Set Weak Evolution and Transverse Field , Variational Applications and Shape Differential Equation

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Abstract: We consider weak eulerian evolution of domains through the convection of a measurable set by a nonsmooth vector field V. The *transverse variation* leads to derivative of functional associated to the evolution*tube* and we propose eulerian variational formulation for several classical problems such as incompressible euler flow (in [20], [23], minimal curves...which turn to be governed by a *geometrical adjoint state* λ which is backward and is obtained with the use of the so-called *tranvserse field* Z introduced in [16]. We also revisit the shape differential equation introduced in 1976 ([2]) and extend it to the level set approach whose speed vector approach was contained in the free boundary modeling in 1980 ([6]).

Key-words: convection, transverse field, shape differential equation, shape gradient, level set

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Evolution et equation differentielle de domaine

Résumé : Résume Nous considérons l'évolution faible de domaine via la convection d'une partie mesurable par un champ V peu régulier. La variation *transverse* conduit aux dérivées de fonctionnelles associées aux tubes d'évolution et nous proposons plusieurs formulations variationnelles eulériennes pour des problèmes classiques tels que le flot incompressible eulérien, les courbes minimales...qui se trouvent alors etre gouvernés par un *état adjoint géométrique* λ qui est solution d'un problème rétrograde en temps obtenu à partir du *champs transverse* Z introduit dans [16]. On revisite également l'equation différentielle de domaine introduite en 1976 ([2]) et on l'étend au cadre des "level sets" dont l'approche par les vitesses étaient contenue dans la modélisation des problèmes à frontières libres proposée en 1980 dans ([6]).

Mots-clés : convection, champ transverse, gradient de forme, eqation différentielle de domaine, "level set"

Contents

1 Introduction

The weak convection of measurable sets ([23], [20]) has been introduced in relation with the shape differential equation and related topics. We enlarge that approach to variational problem related to the analysis and Tubes evolution . We focus on the compacity results and we propose three "families" of "viscosity" constraints on the vector fields V leading to existence results connected to the "parabolic version" of the compactness of the inclusion mapping from bounded variation functions in integrale functions . The last one is based on the use of the "Density Perimeter" properties ([14], [13]).

The so-called "Speed Method" have been developed in relation with the shape optimization governed by Partial Differential Equations ([4],[2],[3],[7]). In the strong version we considered the flow mapping $T_t(V)$ of a smooth vector field $V \in C^0([0,\infty[, C^k(R^N, R^N) \cap L^\infty(R^N, R^N)$, (i.e. V is smoothly globally defined over R^N at all times t).

For any set Ω_0 the set $\Omega_t(V) = T_t(V)(\Omega_0)$ is then defined at any time. The shape differential equation (considered in [2],[7],... [19]) is studied for shape functional governed by several classical boundary value problems (with use of the extractor estimate for the shape gradient [17]), then in [20] for non linear viscous flow.

The characteristic function of the evolution domain, $\xi_t = \xi_{\Omega_t(V)}$, is converted as $\xi_t = \xi_0 o T_t(V)^{-1}$ solves the convection problem

$$\frac{\partial}{\partial t}\xi + \nabla\xi.V = 0, \quad \xi(0) = \xi_{\Omega_0}$$

That this problem have solution when the vector field $V \in L^2$ with $divV \in L^2$ and some growth assumption on the positive part $(divV)^+$. The incompressible situation was already introduced in [17], [20]. We give several continuity and compacity results. Applications are the existence results for variational principle related to the Euler incompressible flow equation and metric on the family of subsets of a bounded domain D

2 Weak Convection of characteristic functions

2.1 Convection

When V is a smooth vector field we denote its flow mapping by $T_t(V)$ and for any measurable set Ω we consider the "perturbated" set $\Omega_t(V) = T_t(V)(\Omega)$. The characteristic functions verifies

$$\xi_{\Omega_t} = \xi_\Omega o T_t(V)^{-1} \tag{1}$$

and it is easily verified that $\xi(t,x) = \xi_{\Omega_t}(x)$ verifies in weak sens the following *convection* equation

$$\frac{\partial}{\partial t}\xi + \nabla\xi.V \ = \ 0, \quad \xi(0) = \xi_\Omega \tag{2}$$

Notice that $\nabla\xi.V = div(\xi V) - \xi divV$ makes sense as a distribution when $\xi \in L^\infty(]0,\tau[\times D)$ and $V \in L^1(]0,\tau[\times D; R^N)$, $divV \in L^1(]0,\tau[\times D)$.

When V is a smooth vector field, say $V \in L^1(0,\tau,W^{1,\infty}(D,R^N))$ where D is a bounded universe (and the boundary condition $< v, n >= 0$ on the boundary ∂D, or any such *viability condition* for D in the case of non smooth boundary) the flow mapping is well defined for any large time $t \in [0,\tau]$ with some slight extra conditions on V, see the section devoted

to smooth tubes and vector field, then (2) and (1) are equivalent. Many questions arise concerning (2) when the vector field is non smooth. Nethertheless we do think that it is the correct approach for many shape problem such as shape evolution, shape identification (or optimization), free boundary problems , coupling fluid structure problems, image analysis....problemes in which the *topology* of the set Ω_t may be evolutive. The problem (2) is equivalent to the following one:

$$\forall\phi \in C^\infty([0,\tau]\times D),\ \phi(\tau,.) = 0,$$

$$\int_0^\tau \int_D \xi(\frac{\partial}{\partial t}\phi + < V, \nabla\phi > + \phi\ divV\ )dxdt \ + \int_\Omega \phi(0,x)dx = 0 \tag{3}$$

For non smooth vector fields we shall distinguish *three nonsmoothness levels.* The first level at which we are able to speak of solutions for (2) , (3) and for the more general "evolution" problem (4) (when right hand side is non zero and initial data non necessary characeristic function) with non smooth initial data is:

Level one: $V \in L^1(0,\tau,L^1(D;R^N))$ and the positive part $(divV)^+ \in L^1(0,\tau,L^\infty(D))$. Also for the "dual evolution" problems (7) we assume the negative part

$$(divV)^- \in L^1(0,\tau,L^\infty(D)).$$

In those cases, using Galerkine approximation and energy estimate, we derive existence of solution for both problems with initial data given in $H^{-1/2}(D)$. Today we have no uniquiness result at that level but only an estimate (5) for such "variational weak solutions", see section bellow. When V is smoother, mainely with respect to t, using the double scale hyperbolic theory by Kato we derive existence and uniquiness of smooth solution to the evolution problem (4) with smooth initial data $\phi \in H^1_0(\Omega)$:

Level two: for $V \in W^{1,\infty}(0, \tau, L^3(D))$ ($divV = 0$) and initial data in $H^1_0(D)$ we get the existence and uniquiness of a solution in

$$C^0([0, \tau], H^1_0(D)) \cap C^1([0, \tau], L^{6/5}(D))$$

.

Level three: for $V \in W^{1,\infty}(0, \tau, L^\infty(D))$ (and $divV = 0$) that solution u also verifies

$$u \in C^1([0, \tau], L^2(D))$$

.

At *Level two* we derive, by transposition technic, the uniquiness of weak solution to (2) and (4) when the initial data is in $L^\infty(D)$.

From that uniquiness property we derive the following monotonicity in the weak convection problem (2) : let two initial conditions verify $0 \le \xi^1_0 \le \xi^2_0 \le M$ the the two solutions verify $0 \le \xi^1 \le \xi^2$

At *Level three*, V and $\dot{V}$ being bounded, we derive the uniquiness of a characterisic solution $\xi = \xi^2$ to (2) when the initial data is itself a characteristic function. The idea is to prove that , ξ being the solution of (2). associated with characteristic initial data, then ξ^2 is also a solution (then from uniquiness the conclusion derives). For doing this we regularise the initial condition by an increasing sequence of positive elements ϕ_n in $H^1_0(D)$.

In order to manage variational problems we need to escape to the previous *Levels 2 and 3*, that is to avoid such regularity on the vector field V (while at those levels the flow mapping is still not defined). When the field and its divergence are simply L^1 functions the notion of weak solutions to the convection problems (2), (3) is not defined then the *modeling tool* for shape evolution is to introduce the *product space* of elements $(\xi = \xi^2, V)$ equipped with a "parabolic" BV like topology for which the "constraint" (2), (3) defines a closed subset $\mathcal{T}_\Omega$ which contains the "weak closure" (that is made precise bellow) of smooth elements

$$\{\, \xi_\Omega o T_t^{-1}(V),\ V) \ \mid V \in C^\infty([0, \tau] \times \bar{D})\ \}.$$

That approach consists in handling characteristic functions $\xi = \xi^2$ which belongs to $L^1(0, \tau, BV(D))$ togather with fields $V \in L^1(0, \tau, L^1(D, R^N))$ verifying (2), (3). For a given element $(\xi, V) \in \mathcal{T}_\Omega$, (we say that ξ is a *tube* (a measurable non cylindrical subset in $(0, \tau) \times D$, defined up to a zero measure set) with bottom Ω), we consider the set of fields W such that $(\xi, W) \in \mathcal{T}_\Omega$. It is a closed convex set $\mathcal{V}_\xi$. We can define the *unique* minimal norm energy element V_ξ in the convex set $\mathcal{V}_\xi$. For a given *tube* ξ, the element $V\xi$ is the unique (minimal norm) vector field associated to ξ via the equation (2). Finally we shall generalised the classicla shape differential equation (which is recalled bellow) in the following a Hamilton Jacobi equation for the characteristic function $\chi \in L^1(0,\tau,BV(D))$:

$$\chi(0)=\chi_{\Omega_0},\ \ \frac{\partial}{\partial t}\chi + <\nabla\chi\ ,\ A^{-1}.(\mathcal{G}(\chi)\nabla\chi)>=0$$

Where A is a *had hoc* duality operator. We shall also discuss that equation on the *level set* setting, introducting operator A_ϵ "without step" in relation with the tomography inverse problem in electromagnetic TM (transverse magnetic) mode.

2.2 The Galerkin Approximation

We consider the evolution problem

$$\frac{\partial}{\partial t}u + \nabla u.V \ = \ f,\quad u(0)=u_0 \tag{4}$$

Proposition 2.1 *Let $V \in L^1(0,\tau,L^2(D,R^3))$ with $divV \in L^1(0,\tau,L^2(D,R^3))$ verifying the following uniform integrability condition :*

$$\textit{There exist}\ \ T_0>0,\ \rho<1,\ s.t.\ \forall a\geq 0,\ \int_a^{a+T_0} ||V(t)||_{L^2(D,R^3)}dt\ \leq\ \rho<1$$

(That property is verified when $V \in L^p(0,\tau,L^2(D,R^3))$ with $p>1$). Moreover we assume that the positive part of the divergence $divV = (divV)^+ - (divV)^-$, verifies $(divV)^+ \in L^1(0,\tau,L^\infty(D,R^3))$, that is :

$$||(divV(t))^+||_{L^\infty(D,R^3)} \in L^1(0,\tau).$$

Then if $<V(t,.),n>=0$ (as an element of $L^1(0,\tau,H^{-\frac{1}{2}}(\partial D))$) , $f \in L^1(0,\tau,L^2(D))$ and initial condition $u_0 \in L^2(D)$, there exists solutions u to the problem (4) verifying

$$u\in\ L^\infty(0,\tau,L^2(D))\cap W^{1,p^*}(0,\tau,\ W^{-1,3}(D))\subset C^0([0,\tau],W^{-\frac{1}{2},\frac{3}{2}}(D)),$$

where $\frac{1}{p}+\frac{1}{p^*}=1$. Moreover there exists a constant M such that :

$$\forall\tau,\ \ ||u||_{L^\infty(0,\tau,L^2(D,R^N))}\leq M\{\ ||\phi||_{L^2(D)}+||f||_{L^1(0,\tau,L^2(D))}\ \} \tag{5}$$
$$\{\ 1+\int_0^\tau(||(divV(s))^+||_{L^\infty(D,R^3)}+||f(s)||_{L^2(D,R^3)})$$
$$\int_s^\tau(||(divV(\sigma))^+||_{L^\infty(D,R^3)}+||f(\sigma)||_{L^2(D,R^3)})d\sigma)ds\ \ \}$$

In order to derive a uniquiness result we shall consider the dual problem for which we need smoother solutions. At that point we need more regular vector fileds V and also free divergence ones in order to apply the so-called *double scale Kato theory.* Possibily the free divergence assumption in what follows could be avoid.

2.3 Time Smooth Speed Field

2.3.1 Bounded Speed Field

Let W be a given element in $L^\infty(D, R^N)$ with $divW = 0$. We consider the unbounded operator A_W in the Hilbert space $H = L^2(D)$,with dense domain $D_W = H_0^1(D)$ and defined by

$$A_W.\phi = W.\nabla\phi \tag{6}$$

It can easily be verifyed that the adjoint unbounded operator verifies

$$A_W^* = -A_W$$

.

Proposition 2.2 *The unbounded operator A_W is the infinitesimal generator of a semi group of contraction in $H = L^2(D)$.*

Proof. As $H_0^1(D) \subset L^5(D)$, with $N = 3$, and as $\frac{1}{2} + \frac{1}{5} + \frac{3}{10} = 1$, let W_n be a sequence in $W^{1,\infty}(D, R^N)$ with $div\, W_n \to 0$ in $L^{\frac{5}{3}}(D)$
and converging to W in $L^{\frac{10}{3}}(D, R^N)$. We get:

$$\int_D A_W.\phi\ \ \phi\, dx = \int_D W.\nabla\phi\ \ \phi\ dx$$

$$= lim\{ \int_D W_n.\nabla\phi\,\phi\ dx\ \} = -\,lim\{ \int_D (\ divW_n\ \ (\phi)^2\ \ + (W_n.\nabla\phi)\ \ \phi\ )dx\ \}$$

from which in the limit we deduce firstly that $\int_D (W.\nabla\phi)\ \ \phi\ dx = 0$ and then that the operator A_W is dissipative: $\int_D (A_W.\phi)\ \phi\ dx = 0$.

We consider now the evolution hyperbolic problem associated to any element

$$V \in W^{1,\infty}(0, \tau, L^\infty(D, R^N))$$

with $div\, V(t, .) = 0\ \ a.e.t.$We consider the unbounded operator $A(t)$ in the Hilbert space $H = L^2(D)$ defined by $A(t).\phi = V(t).\nabla\phi$ with dense domain $D = H_0^1(D)$ which is independant on t. The Triplet{ A(.),H, D } is then a CD-system in the sense of [8] (page 9)as we shall verify the following stability condition: for any times $t_1 < .... < t_k$,

$$||\, R(t_k, \lambda)...R(t_2, \lambda).R(t_1, \lambda)\, || \le M(\lambda - \beta)^{-1}$$

where the resolvant $R(t, \lambda) = (\lambda\ I_d\ + A(t)\ )^{-1}$ exists for any $t > \beta$. That stability condition is obviously verified from the previous contraction property of each operator $A(t_k)$. we obtain the following result ([8], thm1.2 page 11)

Proposition 2.3 *Let $V \in W^{1,\infty}(0, \tau, L^\infty(D, R^N))$, $f \in Lip(0, \tau, L^2(D))$, and $\phi \in H_0^1(D)$, there exists a unique solution*

$$u \in C([0, \tau], H_0^1(D)) \cap C^1([0, \tau], L^2(D))$$

to the evolution problem (4).

2.3.2 Unbounded Speed Field

We directely applied the theory in the Hilbert space $L^2(D)$ and we derived that $V(t,.) \in L^\infty(D)$ was enough to describe the semigroup . From Sobolev embeding inequalities we have $H^1(D) \subset L^p(D)$ for any $p \leq \frac{2N}{N-1}$. Let V be a given element in $L^q(D, R^N)$ with $q > 2 + 2\frac{N-1}{N+1}$ and $divV = 0$.

Let us observe that in dimension 3 the following inclusion holds : $H^1(D) \subset L^6(D)$ then as soon as $V(t,.) \in L^3(D)$ we get $\forall\, \phi \in H^1(D), \quad \phi V(t,.).\nabla\phi \in L^1(D)$.

The semi group is also disspative as well as his adjoint taking the Banach space $H = L^3(D)$ while the dense domain is $\mathcal{D} = H^1_0(D)$. Then we get the

Proposition 2.4 *Let $V \in W^{1,\infty}([0,\tau], L^3(D, R^N))$ with $divV(t,.) = 0$. Let initial data $\phi \in H^1_0(D)$ and $f \in W^{1,\infty}([0,\tau], L^{\frac{6}{5}}(D))$, then dynamical system (2) has a unique solution u verifying*

$$u(0) = \phi, \quad u \in C^0([0,\tau], H^1_0(D)) \cap C^1([0,\tau], L^{\frac{6}{5}}(D)).$$

By transposition technic the previous existence result furnishes an uniquiness results for weak solutions as follows.

Let $V \in L^2(0,\tau, L^2(D, R^N))$ and, for $i = 1,2$, $\xi_i \in L^\infty([0,\tau] \times D) \subset L^2(0,\tau, L^2(D))$ be two solutions to the problem , given $g \in L^2(0,\tau, L^2(D))$,

$$\frac{\partial}{\partial t}\xi_i + div(\xi_i\, V) = g$$

(so $\xi_i \in L^2(0,\tau, L^2(D)) \cap H^1(0,\tau, H^{-1}(D)) \subset C^0([0,\tau], H^{-1/2}(D)))$ with initail condition

$$\xi_i(0) = u_0 \in H^{-1/2}(D)$$

We set $w = \xi_1 - \xi_2$ which verifies the homogeneous version of the previous equation.We get

$$\forall u \in C^0([0,\tau], H^1_0(D)) \cap C^1([0,\tau], L^{\frac{6}{5}}(D)), \quad u(\tau) = 0,$$

$$\int_0^\tau \int_D w(-\frac{\partial}{\partial t}u - <\nabla u, V>)dxdt = 0$$

then, if the vector field V is smoother in time, $V \in W^{1,\infty}(0,\tau, L^3(D))$, from the the previous well posedeness w is orthogonal in $L^2(0,\tau, L^2(D))$ to any element $f \in W^{1,\infty}(0,\tau, L^{6/5}(D))$, wchich is a dense subspace. Then $w = 0$ and we claim the uniquiness of the solution.

