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Abstract: It is well-known that real numbers with a purely periodic decimal expansion are the rationals having a denominator coprime with 10. We are interested in beta-expansions with a non-unit Pisot basis. We give a characterization of real numbers having a purely periodic expansion in such a basis. The characterization is given in terms of an explicit self-similar compact subset of non-zero measure in a product of a Euclidean space and p-adic spaces.

Key-words: expansion in a non-integral basis, Pisot number, beta-shift, periodicity, iterated morphism, generalized Rauzy fractals

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Beta-développements purement périodiques dans le cas Pisot non unitaire

Résumé : On sait depuis longtemps que les réels dont le développement décimal est purement périodique sont les rationnels dont le dénominateur est premier avec 10. On s'intéresse aux beta-développements ayant pour base un nombre de Pisot non unitaire. On caractérise les réels pour lesquels un tel développement est purement périodique. La caractérisation fait intervenir un ensemble auto-similaire compact de mesure non nulle dans un produit d'espaces Euclidien et p-adiques.

Mots-clés : développement en base non entière, nombre de Pisot, périodicité, morphisme itéré, corps p-adique, fractal de Rauzy généralisé

Let β be a Pisot number and $T_\beta : x \mapsto \beta x \pmod 1$ be the associated β-transformation. The aim of this note is to characterize the real numbers x in $\mathbb{Q}(\beta) \cap [0,1]$ having a purely periodic β-expansion in terms of an explicit self-similar compact subset of non-zero measure in the product of a Euclidean space and p-adic spaces inspired by [AI01, Sie02]. The proof is a generalization of the one in [RI02b] of a similar characterization of purely periodic expansions in the unit case.

We first introduce a representation of the two-sided β-shift (X_β, S) in terms of a formal power series with coefficients into the ring $\mathbb{Q}[[\beta]]$ of formal power series with coefficients in $\mathbb{Q}$; we then obtain a representation map by gathering the set of finite values which can be taken for any topology (Archimedean or not) by the formal power series, that is, by taking the completion of $\mathbb{Q}(\beta)$ with respect to all the absolute values on $\mathbb{Q}(\beta)$ which take a value different from 1 on β. This representation is inspired by the geometric representation as generalized Rauzy fractals of substitutive symbolic dynamical systems in the non-unimodular case (see [Sie02]).

1 Representation of β-shifts

Let β be a Pisot number. Let P_β be its minimal polynomial (it is of degree d). Let $(\beta_2, \ldots, \beta_d)$ denote its conjugates with $\beta_1 = \beta$. For $i = 1$ to d, let τ_i be a $\mathbb{Q}$-automorphism of $\mathbb{K} = \mathbb{Q}(\beta_1, \ldots, \beta_d)$ which sends β on its algebraic conjugate β_i.

1.1 β-numeration

The Renyi β-expansion of a real number $x \in [0,1]$ is defined as the sequence $(x_i)_{i \geq 1}$ with values in $\{0, 1, \ldots, \lfloor \beta \rfloor\}$ produced by the β-transformation $T_\beta : x \mapsto \beta x \pmod 1$ as follows

$$\forall i \geq 1,\ x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \text{ and thus } x = \sum_{i \geq 1} x_i \beta^{-i}.$$

Let $d_\beta(1) = (b_i)_{i \geq 1}$ denote the β-expansion of 1. Let $d^*_\beta(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and $d^*_\beta(1) = (t_1 \ldots t_{m-1}(t_m - 1))^\infty$, if $d_\beta(1) = (t_1 \ldots t_{m-1} t_m)$ is finite (with $t_m \neq 0$). Let (X_β, S) denote the two-sided symbolic dynamical system associated with β. The set X_β is the set of two-sided sequences such that each left truncated sequence is less than or equal to $d^*_\beta(1)$. For more details on the β-numeration, see for instance [Fro02, Fro00].

