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Dynamic Layouts for Wireless ATM*

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Abstract: In this paper we present a new model able to combine quality of service and mobility aspects in wireless ATM networks. Namely, besides the hop count and load parameters of the basic ATM layouts, we introduce a new notion of distance, that estimates the time needed to reconstruct the virtual channel of a wireless user when it moves through the network. Quality of service guarantee dictates that the rerouting phase must be imperceptible, that is the maximum distance between two virtual channels must be maintained as low as possible. Therefore, a natural combinatorial problem arises in which suitable trade-offs must be determined between the different performance measures. We first show that deciding the existence of a layout with maximum hop count h, load l and distance d is NP-complete, even in the very restricted case $h = 2$, $l = 1$ and $d = 1$. We then provide optimal layout constructions for basic interconnection networks, such as chains and rings.

Key-words: Routing, ATM networks, Wireless Networks, Mobile Users.

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Chemins virtuels dynamiques pour ATM Sans fil

Résumé : Dans cet article, nous présentons un nouveau modèle capable de combiner la qualité de service et les aspects de mobilité dans les réseaux ATM sans fil. Plus précisément, mis à part le nombre de sauts et la charge dans les réseaux ATM de base, nous introduisons une nouvelle notion de distance, qui estime le temps nécessaire pour reconstruire le chemin virtuel d'un utilisateur lorsqu'il se déplace à travers le réseau. La qualité de service garantit que le calcul de nouvelles routes doit être transparente, c'est à dire que la distance maximale entre les deux chemins virtuels doivent être maintenue aussi basse que possible. Par conséquent, nous avons un problème combinatoire pour lequel des compromis convenables doivent être trouvés entre les diverses mesures de performance. Nous montrons d'abord que décider de l'existence d'un réseau virtuel avec un un nombre de sauts borné h, une charge l et une distance d est les NP Complet, même dans le même cas restraint $h = 2$, $l = 1$ et $d = 1$. Ensuite, nous donnons les constructions optimales pour les réseaux de l'interconnexion de base, tels que les chaînes et les anneaux.

Mots-clés : Routage, Réseaux ATM, Réseaux sans fil, Utilisateurs Mobiles.

1 Introduction

Wireless ATM networks are emerging as one of the most promising technologies able to support users mobility while maintaining the quality of service offered by the classical ATM protocol for Broadband ISDN [2]. The mobility extension of ATM gives rise to two main application scenarios, called respectively *End-to-End* WATM and WATM *Interworking* [13]. While the former provides seamless extension of ATM capabilities to users by allowing ATM connections that extend until the mobile terminals, the latter represents an intermediate solution used primarily for high-speed transport over network backbones by exploiting the basic ATM protocol with additional mobility control capabilities. Wireless independent subnets are connected at the borders of the network backbone by means of specified ATM interface nodes, and users are allowed to move among the different wireless subnets. In both scenarios, the mobility facility requires the efficient solution of several problems, such as handover (users movement), routing, location management, connection control and so forth. A detailed discussion of these and other related issues can be found in [13, 6, 5, 21, 19].

The classical ATM protocol for Broadband ISDN is based on two types of predetermined routes in the network: *virtual paths* or VPs, constituted by a sequence of successive edges or physical links, and *virtual channels* or VCs, each given by the concatenation of a proper sequence of VPs [16, 15, 20]. Routing in virtual paths can be performed very efficiently by dedicated hardware, while a message passing from one virtual path to another one requires more complex and slower elaboration.

A graph theoretical model related to this ATM design problem has been first proposed in [12, 7]. In such a framework, the VP layouts determined by the VPs constructed on the network are evaluated mainly with respect to two different cost measures: the *hop count*, that is the maximum number of VPs belonging to a VC, which represents the number of VP changes of messages along their route to the destination, and the *load*, given by the maximum number of virtual paths sharing an edge, that determines the size of the VP routing tables (see, e.g., [8]). For further details and technical justifications of the model for ATM networks see for instance [1, 12].

While the problem of determining VP layouts with bounded hop count and load is NP-hard under different assumptions [12, 9], many optimal and near optimal constructions have been given for various interconnection networks such as chain, trees, grids and so forth [7, 17, 10, 11, 22, 4] (see [23] for a survey).

