\}$. Finally, if X is a bisemimodule and $\mu, \nu \in \mathcal{K}$, the maps $x \mapsto x\lambda$ and $x \mapsto \nu x$ commute, hence, by (5c), their residuated maps commute, which means that

(12) $$(\nu\backslash x)/\mu = \nu\backslash(x/\mu) \ .$$

Since there is no ambiguity, we shall simply write $\nu\backslash x/\mu$ for (12).

Remark 3. Note that (9a) is dual of the definition (5.1) in Litvinov *et al.*; the latter requires the assumption that the action of vectors on scalars satisfies $x(\bigwedge_{\lambda\in\Lambda} \lambda) = \bigwedge_{\lambda\in\Lambda}(x\lambda)$ — see Litvinov *et al.* (Eq. (4.7)) which is written for right action of vectors on scalars — whereas, in this paper, we stick to the more natural assumption that this property holds with $\bigvee$ instead of $\bigwedge$: this is the case for instance if the underlying semiring is a semiring of formal series, or of matrices, over a complete idempotent semiring.

2.3. **Opposite Semimodules.** If X is a complete right $\mathcal{K}$-semimodule, we call *opposite* semimodule of X the *left* $\mathcal{K}$-semimodule X^{op} with underlying set X, addition $(x, y) \mapsto \bigwedge\{x, y\}$ (the $\bigwedge$ is for the natural order of X) and left action $\mathcal{K} \times X \to X$, $(\lambda, x) \to x/\lambda$. For clarity, we shall sometimes denote by $(\lambda, x) \mapsto \lambda \overset{\mathrm{op}}{\cdot} x = x/\lambda$ the left action of X^{op}. That X^{op} is a semimodule follows from formulæ (11d), (11g), and (11h). In particular, (11g) yields

(13) $$\begin{aligned}(\lambda\mu) \overset{\mathrm{op}}{\cdot} x &= x/(\lambda\mu) = (x/\mu)/\lambda \\ &= \lambda \overset{\mathrm{op}}{\cdot} (\mu \overset{\mathrm{op}}{\cdot} x) \ ,\end{aligned}$$

for all $\lambda, \mu \in \mathcal{K}$ and $x \in X^{\mathrm{op}}$, which shows why X^{op} must be considered as a *left* rather than a *right* semimodule. Indeed, considering $(x, \lambda) \mapsto x/\lambda$ as a right action would require the property symmetrical to (13) to hold, that is, by (8a), $x/(\lambda\mu) = (x/\lambda)/\mu$, but this property need not hold for a semimodule X over a *noncommutative* semiring $\mathcal{K}$.

Denoting by $\overset{\text{op}}{\backslash}$ and $\overset{\text{op}}{/}$ the residuated operations built from $\overset{\text{op}}{\cdot}$, we get from (10),

$$\lambda \overset{\text{op}}{\backslash} x = (L_\lambda^{X^{\text{op}}})^\sharp(x) = \bot\{y \in X \mid y/\lambda \geq x\} \ ,$$

(14a)
$$= x\lambda \ ,$$

$$x \overset{\text{op}}{/} y = (R_y^{X^{\text{op}}})^\sharp(x) = \top\{\lambda \in \mathcal{K} \mid y/\lambda \geq x\} \ ,$$

(14b)
$$= x\backslash y \ .$$

Eqn (14a) is an involutivity property: the residuated law of the residuated law of the right action of X is the right action of X itself. Therefore,

Proposition 4. *For all complete $\mathcal{K}$-semimodules X, $(X^{\text{op}})^{\text{op}} = X$.* □

3. Nonlinear Projectors, Universal Separation Theorem and Hilbert projective metric

3.1. **Nonlinear Projector.** Let V denote a complete subsemimodule of a complete semimodule X over a complete idempotent semiring $\mathcal{K}$. We call *canonical projector* on V the map

$$P_V : X \to V, \quad P_V(x) = \top\{v \in V \mid v \leq x\}$$

(the least upper bound of $\{v \in V \mid v \leq x\}$ belongs to the set because V is complete). Thus, P_V is the residuated map of the canonical injection $i_V : V \to X$. Since $i_V^2 = i_V$, using (5c), we get that $P_V^2 = P_V$, and since i_V is injective, using (6a), we see that $P_V : X \to V$ is surjective. We say that W is a *generating family* of a complete subsemimodule V if any element $v \in V$ can be written as $v = \vee\{w\lambda_w \mid w \in W\}$, for some $\lambda_w \in \mathcal{K}$.

Theorem 5 (Projector Formula)**.** *If V is a complete subsemimodule of X with generating family W, then*

(15)
$$P_V(x) = \bigvee_{w\in W} w(w\backslash x) \ .$$

Proof. We can write $P_V(x) = \vee_{w\in W} w\lambda_w$, for some $\lambda_w \in \mathcal{K}$. From $P_V(x) \leq x$, we get $w\lambda_w \leq x$, or, equivalently, $\lambda_w \leq w\backslash x$. This shows that $P_V(x) \leq \vee_{w\in W} w(w\backslash x)$. But, $\vee_{w\in W} w(w\backslash x)$ is an element of V, which, by (11a), is less than x. This proves (15). □

We may rewrite (15) as $P_V = \vee_{w\in W} P_w$, where P_w denotes the projector on the "one dimensional" space $w\mathcal{K}$. Similar formulæ for the projector appeared in (Moller 1988).

Proposition 6 (Dual characterization of the projector)**.** *Let $V \subset X$ denote a complete subsemimodule with generating family W. Then,*

(16)
$$P_V(x) = \bot\{z \in X \mid w\backslash z \geq w\backslash x, \ \forall w \in W\} \ .$$

Proof. Since $w\backslash z \geq w\backslash x \iff z \geq w(w\backslash x)$, this follows from (15). □

Example 7. We return to the $\overline{\mathbb{R}}_{\max}$-semimodule $\mathcal{F}$ introduced at Example 2 and discuss the application of previous results in this section. First of all, observe that

$$\forall f, g \in \mathcal{F}, \quad f\backslash g = \inf_{u\in\mathcal{U}} \big(g(u) - f(u)\big) ,$$

with the convention here that $+\infty - \infty = +\infty$ (since in any complete idempotent semiring S, $\beta_S\backslash\beta_S = \tau_S\backslash\tau_S = \tau_S$).[1]

Assume now that $\mathcal{U}$ is a topological vector space and consider the complete subsemimodule V generated by the set W of continuous linear functions over $\mathcal{U}$. Clearly, this is the subsemimodule of l.s.c. convex functions over $\mathcal{U}$. For any $f \in \mathcal{F}$, $P_V(f)$, as defined in §3.1, is the classical l.s.c. convex hull of f. For $w \in W$,

$$w\backslash f = \inf_{u\in\mathcal{U}} \big(f(u) - w(u)\big) = -\sup_{u\in\mathcal{U}} \big(w(u) - f(u)\big) ,$$

which is nothing but the opposite of the Fenchel transform f^* of f evaluated at w. Eqn (15) then yields

$$P_V(f)(\cdot) = \bigvee_{w\in W} w(w\backslash f) = \bigvee_{w\in W} \big(w(\cdot) - f^*(w)\big) ,$$

that is to say, the l.s.c. convex hull of f is the Fenchel transform of the Fenchel transform of f.

Finally, Eqn (16) says that the l.s.c. convex hull of f is the least function g in $\mathcal{F}$ such that g^* is less than, or equal to, f^* (pointwise).