We will use the following notation. We denote the elements of X_β as follows. If $(x_i)_{i \in \mathbb{Z}} \in X_\beta$, define $u = (u_i)_{i \geq 1}$ as $u_i = x_i$, for $i \geq 1$, and $w = (w_i)_{i \geq 0}$, as $w_i = x_{-i}$, for $i \geq 0$. One thus gets a sequence of the form

$$\ldots, w_2, w_1, w_0, u_1, u_2, \ldots,$$

and write is as $((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) = (w, u)$. In other words, we will use the letters (w_i) for denoting the "past " and (u_i) for the "future".

Furthermore, we will denote by X_β^l the set of one-sided sequences $w = (w_i)_{i\geq 0}$ such that there exists $u = (u_i)_{i\geq 1}$ with $(w, u) \in X_\beta$. This set will be called the *left one-sided shift*.

If one similarly defines X_β^r as the set of one-sided sequences $u = (u_i)_{i\geq 1}$ such that there exists $w = (w_i)_{i\geq 0}$ with $(w, u) \in X_\beta$, then X_β^r coincides with the usual one-sided shift. We call it the *right one-sided shift*.

Let us note that the right one-sided shift admits as a natural geometric representation the interval $[0, 1]$; indeed one associates with a sequence $(u_i) \in X_\beta^r$ its real value $\sum_{i\geq 1} u_i \beta^{-i}$. We even have a metric isomorphism between X_β^r endowed with the shift, and $[0, 1]$ endowed with the map T_β. The aim now is to give a geometric interpretation of the set X_β, and hence to give a geometric representation of the added part, that is, X_β^l.

1.2 Representation of the left one-sided shift X_β^l

The aim of this section is to introduce first a formal representation of X_β^l and second a geometrical representation as an explicit self-similar compact set of non-zero measure in the product of a Euclidean space and p-adic spaces following [AI01, Sie02]; the primes which appear as p-adic spaces here will be the prime factors of the norm of β.

Formal representation of the symbolic dynamical system X_β^l

Denote by $\varphi_{\mathbb{Q}(\beta)} : X_\beta \to \mathbb{Q}[[\beta]]$ the following map, where $\mathbb{Q}[[\beta]]$ is the ring of formal power series with coefficients in $\mathbb{Q}$:

$$\text{for all } (w_i)_{i\geq 0} \in X_\beta^l, \quad \varphi_{\mathbb{Q}(\beta)}(w_i) = \sum_{i\geq 0} w_i \beta^i \in \mathbb{Q}[[\beta]].$$

One could similarly define for any algebraic conjugate β_j of β, the series $\varphi_{\mathbb{Q}(\beta_j)} := \tau_j(\varphi_{\mathbb{Q}(\beta)})$ which satisfies for any $(w_i)_{i\geq 0}$

$$\varphi_{\mathbb{Q}(\beta_j)}(w_i) = \sum_{i\geq 0} w_i \beta_j^i \in \mathbb{Q}[[\beta_j]].$$

This map $\varphi_{\mathbb{Q}(\beta)}$ can hence be considered as formal in the sense that it does not depend on the choice of the eigenvalue β; we call it *formal representation* of X_β^l.

Topologies over $\mathbb{Q}(\beta)$

Let us now try to associate with these sums values, that is, to find a topological framework in which all the series $\varphi_{\mathbb{Q}(\beta_j)}(w)$, $w \in X_\beta^l$, $1 \le j \le d$, would converge; in fact, this boils down to find all the Archemedean and not Archemedean topologies on $\mathbb{Q}(\beta)$ for which $\varphi_{\mathbb{Q}(\beta)}$ takes finite values. More precisely the metrizable topologies on $\mathbb{Q}(\beta)$ for which the series $\varphi_{\mathbb{Q}(\beta)}$ takes finite values are of two types.