Proposition 2.5 *Let $V \in W^{1,\infty}(0,\tau, L^3(D))$ with $divV = 0$, $g \in L^2(0,\tau, L^2(D))$ and $u_0 \in H^{-1/2}(D)$ there is a unique solution (if it exists) $\xi \in L^\infty(0,\tau, L^\infty(D))$ to the problem*

$$\frac{\partial}{\partial t}\xi + div(\xi\, V) = g, \quad \xi(0) = u_0$$

2.4 Uniquiness with nonsmooth initial data

We consider the dual evolution problem

$$\frac{\partial}{\partial t}u + div(\ u\,V\ )\ =\ g,\quad u(0) = u_0 \tag{7}$$

By transposition technic the previous existence result furnishes an uniquiness results for weak solutions as follows.

Proposition 2.6 *Let $V \in W^{1,\infty}(0,\tau,L^3(D))$ with $divV = 0$, $f(resp.g) \in L^2(0,\tau,L^2(D))$ and $u_0 \in H^{-1/2}(D)$. Then the problem (4) (resp.(7)) has a unique solution in $L^\infty(0,\tau,L^\infty(D))$.*

Let $V \in L^2(0,\tau,L^2(D,R^N))$ and, for $i = 1,2$, $\xi_i \in L^\infty([0,\tau]\times D) \subset L^2(0,\tau,L^2(D))$ be two solutions to the problem , given $g \in L^2(0,\tau,L^2(D))$,

$$\frac{\partial}{\partial t}\xi_i + div(\xi_i\ V) = g$$

(so $\xi_i \in L^2(0,\tau,L^2(D)) \cap H^1(0,\tau,H^{-1}(D)) \subset C^0([0,\tau],H^{-1/2}(D)))$ with initail condition

$$\xi_i(0) = u_0 \in H^{-1/2}(D)$$

We set $w = \xi_1 - \xi_2$ which verifies the homogeneous version of the previous equation.We get

$$\forall u \in C^0([0,\tau],H^1_0(D)) \cap C^1([0,\tau],L^{\frac{6}{5}}(D)),\quad u(\tau)\ =\ 0,$$

$$\int_0^\tau \int_D\ w(-\frac{\partial}{\partial t}u - \ <\nabla u,\ V>\ )dxdt = 0$$

then, if the vector field V is smoother in time, $V \in W^{1,\infty}(0,\tau,L^3(D))$, from the the previous well posedeness w is orthogonal in $L^2(0,\tau,L^2(D))$ to any element $f \in W^{1,\infty}(0,\tau,L^{6/5}(D))$, wchich is a dense subspace. Then $w = 0$ and we claim the uniquiness of the solution.

3 Compacity results

3.1 Vector fields in L^1

Lemma 3.1 *Let $V \in L^1(0,\tau,\ L^1(D,R^N))$ and $\xi \in L^\infty(0,\tau,\ L^\infty(D))$ solution to*

$$\frac{\partial}{\partial t}\xi + div(\xi V) = 0$$

Then we have

$$||\frac{\partial}{\partial t}\xi||_{L^1(0,\tau,W^{1,\infty}(D)')}\ \le\ ||\xi||_{L^\infty([0,\tau]\times D)}||V||_{L^1(0,\tau,L^1(D))} \tag{8}$$

Proof. $\forall \phi \in L^\infty(0,\tau, W_0^{1,\infty}(D))$ we have

$$\int_0^\tau < \frac{\partial}{\partial t}\xi,\, \phi >_{W^{1,\infty}(D)' \times W_0^{1,\infty}(D)} dt = \int_0^\tau \int_D \xi\; V.\nabla\phi\; dxdt$$

Then

$$|\int_0^\tau < \frac{\partial}{\partial t}\xi,\, \phi >_{W^{1,\infty}(D)' \times W_0^{1,\infty}(D)} dt| \leq ||\xi V||_{L^1(I\times D)}\;\; ||\phi||_{L^1(I,W_0^{1,\infty}(D))}$$

3.2 Boundedness of the Perimeter

We deal with elements f_n which are continuous in $H^{-2}(D)$

Proposition 3.1 *Let f_n be a bounded sequence in $L^1(0,\tau, BV(D))$ such that $\frac{\partial}{\partial t} f_n$ is bounded in $L^1(0,\tau, H^{-2}(D))$. Such elements belong to $C^0([0,\tau], H^{-2}(D))$, assume that $f_n(0) = f_0$ is a given element in $BV(D)$. Then there exists a subsequence strongly convergent in $L^1(0,\tau, L^1(D))$.*

We adapt to the present situation the proof of [1] theorem 5.1 page 58 (in the Roger Temam's version given in foot notes).

Lemma 3.2

$$\forall \eta > 0, \textit{there exists a constant } c_\eta \quad \textit{with} \quad \forall\phi \in BV(D),$$

$$||\phi||_{L^1(D)} \leq \eta ||\phi||_{BV(D)} + c_\eta ||\phi||_{H^{-2}(D)}$$

Proof of the lemma: assume that it is wrong. Then, $\forall \eta > 0$, there exists $\phi_n \in BV(D)$ and $c_n \to \infty$ such that

$$||\phi_n||_{L^1(D)} \geq \eta ||\phi_n||_{BV(D)} + c_n ||\phi_n||_{H^{-2}(D)}$$

We introduce $\psi_n = \phi_n / ||\phi_n||_{BV(D)}$, and we derive:

$$||\psi_n||_{L^1(D)} \geq \eta + c_n ||\psi_n||_{H^{-2}(D)} \geq \eta$$

But also $||\psi_n||_{L^1(D)} \leq c\, ||\psi_n||_{BV(D)} = c$, for some constant c. Then : $||\psi_n||_{H^{-2}(D)} \to 0$. But as $||\psi_n||_{BV(D)} = 1$, there exists a subsequence strongly convergent in $L^1(D) \subset H^{-2}(D)$, which turns to be strongly convergent to zero. This is a contradiction with $||\psi_n||_{L^1(D))} \geq \eta$.

Proof of the proposition: From the lemma, $\forall n > 0$, there exists a constant d_n such that

$$\forall f \in L^1(0,\tau, BV(D)),$$

$$||f||_{L^1(0,\tau,L^1(D))} \leq n ||f||_{L^1(0,\tau,BV(D))} + d_n ||f||_{L^1(0,\tau,H^{-2}(D))}$$

We consider now the terms $f_{n,m} = f_n - f_m$, for $m > n$. With the initial condition $f_{n,m}(0) = 0$ Given $\epsilon > 0$, as $||f_{n,m}||_{L^1(0,\tau,BV(D))} \leq M$, if we chose n such that $n\, M \leq 1/2\, \epsilon$, we shall get ;

$$||f_{n,m}||_{L^1(0,\tau,L^1(D))} \leq 1/2\; \epsilon + d_n\; ||f_{n,m}||_{L^1(0,\tau,H^{-2}(D))}$$

. At that point the conclusion will derive if we etablish strong convergence to zero of $f_{n,m}$ in $L^1(0,\tau,H^{-2}(D))$. Now, as $L^1(D) \subset H^{-2}(D)$ (for $N \leq 3$), we get $f_{n,m} \in W^{1,1}(0,\tau,H^{-2}(D)) \subset C^0([0,\tau],H^{-2}(D))$, and

$$\forall t > 0,\ ||f_{n,m}(t)||_{H^{-2}(D)} \ \leq \ M$$

so that by use of Lebesgue dominated convergence theorem it will be sufficient to prove the pointwise convergence of $f_{n,m}(t)$ strongly to zero in $H^{-2}(D)$. We shall prove it for $t = 0$. We have $f_{n,m}(0) = a_n + b_n$, with

$$a_n = 1/s \int_0^s f_{n,m}(t)dt, \quad b_n = -1/s \int_0^s (s-t)f'_{n,m}(t)dt$$

If $\epsilon > 0$ is given we chose s such that

$$||b_n||_{H^{-2}(D)} \leq \int_0^s \ ||f'_{n,m}(t)||_{H^{-2}(D)}\, dt$$

Finally we observe that $a_n \to 0$ weakly in $BV(D)$, then strongly in $H^{-2}(D)$.

3.3 Tube evolution

We derived the existence of tube associated to a vector field V while the flow mapping $T_t(V)$ does not exist. Netherveless we need a regularity such as $V \in W^{1,\infty}(0,\tau,\ L^3(D,R^N))$ which makes problem for many variational applications where the field should be no more regular than $L^2(0,\tau,L^2(D,R^N))$ (and also its divergence). For that reason we develop a setting in which none extra regularity is required on the vector field (and its divergence) but we shall impose a reguarity on the tube boundary. That regularity will be related to the tube through its boundary *perimeter* . At that point there are several possible approaches of that concept. One possibility is to consider the *time-space* perimeter of the lateral boundary Σ of the tube, an other one is to consider the time integral of the spatial perimeter of the moving domain which built the tube. The first approach would furnish immediat acces to the variational properties of the Bounded Variation framework ans specifiquely to the good compacity properties of the family of tubes with bounded perimeters in R^{N+1}. That approach was considered in ([20]) . It led to very havy variational analysis. We consider here the second approach and we adapt the results of ([23]) to the case of vectors fields in $L^1((0,\tau)\times D)$. We handle existence results for solutions of the convection equation, but no uniquiness result, the tool in that approach is to consider the set of pairs (ξ,V) solving the convection from given initial set Ω in D :

Given a measurable subset $\Omega \subset D$, we consider the following sets equipped with respective weak topologies :

$$\mathcal{A} = \{\xi \in L^\infty(0,\tau,L^\infty(D)) \text{ with } \xi^2 = \xi\,,\ \nabla\xi \in L^1(0,\tau,M^1(D,R^N))\} \tag{9}$$

The set $\mathcal{A}$ is equipped with the following weak topology :

$\xi_n \to \xi$ if and only if the two following conditions hold :

$$\forall \phi \in L^1([0,\tau] \times D), \int_0^\tau \int_D (\xi_n - \xi)\phi dx dt \to 0$$

$$\forall g \in L^1(0,\tau, C^0_{comp}(D)), \int_0^\tau \int_D (\nabla \xi_n - \nabla \xi) g \; dt dx \to 0.$$

$$\mathcal{B} = \{ V \in L^1(0,\tau, L^1(D; R^N)), \text{with } divV \in L^1(0,\tau, L^1(D)) \} \tag{10}$$

equipped wit the weak $\sigma(L^1, L^\infty)$ topology:

$$\forall \psi \in L^\infty([0,\tau] \times D; R^N), \int_0^\tau \int_D < (V_n - V), \psi > dx dt \to 0$$

and

$$\forall \psi \in L^\infty([0,\tau] \times D), \int_0^\tau \int_D (divV_n - divV)\psi dx dt \to 0$$

Let us consider the following set:

$$\mathcal{T}_\Omega = \{ (\xi, V) \in \mathcal{A} \times \mathcal{B}, \text{with } \frac{\partial}{\partial t}\xi + \nabla \xi . V = 0, \; \xi(0) = \xi_\Omega \} \tag{11}$$

Theorem 3.1 *The set $\mathcal{T}_\Omega$ is closed in $\mathcal{A} \times \mathcal{B}$.*

Proof:

Let $(\xi_n, V_n) \to (\xi, V)$. If the converging sequence is in $\mathcal{T}_\Omega$ from the Banach Steinhauss theorem we get the boundedness of the following $L^1(0,\tau, BV(D))$ norm:

$$\int_0^\tau \; ||\xi_n||_{M^1(D)} \, dt \; \leq \; M$$

from the previous results we get the strong L^1 convergence of the sequence ξ_n, then the limiting element verifies $\xi^2 = \xi$ and with the weak convergences of both V_n and $divV_n$ we get in the limit in the weak formulation (3) of the convective equation for the pair (ξ, V) which turns to be an element of $\mathcal{T}_\Omega$:

$$\forall \phi \in C^\infty([0,\tau] \times D), \; \phi(\tau, .) = 0,$$

$$\int_0^\tau \int_D \xi_n (\frac{\partial}{\partial t}\phi + < V_n, \nabla \phi > - \phi \; divV_n \,) dx dt \; + \int_\Omega \phi(0,x) dx = 0$$

3.4 Boundedness of the Density Perimeter

3.4.1 Density Perimeter

Following [14] ,[13] , we consider for any closed set A in D the density perimeter associated to any $\gamma > 0$ by the following.

$$P_\gamma(A) = sup_{\epsilon \in (0,\gamma)} \ \left[\frac{meas(A^\epsilon)}{2\,\epsilon}\ \right] \tag{12}$$

Where A^ϵ is the dilation $A^\epsilon = \cup_{x\in A} B(x,\epsilon)$. We recall some main properties:

$$\text{The mapping } \Omega\ \to P_\gamma(\partial\Omega) \text{ is lower-semi continuous in the } H^c\text{-topology}$$

$$\text{The property } P_\gamma(\partial\Omega) < \infty \text{ implies that meas}(\partial\Omega) = 0 \,\text{and}\, \Omega\ -\ \partial\Omega \text{ is open in } D.$$

$$\text{If } \ P_\gamma(\partial\Omega_n) \leq m \text{ and } \Omega_n \text{ converges in the } \ H^c - \text{topology to some open subset}$$

$$\Omega \subset D, \text{then the convergence holds in the } \ L^2(D)\text{topology of the characteristic functions}$$

3.4.2 Clean Open tube

A Clean open tube is a set $\tilde{Q}$ in $]0,\tau[\times D$ such that for $a.e.t$, $\tilde{\Omega}_t = \{x \in D \mid (t,x) \in \tilde{Q}\}$ is an open set in D verifying for almost every t in $(0,\tau)$ the following cleaness property:

$$meas(\partial\tilde{\Omega}_t) = 0, \ \ \tilde{\Omega}_t = \text{ interior of } cl(\tilde{\Omega}_t). \tag{13}$$

Notice that as the previous openess condition holds at almost every time t , the set Q is not necessarelly itself an open subset in $]0,\tau[\times D$. Nethertheless when the field V is smooth the tube $\bigcup_{0<t<\tau}\{t\}\times T_t(V)(\Omega_0)$ is open (resp. open and clean open) when the initial set Ω_0 is open (resp. clean open) in D.

We say that two tubes Q and Q' are equivalent if the characteristic functions are equal as elements in $L^2(0,\tau,L^2(D))$ (i.e. $\xi_Q = \xi_{Q'}$), that is to say that at almost every time t the two sets $\{x \in D \mid (t,x) \in Q\}$ and $\{x \in D \mid (t,x) \in Q'\}$ are the same up to a mesurable subset E of D verifying $meas(E) = 0$.

Lemma 3.3 *Let Q be a mesurable set in $]0,\tau[\times D$, if there exists a clean open tube $\tilde{Q}$ such that $\xi_Q = \xi_{\tilde{Q}}$, then that clean tube is unique.(There exists at most one equivalent clean open tube)*

Proof: assume two such clean tubes $\tilde{Q}$ and $\tilde{Q}'$. Then at a.e. t we have $\tilde{\Omega}_t = \tilde{\Omega}'_t$ up to a mesurable set E_t verifying $meas(E_t) = 0$. As those two open set verify (13) they are equals.

When Ω_0 is a clean open in D and Q is a clean open tube in $]0,\tau[\times D$, with $\xi_Q \in C^0([0,\tau],H^{-1/2}(D))$ and such that there exists a divergence free field V in E such that :

$$\frac{\partial}{\partial t}\xi_Q + \nabla\xi_Q.V = 0, \quad \xi_Q(0) = \xi_{\Omega_0} \tag{14}$$

we say that V builts the tube and we note $Q = Q_V$. Now such field when it exists is not unique. The set of fields that built the clean open tube Q is closed and convex:

$$\mathcal{V}_Q = \{V \in E \ \mid Q_V = Q\} \tag{15}$$

Lemma 3.4 *If the set $\mathcal{V}_Q$ is non empty, then it is closed and convex in E so it contains a unique element V_Q which minimizes the $L^2(0,\tau,L^2(D))$ norm in that class.*

When the tube Q is built by a smooth field $V \in L^1(0,\tau,W_0^{1,\infty}(D,R^N))$, that is $Q = Q_V$ (i.e. $\Omega_t = \{x \in D \mid (t,x) \in Q\} = T_t(V)(\Omega_0)$), obviously the convex set $\mathcal{V}_{Q_V}$ is non empty as $V \in \mathcal{V}_{Q_V}$. But in general V_{Q_V}, the minimum L^2-norm element in $\mathcal{V}_{Q_V}$, is different from V.

We describe now a construction of clean open tubes for which the set $\mathcal{V}_Q$ is non empty.

3.4.3 The "parabolic" situation

We turn to the situation of dynamical domain. One could think to use the time-space perimeter as it was considered in (**??**). For any smooth free divergence vector field, $V \in C^0([0,\tau],W_0^{1,\infty}(D,R^N))$, we consider,

$$\Theta_\gamma(V,\Omega_0) \ = \ Min \ \{ \int_0^\tau (\frac{\partial}{\partial t}\mu)^2 \ dt \,|\, \mu \in \mathcal{M}_\gamma(V,\Omega_0) \ \} \tag{16}$$

Where

$$\mathcal{M}_\gamma(V,\Omega_0) = \{ \ \mu \in H^1(0,\tau),\, P_\gamma(\partial\Omega_t(V)) \leq \mu(t) \ a.e.t, \quad \mu(0) \leq (1+\gamma)P_\gamma(\partial\Omega_0)\,\}$$

Where are many examples in which that set is non empty. When that set is empty we put $\Theta_\gamma(V,\Omega_0) = +\infty$. Notice that even when the mapping $p = (t \longrightarrow P_\gamma(\Omega_t(V))\ )$ is an element of $H^1(0,\tau)$ (then $p \in \mathcal{M}_\gamma(V,\Omega_0)\ \}$), we may have: $\Theta(V,\Omega_0) < ||p'||^2_{L^2(0,\tau)}$ as the minimizer will escape to possible variation of the function p.

Proposition 3.2 *For any smooth free divergence field $V \in C^0([0,\tau],W_0^{1,\infty}(D,R^N))$, we have:*

$$P_\gamma(\partial\Omega_t(V)) \leq 2P_\gamma(\partial\Omega_0) + \sqrt{\tau}\ \ \Theta(V,\Omega_0)^{1/2} \tag{17}$$

proof. As

$$P_\gamma(\partial\Omega_t(V)) \leq \mu(t) \leq 2P_\gamma(\partial\Omega_0) + \int_0^\tau \frac{\partial}{\partial t}\mu(t)dt$$

Then

$$P_\gamma(\partial\Omega_t(V_n)) \leq \mu(t) \leq 2P_\gamma(\partial\Omega_0) + \sqrt{\tau}(\int_0^\tau (\frac{\partial}{\partial t}\mu(t))^{1/2}dt$$

as μ is chosen beeing the minimizer element associated with V, the estimation is proved.