3.2. **Universal Separation Theorem.**

Theorem 8 (Universal Separation Theorem)**.** *Let $V \subset X$ denote a complete subsemimodule, and let $x \in X$. Then,*

$$\forall v \in V, \quad v\backslash P_V(x) = v\backslash x , \tag{17a}$$

and

$$x \in V \iff x\backslash P_V(x) = x\backslash x . \tag{17b}$$

Seeing $y\backslash x$ as a "scalar product", Eqn (17a) says that the vector $(x, P_V(x))$ is "orthogonal" to the semimodule V, and (17b) shows that the "hyperplane" $\{y \mid y\backslash P_V(x) = y\backslash x\}$ separates x from V, if and only if $x \notin V$. This terminology will be justified in §4.

Proof. Since, by definition, the $\perp$ in (16) belongs to the set, we have that $v\backslash P_V(x) \geq v\backslash x$, for all $v \in V$. Using $P_V(x) \leq x$ and the monotonicity of $y \mapsto v\backslash y$, we get the reverse inequality, which shows (17a). If $x \in V$, then $P_V(x) = x$, and $x\backslash P_V(x) = x\backslash x$, trivially. Conversely, if $x\backslash P_V(x) = x\backslash x$, we have, by (10), that $P_V(x) \geq x(x\backslash x)$, and, by (11e), that $x(x\backslash x) = x$, which shows that $P_V(x) \geq x$. Since $P_V(x) \leq x$, we have $x = P_V(x) \in V$. ☐

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[1] Observe however that $\beta_S \otimes \tau_S = \beta_S$, which translates, in $\overline{\mathbb{R}}_{\max}$, as $-\infty + \infty = -\infty$, so that this "rule" written in conventional notation is ambiguous, and one must keep in mind what are the correct algebraic operations hidden behind the conventional notation to apply the rule correctly.

Remark 9. The separating set $H = \{v \in V \mid v \backslash P_V(x) = v \backslash x\}$ is a semimodule. Indeed, by (11h) it is stable by addition and (11g) shows that it is stable by scalar action.

Remark 10. According to the previous remark, it is sufficient to check (17a) only for v ranging in a generating subset W of V.

Example 11. For the semimodule $\mathcal{F}$ introduced at Example 2 and the subsemimodule V of l.s.c. convex functions generated by the subset W of continuous linear functions as discussed at Example 7, the equality (17a) (restricted to $v \in W$ as observed in the previous remark) of the Separation Theorem says that the Fenchel transform of any function f coincides with the Fenchel transform of its l.s.c. convex hull. As for (17b), observe first that $f \backslash f = 0$ unless f assumes only $\pm\infty$ values (in this latter case, $f \backslash f = +\infty$). Let us put aside this singular situation first. Then (17b) says that f coincides with its l.s.c. convex hull at all points if and only if it is itself l.s.c. convex.

In the singular case, and according to (17b), f is l.s.c. convex if and only if $f \backslash P_V(f) = \inf_{u \in \mathcal{U}} \big(P_V(f)(u) - f(u)\big) = +\infty$, that is, $P_V(f)(u) - f(u) = +\infty$ for all u. According to the rule $-\infty + \infty = +\infty$ which applies here, this shows that $f(u) = +\infty$ implies that $P_V(f)(u) = +\infty$. On the other hand, if $f(u) = -\infty$, then $P_V(f)(u) = -\infty$ because $P_V(f) \leq f$ pointwise. Finally, in all cases, we have reached the conclusion that (17b) says that f coincides with its l.s.c. convex hull at all points if and only if it is itself l.s.c. convex.

The "scalar product" $y \backslash x$ separates points, in the following sense:

Proposition 12 (Separation of Points). *If X is a complete $\mathcal{K}$-semimodule, then, for all $x, y \in X$,*

$$(\forall z \in X, \quad x \backslash z = y \backslash z) \implies x = y \ . \tag{18}$$

Proof. If $x \backslash z = y \backslash z$ for all $z \in Z$, taking $z = x$, we get that $e \leq x \backslash x = y \backslash x$, which means that $y \leq x$. By symmetry, $x \leq y$. □

Finally, we note that all the above results have dual versions for the semimodule X^{op}: they are derived readily from Eqn (14). For instance, if $V \subset X^{\mathrm{op}}$ is a complete subsemimodule, we define

$$\begin{aligned} P_V^{\mathrm{op}}(x) &= \bigvee\nolimits^{\mathrm{op}} \{v \in V \mid v \overset{\mathrm{op}}{\leq} x\} \\ &= \bigwedge \{v \in V \mid v \geq x\} \ , \end{aligned} \tag{19}$$

where $\bigvee^{\mathrm{op}} = \bigwedge$ denotes the least upper bound associated with $\overset{\mathrm{op}}{\leq}$, and the dual version of Theorem 8 reads:

Theorem 13 (Dual Separation Theorem). *Let $V \subset X^{\mathrm{op}}$ denote a complete subsemimodule, and let $x \in X$. Then,*

$$\forall v \in V, \quad P_V^{\mathrm{op}}(x) \backslash v = x \backslash v \ , \tag{20a}$$

and

$$x \in V \iff P_V^{\mathrm{op}}(x) \backslash x = x \backslash x \ . \tag{20b}$$

In the same way, dualizing (18), we get the following separation property for points:

$$(\forall z \in X, \quad z\backslash x = z\backslash y) \implies x = y \; . \tag{21}$$

Remark 14. It is natural to ask whether the projector

$$Q_V(x) = \bigwedge\{v \in V \mid v \geq x\}$$

can be defined when V is a subsemimodule of X, rather than a semimodule of X^{op} as in (19). The difficulty is that $Q_V(x)$ need not belong to V. For instance, when $V \subset \overline{\mathbb{R}}^3_{\max}$ is the subsemimodule generated by the columns of the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \; ,$$

$$Q_V \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

does not belong to V. However, in the special case when V is a complete subsemimodule of X stable by arbitrary infs, we have $Q_V(x) \in V$, for all $x \in X$, and Q_V preserves arbitrary sups, whereas P_V need not have this property.

We now derive from Theorem 34 a Hahn-Banach theorem for complete convex subsets, in the spirit of (Cohen *et al.* 2000). We say that a subset C of a complete semimodule over a complete semifield $\mathcal{K}$ is *convex* (resp. *complete convex*) if for all finite (resp. arbitrary) families $\{x_i\}_{i\in I} \subset C$ and $\{\alpha_i\}_{i\in I} \subset \mathcal{K}$, such that $\bigvee_{i\in I} \alpha_i = e$, we have that $\bigvee_{i\in I} \alpha_i x_i \in C$. Theorem 8 has an immediate extension to convex sets.