- Suppose that the topology (with absolute value $|\cdot|$) is Archemedean: its restriction to $\mathbb{Q}$ corresponds to the usual absolute value on $\mathbb{Q}$ and there exists a $\mathbb{Q}$-automorphism τ_i such that $|x| = |\tau_i(x)|_{\mathbb{C}}$. The series $\varphi_{\mathbb{Q}(\beta)}$ converges in $\mathbb{C}$ if and only if τ_i is associated with a conjugate β_i of modulus strictly smaller than one. In other words, there exists a conjugate β_i of β, of modulus strictly smaller than one, such that $|Q(\beta)| = |Q(\beta_i)|_{\mathbb{C}}$, for any $Q \in \mathbb{Q}[X]$.
- Assume that the topology is non-Archimedean: there exists a prime ideal $\mathcal{I}$ of the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ for which the topology coincides with the the $\mathcal{I}$-adic topology; let p be the prime number defined by $\mathcal{I} \cap \mathbb{Z} = p\mathbb{Z}$; the restriction of the topology to $\mathbb{Q}$ is the p-adic topology. The series $\varphi_{\mathbb{Q}(\beta)}$ takes finite values in the completion $\mathbb{K}_{\mathcal{I}}$ of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology if and only if $\beta \in \mathcal{I}$.

Representation space $\mathbb{K}_\beta$ of X_β^l

Let $\beta_2, \ldots, \beta_r$ be the real conjugates of β (they all have modulus strictly smaller than 1, since β is Pisot), and let β_{r+1}, $\overline{\beta_{r+1}}$, $\ldots$, β_{r+s}, $\overline{\beta_{r+s}}$ be its complex conjugates. For $2 \le j \le r+s$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{R}$ or $\mathbb{C}$ depending on the imaginary part of β_j, endowed with the usual topology.

We first gather the complex representations by omitting the ones which are conjugate in the complex case. This representation contains all the possible Archemedean values for $\varphi_{\mathbb{Q}(\beta)}$. It takes values in

$$\mathbb{K}_\infty = \mathbb{K}_{\beta_2} \times \cdots \times \mathbb{K}_{\beta_{r+s}}.$$

Let $\mathcal{I}_1, \ldots, \mathcal{I}_\nu$ be the prime ideals in the integer ring of $\mathbb{Q}(\beta)$ that contain β, that is,

$$\beta\mathcal{O}_{\mathbb{Q}(\beta)} = \prod_{i=1}^{\nu} \mathcal{I}_i^{n_i}. \tag{1}$$

Recall that $\mathbb{K}_{\mathcal{I}}$ denotes the completion of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology. We then gather the representations in the completions of $\mathbb{Q}(\beta)$ for the non-Archemedean topologies.

Hence one defines the *representation space of* X_β as the direct product $\mathbb{K}_\beta$ of all these fields:

$$\mathbb{K}_\beta = \mathbb{K}_{\beta_2} \times \ldots \mathbb{K}_{\beta_{r+s}} \times \mathbb{K}_{\mathcal{I}_1} \times \ldots \mathbb{K}_{\mathcal{I}_\nu} \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{K}_{\mathcal{I}_1} \times \cdots \times \mathbb{K}_{\mathcal{I}_\nu}.$$

Let us note that for a given prime p, the fields $\mathbb{K}_{\mathcal{I}}$ that are p-adic fields are the ones for which $\mathcal{I}$ contains simultaneously p and β. Furthermore, following [Sie02], the prime numbers p for which there exists a prime ideal of $\mathcal{O}_{\mathbb{Q}(\beta)}$ which contains simultaneously p and β are exactly the prime divisors of the constant term of the minimal polynomial of β. In particular, $\mathbb{K}_\beta$ is a Euclidean space if and only if β is a unit.

Endowed with the product of the topologies of each of its elements, $\mathbb{K}_\beta$ is a metric abelian group.