In this paper we mainly focus on handover management issues in wireless ATM. In fact, they are of fundamental importance, as the virtual channels must be continually modified due to the terminals movements during the lifetime of a connection. In particular, we extend the model of [12, 7] in order to combine quality of service and mobility aspects in wireless ATM networks. In such a framework, a subset of the nodes of the physical graph or network backbone corresponds to radio bridges or stations covering cells of the geographic space. A given source node provides high speed services to mobile users residing in the cells and able to move between adjacent cells during an handover phases. Adjacencies are represented by means of a cells adjacency graph in which nodes are cells and there exists an edge between a pair of cells if they are adjacent in the geographic space. Such a graph in general does not coincide with the physical graph. As an example, in nowadays cellular systems like GSM the physical graph G is a tree, cells correspond to its leaves and the adjacency graph is an hexagonal grid (see for instance [18]). When users move from a cell to an adjacent one, the corresponding virtual channels must be reconstructed in order to maintain their connection to the source. This rerouting phase must be as fast as possible in order to maintain the required quality of service provided by the basic ATM protocol. Typical handover managements issues are the path extension scheme, in which a VC is always extended by a virtual path during a handover [5], or the anchor-based rerouting and the nearest common node rerouting [13, 3], that involve the deletion of all the VPs of the old VC and the addition of all the VPs of the new one after a commmon prefix of the two VCs. Other handover strategies can be found in [13, 6, 5].

Starting from the above observations, besides the standard hop count and load performance measures, we introduce the new notion of virtual channel distance, that estimates the time needed to reconstruct a virtual channel during a handover phase. Informally speaking, the channel distance between adjacent cells u and v is the sum of the number of VPs deleted from the virtual channel $VC(u)$ of u and added to obtain $VC(v)$ after the longest commmon prefix of $VC(u)$ and $VC(v)$. In order to make the rerouting phase imperceptible to users and thus to obtain a sufficient quality of service, the maximum distance between two virtual channels must be maintained as low as possible. Therefore, a natural combinatorial problem arises in which suitable trade-offs must be determined between the different performance measures.

The above scenario concerns End-to-End WATM, but it can be directly applied to WATM Interworking just replacing the cells with the interface nodes used at the borders of the ATM backbone to comunicate with the wireless subnets.

We first show that deciding the existence of a layout with maximum hop count h, load l and distance d is NP-complete, even in the very restricted case $h = 2$, $l = 1$ and $d = 1$.

We then provide optimal layout constructions for basic interconnection networks, such as chains and rings. Such results are obtained by means of nice recursive characterizations of the structure of optimal layouts, that is that maximize the size of the covered chain or ring. The solution of the respective arising recurrences corresponds to the maximum size of a chain or ring allowing a layout with bounded values of h, l and d. All the results are shown for two slightly different realistic notions of distance.

The paper is organized as follows. In the next section we introduce the model, the notation and the necessary definitions. In Section 3 we provide the above mentioned hardness results for the layout construction problem. In Section 4 and 5 we provide the optimal layouts for chains and rings, respectively. Finally, in Section 6, we give some conclusive remarks and discuss some open questions.

2 The WATM model

We model the network as an undirected graph $G = (V, E)$, where nodes in V represent switches and edges in E are point-to-point communication links. In G there exists a subset of nodes $U \subseteq V$ constituted by cells with corresponding radio stations, i.e., switches adapted to support mobility and having the additional capability of establishing connections with the mobile terminals. A distinguished source node $s \in V$ provides high speed services to the users moving along the network. We observe that, according to the wireless nature of the system, during the handover phase mobile terminals do not necessarily have to move along the network G, but they can switch directly from one cell to another, provided that they are adjacent in the physical space. It is thus possible to define a (connected) adjacency graph $A = (U, F)$, whose edges in F represent adjacencies between cells.

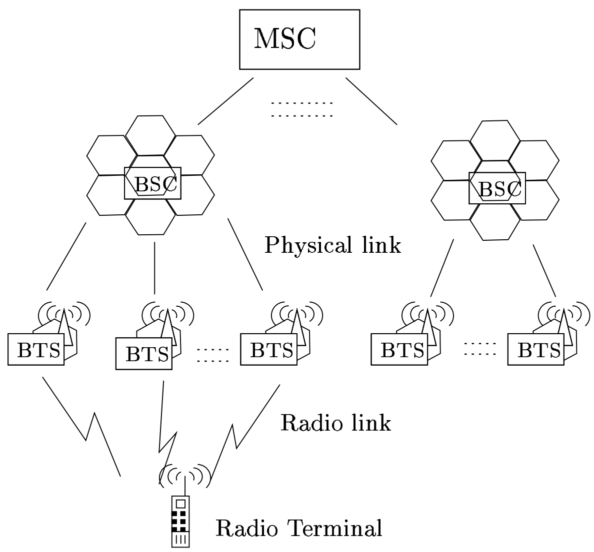

Figure 1: GSM Tree.