Corollary 15 (Separating a Point from a Convex Set)**.** *If C is a complete convex subset of a complete $\mathcal{K}$-semimodule X, and if $x \in X$ is not in C, then we have*

$$v\backslash x \wedge e = v\backslash y \wedge \nu \; , \qquad \forall v \in C \; , \tag{22a}$$

$$x\backslash x \wedge e > x\backslash y \wedge \nu \; , \tag{22b}$$

with

$$\nu = \bigoplus_{v \in C} (v\backslash x \wedge e) \; , \tag{23a}$$

$$y = \bigoplus_{v \in C} v(v\backslash x \wedge e) \; . \tag{23b}$$

Proof. Consider the complete $\mathcal{K}$-semimodule $Y = X \times \mathcal{K}$ and the complete subsemimodule V generated by the vectors (v, λ), where $v \in C$ and $\lambda \in \mathcal{K}$. It is easy to see that (v, e) belongs to V iff v belongs to the complete convex set generated by C, which coincides with C. When $x \notin C$, then $(x, e) \notin V$, and applying Theorem 8, we have that

$$(v, e)\backslash(x, e) = (v, e)\backslash P_V((x, e)), \quad \forall v \in C \; .$$

$$(x, e)\backslash(x, e) > (x, e)\backslash P_V((x, e)) \; .$$

By using this result with

$$(y,\nu) = P_V((x,e)) = \bigoplus_{v\in C}(v,e)\big((v,e)\backslash(x,e)\big)$$

(thanks to (15))

$$= \bigoplus_{v\in C}(v,e)(v\backslash x \wedge e)$$

(since $(a,\lambda)\backslash(b,\lambda') = a\backslash b \wedge \lambda\backslash\lambda'$), the proof is completed. □

Remark 16. Observe that if $x \in C$, then $P_V((x,e)) = (x,e)$, $\nu = e$ and $y = x$. Moreover, if ν is invertible, then it is easy to see that $y\nu^{-1}$ belongs to C and can thus be considered as the projection of x onto the convex subset C. Indeed, setting $P_C(x) = y\nu^{-1}$ (whenever this expression is defined), the image of C by P_C is C and $P_C \circ P_C = P_C$.

When ν is not invertible (in $\overline{\mathbb{R}}_{\max}$, this means that $\nu = \varepsilon$ since ν is not greater than e), we still do have a separating equation but its interpretation in terms of projection onto C is missing. This happens in the following example: $X = \mathcal{K} = \overline{\mathbb{R}}_{\max}$ and $C = (-\infty, +\infty]$. This C is complete convex but not closed in the usual topology. Nevertheless, the previous theory still applies and we can separate $x = -\infty$ from C. Calculations show that $y = \nu = -\infty$ and relations (22) can be checked to be true.

3.3. **Generalized Hilbert projective metric.** Consider $d_H : X \times X \to \mathcal{K}$ defined by $d_H(x,y) = (x\backslash y)(y\backslash x)$. Observe that $d_H(x,y) = d_H(y,x)$ when $\mathcal{K}$ is commutative. When $X = \overline{\mathbb{R}}^n_{\max}$, d_H can be thought of as the opposite of an additive version of the Hilbert projective metric: this is the map

$$\delta_H(x,y) = \max_{1\le i,j\le n} \log\left(\frac{x_i}{y_i}\frac{y_j}{x_j}\right)$$

for x, y ranging in the open positive cone of $\mathbb{R}^n$. When $x, y \in \mathbb{R}^n$,

$$d_H(x,y) = \min_{1\le i,j\le n}(x_i - y_i + y_j - x_j) = -\delta_H(\exp x, \exp y)\ ,$$

where exp operates coordinatewise.

Theorem 17. *The map d_H satisfies the following properties:*

- *anti-triangular inequality (when $\mathcal{K}$ is commutative):*

$$d_H(x,z) \ge d_H(x,y)d_H(y,z)\ ;$$

- *definiteness:*

$$d_H(x,y) = e \Rightarrow x = y\lambda, \quad \lambda \in \mathcal{K}\ ,$$

- *nonpositiveness:*

$$d_H(x,y) \le (x\backslash x) \text{ and } d_H(x,y) \le (x\backslash x) \wedge (y\backslash y) \text{ when } \mathcal{K} \text{ is commutative.}$$

Proof.

- Anti-triangular inequality:

$$(x\backslash y)(y\backslash x)(y\backslash z)(z\backslash y) = (x\backslash y)(y\backslash z)(z\backslash y)(y\backslash x) \\ \leq (x\backslash z)(z\backslash x)$$

  by (11b) and (11a).
- Definiteness: if $d_H(x,y) = e$ we have that

$$x = x(x\backslash y)(y\backslash x) \leq y(y\backslash x) \leq x \ ,$$

  hence $x = y(y\backslash x)$.
- Nonpositiveness:

$$(x\backslash y)(y \ \backslash x) \leq (x\backslash(y(y\backslash x)) \leq x\backslash x \ .$$

□

In conventional Euclidean spaces, the projection of a point onto a subspace minimizes the distance from that point to any point of the subspace. We show here that d_H is maximized by projection.

Theorem 18. *For all $x \in X$ and $v \in V$, where V is a complete subsemimodule of a semimodule X, we have that $d_H(x,v) \leq d_H(x, P_V(x))$.*

Proof.

$$\begin{aligned} d_H(x, P_V(x)) &= (x\backslash P_V(x))(P_V(x)\backslash x) \\ &\geq x\backslash P_V(x) && \text{(because } P_V(x) \leq x) \\ &= x\backslash \Big(\bigvee_{v\in V} (v(v\backslash x)) \Big) \\ &\geq x\backslash(v(v\backslash x)) \ , \quad \forall v \in V \\ &\geq (x\backslash v)(v\backslash x) \ , \quad \forall v \in V && \text{(by (11b))} \\ &= d_H(x,v) \ , \quad \forall v \in V \ . \end{aligned}$$

□

Example 19. Once again we return to our favorite illustration described at Examples 2 and 7. For two functions f and g in $\mathcal{F}$, we consider

$$-d_H(f,g) = \sup_{u\in\mathcal{U}} \big(f(u) - g(u)\big) + \sup_{v\in\mathcal{U}} \big(g(v) - f(v)\big) \ ;$$

it does not depend on the relative vertical position of the graphs of f and g since any constant can be added to either function without affecting that value. It is rather a “form factor” of the difference of those functions based on the sum of the maximal vertical absolute values reached by this difference above and below zero. Then, Theorem 18 says that the l.s.c. convex hull of f is, among all l.s.c. convex functions, one which minimizes this form factor difference with f (but of course not the only one).

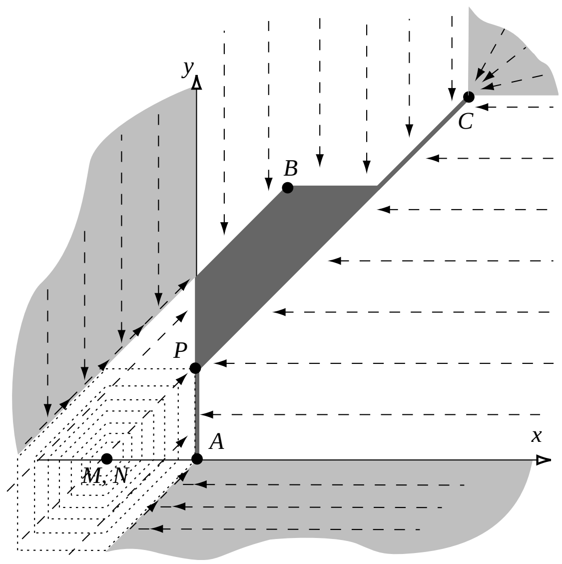

FIGURE 2. The view in the (x, y)-plane

3.4. **A two dimensional example.** We consider the convex generated by points A, B, C of coordinates $(0,0), (1,3)$ and $(3,4)$ in $\mathbb{R}^2_{\max}$. Figure 2 represents these 3 points in this space and the convex is depicted in dark grey (notice it has two "antennas" ending in A and C in addition to the polygon with nonempty interior). Figure 3 is a representation in the 3D projective space where (a fragment of) the subsemimodule V — introduced in Corollary 15 — generated by points A, B, C (now with coordinates $(0,0,0), (1,3,0)$ and $(3,4,0)$) is represented. The intersection of this subsemimodule with the (x, y)-plane is the convex represented in Figure 2. The "cylinder" is parallel to the vector $(1,1,1)$. Figure 4 is a representation of what can be seen by an observer located at a remote point along the vector $(1,1,1)$.