The *canonical embedding* of $\mathbb{Q}(\beta)$ into $\mathbb{K}_\beta$ is defined by the following morphism:

$$\delta_\beta : P(\beta) \in \mathbb{Q}(\beta) \mapsto (\underbrace{P(\beta_2)}_{\in \mathbb{K}_{\beta_2}}, \ldots, \underbrace{P(\beta_{r+s})}_{\in \mathbb{K}_{\beta_{r+s}}}, \underbrace{P(\beta)}_{\in \mathbb{K}_{\mathcal{I}_1}}, \ldots, \underbrace{P(\beta)}_{\in \mathbb{K}_{\mathcal{I}_\nu}}) \in \mathbb{K}_\beta.$$

Definition 1 *The* representation map of X_β^l, *called* one-sided representation map, *is defined by*

$$\varphi_\beta : (w_i)_{i\geq 0} \in X_\beta^l \mapsto \lim_{n\to\infty} \sum_{i=0}^{n} w_i \beta^i \in \mathbb{K}_\beta.$$

We set $\mathcal{R}_\beta^l := \varphi_\beta(X_\beta^l)$ *and call it the* Rauzy geometric representation of the left one-sided shift X_β^l.

The set $\mathcal{R}_\beta^l$ appears to be the Rauzy fractal associated with a β-substitution. Such representations first appeared in [Rau82] for the Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. The definition was then generalized to unimodular substitutions of Pisot type in [Rau88, AI01, CS01, RI02a]. Finally, the definition and properties were extended to non-unimodular substitutions of Pisot type in [Sie02]. A similar approach has also been introduced by S. Akiyama in [Aki98, Aki99] inspired by [Thu89].

In particular, the results proved in [Sie02] hold for these sets:

- The set $\mathcal{R}_\beta^l$ has a self-similar structure.
- The set $\mathcal{R}_\beta^l$ has non-zero measure for the Haar measure over $\mathbb{K}_\beta$.
- The set $\mathcal{R}_\beta^l$ can be divided into a finite number of domains that can be moved up to translation so that the whole set $\mathcal{R}_\beta^l$ is invariant.

1.3 Examples

The golden ratio

Let $\beta = (1+\sqrt{5})/2$ be the golden ratio, that is, the largest root of $X^2 - X - 1$. One has $d_\beta(1) = 11$ and $d_\beta^*(1) = (10)^\infty$. Hence X_β is the set of sequences in $\{0,1\}^\mathbb{Z}$ in which there are no two consecutive 1's. One has $\mathbb{K}_\beta = \mathbb{R}$; the canonical embedding δ_β is reduced to the map $\tau_{(1-\sqrt{5})/2}$, and $\delta_\beta(\mathbb{Q}(\beta)) = \mathbb{Q}(1-\sqrt{5})/2)$. The set $\mathcal{R}_\beta$ is an interval.

The Tribonacci number

Let β be the Tribonacci number, that is, the Pisot root of the polynomial $X^3 - X^2 - X - 1$. One has $d_\beta(1) = 111$ and $d_\beta^*(1) = (110)^\infty$. Hence X_β is the set of sequences in $\{0,1\}^\mathbb{Z}$ in which there are no three consecutive 1's. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is reduced to the map τ_α, where α is one of the complex root of $X^3 - X^2 - X - 1$. The set $\mathcal{R}_\beta$ is a compact subset of $\mathbb{C}$ called the Rauzy fractal. It is shown in Fig. 1. The different shades correspond to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$. There are as many shades as the length of $d_\beta(1)$, and here as many as the degree of β.

The smallest Pisot number

Let β be the Pisot root of $X^3 - X - 1$. One has $d_\beta(1) = 10001$ and $d_\beta^*(1) = (10000)^\infty$. This substitution and some surprising tilings generated by it have been studied in [EI02]. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is also reduced to the map τ_α, where α is one of the complex roots of $X^3 - X - 1$. The set $\mathcal{R}_\beta$ is shown in Fig. 1.

The number of different shades corresponds to the lenght of $d_\beta(1)$, but there are 5 shades whereas the degree of β is 3.

A non-unit example

Let $\beta = 2+\sqrt{2}$ be the dominating root of the polynomial X^2-4X+2. The other root is $2-\sqrt{2}$. One has $d_\beta(1) = d_\beta^*(1) = 31^\infty$. The ideal $2\mathbb{Z}$ is ramified in $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$, that is, $2\mathbb{Z} = \mathcal{I}^2$. Hence there exists only one ideal which contains simultaneously $\sqrt{2}$ and 2; its index of ramification is 2. Its geometric one-sided representation $\mathcal{R}_\beta$ is a subset of $\mathbb{R}_2 \times \mathbb{Q}_2 \times \mathbb{Q}_2$, shown in Fig. 1. There are two different shades.