Proposition 3.3 *Let $V_n \in C^0([0,\tau], W_0^{1,\infty}(D, R^N))$, with the following convergence:*

$$V_n \longrightarrow V \ \ in \ \ L^2((0,\tau) \times D, R^N)$$

and the uniform boundedness :

$$\exists M > 0, \quad \Theta(V_n, \Omega_0) \leq M$$

Then

$$\Theta(V, \Omega_0) \leq \ liminf \ \Theta(V_n, \Omega_0)$$

Proof. With the boundedness assumption :

$$P_\gamma(\partial\Omega_t(V_n)) \leq C = 2P_\gamma(\partial\Omega_0) + \sqrt{\tau} \ \ M^{1/2}$$

Let μ_n be the unique minimizer in $H^1(0,\tau)$ associated with $\Theta(V_n, \Omega_0)$. There exists a subsequence, still denoted μ_n , which weakly converges to an element $\mu \in H^1(0,\tau)$. That convergence holds strongly in $L^2(0,\tau)$, then almost every where. By definition we have

$$P_\gamma(\partial\Omega_t(V_n)) \leq \ \mu_n \ a.e.$$

Then in the limit:

$$P_\gamma(\partial\Omega_t(V)) \leq liminf P_\gamma(\partial\Omega_t(V_n)) \leq \ \mu(t), \ \ a.e.t$$

Also the square of the norm being weakly lower semi continuous we have

$$\int_0^\tau (\frac{\partial}{\partial t}\mu(t))^2 dt \ \leq \ liminf \int_0^\tau (\frac{\partial}{\partial t}\mu_n(t))^2 dt$$

that leads to

$$\Theta(V, \Omega_0) \leq \int_0^\tau (\frac{\partial}{\partial t}\mu(t))^2 dt \leq liminf \ \Theta(V_n, \Omega_0)$$

Proposition 3.4 *Let$V_n \in C^0([0,\tau], W_0^{1,\infty}(D, R^N))$, with the following convergence:*

$$V_n \longrightarrow V \ \ in \ \ L^2((0,\tau) \times D, R^N)$$

and the uniform boundedness :

$$\exists M > 0, \quad \Theta(V_n, \Omega_0) \leq M$$

. We assume that Ω_0 is an open subset in D verifying

$$\Omega_0 \ = \ interior \ of \ \bar{\Omega}_0$$

Then there exists a clean open tube $\tilde{Q} = \bigcup_{0<t<\tau} \{t\} \times \tilde{\Omega}_t$ *such that:*

$$\text{at a.e.}t \in (0,\tau),\ \ \xi_{\Omega_t(V_n)} \to \xi_{\tilde{\Omega}_t} \text{ in } L^2(D),\ \ \Omega_t(V_n) \to \tilde{\Omega}_t \text{ in} H^c \text{ topology.}$$

a.e.$t \in (0,\tau)$, *the set* $\tilde{\Omega}_t$ *is an open set verifying cleanness property (13). Moreover* $\xi_{\tilde{\Omega}_t}$ *is the single a.e.t open set verifying those conditions (13) and whose characteristic function solves the convection problem:*

$$\frac{\partial}{\partial t}\xi_{\tilde{\Omega}_t} + \nabla\xi_{\tilde{\Omega}_t}.V(t) = 0, \quad \xi_{\tilde{\Omega}_0} = \xi_{\Omega_0}$$

That is : $\tilde{\Omega}_t$ *is the unique open family in* D *verifying the previous cleanness property and such that* $\xi_{\tilde{\Omega}_t} = \xi_{\Omega_t(V)}$ *a.e.t.*

Proof We have $\xi_{Q_{V_n}} \to \xi_{Q_V}$ in $L^2(I \times D)$. Then for almost every t we have $\xi_{\Omega_t(V_n)} \to \xi_{\Omega_t(V)}$ in $L^2(D)$. At each t there exist a subsequence (depending on t) which converges in H^c-toplogy to an open set :$\Omega_t(V_{n_k}) \to \omega_t$. Now for a.e.$t$ we know that $\Omega_t(V_n) \to \Omega_t(V)$ in measure (i.e. for the $L^2(D)$-norm of the characteristic functions), then at a.e.t, $\xi_{\omega_t} = \xi_{\Omega_t(V)}$. From the boundedness of $P_\gamma(\Omega_t(V_n))$ we derive that for almost every t , $\omega_t(V)$ is an open set in D and $meas(\partial\omega_t(V)) = 0$. Then we set $\tilde{\Omega}_t = cl(\omega_t) - \partial\omega_t$.

We consider now for any smooth field

Definition 3.1

$$p_\gamma(V,\Omega_0) = \int_0^\tau P_D(\Omega_t(V))dt \ + \gamma\,\Theta_\gamma(V,\Omega_0)$$

Theorem 3.2 *Let* $V_n \in C^0([0,\tau], W_0^{1,\infty}(D,R^N))$ *which weakly converges in* E *to* V *with the boundedness condition :*

$$p_\gamma(V_n,\Omega_0) \le M. \tag{18}$$

Then there exists a clean open tube $\tilde{Q}$ *built by* V *with the following convergence*

$$a.e.t \in (0,\tau), \quad \xi_{V_n}(t) \to \xi_V(t), \quad \Omega_t(V_n) \to \tilde{\Omega}_t \ \text{ in } H^c \text{ topology}$$

and $p_\gamma(V) \le liminf\ p_\gamma(V_n)$.

4 Transverse field

Let $(\xi, V) \in \mathcal{T}_\Omega$ and a vector field W such that for all s, $|s| \le s_1$ there exits $\xi^s = (\xi^s)^2$ with $(\xi^s, V + sW) \in \mathcal{T}_\Omega$. We consider heuristically the term (if it exits)

$$\dot{\xi} = \frac{\partial}{\partial s}(\xi^s)|_{s=0}$$

as a measure over $(0,\tau) \times D$. That measure should solves the evolution problem (4) with a measure as right hand side:

$$\frac{\partial}{\partial t}\dot{\xi} + \nabla\dot{\xi}.V = -\nabla(\xi)\,.W, \qquad \dot{\xi}(0) = 0 \tag{19}$$

4.1 Transverse derivative

We give an existence result for that derivative. Indeed we furnish a functional framework in which $a_s = \frac{\xi^s - \xi}{s}$ does converges in the space of $\mathcal{M}$easures to the element $\dot{\xi}$. From(3) we get

$$\int_0^\tau \int_D (a_s(\frac{\partial}{\partial t}\phi + < V, \nabla\phi > + \phi \; divV) + \xi^s < \nabla\phi, W >)dxdt = 0 \tag{20}$$

Lemma 4.1 *As $s \to 0$ we have $\xi^s \to \xi$ in $L^2((0,\tau) \times D))$*

We set

$$f = \frac{\partial}{\partial t}\phi + < V, \nabla\phi >$$

We assume the field $V \in W^{1,\infty}([0,\tau], L^3(D))$, then for any $f \in W^{1,\infty}([0,\tau], L^{6/5}(D))$, there exist a ϕ with $\phi(\tau) = 0$ solving the previous equation. We conclude :

Proposition 4.1 *Let $V \in W^{1,\infty}([0,\tau], L^3(D))$, the sequence $\frac{\xi^s - \xi}{s}$ converges in*

$$W^{1,\infty}([0,\tau], L^{6/5}(D))'$$

to an element $\dot{\xi}$.

4.2 Derivative of "volume" functional

We consider a speed functional in the form

$$j(V) = \int_0^\tau \int_{\Omega_t(V)} F_{\Omega_t(V)}(x)dx + \int_{\Gamma_t(V)} f_{\Gamma_t(V)}(x)ds_t)dt + \int_{\Omega_\tau} g(\Omega_\tau(V))dx \tag{21}$$

In the case where $f_\Gamma = 0, g = 0$ (the functional is a "distributed one") we would get

$$j'(V,W) = \int_0^\tau \int_D (\dot{\xi} F_{\Omega_t(V)} \; + \; \xi F' \;)dtdx \tag{22}$$

We introduce the following "superficial" non cylindrical adjoint state λ :

$$-\frac{\partial}{\partial t}\lambda - div(\lambda V) = F(\Omega_t(V)), \qquad \lambda(\tau) = 0 \tag{23}$$

To simplify assume that $F' = 0$ we get

$$j'(V,W) = \int_0^\tau \int_D -\dot{\xi}(\frac{\partial}{\partial t}\lambda + div(\lambda V)\,)dtdx$$

that is

$$j'(V,W) = \int_0^\tau \int_D \lambda(\frac{\partial}{\partial t}\dot{\xi} + \nabla\dot{\xi}.\,V\,)dtdx = -\int_0^\tau \int_D \lambda\;\nabla\xi.W\;dxdt$$

$$j'(V,W) = \int_0^\tau \int_D \;\xi div(\lambda W)\,dxdt = \int_0^\tau \int_{\Gamma_t(V)} \lambda < W, n_t > dsdt \tag{24}$$

In several previous works we introduced the *transverse field* Z such that

$$(\frac{\partial}{\partial s}\int_0^\tau \int_{\Omega_t(V+sW)} F(x)\;dxdt)_{s=0}$$

$$= \int_0^\tau \int_{\partial\Omega_t(V)} F(x)\;\; < Z(t,x), n_t(x) > ds_t(x)dt$$

We deduce that as a measure on $]0,\tau[\times D$ the element $\dot{\xi}$ verifies

$$\int_0^\tau \int_d \dot{\xi}\;F\;dxdt = \int_0^\tau \int_{\partial\Omega_t(V)} F\; < Z(t), n_t >\; ds_t dt \tag{25}$$

$$= \int_\Sigma \;< Z(t), n_t > (1 + (< V(t), n_t >)^2)^{-1/2}\;\, d\Sigma$$

So that if $\gamma_\Sigma \in \mathcal{L}(C^0(]0,\tau[\times D), C^0(\Sigma))$ is the trace operator on the lateral boundary Σ of the (non cylindrical) tube $Q = \cup_{0<t<\tau}\{t\} \times \Omega_t(V)$, $\gamma_\Sigma^* \in \mathcal{L}(\mathcal{M}(\Sigma), \mathcal{M}(]0,\tau[\times D))$ its adjoint operator, we get :

$$\dot{\xi} \;=\; \gamma_\Sigma^*.(\,F\;\frac{< Z(t), n_t >}{\sqrt{(1 + (< V(t), n_t >)^2)}}\;\;) \tag{26}$$

By comparison of (24) and (25), we derive:

Proposition 4.2 *Let $V \in W^{1,\infty}(0,\tau,\,L^{6/5}(D,R^3))$, for any $F \in$, λ being the unique solution of the backward problem (**??**), we have :*

$$\int_0^\tau \int_{\Gamma_t(V)} F\; < Z(t), n_t >\; ds_t\,dt = \int_0^\tau \int_{\Gamma_t(V)} \lambda\;\; < W(t), n_t >\; ds_t\,dt \tag{27}$$

4.3 Functional with end term

We consider the same situation but now the end term $g(\Omega)$ in the functional definition is not zero but is the restriction to the final domain Ω_τ of a given (smooth enough) function g, that is

$$g(\Omega_\tau) = g|_{\Omega_\tau},\;\; g = g(x),\;\; x \in D$$

Then we get:

$$j'(V,W) = \int_0^\tau \int_D (\dot{\xi}\, F_{\Omega_t(V)} \;+\; \xi\, F'\;)dtdx + \int_D \dot{\xi}_{\Omega_\tau}(x)g(x)\; dx \tag{28}$$

I

$$j'(V,W) = \int_0^\tau \int_D -\dot{\xi}(\frac{\partial}{\partial t}\bar{\lambda} + div(\bar{\lambda}\, V)\;)dtdx + \int_D \dot{\xi}_{\Omega_\tau}(x)\bar{\lambda}(\tau)(x)\; dx$$

Where $\bar{\lambda}$ is now solution of

$$\bar{\lambda}(\tau) = g,\;\; -\frac{\partial}{\partial t}\bar{\lambda} - div(\bar{\lambda}\, V) = F(\Omega_t(V)) = F|_{\Omega_t(V)} \tag{29}$$

Again we assume here that F is the restriction to the moving domain of a function $F(t,x)$ smoothly defined over R^{N+1}. By "integration by part" we obtain

$$J'(V,W) = \int_0^\tau \int \Gamma_t(V)\;\bar{\lambda}\; < W(t), n_t > \; d\Gamma_t\, dt$$

Using the transverse field Z we would get :

$$J'(V,W) = \int_0^\tau \int_{\Gamma_t(V)} F\; < Z(t), n_t > \; d\Gamma_t(V)\, dt + \int_{\Gamma_\tau(V)} g\; < Z(\tau), n_\tau > \; d\Gamma_\tau(V)$$

In the specific case where $F = 0$ we just deal with the end term g, then $\bar{\lambda}$ is solution for the homogeneous equation with end value $\bar{\lambda}(\tau) = g$ and we get

$$\int_{\Gamma_\tau(V)} g\; < Z(\tau), n_\tau > \; d\Gamma_\tau(V) = \int_0^\tau \int \Gamma_t(V)\;\bar{\lambda}\; < W(t), n_t > \; d\Gamma_t\, dt \tag{30}$$

5 Equations for the term $z =< Z(t), n_t >$, tangential Calculus on Σ

To begin with let us recall the results of ([18]) (see also ([21])): the normal transverse speed z is given by

$$\forall x \in \Gamma, z(t, T_t(V)(x)) = \int_0^t W(\sigma, T_\sigma(V)(x)).n_\sigma(T_\sigma(V)(x) \tag{31}$$

$$exp(\int_\sigma^t < DV(\sigma, (T_r(V)(x)).n_r((T_r(V)(x)),\; n_r((T_r(V)(x)) > dr\;)\, d\sigma$$

When the vectore field V is in the *canonical form*

$$J'(V,W) = V(t,.) = V(t,.)op_t(.)$$

we get $DV.n_t = 0$ on Γ_t so that the previous relation simplifies for:

$$(t,x) \in \Sigma,\ i.e. x \in \Gamma_0, z(t, T_t(V)(x)) = \int_0^t W(\sigma, T_\sigma(V)(x)).n_\sigma(T_\sigma(x))\, d\sigma \tag{32}$$

We consider the time space operators

$$< \mathcal{G}rad\lambda, \mathcal{V} \ >= \frac{\partial}{\partial t}\lambda + \nabla\lambda.V$$

The vector field $\mathcal{V} = (1, V)$ is tangent on the lateral surface Σ as the outgoing (to Q) normal field is

$$\nu = \frac{1}{\sqrt{1 + (< V, n_t >)^2}} \ (- < V(t), n_t >,\ \vec{n}_t\ )$$

Thus we have :

$$\int_\Sigma F \, \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} d\Sigma$$

$$= \int_\Sigma \lambda \frac{< W(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} d\Sigma \tag{33}$$

That is

$$\int_\Sigma (< \mathcal{G}rad\lambda, \mathcal{V} \ > \ + \lambda\, divV(t)) \quad \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ d\Sigma$$

$$= - \int_\Sigma \lambda \ \frac{< W(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ d\Sigma$$

As the product $\lambda \ < Z, n >= 0$ on $\partial\Sigma = \partial\partial Q$, the by part integration on the manifold Σ following the tangential differential operator leads to the following identity :

$$\int_\Sigma \lambda\, divV(t) \quad \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ d\Sigma$$

$$- \int_\Sigma (\lambda \ \mathcal{D}iv_\Sigma(\ \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ \mathcal{V} \ ) \ d\Sigma$$

$$= - \int_\Sigma \lambda \ \frac{< W(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ d\Sigma$$

and we get the

Proposition 5.1

$$- \, divV(t) \quad < Z(t), n_t > + \sqrt{1 + (< V, n_t >)^2}\, \mathcal{D}iv_\Sigma(\ \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ \mathcal{V} \ ) \tag{34}$$

$$=< W(t), n_t >$$

But now we have

$$\mathcal{D}iv_\Sigma E = \mathcal{D}iv E - \ < D_{t,x}E.\nu, \nu \ >$$

Where of course the right hand side is independant on the values of the vector field E outside of the lateral manifold Σ. So that, considering any extension of the concerned quantites outside of Σ, we have:

$$\mathcal{D}iv_\Sigma(\ \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ \mathcal{V} \ \ )$$

$$= \mathcal{D}iv(\ \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ \mathcal{V}) - < D_{t,x}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ \mathcal{V}).\nu, \nu >$$

Finally we get the following identity:

$$\mathcal{D}iv_\Sigma(\ \frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ \mathcal{V} \ \ )$$

$$= \frac{\partial}{\partial t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ ) \ + \ div(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ V \ )$$

$$-\frac{< V(t), n_t >}{1 + (< V, n_t >)^2} \ (\frac{\partial}{\partial n_t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}})$$

$$- < \frac{\partial}{\partial t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} V(t) \ ), \ \ n_t >)$$

$$-\frac{(< V(t), n_t)^2}{1 + (< V, n_t >)^2} \ \ \frac{\partial}{\partial t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}})$$

$$+\frac{1}{1 + (< V(t), n_t >)^2} < D(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} V(t) \ ).n_t, \ n_t >$$

We get the

Proposition 5.2 *$z =< Z(t), n_t >$ solves the following equation :*

$$-\frac{div V(t)}{\sqrt{1 + (< V, n_t >)^2}} \ \ \ < Z(t), n_t >$$

$$+ \frac{\partial}{\partial t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ ) \ + \ div(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} \ V \ )$$

$$- \frac{< V(t), n_t >}{1 + (< V, n_t >)^2} \ (\frac{\partial}{\partial n_t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}})$$

$$- < \frac{\partial}{\partial t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} V(t) \ ), \ n_t >)$$

$$- \frac{(< V(t), n_t)^2}{1 + (< V, n_t >)^2} \ \frac{\partial}{\partial t}(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}})$$

$$+ \frac{1}{1 + (< V(t), n_t >)^2} < D(\frac{< Z(t), n_t >}{\sqrt{1 + (< V, n_t >)^2}} V(t)).n_t, \ n_t >$$

$$= \frac{1}{\sqrt{1 + (< V, n_t >)^2}} \ < W(t), n_t >$$

6 Tube derivative

In that section we consider a tube Q with lateral boundary Σ and an "horizontal perturbation" leading to the perturbated tube Q^s in the following form.