We now consider projecting the point M of coordinates $(-1, 0)$ (in $\mathbb{R}^2_{\max}$) onto the convex. According to Remark 16, this point is first projected on the subsemimodule V at point N of coordinates $(-1, 0, -1)$ in $(\mathbb{R}^3_{\max})$: indeed, this is the "best approximation from below" of M by an element of the subsemimodule. The reader can check this claim by using the provided explicit formulæ (15). Then, N is brought back to $\mathbb{R}^2_{\max}$ by "normalization" of the z-coordinate to 0, yielding the point P of coordinates $(0, 1, 0)$. Points M, N, P are shown in the three figures.

Relations (22) yield the following

$$\min(-1-x, -y, 0) = \min(-1-x, -y, -1), \ \ \forall (x, y) \text{ in the convex;}$$
$$\min(-1-(-1), 0, 0) > \min(-1-(-1), 0, -1) \ \text{ when applied to } M\ .$$

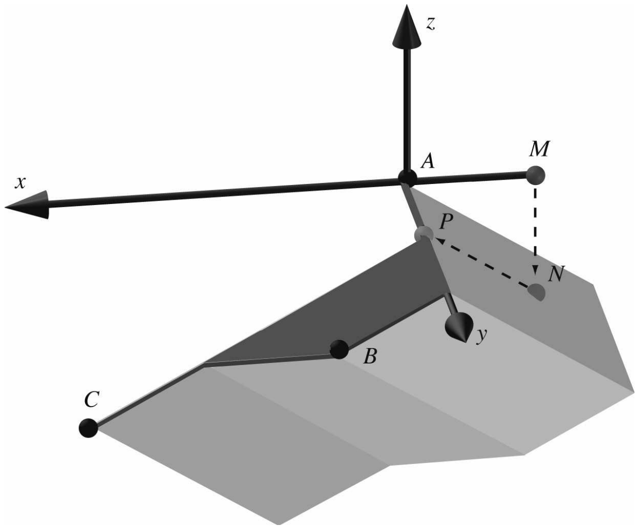

FIGURE 3. The 3D view

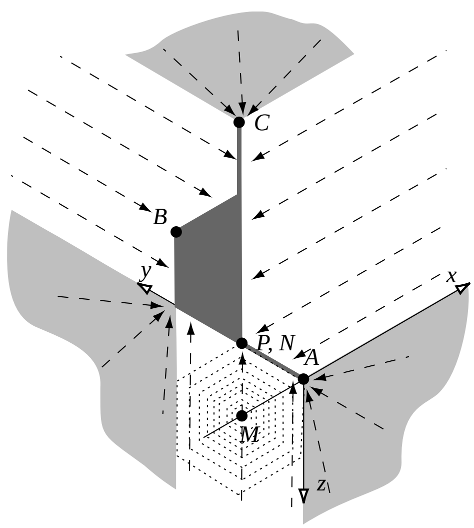

FIGURE 4. The view of an observer located along the vector $(1, 1, 1)$

The former equation simplifies into $\min(-1-x,-y,0) \leq -1$ which says that $-1-x \leq -1$ or $-y \leq -1$: this is the union of two half planes.

Observe that in Figure 2, points M and N are located at the same place because it turns out that they are located on the same vertical line of $\mathbb{R}^3_{\max}$, whereas in Figure 4, points N and P are located at the same place: this is a general fact because normalization always implies a move in the direction in which the observer of this figure is located.

In Figures 2 and 4, several zones around the convex are also shown:

- in light grey conic zones, it turns out that all points project onto a particular "extreme" point of the convex;
  - in the grey zone attached to point C (and in the whole positive orthant ($x \geq 0, y \geq 0$) as well), there is a single move in the (x,y)-plane, that is, the projection onto the subsemimodule coincides with that onto the convex set;
  - in the other two grey zones, there are actually two moves: one caused by the projection onto the subsemimodule, the other one caused by normalization; this is materialized by dotted line arrows in Figure 2; in Figure 4, the latter move (caused by normalization) is not visible for reasons already explained hereabove.
- in the white zones of the positive orthant, as already mentioned, the moves are always one-phase (i.e. horizontal); in the white zone which M belongs to, the former move is vertical (thus it cannot be visualized on Figure 2) and the latter one (normalization) is (as everywhere) along the first diagonal.

Finally, level sets of the generalized Hilbert metric are shown around point M in those figures.

Example 20. It is useful to understand the geometry of affine max-plus hyperplanes of $\mathbb{R}^2_{\max}$, that we shall call *lines.* The general line is defined by an equation of the form

$$ax \oplus by \oplus c = a'x \oplus b'y \oplus c' ,$$

for some $a, b, c, a', b', c' \in \mathbb{R}_{\max}$, but not all the coefficients are needed. For instance, the lines with equations $2x \oplus y = 1x \oplus y \oplus 3$ and $2x \oplus y = y \oplus 3$, coincide. More generally, it is not difficult to see that there are 12 generic shapes of lines, as shown in Figure 5. Indeed, a generic line can be defined by three real numbers a, b, c plus a "sign" information, which tells the side of the equation in which the corresponding coefficients is dominant (say "$\oplus$" for the left hand side, "$\ominus$" for the right and side, and a dot when coefficients on both sides are equal). For instance, the line with equation $ax \oplus c = by \oplus c$ can be denoted $L(\oplus a, \ominus b, \dot{c})$. This notation can be justified by introducing the *symmetrized* max-plus semiring (Max Plus 1990, Gaubert 1992, Baccelli *et al.* 1992). It is fundamental to note that a line with a dotted coefficient has dimension 2 in the usual sense. There is no point to distinguish algebraically between lines and half-planes, since for instance an inequality of the form $x \geq y$ can be written as an equation $x = x \oplus y$.

4. Dual Semimodules and Hahn-Banach Theorems

4.1. **Dual and Predual Pairs.** Given a complete idempotent semiring $\mathcal{K}$, we call *predual pair* a complete right $\mathcal{K}$-semimodule X together with a complete left semimodule Y equipped