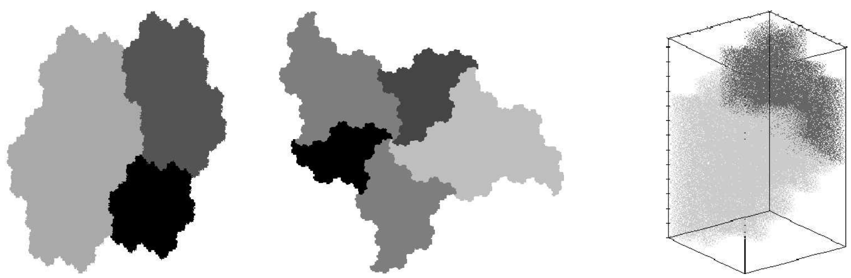

Figure 1: The Rauzy geometric one-sided representation for the Tribonacci-shift, the smallest Pisot number-shift and the $2+\sqrt{2}$-shift.

1.4 Representation of the two-sided shift (X_β, S)

Representation space $\tilde{\mathbb{K}_\beta}$ of X_β and representation map $\tilde{\varphi_\beta}$

We can now define the representation of a point of X_β. The *representation* of $(w,u) = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) \in X_\beta$ takes its values in $\mathbb{R}\times\mathbb{K}_\beta$ and is obtained by associating on the first coordinate with the β-expansion u its real value, and by gathering on the other coordinates the set of finite values which can be taken by the formal power series $\varphi_{\mathbb{Q}(\beta)}(w)$ for all the topologies that exist on $\mathbb{Q}(\beta)$. Hence one defines $\tilde{\mathbb{K}_\beta}$ as $\mathbb{R}\times\mathbb{K}_\beta$. Since the topology on $\mathbb{K}_\beta$ has been chosen so that the formal power series is convergent in $\mathbb{K}_\beta$, one shall define the following:

Definition 2 *The* representation map of X_β, *called* two-sided representation map, *is defined for all* $((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) \in X_\beta$ *by:*

$$\tilde{\varphi_\beta}((w_i),(u_i)) = (\sum_{i\geq 1} u_i\beta^{-i}, -\varphi_\beta(w_i)) = (\sum_{i\geq 1} u_i\beta^{-i}, -\delta_\beta(\sum_{i\geq 0} w_i\beta^i)) \in \mathbb{R}\times\mathbb{K}_\beta.$$

The set $\tilde{\mathcal{R}}_\beta := \tilde{\varphi_\beta}(X_\beta)$ *is called the* Rauzy geometric representation of the two-sided β-shift.

We will see later the interest of introducing the sign minus before δ_β. Let us note that following [Sie02], if β is a Pisot number, the Rauzy geometric representation of the β-shift is a compact set of non-zero measure for the Haar measure over $\mathbb{R} \times \mathbb{K}_\beta$. For more details, see [BS02].

The map $\tilde{T_\beta}$.

One can extend in a natural way the definition of T_β to the product of $\mathbb{R}$ by the representation space $\mathbb{K}_\beta$ as follows. Let h_β denote the multiplication in $\mathbb{K}_\beta$ by the diagonal matrix of diagonal $\delta_\beta(\beta)$; one has

$$\tilde{T_\beta} : (a, b) \in \mathbb{R} \times \mathbb{K}_\beta \mapsto (\beta a - [\beta a], h_\beta(b) - [\beta a]\delta_\beta(1)).$$

In particular, the following commutation relations hold, where Id denotes the identity map over $\mathbb{R}$; recall that S denotes the shift over X_β:

- $\tilde{\varphi_\beta} \circ S = \tilde{T_\beta} \circ \tilde{\varphi_\beta}$ over X_β,
- $\tilde{T_\beta} \circ (Id, \delta_\beta) = (Id, \delta_\beta) \circ T_\beta$ over $\mathbb{Q}(\beta)$.