An example of a GSM-like system is shown in Figure 1. A *layout* Ψ for $G = (V, E)$ with source $s \in V$ is a collection of paths in G, termed *virtual paths* (VPs for short), and a mapping that defines, for each cell $u \in U$, a virtual channel $VC_\Psi(u)$ connecting s to u, i.e., a collection of VPs whose concatenation forms a shortest path in G from s to u.

Definition 2.1 *[12] The* hop count $h_\Psi(u)$ *of a node* $u \in U$ *in a layout* Ψ *is the number of VPs contained in* $VC_\Psi(u)$, *that is* $|VC_\Psi(u)|$. *The* maximal hop count *of* Ψ *is* $\mathcal{H}_{\max}(\Psi) \equiv \max_{u \in U}\{h_\Psi(u)\}$.

Definition 2.2 *[12] The* load $l_\Psi(e)$ *of an edge* $e \in E$ *in a layout* Ψ *is the number of VPs* $\psi \in \Psi$ *that include* e. *The* maximal load $\mathcal{L}_{\max}(\Psi)$ *of* Ψ *is* $\max_{e \in E}\{l_\Psi(e)\}$.

As already observed, when passing from a cell $u \in U$ to an adjacent one $v \in U$, the virtual channel $VC_\Psi(v)$ must be reconstructed from $VC_\Psi(u)$ changing only a limited number of VPs. Once fixed $VC_\Psi(u)$ and $VC_\Psi(v)$, denoted as $VC_\Psi(u,v)$ the set of VPs in the sub-channel corresponding to the longest common prefix of $VC_\Psi(u)$ and $VC_\Psi(v)$, this requires the deletion of all the VPs of $VC_\Psi(u)$ that occur after $VC_\Psi(u,v)$, plus the addition of all the VPs of $VC_\Psi(v)$ after $VC_\Psi(u,v)$. The number of removed and added VPs, denoted as $D(VC_\Psi(u), VC_\Psi(v))$, is called the distance of $VC_\Psi(u)$ and $VC_\Psi(v)$ and naturally defines a channel distance measure d_Ψ between pairs of adjacent nodes in A.

Definition 2.3 *The channel* distance *of two nodes u and v such that $\{u,v\} \in F$ (i.e., adjacent in A) is $d_\Psi(u,v) = D(VC_\Psi(u), VC_\Psi(v)) = h_\Psi(u) + h_\Psi(v) - 2|VC_\Psi(u,v)|$. The* maximal distance *of Ψ is $\mathcal{D}_{\max}(\Psi) \equiv \max_{\{u,v\}\in F}\{d_\Psi(u,v)\}$.*

It is now possible to give the following definition concerning layouts for WATM networks.

Definition 2.4 *A layout Ψ with $\mathcal{H}_{\max}(\Psi) \leq h$, $\mathcal{L}_{\max}(\Psi) \leq l$ and $\mathcal{D}_{\max}(\Psi) \leq d$ is a $\langle h,l,d\rangle$-* layout *for G, s and A.*

In the following, when the layout Ψ is clear from the context, for simplicity we will drop the index Ψ from the notation. Moreover, we will always assume that all the VPs of Ψ are contained in at least one VC. In fact, if such property does not hold, the not used VPs can be simply removed without increasing the performance measures h, l and d.

Notice that an alternative definition of channel distance can be also the symmetric difference between $VC(u)$ and $VC(v)$, i.e., $D_\triangle(VC(u), VC(v)) = |VC(u) \triangle VC(v)|$. This differs from the measure of Definition 2.3 when there exist VPs that occur after $VC(u,v)$ both in $VC(u)$ and $VC(v)$. Such VPs must not be removed nor added when reconstructing $VC(v)$ from $VC(u)$. By definition, $D_\triangle(VC(u), VC(v)) \leq D(VC(u), VC(v))$ always holds, thus any $\langle h,l,d\rangle$-layout under D is also $\langle h,l,d\rangle$-layout under $D_\triangle$.

The distance D seems to be more appropriate than $D_\triangle$, since during the handover phase the control signals must anyway propagate from u back to the end of $VC(u,v)$ and then arrive to v when adding the new VPs (see also [6, 13]), thus yielding a delay proportional to $D(VC(u), VC(v))$. However, all the results in the sequel will be proved under both the two distance measures.

Before concluding the section, let us remark that for practical purposes and quality of services guarantees, it makes sense to consider the case where $d << h$. In fact, while a little communication delay proportional to the hop count in general can be tolerated, connections gaps due to rerouting of virtual channels must not be appreciated by mobile users. On the other hand, when $d \geq 2h$, our model coincides with the classical one presented in [12] for standard ATM networks, since the difference between any two virtual channels is always at most equal to $2h$.