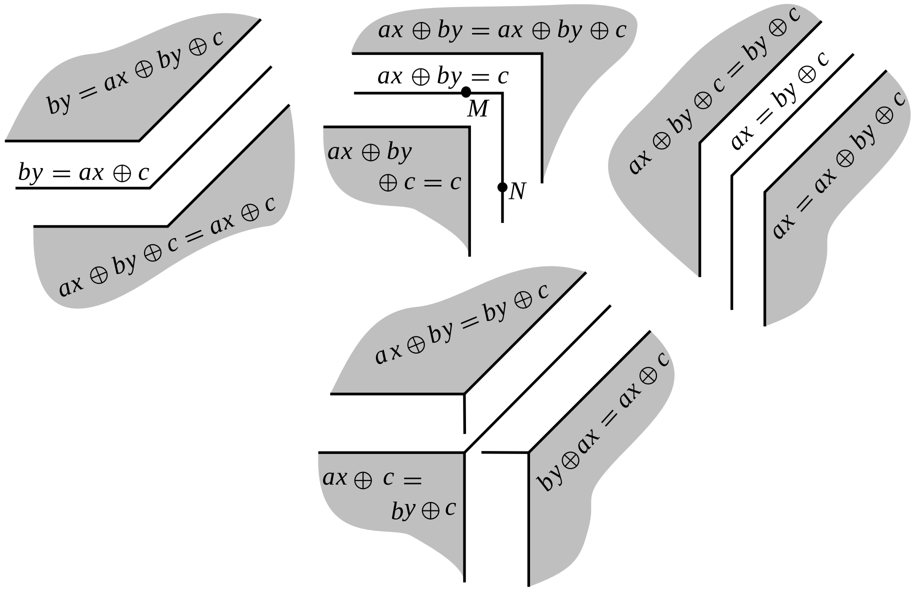

FIGURE 5. The twelve generic lines of $\mathbb{R}^2_{\max}$

with a bracket $\langle \cdot \mid \cdot \rangle$ from $Y \times X$ to a complete $\mathcal{K}$-bisemimodule Z, such that, for all $x \in X$, the maps $\mathsf{R}_x : Y \to Z$, $y \to \langle y \mid x \rangle$ and $\mathsf{L}_y : X \to Z$, $x \to \langle y \mid x \rangle$ are respectively left and right linear, and continuous. The most familiar choice of Z, which corresponds to "classical" bilinear forms, is $Z = \mathcal{K}$. In particular, the semiring $\mathcal{K}$ yields another $\mathcal{K}$-bisemimodule $Z = \mathcal{K}^{\mathrm{op}}$, with addition $(x, y) \mapsto \wedge\{x, y\}$, right action $(x, \lambda) \to \lambda \backslash x$, and left action $(\lambda, x) \to x/\lambda$.

We say that Y *separates* X if

$$(\forall y \in Y, \langle y \mid x_1 \rangle = \langle y \mid x_2 \rangle) \implies x_1 = x_2 \ ,$$

and that X *separates* Y if

$$(\forall x \in X, \langle y_1 \mid x \rangle = \langle y_2 \mid x \rangle) \implies y_1 = y_2 \ .$$

A predual pair X, Y such that X separates Y and Y separates X is a *dual pair*. This notion is inspired by the dual pairs which arise in the theory of topological vectors spaces, see (Bourbaki 1964), or Aliprantis and Border, Chapter 5, (1999).

Example 21. The right semimodule $\mathcal{K}^I$ forms a dual pair with the left semimodule $\mathcal{K}^I$ (both were introduced at Example 2), for the canonical bracket $\langle a \mid b \rangle = \bigvee_{i \in I} a(i)b(i)$.

Theorem 22 (Opposite Dual Pair). *Let X denote a complete right $\mathcal{K}$-semimodule. Then, the semimodules X^{op}, X form a dual pair for the bracket $X^{\mathrm{op}} \times X \to \mathcal{K}^{\mathrm{op}}$, $(y, x) \mapsto \langle y \mid x \rangle = x \backslash y$.*

Proof. The bilinearity and continuity of $\langle \cdot \mid \cdot \rangle$ follows from (11d), (11g), (11h), and (12). Eqn (18) shows that X^{op} separates X, and Eqn (21) shows that X separates X^{op}. □

A different example of predual pair arises when considering the (topological) ***dual*** X' of a complete semimodule X, which is the set of linear continuous maps $y : X \to \mathcal{K}$. The spaces X', X form a predual pair for the bracket $\langle y \mid x \rangle = y(x)$, and X trivially separates X', but X' need not separate X (see Example 38 below).

Example 23. Consider again the dual pair $(\mathcal{K}^I, \mathcal{K}^I)$ of Example 21. With any element $a \in \mathcal{K}^I$ is associated an element of the dual, $\mathsf{L}_a : (\mathcal{K}^I)'$, $b \mapsto \langle a \mid b \rangle$, and any element of the dual is of this form. Thus, $(\mathcal{K}^I)'$ can be identified to $\mathcal{K}^I$, and $(\mathcal{K}^I)'$ trivially separates $\mathcal{K}^I$ (indeed, if $b, c \in \mathcal{K}^I$ are such that $b(i) \neq c(i)$ for some $i \in I$, the Dirac function at point i, $\delta_i \in (\mathcal{K}^I)', \delta_i(d) = d(i)$, separates b from c).

4.2. **Involutions.** Given a bracket $\langle \cdot \mid \cdot \rangle$ from $X \times Y$ to a complete $\mathcal{K}$-bisemimodule Z, and an arbitrary element $\varphi \in Z$, we define the maps:

$$X \to Y,\ x \mapsto {}^{-}x = \top\{y \in Y \mid \langle y \mid x \rangle \leq \varphi\},$$
$$Y \to X,\ y \mapsto y^{-} = \top\{x \in X \mid \langle y \mid x \rangle \leq \varphi\} \ .$$

Thus, ${}^{-}x = \mathsf{R}_x^\sharp(\varphi)$ and $y^{-} = \mathsf{L}_y^\sharp(\varphi)$.

Proposition 24. *If (Y, X) is a predual pair, then*

(24a) $$({}^{-}x)^{-} \geq x \ , \qquad {}^{-}(({}^{-}x)^{-}) = {}^{-}x, \ \forall x \in X \ ,$$

(24b) $$ {}^{-}(y^{-}) \geq y \ , \qquad ({}^{-}(y^{-}))^{-} = y^{-} \ , \ \forall y \in Y \ .$$

Proof. We have

(25) $$x \leq y^{-} \iff \langle y \mid x \rangle \leq \varphi \iff y \leq {}^{-}x \ .$$

Consider now the maps $\iota_l : Y \to X,\ y \mapsto y^{-}$ and $\iota_r : X \to Y,\ x \mapsto {}^{-}x$. Eqn (25) shows that $\iota_l : (Y, \leq) \mapsto (X, \overset{\mathrm{op}}{\leq})$ is residuated, with $\iota_l{}^\sharp = \iota_r$. Thus, (24a) and (24b) follow from (3) and (5). □

We call *closed* the elements of X and Y of the form y^{-} and ${}^{-}x$, respectively. We set $\overline{X} = \{y^{-} \mid y \in Y\}$ and $\overline{Y} = \{{}^{-}x \mid x \in X\}$.

Proposition 25. *The sets of closed elements $\overline{X}$ and $\overline{Y}$ are complete inf-subsemilattices of X and Y, respectively,*

Proof. The set $\overline{X}$ is the image of the map $\iota_l : (Y, \leq) \to (X, \overset{\mathrm{op}}{\leq})$ which is residuated, and, by Lemma 1, this image must be a complete sup-semilattice for the order $\overset{\mathrm{op}}{\leq}$, i.e., a complete inf-semilattice for the order $\leq$. □

It follows from (24) that $({}^{-}x)^{-} = x$ (resp. ${}^{-}(y^{-}) = y$) if and only if x (resp. y) is closed, hence:

Proposition 26. *The map $x \mapsto {}^{-}x$ is an anti-isomorphism of complete lattices $\overline{X} \to \overline{Y}$, with inverse $y \to y^{-}$.*

(We warn the reader that the sup laws of $\overline{X}$ and $\overline{Y}$ do not coincide with those of X and Y, in general.) Recall that if S, T are complete lattices, a map $f : S \to T$ is an anti-isomorphism if, for all $U \subset S$, $f(\bigvee U) = \bigwedge f(U)$ and $f(\bigwedge U) = \bigvee f(U)$, and that a map f is *antitone* if $s \leq s' \implies f(s) \geq f(s')$.