Examples

The geometric representation of the golden ratio-shift maps to $\mathbb{R}^2$. The ones for the Tribonacci number and the smallest Pisot number map to $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$. They are shown (up to a change of coordinates) in Fig. 2.

2 Characterization of purely periodic points

We can now state the main theorem of this note.

Theorem 1 *For all* $x \in \mathbb{Q}(\beta) \cap [0, 1]$*, the* β*-expansion of* x *is purely periodic if and only if* $(x, \delta_\beta(x)) \in \tilde{\varphi_\beta}(X_\beta)$.

Proof

Let us assume that the β-expansion of x is purely periodic. Write x as $x = 0.\overline{a_1 \dots a_L}$ and set $w = \dots \overline{a_1 \dots a_L}$ and $u = \overline{a_1 \dots a_L} \dots$. One has $(w, u) \in X_\beta$. Let us compute $\tilde{\varphi_\beta}(w, u)$. Note that the first coordinate of $\tilde{\varphi_\beta}(w, u)$ is x by definition,

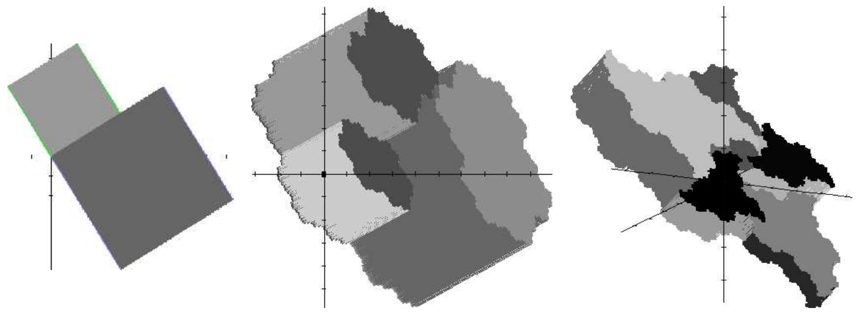

Figure 2: The Rauzy geometric two-sided representation for the Fibonacci-shift, the Tribonacci-shift, and the smallest Pisot number-shift.

and that the space $\mathbb{K}_\beta$ was built so that $\lim_{n\to\infty} \delta_\beta(\beta^n) = 0$. Since we also know that the first coordinate of $\tilde{\varphi_\beta}(w,u)$ is $\sum_{i\geq 1} u_i\beta^{-i} = \frac{a_1\beta^{L-1}+\cdots+a_L}{\beta^L-1} = x$ by definition, we have

$$\begin{aligned}
\tilde{\varphi_\beta}(w,u) &= \left(\sum_{i\geq 1} u_i\beta^{-i}, -\delta_\beta(\sum_{i\geq 0} w_i\beta^i)\right)\\
&= \left(x, -\lim_{n\to\infty} \delta_\beta((a_L+\cdots+a_1\beta^{L-1})(1+\beta^L+\cdots+\beta^{nL}))\right)\\
&= \left(x, \lim_{n\to\infty} \delta_\beta\left(-(a_1\beta^{L-1}+\cdots+a_L)\frac{1-\beta^{nL}}{1-\beta^L}\right)\right)\\
&= \left(x, \delta_\beta\left(\frac{a_1\beta^{L-1}+\cdots+a_L}{\beta^L-1}\right)\right)\\
&= (x, \delta_\beta(x)),
\end{aligned}$$

hence $(x,\delta_\beta(x)) \in \tilde{\mathcal{R}_\beta}$.

Consider now the reciprocal and let $x \in \mathbb{Q}(\beta)$ with $(x,\delta_\beta(x)) \in \tilde{\mathcal{R}_\beta}$.