An horizontal field $\tilde{Z} = (0, Z(t, x))$ is an autonomeous vector field on $R^{N+1} = R_t \times R_x^N$. We consider its flow mapping $\mathcal{T}_s(\tilde{Z})$ over R^{N+1}:

$$\mathcal{T}_s(\tilde{Z}) : \ (t, x) \ - -- > \ (t, \ T_s(Z)(x) \,)$$

We designate by $\tilde{Z}_\Sigma$ the tangential componant of the vector field $\tilde{Z}$ to the lateral surface Σ. From the expression of the normal field ν we easily derive

$$\tilde{Z}_\Sigma = (\frac{vz}{1 + v^2}, \ Z - \frac{z}{1 + v^2} \, n_t \,) \tag{35}$$

Where

$$v =< V(t), n_t >, \ \ z =< Z(t), n_t >$$

The pertube tube is the given as $Q^s = \mathcal{T}_s(Q)$. We consider

Here ν is understood as any extension of the normal field to a neiboorhoud of the lateral boundary of the tube. For example we can choose $n_t = \nabla b_{\Omega_t}$ (the oriented distance to the section of the tube, at time t) and v can be understood as vop_t where p_t is the R^N projection on $\Gamma_t = \partial \Omega_t$.

6.1 Mean curvature of the lateral time-space boundary

The term v can be chosen as $v =< V(t), \nabla b_{\Omega_t(V)} >$ and

$$\frac{\partial}{\partial t}(\frac{v}{\sqrt{1 + v^2}}) = \frac{1}{(\sqrt{1 + v^2})^3} \ \frac{\partial}{\partial t} v$$

But

$$\frac{\partial}{\partial t} v =< \frac{\partial}{\partial t} V, \ \nabla b > \ + \ < \frac{\partial}{\partial t} \nabla b, \ V >$$

Now we have

$$\frac{\partial}{\partial t} b_{\Omega_t(V)} = - < V(t), n_t > op_t \tag{36}$$

Where p_t is the projection onto the boundary $\Gamma_t(V) = \partial\Omega_t(V)$. And

$$\frac{\partial}{\partial t} \nabla b_{\Omega_t(V)} = -(\nabla_{\Gamma_t} < V(t), n_t >) op_t$$

then we get :

$$\frac{\partial}{\partial t} v =< \frac{\partial}{\partial t} V(t), n_t > - < (\nabla_{\Gamma_t} < V(t), n_t >) op_t, V_{\Gamma_t} >$$

$$\frac{\partial}{\partial t}(\frac{v}{\sqrt{1+v^2}}) = \frac{1}{(\sqrt{1+v^2})^3}(< \frac{\partial}{\partial t} V(t), n_t > - < \nabla_{\Gamma_t} op_t, V_{\Gamma_t} >$$

On the other hand we have :

$$div(\frac{1}{(\sqrt{1+v^2})} n) = - < \nabla(\frac{1}{(\sqrt{1+v^2}}, n > + \frac{1}{(\sqrt{1+v^2})^3}\ divn$$

so that we get

$$div(\frac{1}{\sqrt{1+v^2}} n) = - \frac{1}{(\sqrt{1+v^2})^3}\ < \epsilon(V).n_t, n_t > + \frac{H_t}{\sqrt{1+v^2}}$$

Where $\epsilon(V) = 1/2\,(DV + DV^*)$ is the deformation tensor .

We condider the situation in which the field V verifies the following property:

$$V(t) = V(t) op_t \text{ in a neighbourhood of } \Gamma_t \tag{37}$$

Where p_t is theR^N projection mapping onto Γ_t ("horizontal" projection). Then we get :

$$p_t = I_d - b_{\Omega_t(V)} \nabla b_{\Omega_t(V)}$$

and

$$\frac{\partial}{\partial t} p_t = - \frac{\partial}{\partial t} b_{\Omega_t(V)} \nabla b_{\Omega_t(V)} - b_{\Omega_t(V)} \nabla(\frac{\partial}{\partial t} b_{\Omega_t(V)})$$

The restriction to the boundary Γ_t leads to the distance $b_{\Omega_t(V)} = 0$ so the expressions simplify as follows (also we shall now denote by b_t that distance function) :

$$\frac{\partial}{\partial t} p_t|_{\Gamma_t} =< V(t), n_t > n_t$$

and on the boundary $\Gamma_t(V)$ we get

$$DV(t).n_t = 0,,$$

Proposition 6.1 *Assume that the field V verifies for each t:*

$$V(t) = V(t)op_t$$

Then on the boundary $\Gamma_t(V)$ we have :

$$\mathcal{D}iv_{t,x}\nu = -\frac{1}{(\sqrt{1+v^2})^3}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t}(< V(t), n_t >), V(t)_{\Gamma_t} >)$$

$$+ \frac{1}{\sqrt{1+v^2}} H_t$$

The time-space mean curvature of the lateral boundary Σ is given by:

$$\mathcal{H} = -\frac{1}{(\sqrt{1+v^2})^3}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t}(< V(t), n_t >), V(t)_{\Gamma_t} >) \tag{38}$$

$$+ \frac{1}{\sqrt{1+v^2}} H_t$$

The normal componant of the horizontal field is given by :

$$< \tilde{Z}, \nu >= \frac{1}{\sqrt{1+v^2}} < Z, n_t >$$

If $f(\Sigma)$ is the restriction to the lateral boundary Σ of a function $F(t,x)$ defined over R^{N+1}, we get the (lateral) shape boundary derivative $f'_{Sigma}(\tilde{Z})$ in the direction of the horizontal field $\tilde{Z}$ as follows :

$$f'_{\Sigma}(\tilde{Z}) = \frac{\partial}{\partial \nu}F$$

In a general setting we recall that

$$f'_{\Sigma}(\tilde{Z}) = (\frac{d}{ds}(f(\Sigma_s)o\mathcal{T}_s(\tilde{Z}))_{s=0} - < \nabla_{\Sigma} f(\Sigma), \tilde{Z}_{\Sigma} >$$

Notice that the operator ∇_{Σ}, as a tangential differential operator of the space time surface Σ is itself a time-space manifold and we get

$$f'_{\Sigma}(\tilde{Z}) = \dot{f}(\Sigma, \tilde{Z}) - \frac{vz}{1+v^2}\frac{\partial}{\partial t}f - < Z - \frac{z}{1+v^2}n_t, \nabla f >$$

6.2 Lateral Boundary Derivative

Consider a given function $F \in C^1([0,\tau] \times \bar{D})$ In a first step we assume that F is zero in the neighbourhood of $t = \tau$ so that the following derivative of the lateral boundary integral could be considered as derivative of integral on the total boundary of the tube (as it will generate no term on the top $t=\tau$ of the tube). Then the usual derivative expressions apply : we consider the derivative of the lateral integral.

$$\Sigma^s = \{\ (t, T_t(V+sW)(x))\ \mid\ x \in \partial\Omega_0\ \}$$

$$\frac{\partial}{\partial s}_{s=0}\left(\int_{\Sigma^s} F\ d\Sigma^s\right) = \int_\Sigma \left(\ \frac{\partial}{\partial \nu}F + \mathcal{H}_\Sigma F\ \right)\ < \mathcal{Z}, \nu >_{R^{N+1}}\ )\, d\Sigma$$

Where $\mathcal{H}_\Sigma$ is the mean curvature of the lateral boundary of the tube.

At each point $(t,x) \in \Sigma$ we have :

$$< \mathcal{Z}(t,x), \nu(t,x) >_{R^{N+1}} = \frac{1}{\sqrt{1+ < V(t), n_t >^2}}\ < Z(t), n_t >$$

Moreover

$$\frac{\partial}{\partial \nu}F\ =\ \frac{1}{\sqrt{1+(< V(t), n_t >)^2}}(- < V(t), n_t >\ \frac{\partial}{\partial t}F\ +\ \frac{\partial}{\partial n_t}F\ )$$

Then

$$\begin{aligned}\frac{\partial}{\partial s}_{s=0}\left(\int_{\Sigma^s} F\ d\Sigma^s\right) = \int_\Sigma [\frac{1}{\sqrt{1+v^2}}(-v\ \frac{\partial}{\partial t}F\ +\ \frac{\partial}{\partial n_t}F\ ) \\ (-\frac{1}{(\sqrt{1+v^2})^3}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\ V(t)_{\Gamma_t}\ >\ ) \\ +\frac{1}{\sqrt{1+v^2}}\ H_t)\ F]\ \frac{1}{\sqrt{1+v^2}}\ < Z, n >_{R^N}\ \ d\Sigma\end{aligned} \tag{39}$$

Proposition 6.2 *Assume the vector field V in the canonical form $V(t) = V(t)op_t$ in a neigbourhood of the lateral boundary Σ and let $v =< V(t), n_t >$ on Γ_t then we have :*

$$\begin{aligned}\frac{\partial}{\partial s}_{s=0}\left(\int_{\Sigma^s} F\ d\Sigma^s\right) = \int_0^\tau \int_{\Gamma_t} [\frac{1}{\sqrt{1+v^2}}(-v\ \frac{\partial}{\partial t}F\ +\ \frac{\partial}{\partial n_t}F\ ) \\ +F(\,-\frac{1}{(\sqrt{1+v^2})^3}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\ V(t)_{\Gamma_t}\ >\ ) \\ +\frac{1}{\sqrt{1+v^2}}\ H_t)\,]\ < Z, n >_{R^N}\ \ d\Gamma_t\, dt\end{aligned} \tag{40}$$

In the specific case where $F=1$ all the derivatives of F cancell and we have the derivative of the lateral surface of the tube :

$$\begin{aligned}\frac{\partial}{\partial s}_{s=0}\left(\int_{\Sigma^s} d\Sigma^s\right) = \int_\Sigma [\,-\frac{1}{(1+v^2)^2}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v,\ V(t)_{\Gamma_t}\ >) \\ +\frac{1}{1+v^2}\ H_t)\,]\ < Z, n >_{R^N}\ \ d\Sigma\end{aligned} \tag{41}$$

The optimality condition for a minimal surface tube is easily obtained via the adjoint problem solution λ as

$$\frac{\partial}{\partial s}_{s=0}(\int_{\Sigma^s} d\Sigma^s) = \int_{\Sigma} \lambda \ < W, n_t > \ d\Sigma \tag{42}$$

Where λ solves :

$$\lambda(\tau) = 0, \ \ -\frac{\partial}{\partial t}\lambda - div(\lambda V) \tag{43}$$

$$= -\frac{1}{(1+v^2)^2}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v, \ V(t)_{\Gamma_t} \ >) + \frac{1}{1+v^2} \ H_t$$

The optimality condition for a tube with minimal lateral surface would be

$$-\frac{1}{(1+v^2)}(< \frac{\partial}{\partial t}V, n_t > - < \nabla_{\Gamma_t} v, \ V(t)_{\Gamma_t} \ >) + \ \ H_t = 0 \tag{44}$$

7 Optimal trajectory

This kind of optimality analysis applies in trajectories problems. Let D be a compact domain in R^N with a smooth boundary. Let $a \in D$ and for any point $x \in \partial D$ we consider the family of C^1 curves which joint these two point in D. Let

$$\mathcal{C}_{a,x} = \{C^1 \ \ \text{curves } C_{a,x} \subset D, \ \text{with extremities } a \text{and } x \ \}$$

. For any such curve $C_{a,x}$ there exists an injective C^1 mapping

$$\gamma \in C^1([0,1], D), \ \ C_{a,x} = \gamma([0,1]), \gamma(0) = a, \ \gamma(1) = x$$

For any $C_{a,x}$ in $\mathcal{C}_{a,x}$ let g_C be an element of $L^1(C_{a,x})$. The objective is to minimize with respect to the curves $C_{a,x}$ the integrals $\int_{C_{a,x}} g_C \ dC$.

Let $V(t,x)$ be a vector field in $L^1([0,1], C^1(\bar{D}, R^N))$ with $< V, n >= 0$ on the boundary ∂D. Then $C^V_{a,x} = T_1(V)(C^0_{a,x}) \in \mathcal{C}_{a,x}$ and the claim is that from any element $C^0_{a,x}$ in $\mathcal{C}_{a,x}$ the elements $C^V_{a,x}$ furnishes all the curves when V described the linear space:

$$\mathcal{C}_{a,x} = \{ \ C^V_{a,x} \ \ | \ V \in L^1(0,1,C^1(\bar{D}, R^N)), \ < V, n >= 0 \ \text{ on } \partial D \ \}$$

The problem under consideration is then

$$MIN \{ \ J(V) = \int_{C^V_{a,x}} g_{C^V_{a,x}} \ dc^V_{a,x} \ \ | \ V \in L^1(0,1,C^1(\bar{D}, R^N)), \ < V, n >= 0 \ \}$$

We consider the derivative $J'(V,W) = (\frac{\partial}{\partial s}J(V+sW)\,)_{s=0}$ in the direction of any admissible field W in the same space as V. To begin with we consider the simple case in which the "density" function g_C is the restriction of a given function $G \in C^1(\bar{D})$, that is $g_C = G|_C$.

In that case the parameter s of "perturbations" takes place only in the measure element on the curve. Let γ be a parametrization of the reference curve $C^0_{a,x}$, we have:

$$J(V+sW) = \int_0^1 \; Go(\,T_1(V+sW)o\gamma(\sigma)\;||(DT_1(V+sW)o\gamma.\gamma')(\sigma)||\,d\sigma$$

We introduce

$$S_t(V;W) = (\frac{d}{ds}T_t(V+sW))_{s=0}$$

. We have

$$\frac{d}{ds}(\;||DT_1(V+sW)(\gamma(\sigma)).\gamma'(\sigma)||^2\;)_{s=0}$$

$$= 2\;<DS_1(V;W)(\gamma(\sigma)).\gamma'(\sigma), DT_1(V)(\gamma(\sigma)).\gamma'(\sigma)\;>$$

So that

$$\frac{d}{ds}(\;||DT_1(V+sW)(\gamma(\sigma)).\gamma'(\sigma)||\;)_{s=0} =< DS_1(V;W)(\gamma(\sigma)).\gamma'(\sigma), \tau(\sigma))\;>$$

$$= (\;<\;DS_1(V;W).\tau\,,\tau_1 o T_1(V)\;>)(\gamma(\sigma))\,||\gamma'||$$

Where τ_1 is the unitary tangential vector on the curve $C^V_{a,x}$ while τ is the unitary tangential vector to the reference curve $C^0_{a,x}$. For $0 \le t \le 1$, S_t is the solution of the following dynamical system

$$S(0)=0,\;\; \frac{\partial}{\partial t}S_t - DV(t)oT_t(V).S_t = W(t)oT_t(V) \tag{45}$$

we get

$$J'(V,W) = \int_0^1 (\;<\;\nabla G(T_1(V)),\,S_1\;>(\gamma(\sigma))$$

$$+G(T_1(V)(\gamma(\sigma)) < DS_1(\gamma(\sigma)).\tau(\gamma(\sigma)), \tau_1(T_1(\gamma(\sigma)))>)\,||\gamma'(\sigma)||\,d\sigma$$

Txc hat is

$$J'(V,W) = \int_{C^0_{a,x}} (\;<\;\nabla G(T_1(V),\,S_1\;> + G(T_1(V)))\;\;< DS_1.\tau, \tau_1 o T_1(V)\;>)\;dC \tag{46}$$

Obviously $J'(V,W)$ depends linearly on W through the term S_1, the final time term of the previous dynamical system. The jacobian matrix is itself solution of the following dynamical system (for which we assume $V \in C^0([0,1], C^2(D,R^N))$

$$DS(0) = 0,$$

$$\frac{\partial}{\partial t}DS_t\;-\;(D^2VoT_t(V).DT_t(V)\;).S_t\;-DVoT_t(V).DS_t = \;D(WoT_t(V)) \tag{47}$$

The couple (S_t, DS_t) is solution of the dynamical system (45), (47), system to which we shall now introduce the *backward* adjoint dynamical system whose solution (θ, A) will permit to explicit the linear contribution of the field W in the expression of $J'(V, W)$.The vector θ and the matrix A are defined on $[0, 1] \times C^0_{a,x}$ in the variables $(t, x = \gamma(\sigma)$ where the parameter σ also lies in some interval. Of course σ could be here the arc length of the reference curve $C^0_{a,x}$ so that we would have $||\gamma''(\sigma)|| = 1$ in all what follows.

$$\theta(1, \gamma(\sigma)) = \nabla G(T_1(V)(\gamma(\sigma))$$

$$-\frac{\partial}{\partial t}\theta - DV(t)^* oT_t(V).\theta = 0 \tag{48}$$

The matrix A verifies :

$$A(1, \gamma(\sigma)) = \; G(T_1(V))(\gamma(\sigma)) \; \tau(\gamma(\sigma)).\tau_1^*(T_1(V)(\gamma(\sigma)))$$

$$-\frac{\partial}{\partial t}A - (D^2VoT_t(V).DT_t(V))^*.S_t \; - DVoT_t(V)^*.A = 0 \tag{49}$$

The derivative takes the following form :

$$J'(V, W) = \int_0^1 (\; < \theta(1), \; S_1 \; > + A(1)..DS_1 \;\;)o\gamma(\sigma) \, ||\gamma'(\sigma)||d\sigma$$

$$= \int_{C^0_{a,x}} (\; < \theta(1), \; S_1 \; > + A(1)..DS_1 \;) \; dC$$

$$= \int_0^1 \; dt(\int_0^1 \; (\theta(t)W(t)oT_t(V) + D\theta..D(WoT_t(V)) \,)o\gamma(\sigma)d\sigma$$

Then we get a backward calculus for the optimal field V along the reference trajectory $C^0_{a,x}$ parametrized by γ (σ being here the parameter), as we shall verify the following eulerian approach, which is here developped for planar curves lead to a more explicit expression for the functional derivative.

7.1 Eulerian approach

We consider now a calculus which shall never refer to the reference curve $C^0_{a,x}$ but only the "moving curves" $C^V_{a,x}$ and $C^{V+sW}_{a,x}$.