Proof. We know already that $x \mapsto {}^{-}x$ is an antitone bijection from $\overline{X}$ to $\overline{Y}$ with inverse $y \mapsto y^{-}$. A bijective antitone map between complete lattices whose inverse is antitone is automatically an anti-isomorphism of complete lattices. □

Since Z is a complete $\mathcal{K}$-bisemimodule, $\lambda\backslash\mu$, and, dually, μ/ν are well defined for $\mu \in Z$ and $\lambda, \nu \in \mathcal{K}$. Considering the predual pair $(\mathcal{K}, Z)$ for the bracket $\langle \lambda \mid \mu \rangle = \lambda\mu$ allows us to define $\lambda^{-} = \lambda\backslash\varphi$. We define dually ${}^{-}\nu = \varphi/\nu$.

Proposition 27. *If $x \in X$ and $y \in Y$ are closed, then*

$$z\backslash x = \langle {}^{-}x \mid z \rangle^{-} , \quad \forall z \in X , \tag{26a}$$

$$y/t = {}^{-}\langle t \mid y^{-} \rangle , \quad \forall t \in Y . \tag{26b}$$

Proof. For all $x \in X$ and $\nu \in Z$, consider the maps $L_x^X : \mathcal{K} \to X, \mu \to x\mu$ and $L_\nu^Z : \mathcal{K} \to Z$, $\mu \to \nu\mu$. We have $\mathsf{L}_y \circ L_x^X(\lambda) = \langle y \mid x\lambda \rangle = \langle y \mid x \rangle\lambda$, for all $\lambda \in \mathcal{K}$, that is:

$$\mathsf{L}_y \circ L_x^X = L^Z_{\langle y|x \rangle} , \quad \forall y \in Y , x \in X .$$

Now, if x is closed, we have $x = y^{-}$ for some $y \in Y$, i.e., $x = \mathsf{L}_y^\sharp(\varphi)$. Hence, $z\backslash x = z\backslash y^{-} = (L_z^X)^\sharp \circ \mathsf{L}_y^\sharp(\varphi) = (\mathsf{L}_y \circ L_z^X)^\sharp(\varphi) = (L^Z_{\langle y|z \rangle})^\sharp(\varphi) = \langle y \mid z \rangle^{-}$, which shows (26a). We have proved in passing the following identity, that we tabulate for further use:

$$\forall y \in Y , \ z \in X , \quad z\backslash y^{-} = \langle y \mid z \rangle^{-} . \tag{27}$$

The proof of (26b) is dual. □

4.3. **Reflexive Semirings.** We say that a complete idempotent semiring $\mathcal{K}$ equipped with a distinguished element φ is left (resp. right) *reflexive* if ${}^{-}(\lambda^{-}) = \lambda$ (resp. $({}^{-}\lambda)^{-} = \lambda$), for all $\lambda \in \mathcal{K}$. (The element φ need not be unique; indeed, if $\mathcal{K}$ is left, or right, reflexive for φ, and if λ is invertible, it is not difficult to check that $\mathcal{K}$ is also left (or right) reflexive for $\varphi\lambda$ and $\lambda\varphi$. We shall sometimes write, more properly, that $(\mathcal{K}, \varphi)$ is reflexive.)

Using (6a), together with $\mu^{-} = \iota_l(\mu)$ and ${}^{-}\lambda = \iota_r(\lambda) = \iota_l{}^\sharp(\lambda)$, we get

$$\begin{aligned} \lambda \mapsto \lambda^{-} \text{ is injective} &\Leftrightarrow \mathcal{K} \text{ is left reflexive,} \\ &\Leftrightarrow \lambda \mapsto {}^{-}\lambda \text{ is surjective,} \end{aligned} \tag{28a}$$

$$\begin{aligned} \lambda \mapsto {}^{-}\lambda \text{ is injective,} &\Leftrightarrow \mathcal{K} \text{ is right reflexive,} \\ &\Leftrightarrow \lambda \mapsto \lambda^{-} \text{ is surjective.} \end{aligned} \tag{28b}$$

The interest in reflexive semirings stems in particular from the following result.

Proposition 28. *If $\mathcal{K}$ is right reflexive, then the set of closed elements $\overline{X}$ is a complete subsemimodule of X^{op}.*

Proof. We know from Proposition 25 that $\overline{X}$ is stable by arbitrary sups for $\stackrel{\text{op}}{\leq}$. It remains to check that for all $x \in \overline{X}$ and $\lambda \in \mathcal{K}$, $\lambda \stackrel{\text{op}}{\cdot} x = x/\lambda \in \overline{X}$. By definition of $\overline{X}$, we have $x = y^- = \mathsf{L}_y^\sharp(\varphi)$ for some $y \in Y$. Using (5c) and the right linearity of $\langle \cdot \mid \cdot \rangle$, we get $\mathsf{L}_y \circ R_\lambda^X = R_\lambda^{\mathcal{K}} \circ \mathsf{L}_y \implies (R_\lambda^X)^\sharp \circ \mathsf{L}_y^\sharp = \mathsf{L}_y^\sharp \circ (R_\lambda^{\mathcal{K}})^\sharp$, hence $x/\lambda = (R_\lambda^X)^\sharp \circ \mathsf{L}_y^\sharp(\varphi) = \mathsf{L}_y^\sharp \circ (R_\lambda^{\mathcal{K}})^\sharp(\varphi) = \mathsf{L}_y^\sharp(\varphi/\lambda) = \mathsf{L}_y^\sharp(\mu\backslash\varphi)$ for some $\mu \in \mathcal{K}$, since, by (28b), $\mu \mapsto \mu^- = \mu\backslash\varphi$ is surjective. Using (5c) again, $x/\lambda = \mathsf{L}_y^\sharp \circ (L_\mu^{\mathcal{K}})^\sharp(\varphi) = (L_\mu^{\mathcal{K}} \circ \mathsf{L}_y)^\sharp(\varphi) = \mathsf{L}_{\mu y}^\sharp(\varphi) = (\mu y)^-$, which shows that $x/\lambda \in \overline{X}$. □

Example 29. Let us consider once again the dual pair $(\mathcal{K}^I, \mathcal{K}^I)$ of Examples 2–21–23. Since ${}^-d(i) = \varphi/d(i)$, and since $a^-(i) = a(i)\backslash\varphi$, we see that all the elements of the right (resp. left) semimodule $\mathcal{K}^I$ are closed as soon as $\mathcal{K}$ is right (resp. left) reflexive.

We next exhibit a fundamental class of reflexive idempotent semirings. We say that a (non necessarily commutative) semiring is a ***semifield*** if its non-zero elements have a multiplicative inverse. A complete idempotent semiring $\mathcal{K}$ is never a semifield (unless $\mathcal{K} = \{\varepsilon, e\}$), because the maximal element of $\mathcal{K}$, τ, satisfies $\tau^2 = \tau$. For this reason, we shall call (in a slightly abusive way) ***complete semifield*** a complete semiring $\mathcal{K}$ such that all elements except ε and τ have a multiplicative inverse. For instance, $\overline{\mathbb{R}}_{\max} = (\mathbb{R} \cup \{\pm\infty\}, \max, +)$ is a complete semifield.