The sketch of the proof is the following and is inspired by a similar discrete argument in [RI02b]. We will first introduce a finite subset $\mathcal{T}_x$ of $\tilde{\mathcal{R}_\beta}$, which depends on x, and which is stable under the action of the map $\tilde{T_\beta}$ which is onto on it. In order to define this finite set, we will take into account all the $\mathcal{I}$-adic topologies which correspond to prime ideals $\mathcal{I}$ which do not appear in the decomposition (1). These topologies are the extensions on $\mathbb{Q}(\beta)$ of p-adic topologies for the primes p which do not divide the constant term of the minimal polynomial P_β [Sie02]. Roughly speaking, one introduces further restrictions over $\mathbb{Q}(\beta)$ which involve the primes that were not already considered in $\mathbb{K}_\beta$. We will prove that the set $\mathcal{T}_x$ is finite since its first coordinate in $\mathbb{Q}(\beta)$ is bounded for all the topologies on $\mathbb{Q}(\beta)$.

Let us first introduce the following set $\mathcal{S}_x$:

$$\mathcal{S}_x = \{z \in \mathbb{Q}(\beta), \\ |z|_\mathcal{I} \leq \max(\,|x|_\mathcal{I}, 1\,), \text{ for every prime ideal } \mathcal{I} \text{ such that } |\beta|_\mathcal{I} = 1\}.$$

One has

- $x \in \mathcal{S}_x$ since $|x|_\mathcal{I} \leq \max(\,|x|_\mathcal{I}, 1\,)$ for every prime ideal $\mathcal{I}$.
- $\mathbb{Z} \subset \mathcal{S}_x$ since for every integer N and for every ideal $\mathcal{I}$, one has $|N|_\mathcal{I} \leq 1 \leq \max(\,|x|_\mathcal{I}, 1\,)$.
- $\beta\mathcal{S}_x \subset \mathcal{S}_x$ and $\beta^{-1}\mathcal{S}_x \subset \mathcal{S}_x$ since $|\beta z|_\mathcal{I} = |\beta|_\mathcal{I}\,|z|_\mathcal{I} = |z|_\mathcal{I}$ if $|\beta|_\mathcal{I} = 1$, and as well $|\beta^{-1}z|_\mathcal{I} = |z|_\mathcal{I}$.
- $(\mathcal{S}_x, +)$ is a group since the $\mathcal{I}$-adic valuations are ultrametric.

Hence one deduces that:

- $\mathcal{S}_x$ is stable under the action of T_β; indeed
$$T_\beta \mathcal{S}_x \subset \beta\mathcal{S}_x - [\mathcal{S}_x] \subset \mathcal{S}_x - \mathbb{Z} \subset \mathcal{S}_x - \mathcal{S}_x \subset \mathcal{S}_x.$$
- For every integer N, one has $\beta^{-1}(\mathcal{S}_x + N) \subset \beta^{-1}\mathcal{S}_x \subset \mathcal{S}_x$.

The keypoint of the proof is that $\mathcal{S}_x$ is not only stable under the action of T_β but also under the multiplication by $1/\beta$, even when β is not a unit.

Let us consider now the following subset of $\tilde{\mathcal{R}}_\beta$ obtained by first embeding the points of $\mathcal{S}_x$ into $\mathbb{R} \times \mathbb{K}_\beta$, and then intersecting it with the compact set $\tilde{\mathcal{R}}_\beta$:

$$\mathcal{T}_x = (Id, \delta_\beta)(\mathcal{S}_x) \cap \tilde{\mathcal{R}}_\beta.$$

- **The set $\mathcal{T}_x$ is finite**. Indeed let us prove that all the absolute values of $\mathbb{Q}(\beta)$ are bounded on the first coordinate of the elements of $\mathcal{T}_x$. Let $x \in \mathcal{S}_x$ such that $(x, \delta_\beta(x)) \in \tilde{\mathcal{R}}_\beta$. Note first that the usual absolute value of x is bounded. Furthermore

- if $|\cdot|$ is an Archimedean valuation or if $|\cdot|_{\mathcal{I}}$ is an ultrametric valuation satisfying $|\beta|_{\mathcal{I}} \neq 1$, this valuation appears by construction in $\mathbb{K}_\beta$, and thus $|x|$ is bounded since $(x, \delta_\beta(x))$ belongs to the compact set $\tilde{\mathcal{R}_\beta}$.
- if $|\cdot|_{\mathcal{I}}$ is an ultrametric valuation satisfying $|\beta|_{\mathcal{I}} = 1$, then $|x|_{\mathcal{I}}$ is bounded by definition of $\mathcal{S}_x$.