Let

$$\mathcal{T}_s^t = T_t(V + sW)oT_t(V)^{-1} \tag{50}$$

That mapping send the curve $C^V_{a,x}$ onto $C^s = C^{V+sW}_{a,x}$ Then

$$J(V + sW) = \int_{C^s} g_{C^s} \, dC^s$$

Here s is understood as the "shape" perturbation parameter in the "classical" setting so that we consider the Speed vector associated

$$\mathcal{Z}^t(s,x) = \frac{\partial}{\partial s}\mathcal{T}_s^t o (\mathcal{T}_s^t)^{-1}$$

So that the moving curve C^s is obtain from the fixed one through the flow mapping of that field (flow with respect to the parameter s for fixed t, here $t=1$). We introduced the terminolgy *transverse* flow.

$$C^s = T_s(\mathcal{Z}^t)(C^V_{a,x})$$

And from usual differentiation of boundary integral (in fact we assume here that the dimension is $N=2$ so that the curves can be considered as a part of the boundary of a moving set) we obtain :

$$J'(V,W) = \int_{C^V_{a,x}} (\, g'_{C;Z} \, + \, H \, < Z(1), n >) \, dC^V_{a,x}$$

Where we set

$$Z(t,x) = \mathcal{Z}^t(0,x), \quad \text{then} \quad Z(1) = \mathcal{Z}^1(0,.)$$

and where H stands for the curvature of $C^V_{a,x}$.

The term $g'_{C;Z}$ is the so-called *boundary shape derivative* of the function $g(C)$ on $C^V_{a,x}$ in the direction of the vector field $Z(1,.)$. We recall here the very definition:

$$g'_{C;Z} = (\, \frac{d}{ds} g_{C^s} o \mathcal{T}_s^1 \,)_{s=0} - < \nabla_\tau g_{C^V_{a,x}}, Z(1) > \tag{51}$$

Here $\nabla_\tau g = \nabla G - \frac{\partial}{\partial n} G \, n$ on the curve C, is independant on the choice of the (smooth enough) extension G of g_C outside of the curve, it is the tangential derivative of g_C along the curve.

7.2 g is the restriction to the curve of G defined over D

In the very simple case where the function g_C is the restriction to the curve C of a smooth (enough) function :$g_C = G|_C$, we get

$$g'_{C;Z} = \frac{\partial}{\partial n} G \;\; < Z(1), n > \quad \text{on the curve} \;\; C^V_{a,x}.$$

So that we get

$$J'(V,W) = \int_{C^V_{a,x}} (\frac{\partial}{\partial n} G \, + \, H \,) \, < Z(1), n > \;\; dC^V_{a,x} \tag{52}$$

We recall that the transverse field Z solves the following dynamical system

$$Z(0) = 0$$

$$\frac{\partial}{\partial t} Z + [Z\,,V] \;=\; W,\;\; \text{Lie bracket being:} [Z,V] = DZ.V - DV.Z \tag{53}$$

We introduce the adjoint state

$$\Lambda(1) = \gamma^*_{C^V_{a,x}}.(\,(\frac{\partial}{\partial n} G \,+\, H\,)\,n\,)$$

Is a measure supported by the curve $C^V_{a,x}$ where $\gamma_{C^V_{a,x}}$ is the trace operator ($\gamma_{C^V_{a,x}} \in \mathcal{L}(C^0(\bar{D}), C^V_{a,x})$).

$$-\frac{\partial}{\partial t}\Lambda - DV.\Lambda - D^*\Lambda.V + divV\,\Lambda(??) = 0 \tag{54}$$

and we derive

$$J'(V,W) = <\Lambda(1),\; Z(1)\; >_{\mathcal{M}(D)\times C^0(D)}$$

By chose of Λ being solution of the adjoint we have the identity:

$$\int_0^1 \;<\Lambda, \frac{\partial}{\partial t} Z + \,[Z\,,V]\,>_{\mathcal{M}(D)\times C^0(D)}\;dt \tag{55}$$

$$= \int_0^1 \;< -\frac{\partial}{\partial t}\; <\;\Lambda - DV.\Lambda - D^*\Lambda.V + divV\,\Lambda,\; Z >_{\mathcal{M}(D)\times C^0(D)}\;dt$$

$$+\;<\Lambda(1),\; Z(1)\; >_{\mathcal{M}(D)\times C^0(D)} - <\Lambda(0),\;\; Z(0)\; >_{\mathcal{M}(D)\times C^0(D)}$$

As $Z(0) = 0$ we obtain

$$J'(V,W) = \int_0^1 \;<\Lambda, \frac{\partial}{\partial t} Z + \,[Z\,,V]\,>_{\mathcal{M}(D)\times C^0(D)}\;dt \tag{56}$$

$$= \int_0^1 \;<\Lambda, W>_{\mathcal{M}(D)\times C^0(D)}\;dt$$

It can be verified that the measure Λ is in the form

$$0 \le t \le 1,\;\; \Lambda(t) = \gamma^*_{T_t(V)(C^0_{a,x})}.(\tilde{\lambda}\,n_t)$$

We can also observe that directely from (52), using (30)that we also have :

$$J'(V,W) = \int_0^\tau \int_{\Gamma_t(V)} \bar{\lambda}\;<W(t), n_t>\; d\Gamma_t(V)\,dt \tag{57}$$

Where $\bar{\lambda}$ solves the backward problem :

$$\bar{\lambda}(\tau) = \tilde{g}, \qquad \frac{\partial}{\partial t}\bar{\lambda} + div(\bar{\lambda}\,V) = 0 \tag{58}$$

Where $\tilde{g}$ is any extension of the function $g = \frac{\partial}{\partial n}G + H_\tau$ to the domain D. If $divV = 0$, then the equation for λ turns to be a (backward) convection of the ending term:

With $divV = 0$ let $\tilde{\lambda}(t) = \bar{\lambda}(\tau - t)$ and $\tilde{V}(t) = -V(\tau - t)$ then we get

$$\frac{\partial}{\partial t}\tilde{\lambda} + \nabla\tilde{\lambda}.\tilde{V} = 0, \quad \tilde{\lambda}(0) = g$$

If V is smooth enough so that a flow mapping exists we get

$$\tilde{\lambda}(t) = goT_t(\tilde{V})^{-1}$$

that is

$$\bar{\lambda}(t) = \tilde{\lambda}(\tau - t) = goT_{\tau - t}(V)$$

Then we get :

$$J'(V,W) = \int_0^\tau \int_{\Gamma_t(V)} goT_{\tau-t}(V) \, < W(t), n_t > \, d\Gamma_t(V)\, dt \tag{59}$$

7.3 g is a function of the curvature H

Assume now that the function g effectively depends on the curve, a first example is $g = \bar{g}(H(x), x)$ where $H(x)$ is the curvature at the point X. Then we get

$$g'_{C;Z(1)} = \frac{\partial}{\partial H}\bar{g}(H(x),x) \; H'_{C.Z(1)}(x) + < \nabla_x \bar{g}(H(x),x), n(x) > < Z(1), n >$$

As $H(x) = \Delta b$ we get

$$H'_{C;Z(1)} = -\Delta_C(< Z(1), n >)$$

So that

$$J'(V,W) = \int_{C^V_{a,x}} [-\frac{\partial}{\partial H}\bar{g}(H(x),x)\, \Delta_C\, (< Z(1), n >)$$
$$+(g(H(x),x)\, H \, + < \nabla_x \bar{g}(H(x),x), n(x) >) < Z(1), n >]\, dC$$

from classical tangential "by parts integration" we obtain:

$$J'(V,W) = \int_{C^V_{a,x}} [-\Delta_C(\frac{\partial}{\partial H}\bar{g}(H(x),x)\,)$$
$$+g(H(x),x)\, H \, + < \nabla_x \bar{g}(H(x),x), n(x) >] < Z(1), n > \; dC$$
$$+\frac{\partial}{\partial \tau}(\frac{\partial}{\partial H}\bar{g}(H(x),x)) < Z(1)(x), n(x) > -\frac{\partial}{\partial \tau}(\frac{\partial}{\partial H}\bar{g}(H(a),a)) < Z(1)(a), n(a) >$$
$$-\frac{\partial}{\partial H}\bar{g}(H(x),x)\frac{\partial}{\partial \tau}(< Z(1)(x), n(x) >) + \frac{\partial}{\partial H}\bar{g}(H(a),a)\frac{\partial}{\partial \tau}(< Z(1)(a), n(a) >)$$

The extremities x and a of the curve being fixed we take

$$V(a) = V(x) = W(a) = W(x) = 0$$

which implies that $\mathcal{Z}^t(s,x) = \mathcal{Z}^t(s,a) = 0$ at any t and s so that $Z(1)(.) = \mathcal{Z}^1(0,.)$ verifies $Z(1)(a) = Z(1)(x) = 0$ and the previous derivative simplifies.

8 Shape Differential Equation, General Setting

In that section we recall the concept of shape differential equation developed in ([4]) and ([2]). We present here a simplified version and at the end we give application in dimension 2 which enables us to reach the time asymptotic result.We assume given a shape functional J wich is shape differentiable in $\mathcal{O}_k$ with respect to $\mathcal{V}_k$. We denote $\nabla J(\Omega)$ its gradient, considered as a distribution in $\mathcal{A}'_k$. For any Ω_0 in $\mathcal{O}_k$ and V in $\mathcal{V}_k$, the absolute continuity of J is written, for all $s \geqslant 0$

$$J(\Omega_s(V)) - J(\Omega_0) = \int_0^s \langle \nabla J(\Omega_t(V)) \, , \, V(t) \rangle_{\mathcal{A}'_k \times \, \mathcal{A}_k} \, dt \tag{60}$$

This classical situation of gradient-method in optimization allows to control the variations of J with respect to the domain. Considering the problem $\min_\Omega J(\Omega)$, we want to give a constructive way to decrease the functional "following the gradient". This may be done by solving the non-linear equation for large evolution of the domain

$$\forall t \geqslant 0, \quad \nabla J(\Omega_t(V)) + A(V(t)) = 0 \tag{61}$$

where A is an *ad hoc* duality operator.

From the structure theorem for shape gradient, we have (under some regularity assumptions which are fullfiled for a large class of problems):

$$\nabla J(\Omega) = \gamma_\Gamma^* . (g\vec{n})$$

Where g, the shape density gradient is a distribution on the boundary Γ. Usually it is a function on Γ so that we consider *any extension* $\mathcal{G}$ of g to the domain D (or at least at a neighborhooud of the boundary Γ). So that the shape diferential equation turns into a Hamilton Jacobi equation for the characteristic function χ:

$$\chi(0) = \chi_{\Omega_0}, \; \frac{\partial}{\partial t}\chi + < \nabla\chi \, , \, A^{-1}.(\mathcal{G}(\chi)\nabla\chi) >= 0 \tag{62}$$

We shall see bellow that this equation, in *level set formulation* will be weakened in the following one(see(79),

$$\Phi(0) = \Phi_0, \; \frac{\partial}{\partial t}\Phi - < \nabla\Phi \, A^{-1}.(\mathcal{G}(\chi_t(\Phi))\nabla\chi_t(\Phi)) >= 0 \tag{63}$$

Where $\chi_t(\Phi) = \{x \in D \;\; | \;\; \Phi(t,x) > 0 \;\; \}$. We are going to recall the constructive proof of the existence of a V satisfying (61) and investigate the asymptotic behaviour of the method. The existence of a solution for this so-called *shape differential equation* has been in [4]in a larger setting . It holds for shape differentiable functional whose gradient is continuous and bounded on m $\mathcal{O}_k$, endowed with the Courant's metric topology, ranging in a Sobolev space of Distributions .

8.1 Classical Shape Differential Equation setting

We recall here the material introduce in 1976 when solving the so-called shape differential equation. We denote $\mathcal{T}_k$ the subset of $\mathcal{C}^k(\bar{D}, \mathbb{R}^N)$ whose elements are $\mathcal{C}^k$-diffeomorphism of $\bar{D}$. It is endowed with the Courant metric $\mathfrak{d}_k$ (for which we refer to the book ([24])) which is defined on the family of images of a given domain:

For any now fixed Ω_0 in $\mathcal{O}_k$

$$\mathcal{O}_k(\Omega_0) = \left\{ \Omega \in \mathcal{O}_k \;\middle|\; \exists T \in \mathcal{T}_k,\ \Omega = T(\Omega_0) \right\}$$

Endowed with $\mathfrak{d}_k$, $\mathcal{O}_k(\Omega_0)$ is a complete metric space.

For bounded universe D the following compacity results holds (see(**??**)):

Proposition 8.1

The inclusion $(\mathcal{O}_{k+1}(\Omega_0), \mathfrak{d}_{k+1}) \hookrightarrow (\mathcal{O}_k(\Omega_0), \mathfrak{d}_k)$ *is compact.*

Also for bounded universe D, from (**??**),(**??**) we quote :

Theorem 8.1 *The mapping*

$$\begin{array}{rcl} \mathcal{V}_k & \to & \mathcal{C}^0(I, \mathcal{O}_k(\Omega_0)) \\ V & \mapsto & [t \mapsto \Omega_t(V) = T_t(V)(\Omega_0)] \end{array}$$

is continuous and maps bounded subsets on equicontinuous parts.

Lemma 8.1 *The mappings*

$$\begin{array}{rcl} \mathcal{V}_k(I) & \to & \mathcal{C}^1(I, \mathcal{C}^k(\bar{D}, \mathbb{R}^N)) \\ V & \mapsto & [t \mapsto T_t(V)] \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathcal{V}_k(I) & \to & \mathcal{C}^1(I, \mathcal{C}^k(\bar{D}, \mathbb{R}^N)) \\ V & \mapsto & [t \mapsto T_t^{-1}(V)] \end{array}$$

are continuous.

Although $\mathcal{T}_k$ is not a vector-space, we will write, for shortness

$$\mathcal{C}^1(I, \mathcal{T}_k) = \left\{ T \in \mathcal{C}^0(I, \mathcal{T}_k) \;\middle|\; T' \in \mathcal{C}^0(I, \mathcal{T}_k) \right\}$$

This space is endowed with the canonical norm

$$\|T\|_{\mathcal{C}^1(I, \mathcal{T}_k)} = \sup_{s \in I} \|T(s)\|_{\mathcal{T}_k} + \sup_{s \in I} \|T'(s)\|_{\mathcal{T}_k}$$

For $(\mathcal{T}_k, d_k)$, we have a result similar to theorem 8.1.

Theorem 8.2 *i) The mapping*

$$\begin{array}{rcl} \mathcal{V}_k(I) & \to & \mathcal{C}^1(I, \mathcal{T}_k) \\ V & \mapsto & [t \mapsto T_t(V)] \end{array}$$

is surjective, continuous and maps bounded subsets on equicontinuous parts.

We have the following characterization of the shape continuity.

Corollary 8.1 *Let G be a shape functional defined on $\mathcal{O}_k$ with values in a fixed Banach space $\mathcal{B}$. The followings are equivalent.*

i) G is shape continuous with respect to $\mathcal{V}_k(I)$: for any initial domain Ω_0, for all $V \in \mathcal{V}_k(I)$, $s \mapsto G(\Omega_s(V))$ belongs to $\mathcal{C}^0(I, \mathcal{B})$.

ii) for any initial domain Ω_0, for any T in $\mathcal{C}^1(I, \mathcal{T}_k)$, $s \mapsto G\big(T(s)(\Omega_0)\big)$ belongs to $\mathcal{C}^0(I, \mathcal{B})$.

It is important to notice how easy it is to characterize the shape continuity *via* the space $\mathcal{T}_k$. A characterization involving $\mathcal{O}_k$ would be more elegant, since the "real objects" are the domains, not the diffeomorphism. It is known (see [4] for instance) that a shape functional G defined on $\mathcal{O}_k$ with values in a fixed Banach space $\mathcal{B}$ is shape continuous (in the usual sense) as soon as $[s \mapsto G(\Omega(s))] \in \mathcal{C}^0(I, \mathcal{B})$ is continuous for any $[s \mapsto \Omega(s)] \in \mathcal{C}^0(I, \mathcal{O}_k)$. This condition being necessary is, as far as we know, an open problem.

8.2 Deformation of the Domain

This section aims at proving the following theorem, using a solution of equation (61).

Theorem 8.3 *Let J be a shape functional wich is differentiable in $\mathcal{O}_k$ with respect to $\mathcal{V}_k$. Assume both J and ∇J are uniformly bounded on $\mathcal{O}_{k+1}$ (respectively in $\mathbb{R}$ and $\mathcal{A}'_{k+1}$) and ∇J is shape-continuous on $\mathcal{O}_{k+1}$, in $\mathcal{A}'_k$, with respect to $\mathcal{V}_{k+1}$.*

Then there exists $V \in \mathcal{V}_{k+1} \cap \mathrm{L}^2(\mathbb{R}^+, \mathcal{A}_{k+1})$ such that, for any $s \geqslant 0$,

$$E(\Omega_s(V)) - E(\Omega_0(V)) = -\int_0^s \|V_t\|^2 \, dt = -c \int_0^s \|\nabla J(\Omega_t(V))\|^2 \, dt$$

Provided the duality operator A of equation (61) exists, a solution of this equation is convenient for the theorem. We are going to use a Sobolev space embeded in $\mathcal{A}_k$ to ensure the existence (and "good properties") of this operator, and give a constructive proof of the existence of a solution of (61).

We fix $\kappa > 1$ such that

$$\mathcal{H} = \left\{ V \in \mathrm{H}^\kappa(D, \mathbb{R}^N) \;\middle|\; \langle V \, , \, n \rangle_{\mathbb{R}^N} = 0 \text{ on } \partial D \right\}$$

satisfies

$$\mathcal{H} \hookrightarrow \mathcal{A}_{k+1} \hookrightarrow \mathcal{A}_k \tag{64}$$

We denote A the (linear and continuous) duality operator from $\mathcal{H}$ to its dual $\mathcal{H}'$. The domain Ω_0 being fixed in $\mathcal{O}_{k+1}$, we consider an arbitrary interval I of $\mathbb{R}^+$ which contains 0. Let G_I be the mapping defined by

$$G_I(V) = \begin{array}{ccl} I & \rightarrow & \mathcal{H} \\ s & \mapsto & -A^{-1}\big(\nabla J(\Omega_s(V))\big) \end{array} \tag{65}$$

for $V \in \mathcal{C}^0(I,\mathcal{H}) \subset \mathcal{V}_{k+1}$. Since we assumed the shape continuity of ∇J, $G_I(V) \in \mathcal{C}^0(I,\mathcal{H})$. We are going to prove that G_I has a fixed point : *i.e.* there exists a solution for (61).