Proposition 30. *A complete idempotent semifield $\mathcal{K}$ is reflexive: if $\mathcal{K} = \{\varepsilon, e\}$, one must take $\varphi = \varepsilon$, otherwise, one may take any invertible φ.*

Proof. This follows readily from ${}^-x = \varphi x^{-1}$, $x^- = x^{-1}\varphi$, for $x \notin \{\varepsilon, \tau\}$, ${}^-\varepsilon = \varepsilon^- = \tau$, ${}^-\tau = \tau^- = \varepsilon$. □

We denote by $\mathcal{K}[[G]]$ the complete group $\mathcal{K}$-semialgebra over G, i.e. the free complete $\mathcal{K}$-semimodule $\mathcal{K}^G$, whose elements are denoted as formal sums $\bigoplus_{g\in G} s_g g$ where $\{s_g\}_{g\in G}$ is a family of elements of $\mathcal{K}$, equipped with the Cauchy product

$$(st)_u = \bigoplus_{\substack{gh=u\\ g,h\in G}} s_g t_h \enspace .$$

If φ is an element of $\mathcal{K}$, we denote by $\varphi_{\mathcal{K}[[G]]}$ the element of $\mathcal{K}[[G]]$ whose coefficients all are equal to τ, except the coefficient of the unit, which is equal to $\varphi_{\mathcal{K}}$. We also denote by $\varphi_{nn} \in \mathcal{K}^{n\times n}$ the matrix whose diagonal entries are equal to $\varphi_{\mathcal{K}}$ and whose out-diagonal entries are equal to τ.

The abundance of reflexive semirings is shown by the following kind of properties.

Proposition 31 (Transfer Property)**.** *Let G denote a group. If $(\mathcal{K}, \varphi_{\mathcal{K}})$ is a left (or right) reflexive complete idempotent semiring, then so are $(\mathcal{K}^{n\times n}, \varphi_{nn})$ and $(\mathcal{K}[[G]], \varphi_{\mathcal{K}[[G]]})$.* □

Proposition 32. *If $\mathcal{K}$ is reflexive and if X, Y form a predual pair for which Y separates X, then, all the elements of X are closed.*

Proof. Since $\mathcal{K}$ is right reflexive, $\overline{X} = \{y^- \mid y \in Y\}$ is a complete subsemimodule of X^{op} (Proposition 28), hence, applying the Dual Separation Theorem (Eqn (20a)) to $V = \overline{X} \subset X^{\mathrm{op}}$ and to an arbitrary $x \in X^{\mathrm{op}}$, we get, $\forall y \in Y, \quad P^{\mathrm{op}}_{\overline{X}}(x)\backslash y^- = x\backslash y^-$, and, using (27),

$$\forall y \in Y, \quad \langle y \mid P^{\mathrm{op}}_{\overline{X}}(x)\rangle^- = \langle y \mid x\rangle^- \ . \tag{29}$$

Since $\mathcal{K}$ is left reflexive, by (28a), $\lambda \to \lambda^-$ is injective, and, using (29), we get $\forall y \in Y, \quad \langle y \mid P^{\mathrm{op}}_{\overline{X}}(x)\rangle = \langle y \mid x\rangle$. Since Y separates X, $P^{\mathrm{op}}_{\overline{X}}(x) = x$, which shows that $x \in \overline{X}$. Thus, $X = \overline{X}$. □

Gathering Proposition 26 and Proposition 32 together with the symmetric result to Proposition 32, we get:

Corollary 33. *If X, Y is a dual pair for a reflexive semiring $\mathcal{K}$, then the map $x \mapsto {}^-x$, together with its inverse $y \mapsto y^-$, are anti-isomorphisms of lattices between X and Y.*

Theorem 34 (Hahn-Banach Theorem, Geometric Form)**.** *Let X, Y denote a predual pair for a left reflexive semiring $\mathcal{K}$. If $V \subset X$ is a complete subsemimodule whose elements are all closed, and if x is closed, then,*

$$\langle {}^-P_V(x) \mid v\rangle = \langle {}^-x \mid v\rangle \ , \quad \forall v \in V \ , \tag{30a}$$

and

$$\langle {}^-P_V(x) \mid x\rangle = \langle {}^-x \mid x\rangle \Leftrightarrow x \in V \ . \tag{30b}$$

Proof. Using (26a), we rewrite the universal separation property (Eqn 17) as:

$$\forall v \in V \ , \quad \langle {}^-P_V(x) \mid v\rangle^- = \langle {}^-x \mid v\rangle^- \ , \tag{31a}$$

and

$$x \in V \Leftrightarrow \langle {}^-P_V(x) \mid x\rangle^- = \langle {}^-x \mid x\rangle^- \ . \tag{31b}$$

Since $\mathcal{K}$ is left reflexive, as noted in (28a), $\lambda \to \lambda^-$ is injective, hence, (31) implies (30). □

A weaker statement, which is easier to remember, is the following.

Corollary 35. *If X, Y is a predual pair for a reflexive complete semiring $\mathcal{K}$ such that Y separates X, if V is a complete subsemimodule of X, and if $x \in X$, then, the Hahn-Banach type property* (30) *holds.* □

4.4. **Representation of Linear Forms.** We now study the dual pair X, X'. The following result characterizes the linear form ${}^-x$.

Theorem 36. *Let $\mathcal{K}$ be a complete idempotent reflexive semiring, let X be a complete $\mathcal{K}$ semimodule, and consider the dual pair X', X equipped with its canonical bracket. Then,*

$${}^-x(y) = \varphi/(y\backslash x) \ , \quad \forall x, y \in X \ .$$

Proof. If $f \in X'$ is such that $f(x) \leq \varphi$, we get from $x \geq y(y\backslash x)$ that $\varphi \geq f(x) \geq f(y)(y\backslash x)$, hence $f(y) \leq \varphi/(y\backslash x)$, for all $y \in X$. Thus, ${}^{-}x(y) \leq \varphi/(y\backslash x)$. To show that the equality holds, it suffices to show that the map $g : X \to \mathcal{K}$, $y \mapsto \varphi/(y\backslash x)$ is linear continuous and satisfies $g(x) \leq \varphi$. Since $g(x) = \varphi/(x\backslash x) \leq \varphi/e = \varphi$, the latter condition is satisfied. If $\mathcal{K}$ is reflexive, the map $\mathcal{K} \to \mathcal{K}$, $\lambda \mapsto \varphi/\lambda$, which is an anti-isomorphism of lattices, sends arbitrary infs to arbitrary sups, and conversely:

$$\varphi/(\bigwedge \Lambda) = \bigvee(\varphi/\Lambda) , \quad \forall \Lambda \subset \mathcal{K} , \tag{32a}$$

$$\varphi/(\bigvee \Lambda) = \bigwedge(\varphi/\Lambda) , \quad \forall \Lambda \subset \mathcal{K} , \tag{32b}$$

(the residuation equality (32b) holds even if the complete idempotent semiring $\mathcal{K}$ is not reflexive). Using (32), we get that for all $V \subset X$,

$$\varphi/((\bigvee V)\backslash x) = \varphi/(\bigwedge(V\backslash x)) = \bigvee(\varphi/(V\backslash x)) ,$$

which shows that g preserves arbitrary sups. It remains to show that $g(y\lambda) = g(y)\lambda$, for all $y \in X$, $\lambda \in \mathcal{K}$. Since

$$g(y\lambda) = \varphi/((y\lambda)\backslash x) = \varphi/(\lambda\backslash(y\backslash x)) ,$$

it suffices to show that $\varphi/(\lambda\backslash\alpha) = (\varphi/\alpha)\lambda$ holds for all $\alpha \in \mathcal{K}$. Since $\mathcal{K}$ is reflexive, we can write $\alpha = \beta\backslash\varphi$, with $\beta = \varphi/\alpha$, hence, $\varphi/(\lambda\backslash\alpha) = \varphi/(\lambda\backslash(\beta\backslash\varphi)) = \varphi/((\beta\lambda)\backslash\varphi) = \beta\lambda = (\varphi/\alpha)\lambda$. □

Corollary 37 (X' separates X). *If $\mathcal{K}$ is a complete idempotent reflexive semiring and if X is a complete $\mathcal{K}$ semimodule, then X' separates X.*

Proof. Let $x, y \in X$. If $f(x) = f(y)$ for all $f \in X'$, we have in particular, ${}^{-}x(x) = {}^{-}x(y)$. Since $\lambda \mapsto \varphi/\lambda, \mathcal{K} \to \mathcal{K}$ is injective, we get $x\backslash x = y\backslash x$, hence $x \geq y(y\backslash x) = y(x\backslash x) \geq y$, which shows that $x \geq y$. By symmetry, $y \geq x$. □

Example 38. The following counterexample shows that, when $\mathcal{K}$ is not reflexive, X' need not separate X.