- **The map $\tilde{T_\beta}$ is onto over $\mathcal{T}_x$.** Let $(z, \delta_\beta(z)) \in \mathcal{T}_x$, with $z \in \mathcal{S}_x$. By definition, there exists $(w, u) = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) \in X_\beta$ such that $\tilde{\varphi_\beta}(w, u) = (z, \delta_\beta(z))$. An easy computation shows that

$$\tilde{\varphi_\beta} \circ S^{-1}(w, u) = \left(\frac{z + w_0}{\beta}, \delta_\beta\left(\frac{z + w_0}{\beta}\right)\right).$$

Let us note that this computation works thanks to the introduction of a sign minus before δ_β in the expression of $\tilde{\varphi_\beta}$.

One thus deduces that

$$\begin{aligned}(z, \delta_\beta(z)) = \tilde{\varphi_\beta}(w, u) = \tilde{\varphi_\beta} \circ S \circ S^{-1}(w, u) &= \tilde{T_\beta} \circ \tilde{\varphi_\beta} \circ S^{-1}(w, u) \\ &= \tilde{T_\beta} \circ (Id, \delta_\beta)\left(\frac{z + w_0}{\beta}\right).\end{aligned}$$

On one hand $(Id, \delta_\beta)\left(\frac{z+w_0}{\beta}\right) = \tilde{\varphi_\beta} \circ S^{-1}(w, u) \in \tilde{\mathcal{R}_\beta}$, and on the other hand $\frac{z+w_0}{\beta} \in \frac{\mathcal{S}_x + w_0}{\beta} \subset \beta^{-1}(\mathcal{S}_x + \mathbb{Z}) \subset \mathcal{S}_x$. Hence,

$$(z, \delta_\beta(z)) \in \tilde{T_\beta}\mathcal{T}_x.$$

- **The set $\mathcal{T}_x$ is stable by $\tilde{T_\beta}$.**

$$\begin{aligned}\tilde{T_\beta}((Id, \delta_\beta)(\mathcal{S}_x) \cap \tilde{\mathcal{R}_\beta}) &\subset \tilde{T_\beta} \circ (Id, \delta_\beta)\mathcal{S}_x \cap \tilde{T_\beta}(\tilde{\mathcal{R}_\beta}) \\ &= (Id, \delta_\beta) \circ T_\beta(\mathcal{S}_x) \cap \tilde{\varphi_\beta} \circ S(X_\beta) \\ &\subset (Id, \delta_\beta)\mathcal{S}_x \cap \tilde{\varphi_\beta}(X_\beta) = \mathcal{T}_x.\end{aligned}$$

Hence, since $\tilde{T_\beta}$ is onto over a stable finite set, it is one-to-one over this set $\mathcal{T}_x$. By construction, $(x, \delta_\beta(x)) \in \mathcal{T}_x$. Hence there exists an integer n such that

$$(x, \delta_\beta(x)) = \tilde{T_\beta}^n(x, \delta_\beta(x)) = ({T_\beta}^n(x), \delta_\beta({T_\beta}^n(x))).$$

We thus deduce that the expansion of x is purely periodic. ■

3 Additional remarks

A similar result holds in the non-Pisot case by introducing in the representation map $\tilde{\varphi}_\beta$ for the conjugates λ of modulus strictly larger that 1, the coordinate $\sum_{i\geq 1} u_i \lambda^{-i}$.

In the original proof of [RI02b], the analogous of the set $\mathcal{R}_x$ is $\mathbb{Z}[\beta]/q$ for an integer q which depends on x. This set in no more stable under the multiplication by $1/\beta$ in the non-unit case. The idea here has been to introduce a set stable under the multiplication by $1/\beta$ by allowing the division by the primes which do divide the norm of β.

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