Lemma 8.2 *There exists $m > 0$ such that*

$$B_{k,m} = \left\{ V \in \mathcal{V}_{k+1} \;\middle|\; \sup_{s\in I} \|V(s)\|_{\mathcal{A}_k} \right\} \supset G_I(B_{k,m})$$

Proof:

Due to the boundedness of ∇J, there exists m_1 (which may depends on Ω_0) such that for any Ω in $\mathcal{O}_k(\Omega_0)$, $\|\nabla J(\Omega)\|_{\mathcal{A}'_k} \leq m_1$. It follows that

$$\|G_I(V)(s)\|_{\mathcal{H}} \leq m_1 \|A^{-1}\|_{\mathcal{L}(\mathcal{H}',\mathcal{H})} = m_1$$

The choice $m = m_1$ is convenient.

Lemma 8.3 *The mapping G_I is continuous. Provided I is compact, G is compact.*

Proof:

Since G_I can be splitted in $G_3 \circ G_2 \circ G_1$ with:

$$\begin{array}{ccccccc} \mathcal{C}^0(I,\mathcal{H}) & \overset{G_1}{\to} & \mathcal{C}^0(I,\mathcal{T}_k) & \quad & \mathcal{C}^0(I,\mathcal{T}_k) & \overset{G_2}{\to} & \mathcal{C}^0(I,\mathcal{A}'_k) \\ V & \mapsto & [s \mapsto T_s(V)] & & T & \mapsto & \nabla J(T(.)(\Omega_0)) \end{array}$$

$$\begin{array}{ccccccc} G_3: & \mathcal{C}^0(I,\mathcal{A}'_k) & \to & \mathcal{C}^0(I,\mathcal{H}') & \to & \mathcal{C}^0(I,\mathcal{H}) \\ & g & \mapsto & g & \mapsto & -A^{-1}g \end{array}$$

Theorem 8.2 provides the continuity of G_1. By the corollary 8.1, the continuity of G_2 is equivalent to the shape continuity of the ∇J. Since the continuity of G_3 is clear, G_I is continuous.

We suppose I is compact. By theorem 8.2, a bounded subset $B \subset \mathcal{C}^0(I,\mathcal{A}_{k+1})$ is mapped by G_1 on a equicontinuous part of $\mathcal{C}^0(I,\mathcal{T}_{k+1}(\Omega_0))$. By Ascoli's theorem and the compacity of the inclusion of $\mathcal{T}_{k+1}(\Omega_0)$ in $\mathcal{T}_k(\Omega_0)$ (theorem 8.1), the image of B is para-compact in $\mathcal{C}^0(I,\mathcal{T}_k)$. Accordingly, G_1 is a compact mapping, and so is G_I.

Applying Leray-Shauder's fixed point theorem, we can conclude that for any initial domain Ω_0 there exists V in $\mathcal{C}^0([0,1],\mathcal{H})$ with $G_{[0,1]}(V) = V$.

An infinite evolution of the domain Ω_0 may be deduced. It will follow the gradient of the shape functional J and provide a optmization method. Let us define $V \in \mathcal{C}^0(\mathbb{R}^+,\mathcal{H})$ and $(\Omega^n)_{n\in\mathbb{N}} \subset \mathcal{O}_{k+1}(\Omega_0)$ by

$$\forall n \in \mathbb{N}\,, \quad \left\{ \begin{array}{rcll} G_{[n,n+1]}(V_{|[n,n+1]}) & = & V_{|[n,n+1]} & \text{on } D \\ \Omega_n(V_n) & = & \Omega^n & \\ \Omega_{n+1}(V_n) & = & \Omega^{n+1} & \\ \Omega_0(V_0) & = & \Omega_0 & \end{array} \right.$$

The continuity at integer points comes from equation (61). We have, for any $s \geqslant 0$, and any n,

$$J(\Omega^{n+1}) - J(\Omega^n) = -\int_n^{n+1} \|V(t)\|^2_{\mathcal{H}}\, dt = -\int_n^{n+1} \|\nabla J(\Omega_t(V)\|^2_{\mathcal{H}'}\, dt \qquad (66)$$

This so-built field V satisfies theorem 8.3.

Since we assumed J is bounded, $s \mapsto J(\Omega^n)$ is bounded decreasing, hence has a limit and so does $\int_0^s \|V(t)\|^2_{\mathcal{H}}\, dt$, wich proves $V \in \mathrm{L}^2(\mathbb{R}^+, \mathcal{A}_{k+1})$.

8.3 Asymptotic Behaviour

If V is given by theorem 8.3, there exists a non-decreasing sequence $(s_n)_{n \geqslant 0}$ such that $V(s_n) \to 0$. We denote $\Omega^n = \Omega_{s_n}(V) \subset \mathcal{O}_{k+1}(\Omega_0)$. The sequence (Ω^n) may not be bounded in $\mathcal{O}_k(\Omega_0)$ or $\mathcal{O}_{k+1}(\Omega_0)$, since the L^2 convergence of the speed given by this method is not sufficient in general (a L^1 convergence would be). Nevertheless, we can use a weaker topology on the space of domains. We denote $\mathcal{O}_{\mathrm{op}}$ the family of all open subsets of D. In [24] it is proved to be a compact metric space for the Hausdorff metric

$$d(\Omega_1, \Omega_2) = \max \left\{ \sup_{x_1 \in D \setminus \Omega_1} \inf_{x_2 \in \mathsf{D} \setminus \Omega_2} |x_1 - x_2| \, , \, \sup_{x_2 \in D \setminus \Omega_2} \inf_{x_1 \in \mathsf{D} \setminus \Omega_1} |x_1 - x_2| \right\} \qquad (67)$$

Lemma 8.4 *Assume the shape functional J verifies the assumptions of theorem 8.3 and J is defined and continuous in $\mathcal{O}_{\mathrm{op}}$ and ∇J is continuous for Hausorff-complementary topology on $\mathcal{O}_k(\Omega_0)$.*

Then (Ω^n) has cluster points in $\mathcal{O}_{\mathrm{op}}$. If Ω^* is one then

$$\Omega^n \to \Omega^* \text{ in } \mathcal{O}_{\mathrm{op}} \text{ and } \nabla J(\Omega^n) \to 0 \text{ and } J(\Omega^n) \to J(\Omega^*)$$

Proof:

We use the notations of the introduction. The sequence (Ω^n) may be regarded as a sequence in the compact space $\mathcal{O}_{\mathrm{op}}$. Hence passing to a subsequence, it converges towards an open subset Ω^* of D. The gradient $\nabla J(\Omega^*)$ is not *a priori* defined, since the limit set has not enough regularity. Nevertheless since V satisfies (61), $\|V(n)\|_{\mathcal{H}} = \|\nabla J(\Omega^n)\|$ hence $\nabla J(\Omega^n) \to 0$.

9 Shape differential equation for Laplace Equation

9.1 Laplace Equation

We first introduce some notations used in the following. We fix a smooth bounded hold-all D in R^N and and non-negative integer k. We denote by $\mathcal{O}_k$ the set of all open $\mathcal{C}^k$-submanifold of D, and $\mathcal{O}_{\rm lip}$ the set of Lipschitz open subset of D. We are going to use the following spaces

$$\mathcal{A}_k = \left\{ V \in \mathcal{C}^k(D, R^N) \;\middle|\; \langle V \ , \ \nu\rangle_{R^N} = 0 \right\} \quad \mathcal{V}_k(I) = V \in \mathcal{C}^0(I, \mathcal{A}_k) \tag{68}$$

where I is a interval of $\mathbb{R}^+$ which contains 0. For $I = R^+$ we simply denote $\mathcal{V}_k = \mathcal{V}_k(R^+)$.

In this section, we are given a family $g = (g_\Omega)_{\Omega\in\mathcal{O}_{\rm lip}}$ such that for any $\Omega \in \mathcal{O}_{\rm lip}$, $g_\Omega \in \mathrm{H}^{-1}(\Omega)$. We consider the Dirichlet problem

$$\mathcal{P}(\Omega, g) \quad \left\{ \begin{array}{rcll} -\Delta y & = & g_\Omega & \text{on } \Omega \\ y & = & 0 & \text{on } \Gamma = \partial\Omega \end{array} \right.$$

which has a unique solution $y(\Omega, g)$ in $\mathrm{H}^1_0(\Omega)$ which is endowed with the norm $\|z\|^2_\Omega = \displaystyle\int_\Omega |\nabla z|^2$.

9.1.1 A Priori Estimates

An *a priori* estimate for solution $y(\Omega, g)$ of $\mathcal{P}(\Omega, g)$ is derived from the variational formulation of the problem: $y(\Omega, g)$ is the unique minimum of the functional $E_{\Omega,g}$ defined on $\mathrm{H}^1_0(\Omega)$ by

$$E_{\Omega,g}(z) = -\langle g_\Omega \ , \ z\rangle_{\mathrm{H}^{-1}(\Omega)\times\mathrm{H}^1_0(\Omega)} + \int_\Omega \frac{1}{2}|\nabla z|^2 \tag{69}$$

where $|.|$ denotes the euclidean norm in $\mathbb{R}^N$. Accordingly, $E_{\Omega,g}(y(\Omega, g)) \leq 0$. Thus we come to $\frac{1}{2}\|y(\Omega, g)\|^2_\Omega \leq \|g_\Omega\|_{\Omega,*}\|y(\Omega, g)\|_\Omega$ where $\|.\|_{\Omega,*}$ denotes the norm of in the dual space $\mathrm{H}^{-1}(\Omega)$. This yields:

$$\|y(\Omega)\|_\Omega \leq 2\|g_\Omega\|_{\Omega,*} \tag{70}$$

A mere consequence of this estimate is the following uniform boundedness result.

Lemma 9.1 *Let O be a subset of $\mathcal{O}_{\rm lip}$ such that $\{\|g_\Omega\|_{\Omega,*} | \Omega \in O\}$ is bounded.*
Then $\{\|y(\Omega, g)\|_\Omega | \Omega \in O\}$ is bounded.

In the sequel, we shall use the family $(f_{|\Omega})_{\Omega\in\mathcal{O}_{\rm lip}}$ with $f \in \mathrm{L}^2(D)$. Since

$$\|f_{|\Omega}\|_{\Omega,*} \leq \|f\|_{\mathrm{L}^2(\mathsf{D})} \sup_{\substack{z\in\mathrm{H}^1_0(\Omega) \\ \|z\|_\Omega\leq 1}} \leq c_P(\Omega)\|f\|_{\mathrm{L}^2(\mathsf{D})}$$

where $c_P(\Omega)$ is the Poincaré's constant for the domain Ω. It may be defined *via* Rayleigh-quotient:

$$c_P(\Omega)^{-1} = \inf_{\substack{z\in\mathrm{H}^1_0(\Omega) \\ z\neq 0}} \frac{\|\nabla z\|_{\mathrm{L}^2(\Omega)^N}}{\|z\|_{\mathrm{L}^2(\Omega)}}$$

Accordingly, the uniform boundedness property of the solutions of $\mathcal{P}(\Omega, f)$[1] will arise from the following uniform boundedness of Poincaré's constant.

[1] The accurate notation for this is $\mathcal{P}(\Omega, (f_{|\Omega})_{\Omega\in\mathcal{O}_{\rm lip}})$.

Lemma 9.2 *There exist a constant $c_P > 0$ such that*

$$\forall \Omega \in \mathcal{O}_{\text{lip}}\,, \quad c_p(\Omega) \leq c_P$$

Proof.

It is classical that for any $\Omega \in \mathcal{O}_{\text{lip}}$, there exists z_Ω in $\mathrm{H}^1_0(\Omega)$ with $c_P(\Omega)^{-1} = \|\nabla z_\Omega\|_{\mathrm{L}^2(\Omega)^N}$. Extending z_Ω by 0 provides a $\tilde{z_\Omega}$ in $\mathrm{H}^1_0(D)$ such that $c_P(D)^{-1} \leq \|\nabla \tilde{z_\Omega}\|_{\mathrm{L}^2(\Omega)^N}$. Thus $c_P(D) \geqslant c_P(\Omega)$ so $c_P = c_P(D)$ is convenient.

Eventually, we have proved the uniform boundedness of the solutions of $\mathcal{P}(\Omega, f)$ with respect to the domain.

9.1.2 Strong Shape Continuity

Let Ω_0 be a fixed initial domain in $\mathcal{O}_{\text{lip}}$. We assume the following :

We now suppose that the family g is *shape continuous*: for any V in $\mathcal{V}_k$, the mapping $s \mapsto g_{\Omega_s(V)} \star T_s$ where $g_{\Omega_s(V)} \star T_s$ is the element of $\mathrm{H}^{-1}(\Omega_0)$ given by, for any z in $\mathrm{H}^1_0(\Omega_0)$,

$$\left\langle g_{\Omega_s(V)} \star T_s\,,\, z\right\rangle_{\mathrm{H}^{-1}(\Omega_0)\times \mathrm{H}^1_0(\Omega_0)} = \left\langle g_{\Omega_s(V)}\,,\ \gamma(s)^{-1} z \circ T_s^{-1}\right\rangle_{\mathrm{H}^{-1}(\Omega_s(V))\times \mathrm{H}^1_0(\Omega_s(V))}$$

is continuous from $\mathbb{R}^+$ to $\mathrm{H}^{-1}(\Omega_0)$. We also assume that, for any V in $\mathcal{V}_k$,

$$\left[s \mapsto \|g_{\Omega_s(V)}\|_{\mathrm{H}^{-1}(\Omega_s)}\right] \in \mathrm{L}^{\infty}{}_{\text{loc}}(\mathbb{R}^+)$$

Theorem 9.1 *Under assumption* **??**, *if $k \geqslant 2$,*

i) the transported solution of $\mathcal{P}(\Omega_0, g)$ is shape continuous:

$$\forall V \in \mathcal{V}_k, \quad y(\Omega_s(V), g) \circ T_s \to y(\Omega_0, g) \text{ in } \mathrm{H}^1_0(\Omega_0) \text{ as } s \to 0$$

ii) the "energy functional"

$$E(\Omega_0, g) = E_{\Omega_0, g}(y(\Omega_0, g))$$

is continuous

iii) the extended solution of $\mathcal{P}(\Omega_0, g)$ is shape continuous:

$$\forall V \in \mathcal{V}_k, \quad \tilde{y}(\Omega_s(V), g) \to \tilde{y}(\Omega_0, g) \text{ in } \mathrm{H}^1_0(D) \text{ as } s \to 0$$

Moreover, those points hold for $k = 1$ provided for any Ω in $\mathcal{O}_{\text{lip}}$, $g_\Omega \in \mathrm{L}^2(\Omega)$.

For simplicity, we denote $y_s = y(\Omega_s(V), g)$ and $y^s = y_s \circ T_s$. We have $y_0 = y^0$. Due to the local boundedness property of assumption **??**, there exists ϵ such that $(\|y_s\|_{\Omega_s})_{0 \leq s \leq \epsilon}$ is uniformly bounded. Since

$$\|y^s\|^2_{\Omega_0} \leq \|\gamma(s)^{-1} |\mathrm{D}(T_s^{-1})^{-1}|\|_{\mathrm{L}^\infty(D)} \|y_s\|_{\Omega_s}$$

the family $(y^s)_{0\leq s\leq\epsilon}$ is uniformly bounded in $\mathrm{H}^1_0(\Omega_0)$. Up to passing to a subsequence, a sequence $(y^{s_n})_{n\geqslant 0}$ where $0 \leq s_n \leq \epsilon$ and $s_n \to 0$, converges towards a y^*, weakly in $\mathrm{H}^1_0(\Omega_0)$.

We denote $E^{\Omega_s,g}$ the functional defined on $\mathrm{H}^1_0(\Omega_0)$ by

$$E^{\Omega_s,g}(z) = E_{\Omega_s,g}(z \circ T_s^{-1})$$
$$= -\left\langle g_{\Omega_s} \star T_s \ , \ \gamma(s)z\right\rangle_{\mathrm{H}^{-1}(\Omega_0)\times\mathrm{H}^1_0(\Omega_0)} + \int_{\Omega_0} \frac{1}{2}|\mathsf{D}T_s^{-*}.\nabla z|^2\gamma(s)$$

and we have

$$\min E^{\Omega_s,g} = E^{\Omega_s,g}(y^s) = E_{\Omega_s,g}(y_s) = \min E_{\Omega_s,g} \tag{71}$$

Since $k \geqslant 2$ the jacobians $\gamma(s_n)$ converges towards $\gamma(0) \equiv 1$ in $\mathcal{C}^1(D,\mathbb{R}^N)$. This is sufficent for

$$\left\langle g_{\Omega_{s_n}} \star T_{s_n} \ , \ \gamma(s_n)y^{s_n}\right\rangle_{\mathrm{H}^{-1}(\Omega_0)\times\mathrm{H}^1_0(\Omega_0)} \to \left\langle g_{\Omega_0} \ , \ y^0\right\rangle_{\mathrm{H}^{-1}(\Omega_0)\times\mathrm{H}^1_0(\Omega_0)} \tag{72}$$

The convergence of $\gamma(s_n)(\mathsf{D}T_{s_n})^{-1}$ towards the identity in $\mathcal{C}^0(D,\mathbb{R}^N)$ yields the weak convergence of $(\gamma(s_n)\mathsf{D}T_{s_n}^{-*}).\nabla y^{s_n}$ towards ∇y^* in $\mathrm{L}^2(\Omega_0)^N$. Due to the weak-lower semi-continuity of the L^2-norm, we have

$$\int_{\Omega_0} \frac{1}{2}|\nabla y^*|^2 \leq \liminf_{n\to\infty} \int_{\Omega_0} \frac{1}{2}|\mathsf{D}T_{s_n}^{-*}.\nabla y^{s_n}|^2\gamma(s_n)$$

Thus we have proved the weak-lower semi-continuity of $(s,z) \mapsto E^{\Omega_s,g}(z)$ on $\mathbb{R}^+ \times \mathrm{H}^1_0(\Omega_0)$ at $(0,z)$ for any z. This proves for any z

$$E^{\Omega_0,g}(y^*) \leq \liminf_{n\to\infty} E^{\Omega_{s_n},g}(y^{s_n}) \leq \liminf_{n\to\infty} E^{\Omega_{s_n},g}(z)$$

But $s \mapsto E^{\Omega_s,g}(z)$ is continuous for any z. Hence $y^* = y^0 = y_0$, which proves a weak shape-continuity for the transported solutions of $\mathcal{P}(\Omega,g)$.