Clearly, in general a low distance d requires high hop count h and load l. Similarly, a low h or l causes an increase on the other two parameters. Hence, every measure can be traded off for the others.

3 Hardness of construction

In this section we show that constructing optimal dynamic layouts is in general an NP-hard problem, even for the very simple case $h = 2$ and $l = d = 1$.

Notice that when $d = 1$, for any two cells $u, v \in U$ adjacent in $A = (U, F)$, during an handover from u to v by definition only one VP can be modified. This means that in every $\langle h, l, 1\rangle$-layout Ψ, either $VC(v)$ is a prefix of $VC(u)$ and thus $VC(v)$ is obtained from $VC(u)$ by adding a new VP from u to v, or vice versa. In any case, a VP between u and v must be contained in Ψ. As a direct consequence, the virtual topology defined by the VPs of Ψ coincides with the adjacency graph A. Moreover, A must be acyclic. In fact, when moving in one direction along a cycle, it is not possible to rebuild the virtual channel of the initial node when it is reached twice. Since the distances D and $D_{\triangle}$ coincide in layout inducing trees, all the results for $\langle h, l, 1\rangle$-layouts hold for both the two measures.

Theorem 3.1 *Given a network $G = (V, E)$, a source $s \in V$ and an adjacency graph $A = (U, F)$, deciding the existence of a $\langle 2, 1, 1\rangle$-layout for G, s and A is an NP-complete problem.*

Proof. First of all observe that, for any h, l, d, the problem of deciding the existence of a $\langle h, l, d\rangle$-layout is in NP, as given $G = (V, E)$, s, $A = (U, F)$ and a layout Ψ, it is possible to check in polynomial time whether $\mathcal{H}_{\max}(\Psi) \leq h$, $\mathcal{L}_{\max}(\Psi) \leq l$ and $\mathcal{D}_{\max}(\Psi) \leq d$.

We prove the claim by providing a polynomial time reduction from *Disjoint Shortest Paths* (DSP), a well known NP-complete problem. An instance of DSP consists of a graph G and k source destination pairs $(s_1, t_1), \ldots, (s_k, t_k)$ for an integer $k > 0$. We want to determine whether there exist k edge-disjoint shortest paths in G, each connecting a different pair (s_i, t_i), $1 \leq i \leq k$.

Without loss of generality, it is possible to assume that all the pairs (s_i, t_i), $1 \leq i \leq k$, are disjoint, i.e., all nodes $s_1, \ldots, s_k, t_1, \ldots, t_k$ are different. In fact, any instance not satisfying this property can be trivially modified into an equivalent one in which every node v occurring in $k' \leq k$ pairs is connected in G to k' new nodes $v_1, \ldots, v_{k'}$ and the k' pairs contain in the order $v_1, \ldots, v_{k'}$ instead of v.

Starting from an instance of DSP, we construct a network $G' = (V', E')$, a source $s \in V'$ and an adjacency graph $A = (U, F)$ that admit a $\langle 2, 1, 1\rangle$-layout if and only if there exist the requested k edge-disjoint shortest paths in the instance of DSP.

Let $G' = (V', E')$ be such that, given a new source node s not contained in the initial graph G, $V' = V \cup \{s\}$ and $E' = E \cup \{\{s, s_i\} | 1 \leq i \leq k\}$. Concerning the adjacency graph $A = (U, F)$, let $U = \{s, s_1, \ldots, s_k, t_1, \ldots, t_k\}$ and $F = \{\{s, s_i\} | 1 \leq i \leq k\} \cup \{\{s_i, t_i\} | 1 \leq i \leq k\}$

Given any layout for G', s and A, since each VC must induce a shortest path in G', each virtual channel $VC(s_i) = \langle s, s_i\rangle$, $1 \leq i \leq k$, consists of the single edge $\{s, s_i\}$. Moreover, if we restrict to maximum distance 1, each $VC(t_i)$, $1 \leq i \leq k$, is the concatenation of the VP $\langle s, s_i\rangle$ and another VP $\langle s_i, t_i\rangle$ corresponding to a shortest path from s_i to t_i in G' and thus in G. Therefore, there exists a layout with maximum load 1 and maximum distance 1 in G' if and only if there exist the requested k edge-disjoint shortest paths in the instance of

DSP. The theorem follows by observing that any such layout must clearly have maximum hop count 2, since each VC is the concatenation of at most 2 VPs. □

For $h = 1$, any l and any d, the layout construction problem can be solved in polynomial time by exploiting suitable flow constrictions like the ones presented in [9].