Indeed consider the semiring $\overline{\mathbb{N}}_{\max} = \{\mathbb{N} \cup \{-\infty, +\infty\}, \max, +, 0, -\infty\}$ which is complete. $X = \{\mathbb{Z} \cup \{-\infty, +\infty\}, \max\}$ is a complete $\overline{\mathbb{N}}_{\max}$-semimodule for the action $(x, \lambda) \mapsto x + \lambda$ (with the convention $-\infty + \infty = -\infty$). Let us prove that X', the set of $\overline{\mathbb{N}}_{\max}$-linear maps from X to $\overline{\mathbb{N}}_{\max}$, consists only of the two following elements :

1. $x \in X \mapsto -\infty$;
2. $x \in X \mapsto x + \infty$.

Let $\phi \in X'$ a linear map and let us assume that it takes only finite values. Then

$$\phi(p) = \phi(p - n) + n \geq n , \quad \forall n \in \overline{\mathbb{N}}_{\max} ,$$

therefore $\phi(p) \geq \bigvee n = +\infty$ which is a contradiction.

Let us assume that there exists p such that $\phi(p) = -\infty$. By monotony of ϕ, $\phi(q) = -\infty$ for all $q \leq p$. Moreover $\phi(p + n) = \phi(p) + n = -\infty$ which implies $\phi(x) = -\infty$ for all x.

Let us assume that there exists p such that $\phi(p) = +\infty$. By monotony of ϕ, $\phi(q) = +\infty$ for all $q \geq p$. Moreover $\phi(p) = \phi(p-n) + n = +\infty$ which shows that $\phi(x) = +\infty$ for all $x \in \mathbb{Z}$ and $\phi(-\infty) = +\infty - \infty = -\infty$ which can be written $\phi(x) = x + \infty$ for all $x \in \overline{\mathbb{N}}_{\max}$.

Then the set $Y = \{\mathbb{N} \cup \{-\infty, \infty\}\}$ is a subsemimodule of X which cannot be separated from -1 by an element of X'.

Since X' separates X and X separates X', we get as an immediate corollary of Theorem 36, and Corollary 33, the following Riesz representation theorem, which extends (Litvinov *et al.* 2001, Theorem 5.2).

Corollary 39 (Riesz Representation Theorem). *Let $\mathcal{K}$ denote a complete idempotent reflexive semiring, and X a complete $\mathcal{K}$-semimodule. Then, any continuous linear form $f \in X'$ can be represented as*

$$f(y) = {}^{-}x(y) = \varphi/(y\backslash x) , \quad \forall y \in X , \tag{33}$$

for some $x \in X$, and the unique $x \in X$ which satisfies (33) *is equal to f^{-}.*

We get as a last, immediate corollary, the following extension of (Litvinov *et al.* 2001, Theorem 5.3).

Corollary 40 (Hahn-Banach Theorem, Analytic Form). *If $\mathcal{K}$ is a complete idempotent reflexive semiring, and if V is a complete subsemimodule of a complete $\mathcal{K}$-semimodule X, then any continuous linear form defined on V has a continuous extension to X.*

Example 41 (Complete Semilattices). A complete sup-semilattice $(X, \leq)$ can be thought of as a complete semimodule over the Boolean semiring $\mathbb{B} = \{\varepsilon, e\}$, with addition $(x, y) \mapsto \vee\{x, y\}$ and action $xe = x$ and $x\varepsilon = \beta_X$. The dual X' is the set of maps $x' : X \to \{\varepsilon, e\}$ which preserve arbitrary sups. Let us take $\varphi = \varepsilon$, together with the bracket $\langle x' \mid x \rangle = x'(x)$ (as noted in Proposition 30, the Boolean semiring has the exceptional feature of being reflexive for $\varphi = \varepsilon$). For any $a \in X$, we have ${}^{-}a = \vee\{x' \in X' \mid x'(a) = \varepsilon\}$, and it is not difficult to see that ${}^{-}a(x) = \varepsilon$ if $x \leq a$, and ${}^{-}a(x) = e$, otherwise. By Corollary 37, X' separates X and, by Corollary 33, $x \mapsto {}^{-}x$ establishes an anti-isomorphism between the lattices X and X'. An equivalent property was already noticed by Wagneur (1991*b*).

4.5. **Application: duality between row and column spaces.** Let $\mathcal{K}$ denote a complete reflexive semiring, and let $A \in \mathcal{K}^{n\times p}$. The free complete semimodules $X = \mathcal{K}^{p\times 1}$ and $Y = \mathcal{K}^{1\times n}$ form a predual pair for the bracket $\langle y \mid x \rangle = yAx$. We have $y^{-} = \top\{x \mid yAx \leq \varphi\} = (yA)\backslash\varphi$, and dually , ${}^{-}x = \varphi/(Ax)$. Hence,

$$\overline{X} = \{(yA)\backslash\varphi \mid y \in Y\} , \tag{34a}$$

$$\overline{Y} = \{\varphi/(Ax) \mid x \in X\} . \tag{34b}$$

Let $\mathcal{R}(A) = \{yA \mid y \in Y\}$ denote the *row space* of A, i.e., the left $\mathcal{K}$-subsemimodule of $\mathcal{K}^{1\times p}$ generated by the rows of A, and, dually, let $\mathcal{C}(A) = \{Ax \mid x \in X\}$ denote the *column space* of A. Since $\mathcal{K}$ is reflexive, the maps $z \mapsto z\backslash\varphi = (z_i\backslash\varphi)_{1\leq i\leq p}$ and $z \mapsto \varphi/z = (\varphi/z_i)_{1\leq i\leq p}$ are mutually inverse antitone bijections between $\mathcal{K}^{1\times p}$ and $\mathcal{K}^{p\times 1} = X$. By (34a), $z \mapsto z\backslash\varphi$ sends

$\mathcal{R}(A)$ to $\overline{X}$, hence, $\mathcal{R}(A)$ and $\overline{X}$ are anti-isomorphic lattices. Dually, $\mathcal{C}(A)$ and $\overline{Y}$ are anti-isomorphic lattices. By Proposition 26, $\overline{X}$ and $\overline{Y}$ are anti-isomorphic lattices. Composing anti-isomorphisms, we see that the map:

$$\mathcal{R}(A) \to \mathcal{C}(A) , \quad z \mapsto \big[\varphi/\big(A(z\backslash\varphi)\big)\big]\backslash\varphi = A(z\backslash\varphi)$$

is an anti-isomorphism of lattices. We have proved the following result, which extends a theorem of Markowsky (see (Kim 1982, Theorem 1.2.3)) for Boolean matrices.

Theorem 42. *The row space and column space of a matrix with entries in a complete idempotent reflexive semiring are anti-isomorphic lattices.* □

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