The strong continuity will arise from a continuity of the norms, *via* the so-called compliance equality, and a compactness argument for the y_s. For any Ω in $\mathcal{O}_{\text{lip}}$, the necessary (and sufficient) condition of optimality for $E_{\Omega,g}$ is written

$$\forall z \in \mathrm{H}^1_0(\Omega), \quad \int_\Omega \left\langle \nabla y(\Omega,g) \ , \ \nabla z\right\rangle_{\mathbb{R}^N} = \left\langle g_\Omega \ , \ z\right\rangle_{\mathrm{H}^{-1}(\Omega)\times\mathrm{H}^1_0(\Omega)}$$

Setting $z = y(\Omega,g)$ we come to

$$E_{\Omega,g}(y(\Omega,g)) = -\frac{1}{2}\|y(\Omega,g)\|^2_\Omega = -\frac{1}{2}\left\langle g_\Omega \ , \ y(\Omega,g)\right\rangle_{\mathrm{H}^{-1}(\Omega)\times\mathrm{H}^1_0(\Omega)} \tag{73}$$

which leads together with equation (72) and the weak continuity of $s \mapsto y^s$, that $s \mapsto E_{\Omega_s,g}(y(\Omega_s,g))$ is continuous.

Since the sequence $(\|y_{s_n}\|_\Omega) = (\|\tilde{y_{s_n}}\|_D)$, where $\tilde{\ }$ denote the extension to D with 0, there exists a y_* in $\mathrm{H}^1_0(D)$ such that $\tilde{y_{s_n}} \rightharpoonup y_*$ in $\mathrm{H}^1_0(D)$. Since the sequence (Ω_{s_n}) converges towards Ω_0 for the Hausdorff complementary topology, y_* has support in $\bar{\Omega}_0$ and may be written $\tilde{y_\sharp}$ with $y_\sharp \in \mathrm{H}^1_0(\Omega_0)$, due to $\partial\Omega_0$ has non-zero capacity. Using arguments similar to the ones which established (72), we have

$$\tilde{y_{s_n}} \circ T_{s_n}^{-1} = \widetilde{y_{s_n} \circ T_{s_n}^{-1}} \rightharpoonup \tilde{y_\sharp} \text{ in } \mathrm{H}^1_0(D)$$

Accordingly, $y^{s_n} \rightharpoonup y_\sharp$ in $\mathrm{H}^1_0(\Omega_0)$ and $y_\sharp = y_0$. This eventually proves that

$$\tilde{y_{s_n}} \rightharpoonup \tilde{y_0} \text{ in } \mathrm{H}^1_0(D)$$

But due to the continuity of $s \mapsto E_{\Omega_s,g}(y(\Omega_s, g))$ and equation (73),

$$\|\tilde{y_{s_n}}\|_D = \|y_{s_n}\|_{\Omega_{s_n}} \to \|\tilde{y_0}\|_D = \|y_0\|_{\Omega_0}$$

and this is sufficient for

$$\tilde{y_{s_n}} \to \tilde{y_0} \text{ (strongly) in } \mathrm{H}^1_0(D)$$

and

$$y^{s_n} \to y^0 \text{ (strongly) in } \mathrm{H}^1_0(\Omega_0)$$

In this proof, the assumption $k \geqslant 2$ is needed to prove the convergence (72). In the case where g_Ω lays in $\mathrm{L}^2(\Omega)$ for any $\Omega \in \mathcal{O}_{\text{lip}}$, this assumption is not needed anymore.

The following differentiability result is well-known when the right-handside is fixed and in $\mathrm{H}^1(D)$. It may easily be extended to a domain-dependant right-handside. The be found in [25] for instance.

Theorem 9.2 *Assume that for any Ω_0 in $\mathcal{O}_k$, g_{Ω_0} is in $\mathrm{L}^2(\Omega_0)$ and for any V in$\mathcal{V}_k$ the mapping $s \mapsto g_{\Omega_s(V)} \circ T_s$ is strongly differentiable at $s = 0$ in $\mathrm{H}^{-1}(\Omega_0)$ with derivative $\dot{g}_{\Omega_0;V}$.*

Then the solution $y(\Omega)$ of problem $\mathcal{P}(\Omega, f)$ has a material derivative $\dot{y}(\Omega; V)$ in $\mathrm{H}^1_0(\Omega)$ for any speed-field $V \in \mathcal{V}_k$. Moreover, for any $\phi \in \mathrm{H}^1_0(\Omega)$,

$$\int_\Omega \langle \nabla \dot{y}(\Omega; V)\ ,\ \nabla\phi\rangle =$$

$$\int_\Omega -\left\langle [\frac{1}{2}\text{div } V(0) Id - \varepsilon(V(0))].\nabla y(\Omega)\ ,\ \nabla\phi \right\rangle_{\mathbb{R}^N} + [g_\Omega \text{div } (V(0)) + \dot{g}_{\Omega_0;V}]\phi$$

9.2 Shape Gradient for Cost Functionals

We want to control the shape functional:

$$J(\Omega) = \int_\Omega (y(\Omega) - Y)^2$$

where $Y \in \mathrm{H}^1_0(D)$ is given.

9.2.1 Shape Gradient

Proposition 9.1 *For any domain Ω in $\mathcal{O}_{\mathrm{lip}}$, and any field V in $\mathcal{V}_k$ $(k \le 1)$, the functional J has an eulerian derivative $dJ(\Omega;V)$*

$$\int_\Omega (y-Y)[\mathrm{div}\ V(0)(y-Y) - \mathsf{D}Y.V(0)] + 2\,\langle A'\nabla y\ ,\ \nabla p\rangle - 2\,\langle \mathrm{div}\ (fV)\ ,\ p\rangle$$

where p is the solution of the (adjoint) problem $\mathcal{P}(\Omega, y(\tilde{\Omega}) - Y)$.

If T_s is the flow-mapping of V, a change of variable yields

$$J(\Omega_s) = \int_{\Omega_s} (y_s - Y)^2 = \int_\Omega \gamma(s)(y^s - Y\circ T_s)^2$$

Since $s \mapsto T_s$ is of class $\mathcal{C}^1$, we get

$$\partial_s J(\Omega_s) = \int_\Omega \gamma'(s)(y^s - Y\circ T_s)^2 - 2\gamma(s)(y^s - Y\circ T_s)(\partial_s y^s - \mathsf{D}y\circ T_s\partial_s T_s)$$

Accordingly,

$$dJ(\Omega;V) = \int_\Omega \mathrm{div}\ V(0)(y-Y)^2 - 2(y-Y)(\dot{y} - \mathsf{D}Y.V(0))$$

We consider the adjoint state

$$\mathcal{P}(\Omega, y(\tilde{\Omega}) - Y)\quad \begin{cases} -\Delta p &= y - Y \quad \text{on } \Omega \\ p &= 0 \quad \text{on } \Gamma = \partial\Omega \end{cases} \tag{74}$$

Since (theorem 9.2) the strong material derivative of $y(\Omega)$ satisfies

$$\forall \phi \in \mathrm{H}^1_0(\Omega)\ ,\quad \int_\Omega \langle \nabla\dot{y}\ ,\ \nabla\phi\rangle = -\int_\Omega \langle A'(V).\nabla y\ ,\ \nabla\phi\rangle\ + \langle \mathrm{div}\ (fV(0))\ ,\ \phi\rangle$$

with

$$A'(V(0)) = \mathrm{div}\ V(0)\, Id - 2\varepsilon(V(0))$$

we have

$$\begin{aligned} dJ(\Omega;V) &= \int_\Omega \mathrm{div}\ V(0)(y-Y)^2 + 2(\Delta p)(\dot{y} - \mathsf{D}Y.V(0)) \\ &= \int_\Omega \mathrm{div}\ V(0)(y-Y)^2 - \mathsf{D}Y.V(0)) - 2\nabla p\nabla\dot{y} \\ &= \int_\Omega (y-Y)[\mathrm{div}\ V(0)(y-Y) - \mathsf{D}Y.V(0)] \\ &\quad +2\,\langle A'(V(0))\nabla y\ ,\ \nabla p\rangle - 2\,\langle \mathrm{div}\ (fV)\ ,\ p\rangle \end{aligned}$$

When the domain Ω is fixed, the mapping $V \mapsto dJ(\Omega; V)$ is linear and continuous. We consider the element $\nabla J(\Omega)$ of $\mathcal{A}'_k$ given for all $V \in \mathcal{A}_k$ by

$$\begin{aligned} &\langle \nabla J(\Omega) \ , \ V \rangle_{\mathcal{A}'_k \times \mathcal{A}_k} = \\ &\int_\Omega (y(\Omega) - Y)[\text{div } V(y(\Omega) - Y) - \mathsf{D}Y.V] + 2 \langle A'(V) \nabla y(\Omega) \ , \ \nabla p \rangle \\ &-2 \int_\Omega \langle \text{div } (fV) \ , \ p(\Omega) \rangle \end{aligned} \tag{75}$$

9.2.2 Uniform boundedness

Proposition 9.2 *There exists a constant $M > 0$ such that*

$$\forall \Omega \in \mathcal{O}_{\text{lip}}, \quad \|\nabla J(\Omega)\|_{\mathcal{A}'_k} \leq M$$

Proof.

$$\begin{aligned} \|\nabla J(\Omega)\|_{\mathcal{A}'_k} &= \sup_{V \in \mathcal{A}_k} \langle \nabla J(\Omega) \ , \ V \rangle_{\mathcal{A}'_k \times \mathcal{A}_k} \\ &\leq \|\text{div } V\|_{\mathrm{L}^\infty} \|y - Y\|^2_{\mathrm{L}^2} + \|\mathsf{D}Y(y - Y)\|_{\mathrm{L}^1} \|V\|_{\mathrm{L}^\infty} \\ &+2 \|A'(V)\|_{\mathrm{L}^\infty} \|\langle \nabla y \ , \ \nabla p \rangle\|_{\mathrm{L}^1} \|f |\nabla p|\|_{\mathrm{L}^1} \|V\|_{\mathrm{L}^\infty} \end{aligned}$$

so there exists a constant $m > 0$ such that

$$\|\nabla J(\Omega)\|_{\mathcal{A}'_k} \leq m \|y\|_\Omega \|p\|_\Omega \|V\|_{\mathcal{A}_k}$$

and the uniform boundedness of $\|y\|_\Omega$ and $\|p\|_\Omega$ yields the result.

9.2.3 Shape Continuity of the Gradient

Theorem 9.3 *The gradient distribution ∇J is continuous in $\mathcal{A}'_k$.*

Proof.

The gradient $\nabla J(\Omega)$ is a continuous function $G(\xi_\Omega, \tilde{y}(\Omega), \tilde{p}(\Omega))$ which is continuous from $\mathrm{L}^2(D) \times \mathrm{H}^1_0(D) \times \mathrm{H}^1_0$ to $\mathcal{A}'_k$ with

$$\langle G(\xi, y, p) \ , \ V \rangle_{\mathcal{A}'_k \times \mathcal{A}_k} =$$

$$\int_D \xi(y - Y)[\text{div } V(y - Y) - \mathsf{D}Y.V] + 2\xi \langle A'(V) \nabla y \ , \ \nabla p \rangle - 2\xi \langle \text{div } (fV) \ , \ p \rangle$$

Theorem 9.1 provides that the mapping $s \mapsto \tilde{y}(\Omega_s(V))$, $s \mapsto \tilde{p}(\Omega_s(V))$ are continuous for any V in $\mathcal{V}_k$. Consequently, so is $s \mapsto \nabla J(\Omega_s(V))$ is continuous.

9.3 Shape Control - Asymptotic Result for 2D case

We can apply theorem 8.3 for our problem.

Proposition 9.3 *For any Ω_0 in $\mathcal{O}_{k+1}$ there exists a V in $\mathcal{V}_{k+1}$ and an open subset Ω_* od $D \subset \mathbb{R}^2$ such that*

i) $J(\Omega_s(V)) - J(\Omega_0) = \int_0^s \|V(t)\|^2 \, dt$

ii) for any sequence $(s_n)_{n \geqslant 0}$ with $s_n \to \infty$, $\Omega_{s_n}(V) \to \Omega_*$ for Hausdorff complementary topology, $J(\Omega_{s_n}(V)) \to J(\Omega_*)$ and $\nabla J(\Omega_{s_n}(V)) \to 0$.

The general asymptotic behaviour of section 8.3 may be developed in the $N = 2$ case. Indeed, the continuity of $\Omega \mapsto \tilde{y}(\Omega)$ for Hausdorff-complementary topology does not holds in general. Nevertheless, this continuity holds under capacity constraints([15], [12]). In the 2-dimensional case, Sverak proved in [11] ,[10] the convergence of $\tilde{y}(\Omega_n)$ towards $\tilde{y}(\Omega)$, provided $(\complement\Omega_n)_n$ converges to $\complement\Omega$ for Hausdorff topology, with $\sharp\complement\Omega_n$ uniformly bounded.

10 Level Set Formulation for the Shape Differential Equation

In ([6],([25]), ([24]), we considered the domains family parametrized as follows:

$$\Omega_t = \Omega_t(\Phi) = \{x \in D \mid \Phi(t,x) > 0 \}, \; \Gamma_t = \{x \in D \mid \Phi(t,x) = 0 \} \tag{76}$$

Where $\Phi(t,.)$ is a function defined on D verifying a negative condition at the boundary of D, say $\Phi = -1$ on the boundary ∂D. The singular points are those in D at which the gradient of $\Phi(t,.)$ vanishes. We assume that no such points lies onΓ_t so that in a neighbourhood of the lateral boundary Σ of the tube we have the speed vector field defined by

$$V(t,x) = -\frac{\partial}{\partial t}\Phi(t,x) \quad \frac{\nabla\Phi(t,x)}{||\nabla\Phi(t,x)||^2} \tag{77}$$

The result is that if V is smooth enough we get

$$\Omega_t = T_t(V)(\Omega_0)$$

The shape differential equation turns then to be

$$A^{-1}.(\nabla J(\Omega_t(V)) - \frac{\partial}{\partial t}\Phi(t,x) \quad \frac{\nabla\Phi(t,x)}{||\nabla\Phi(t,x)||^2} = 0 \tag{78}$$

Which would implies, with the notation $\Omega_t(\Phi) = \Omega_t(V)$ (V previously defined), the following Hamilton Jacobi equation for the function Φ:

$$\frac{\partial}{\partial t}\Phi(t,x) - < \nabla\Phi(t,x), \quad A^{-1}.(\nabla J(\Omega_t(\Phi)) \; > = 0 \tag{79}$$

Notice that the Hamilton-Jacobi versions (78) and (**??**)are not equivalents. They are booth vector equations and would be merely equivalent if $A^{-1}.(\nabla J(\Omega_t(\Phi))$ was proportional to $\nabla\Phi(t,x)$. In order to bypass that point we consider the "scalar shape differential equation". From the general strucure theorem for shape gradient :

$$G = \nabla J(\Omega) = \gamma_\Gamma^*.(g\ \vec{n}) = \mathcal{G}(\Omega)\, \nabla\chi_\Omega$$

Where n is the normal field to the boundary Γ while g is a scalar distribution on the boundary. In almost classical regular problems g turns to be a function defined on the boundary with $g \in L^2(\Gamma)$. We chose " normal" scalar shape differential equation :

$$v = < V(t,.), n_t > = -\, A_{\Gamma_t}^{-1}.g(\Gamma_t) \ \text{ on } \ \Gamma_t \tag{80}$$

which in term of level set modeling , with $n_t = \frac{\nabla\Phi(t.)}{||\nabla\Phi(t,.)||}$ on Γ_t leads to the following "normal-level" shape differential equation :

$$-\frac{\partial\Phi}{\partial t} \,/\, ||\nabla\Phi|| = -\, A_{\Gamma_t}^{-1}.g(\Gamma_t) \tag{81}$$

Which *implies* the following "normal Hamilton-Jacobi" level set equation in the whole domain D :

$$-\frac{\partial\Phi}{\partial t} \ + \ A_{\Gamma_t}^{-1}.\mathcal{G}\ ||\nabla\Phi|| \ = \ 0 \tag{82}$$

Assuming that Φ would be a solution to that Hamilton Jacobi equation, in order to derive a solution to the previous normal scalar shape differential equation we need to divide by $||\nabla\Phi||$. For that reason we do now an emphasis on a class of functions Φ without step so that $||\nabla\Phi||$ is different of zero almost every where in D :

10.1 Solutions without step

we are intersted in function $\Phi(t,.)$ without steps. We say that a function f defined on a set D as a step t if

$$meas(\{x \in D \ \mid f(x) = t\ \}) > 0$$

A construction of function without step derived from a technic we introduced in modeling of free boundary value problem which arosed in plasmas physic (in the so-called Harold Grad Adiabatic equation of plasmas at equilibrium in the Tokomak ([5]),([6])), Consider, for any $\epsilon > 0$, $g \in H^{-1}(D)$, the variational problem

$$z = argmin\{ \int_D \ (1/2\ ||\nabla u||^2 + (\epsilon|D| - g)\, u\,)dx - \epsilon/2\ \int_D \int_D (u(x) - u(y))^+ \ \ dxdy \ \ \mid u \in H_0^1(D)\ \}$$

from which we get solutions to the problem

$$-\Delta z = \epsilon\beta(z) + \ g \ \text{ in } \ D, \ \ \beta(z)(x) = meas(\{y \in D \ \mid \ z(y) < z(x)\ \}) \tag{83}$$

verifying the extra “no step” condition :

$$\forall t \in R, \quad meas(\{x \in D \ \mid z(x) = t\ \}) = 0 \tag{84}$$

That technic can be limited to “zero step” functions : We can consider the simplier problem

$$z = argmin\{\int_D \ (1/2\ ||\nabla u||^2 - \ gu\ )dx\ - \epsilon \int_D (u(x))^+ \ \ dx\ , \ \ u \in H_0^1(D)\ \}$$

from which we get solutions to the problem

$$-\Delta z = \ \epsilon\ \beta_0(z) + \ g \ \text{ in } \ D, \quad \beta_0(z)(x) = \chi_{\{x \in D \ \mid \ z(x) > 0\ \}} \tag{85}$$

verifying the extra “no zero step” condition :

$$meas(\{x \in D \ \mid z(x) = 0\ \}) = 0 \tag{86}$$

10.2 Iterative Scheme

Obvioulsy the function χ must verifies $\chi^2 = \chi$ while the “level set” function Phi need not. Then, immediately from the previous study, we understand that in the second Hamilton Jacobi equation the speed vector $V = \ A^{-1}.(\nabla J(\Omega_t(\Phi))$ needs only to be in $L^2(0, \tau, L^2(D)^3)$ with its divergence too. In order to perform a fixed point in an iterative approximation scheme in the following form :

$$\frac{\partial}{\partial t}\Phi^n(t,x) - < \nabla\Phi^n(t,x), \quad A^{-1}.(\nabla J(\Omega_t(\Phi^{n-1})) \ > = \ 0$$

With

$$V^{n-1} = \ A^{-1}.(\nabla J(\Omega_t(\Phi^{n-1}))$$

We need only $V^n \in L^2(0, \tau, L^2(D)^3)$ and $div\, V^n \in L^2(0, \tau, L^\infty(D))$. The idea is to chose the (non necesseraly linear) operator A “powerfull enough” so that A^{-1} would map compactely and continuously the gradient space in that speed Sobolev space. We already have recall that in almost all “smooth problems” (following general assumptions of the structure derivative theorem) that gradient takes the following form:

$$G = \nabla J(\Omega_t(\Phi)) = \mathcal{G}(\chi_t(\Phi))\nabla\chi_t(\Phi)$$

So that G is a distribution(supported by the boundary Γ and with zero transverse order) in some negative Sobolev space over D while $\mathcal{G} \in W^{1,1}(D)$ (is non uniquely determined, only its trace g on the boundary Γ, if smooth enough, is intrinsiquely dermined, g is called the density gradient, it is a scalar distribution on the “manifold” Γ, when Γ is smooth in order to make sense).

We consider

$$(A_\epsilon)^{-1}(f) = \{\ z \ \text{ solutions to } \ (83), 84)\ \ \}$$
$$(A_\epsilon^0)^{-1}(f) = \{\ z \ \text{ solutions to } \ (85), 86)\ \ \}$$

Lemma 10.1 *Let $f_n \longrightarrow f$ in $H^{-1}(D)$ and $z_n \in (A_\epsilon)^{-1}(f_n)$ (resp. $z_n \in (A^0_\epsilon)^{-1}(f_n)$). Then there exists a subsequence and a limiting element z such that:*

$$z_{n_k} \longrightarrow z \text{ weakly in } H^1_0(D)$$

Moreover any such element z verifies $z \in (A_\epsilon)^{-1}(f)$ (resp. $z \in (A^0_\epsilon)^{-1}(f)$.

The most important result concerning such elements without zero step is in the following

Lemma 10.2 *Let $\Phi_n,\ \Phi \in L^2(D)$. Assume that $\Phi_n \longrightarrow \Phi$ strongly in $L^2(D)$ and Φ is without zero step :*

$$meas(\{x \in D \ \mid \ \Phi(x) = 0\ \}) = 0$$

Then let $\Omega_n = \{x \in D \mid \Phi_n(x) > 0\ \}$, $\Omega = \{x \in D \mid \Phi(x) > 0\ \}$. We denote by χ_n and χ the respective characteristic functions of those subsets. Then $\chi_n \longrightarrow \chi$ stongly in $L^2(D)$.

The proof is immediate: obviously we have

$$\chi_n\, \Phi_n \ = \ (\Phi_n)^+$$

there exists a subsequence and an element ζ, $0 \ \leq \zeta \ \leq 1$ such that χ_n weakly converges to ζ in $L^2(D)$. In the limit, as $|(\Phi_n)^+(x) - (\Phi)^+(x)| \leq |\Phi_n(x) - \Phi(x)|$ we get :

$$\zeta\ \Phi \ = \ (\Phi)^+$$

So that $\zeta \ = \ 1$ a.e in Ω, $= 0$ a.e. in $D - \Omega$, as Φ has no zero step we conclude that $\zeta \ = \ \chi$, so that the sequence converges strongly in $L^2(D)$.

At that point, it is obvious that with some continuity assumption on the gradient $G(\chi)$ (those hypothesis will be fullfield in the following example) , with respect to the characteristic function, the previous iterative construction V^n will converges and we will derive the existence of solutions to the Hamilton -Jacobi equation for the Level set function associated to the multivalued operators A_ϵ and A^0_ϵ.

10.2.1 A Transverse Magnetic like Inverse Problem

As an illustration of the construction of solution to the Hamilton Jacobi Equation we consider a simplified version of the celebrate transverse magnetic inverse problem. It is simplified mainely in the fact that we consider a bounded universe D. Let $y(\chi) = y \in H^1_0(D)$ be the solution to the problem :

$$-\Delta y \ + \ k\, \chi\, y \ = f$$

Where k is the contrast parameter while f is given in $L^2(D)$. We introduce the observability functional :

$$J(\chi) \ = \ 1/2 \ \int_E \ (y - Y_d)^2\ dx$$

Then the shape gradient is given by

$$G(\Omega) = \gamma_\Gamma^*(yp\ \vec{n})$$

so that the density gradient is $g(\Gamma) = (yp)|_\Gamma$ (restriction, or trace on Γ of the element yp). We consider the non unique extension $\mathcal{G}$ of g to D :

$$\mathcal{G}(\chi) = \ y\ p\ \in W^{2,1}(D)$$

Where p is the solution to the adjoint equation :

$$p \in H_0^1(D), \quad -\Delta p +\ k\ \chi p = \chi_E\ (y - Y_d\ )$$

We verify very easily the continuity of the mapping

$$\chi \in L^2(D) \longrightarrow \mathcal{G}(\chi) \in W^{2,1}(D)$$

At that point it is very interesting on that example to understand that the shape derivative of that functional $J(\Omega)$ can be relaxed to a *Set Derivative Setting* which coincide in smooth situation with the classical shape derivative and shape gradient analysis . We introduce the usual Lagrangian

$$\mathcal{L}(\Omega,\phi,\psi) = \int_D (\nabla\phi.\nabla\psi + \ k\,\chi_\Omega\ \phi\ \psi\ -\ f\,\psi\ + \chi_E\ 1/2\,(\phi\ -\ Y_d\,)^2\ )\,dx \qquad (87)$$

Then

$$J(\Omega) = MIN_{\phi\in H^2(D)}\ MAX_{\psi\in H^2(D)}\mathcal{L}(\Omega,\phi,\psi)$$

with unique saddle point (y,p). The Lagrangian $\mathcal{L}$ beeing partially concave-convex, weakly lower semicontinous and upper weakly semicontinuous , considering any evolutive characteristic function $\chi(t)$ we get (see([9]) :

$$\frac{\partial}{\partial t}J(\chi)|_{\{t=0\}}\ = \frac{\partial}{\partial t}\mathcal{L}(\chi(t),y_0,p_0))|_{\{t=0\}}$$

Where y_0, p_0 are the solution at $t = 0$ That derivative makes sens as soon as the right hand side derivative dœs.

Assume for example that Ω_t is a vanishing sequence, as $t\ \rightarrow\ 0$ of measurable subsets in $D \subset R^N$ verifying:

$$\Omega_n\ \rightarrow\ \{x_0\}\ \text{ in Hausdorf topology}$$

$$(\,meas(\Omega_t)\,/\,t^N)\ \rightarrow\ 0\ \text{ as }\ t \rightarrow\ 0$$

Then, assuming $N = 2$ so that $y,p \in H^2(D) \subset C^0(\bar{D})$, we get :

$$\frac{\partial}{\partial t}J(\chi(t))_{\{t=0\}}\ =\ k\ (yp)(x_0)$$

That set derivative works through the simple fact that $\frac{\chi(t)}{t} \rightarrow \delta_{x_0}$ (Dirac measure at the point x_0), as $t \rightarrow 0$. Then we understand that the important concept in that situation is the derivative, in $\mathcal{M}$easure space of the mapping $t \rightarrow \chi(t)$ at $t = 0$. In that direction we shall consider, when it exists, the following element :

$$\dot{\chi}(t) = lim_{\{\epsilon \rightarrow 0\}}(\chi(t+\epsilon) - \chi(t))/\,\epsilon\,, \quad \text{in } \mathcal{M}(D) \text{ topology}$$

If the saddle point (y_t, p_t) verifies the following continuity :

$$t \longrightarrow (y_t, p_t) \text{ continuous in } H(D) \subset C^0(\bar{D})$$

We get the notion of *Set Derivative For the Set Functional* $J(\Omega)$:

$$\frac{\partial}{\partial t} J(\chi(t)) \;= <\, \dot{\chi}(t),\; \mathcal{G}(\chi(t)) \;> \tag{88}$$

In that situation we had $\mathcal{G}(\chi) = yp$.

Such an example is considered in ([22]) for a "transverse magnetic "inverse shape problem" in burried obstacle reconstruction.

10.3 Shape functional with Dirichlet boundary conditions are not "hole-differentiable"

Let $\Omega = \{x \in R^2 \mid ||x|| < 1\}$ and consider the "perforated" subdomain $\Omega_\epsilon = \{x \in \Omega \;\; s.t. \; ||x|| > \epsilon \;\}$ and the solution y_ϵ to the problem :

$$\Delta y_\epsilon = 0 \; in \; \Omega_\epsilon, \;\; y_\epsilon(x) = 0, \; ||x|| = \epsilon, \quad y_\epsilon(x) = 1, \; ||x|| = 1$$

Obvioulsly we get

$$\epsilon > 0, r = ||x||, \quad y_\epsilon(x) = -\frac{lnr}{ln\epsilon} + 1, \quad y_0(x) = 1$$

Consider the functional

$$j(\epsilon) = \int_{||x||=1} (\frac{\partial}{\partial n} y_\epsilon)^2 \; ds$$

We get

$$j(\epsilon) - j(0) = 2\pi \frac{1}{(ln\epsilon)^2} \rightarrow 0, \;\; \epsilon \rightarrow 0$$

And we see that for any positive number $r > 0$ we get

$$lim \; \frac{j(\epsilon) - j(0)}{\epsilon^r} = +\infty$$

Now the derivative when ϵ goes to zero has no finite limit :

$$j'(\epsilon) = -\frac{4\pi}{\epsilon\,(ln\epsilon)^3} \rightarrow +\infty, \;\; \epsilon \rightarrow 0$$

s

11 Proof of the proposition 2.1

Let us consider $V \in L^2(0,\tau,\ H^1_0(D))$ and a dense family $e_1, ...e_m, ...$ in $H^1_0(D)$ with each $e_i \in C^\infty_{comp}(D, R^3)$. Consider the approximated solution

$$u^m(t,x) = \Sigma_{i=1,\ldots,m} \quad u^m_i(t)\, e_i(x)$$

with $U^m = (u^m_1, ..., u^m_m)$ solution of the following linear ordinary differential system:

$$\forall t,\ \int_D (\frac{\partial}{\partial t}u^m(t) + < V(t), \nabla u^m(t) >)\, e_j(x)\, dx = \int_D f(t,x) e_j(x)\ dx,\ \ j = 1, ..., m$$

That is

$$\frac{\partial}{\partial t}U^m(t) + M^{-1}.A(t).U^m(t)\ =\ F(t) \tag{89}$$

where

$$M_{i,j} = \int_\Omega\ e_i(x)\, e_j(x)\, dx$$

$$A_{i,j}(t) = \int_D\ < V(t), \nabla e_i(x) >\ e_j(x)\, dx$$

That is an ordinary linear differential systems possessing a global solution when $V \in L^p(0,\tau,\ L^2(D, R^N))$ for some $p,\ p > 1$. By classical energy estimate

$$\int_D < V(t), \nabla u^m(t) > u^m(t)\, dx = -\frac{1}{2}\int_D\ < u^m(t), u^m(t) >\ div V(t)\, dx \tag{90}$$

we get :

$$\forall \tau,\ \tau \le T,\ \ ||u^m(\tau)||^2_{L^2(D)}\ \le\ ||u^m(0)||^2_{L^2(\Omega)}$$

$$+\int_0^\tau \int_D < u^m(t,x), u^m(t,x) > (div V(t,x))^+\ dt dx$$

$$+\,2\int_0^\tau \int_D f(t,x) u(t,x) dt dx$$

Setting

$$\psi(t) = ||(div V(t,.))^+||_{L^\infty(D,R^3)}$$

When $f = 0$,

$$\frac{1}{2}\int_D u^m(t,x)^2 dx \le \frac{1}{2}\int_D u^m(0,x)^2 dx$$

$$+\frac{1}{2}\int_0^t \psi(s) \int_D\ ||u^m(t,x)||^2\, dx$$

by use of the Gronwall's lemma we get :

$$\int_D u^m(t,x)^2 dx \le \int_D u^m(0,x)^2 dx\ (1\ +\ \int_0^t \psi(s) exp\{ \int_s^t \psi(\sigma) d\sigma\ \}\, ds\ \ )$$

By the choice of the initial conditions in the ordinary differential system we get

$$M > 0, s.t. \forall \tau, \leq T, \;\; ||u^m(\tau)||_{L^2(D)}$$

$$\leq \; M \; ||\phi||_{L^2(D)} \; (\, 1 + \int_0^t \psi(s) exp\{ \int_s^t \psi(\sigma) d\sigma \; \} \, ds \;\;)$$

When $\psi = 0$, we get

$$\forall \tau, \; \tau \leq T, \;\; ||u^m(\tau)||^2_{L^2(D)} \; \leq \; ||u^m(0)||^2_{L^2(\Omega)}$$

$$+ 2 \int_0^\tau \int_D f(t,x) u(t,x) dt dx$$

In the general case, we use

$$||u^m|| \; \leq \; 1 + \; ||u^m||^2$$

and we derive the following estimate :

$$\forall \tau, \; \tau \leq T, \;\; ||u^m(\tau)||^2_{L^2(D)} \; \leq \; ||u^m(0)||^2_{L^2(D,R^3)}$$

$$+ \int_0^\tau \int_D < u^m(t,x), u^m(t,x) > (divV(t,x))^+ \; dt dx$$

$$+ 2 \int_0^\tau \int_D f(t,x) u(t,x) dt dx$$

$$\leq \;\; ||u^m(0)||^2_{L^2(D,R^3)} \; + \; \int_0^\tau \; ||f(t)||_{L^2(D,R^3)} \, dt$$

$$+ \int_0^\tau x < u^m(t,x), u^m(t,x) > (\psi(t) + ||f(t)||_{L^2(D,R^3)} \,) \, dt dx$$

$$\leq \; M \, (||u_0||^2_{L^2(D,R^3)} + ||f||_{L^1(0,\tau,L^2(D,R^3)} \;)$$

$$+ \int_0^t \;\; (\psi(t) + ||f(t)||_{L^2(D,R^3)}) ||u^m(s)||^2_{L^2(D)} ds$$

From Gronwall's inequality we derive:

$$||u^m(\tau)||^2_{L^2(D)} \; \leq M (||u_0||^2_{L^2(D,R^3)} + ||f||_{L^1(0,\tau,L^2(D,R^3)} \;)$$

$$\{ \; 1 + \int_0^t \; [\;\; (\psi(s) + ||f(s)||_{L^2(D,R^3)}) \int_s^t (\psi(\sigma) + ||f(\sigma)||_{L^2(D,R^3)}) d\sigma \;] ds \;\; \}$$

In all cases n u^m remains bounded in $L^\infty(0,\tau, L^2(D,R^N))$ and there exists an element u in that space and a subsequence still denoted u^m which weakly-* converges to u. In the limit u itself verifies the previous estimate . It can be verified that u solves the problem in distribution sense . That is

$$\forall\phi \in H_0^1(0,\tau,L^2(D,R^3)) \cap L^2(0,\tau,H_0^1(D,R^3)),\ \ \phi(\tau)=0,$$

$$-\int_0^\tau\int_D u(\frac{\partial}{\partial t}\phi + div(\ \phi V)\ )dxdt = \int_D \phi(0)\,u_0\,dx\ + \int_0^\tau\int_D\ < f,\phi > dxdt$$

When $V(t) \in L^2(D,R^3)$ the duality brackets $< \frac{\partial}{\partial t}u,\phi >$ are defined. If $\nabla\phi$ belongs to $L^\infty(D,R^3)$, this is verified, for example, when $\phi \in H_0^3(D)$ so that u_t is identified to an element of the dual space $H^{-3}(D)$. When $V(t) \in H^1(D,R^3)$ we get $a.e.t,\ u(t,.) \in L^2(D),\ V(t,.) \in L^6(D),$ then $\nabla\phi(t)$ should be in $L^3(D,R^3)$,that is $\phi(t) \in W_0^{1,3}(D)$ and then, for $a.e.t$, the element u_t is in the dual space $W^{-1,\frac{3}{2}}(D)$ while $u_t \in L^{p^*}(0,\tau,\ W^{-1,\frac{3}{2}}(D))$ and $u \in W^{1,p^*}(0,\tau,\ W^{-1,\frac{3}{2}}(D))$. Then we have $u \in L^\infty(0,\tau,L^2(D))\cap$

$$\cap W^{1,p^*}(0,\tau\,W^{-1,\frac{3}{2}}(D)) \subset L^2(0,\tau,W^{0,\frac{3}{2}}(D)) \cap W^{1,p^*}(0,\tau,\ W^{-1,\frac{3}{2}}(D))$$

$$\subset C^0([0,\tau],W^{-\frac{1}{2},\frac{3}{2}}(D))$$

Notice that the energy estimate (90) can also be written in the following form

$$\int_D div(\,V(t)\,u^m(t))\,u^m(t)\,dx = +\frac{1}{2}\int_D\ < u^m(t),u^m(t) >\ divV(t)\,dx \tag{91}$$

so that when $(divV)^+$ is turned in $(divV)^-$ the previous existence results applies for the following evolution problem

$$u(0)=u_0,\ \ \frac{\partial}{\partial t}u(t) + div(\,u(t)\,V(t)\,)\ =\ g \tag{92}$$

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