4 Optimal layouts for chain networks

In this section we provide optimal layouts for chain networks. More precisely, we consider the case in which the physical graph is a chain C_n of n nodes, that is $V = \{1, 2, \ldots, n\}$ and $E = \{\{v, v+1\} | 1 \leq v \leq n-1\}$, and the adjacency graph A coincides with C_n. Moreover, without loss of generality, we take the leftmost node of the chain as the source, i.e. $s = 1$, as otherwise we can split the layout construction problem into two equivalent independent subproblems for the left and the right hand sides of the source, respectively. Finally, we always assume $d > 1$, as by the same considerations of the previous section the virtual topology induced by the VPs of any $\langle h, l, 1\rangle$-layout Ψ coincides with the adjacency graph A and thus with C_n. Therefore, the largest chain admitting a $\langle h, l, 1\rangle$-layout is such that $n = h + 1$.

In the following we denote by $\langle u, v\rangle$ the unique VP corresponding to the shortest path from u to v in C_n and by $\langle\langle s, v_1\rangle\langle v_1, v_2\rangle \ldots \langle v_k, v\rangle\rangle$ or simply $\langle s, v_1, v_2, \ldots, v_k, v\rangle$ the virtual channel $VC(v)$ of v given by the concatenation of the VPs $\langle s, v_1\rangle$, $\langle v_1, v_2\rangle$, ..., $\langle v_k, v\rangle$. Clearly, $s < v_1 < v_2 < \ldots < v_k < v$.

Definition 4.1 *Two VPs $\langle u_1, v_1\rangle$ and $\langle u_2, v_2\rangle$ are crossing if $u_1 < u_2 < v_1 < v_2$. A layout Ψ is* crossing-free if it does not contain any pair of crossing VPs.

Definition 4.2 *A layout Ψ is canonic if it is crossing-free and the virtual topology induced by its VPs is a tree.*

According to the following definition, a $\langle h, l, d\rangle$-layout for chains is optimal if it reaches the maximum number of nodes.

Definition 4.3 *Given fixed h,l,d and a $\langle h, l, d\rangle$-layout Ψ for a chain C_n, Ψ is optimal if no $\langle h, l, d\rangle$-layout exists for any chain C_m with $m > n$.*

We now prove that for every *h*,*l*,*d*, the determination of an optimal $\langle h, l, d\rangle$-layout can be restricted to the class of the canonic layouts.

Theorem 4.4 *For every h, l, d, any optimal $\langle h, l, d\rangle$-layout for a chain is canonic.*

Proof. We show that the claim holds under the $D_\triangle$ channel distance, since this directly implies the theorem also for D. In fact, assume that any optimal $\langle h, l, d\rangle$-layout under $D_\triangle$ is canonic and let Ψ be an optimal $\langle h, l, d\rangle$-layout for a chain C_n under $D_\triangle$. Clearly, since Ψ

induces a tree, Ψ is a $\langle h,l,d\rangle$-layout also under D. Then, if Φ is an optimal $\langle h,l,d\rangle$-layout for a chain C_m under D, $m \geq n$. By the definition of the distances D and $D_\triangle$, Φ is also a $\langle h,l,d\rangle$-layout for C_m under $D_\triangle$. Thus, $m = n$ and Φ is canonic.

Assume by contradiction that there exists an optimal $\langle h,l,d\rangle$-layout Ψ for a chain C_n containing crossings or such that a vertex v is the right endpoint of more than one VP, that is Ψ contains cycles and thus it does not induce a tree. We now show that it is possible construct a $\langle h,l,d\rangle$-layout Φ for a chain C_m with $m > n$, thus contradicting the optimality of Ψ.

By hypothesis, there must exist v, $1 < v < n$, such that the following two properties are satisfied:

1. The VPs of Ψ used to reach the nodes in the subchain $[1, v]$ induce a tree and do not form crossings with any other VP, included the ones terminating after v;

2. Property 1. is not true for the subchain $[1, v+1]$.

This means that every two VPs $\langle u_1, v_1\rangle$, $\langle u_2, v_2\rangle \in \Psi$ with $v_1 \leq v$ are not crossing, all VPs $\langle w, z\rangle \in \Psi$ with $z \leq v$ induce a tree and finally node $v+1$ is the right endpoint of more than one VP or at least one VP $\langle u, v+1\rangle$ entering in $v+1$ forms a crossing with some other VP terminating after v. Notice that $v \geq 2$, as between the first two nodes of the chain there can be a unique VP that cannot be crossed by any other VP and clearly induces a tree. Moreover, $v < n$, as otherwise Ψ would be canonic against the hypothesis.

Given any two nodes u, w such that $u \leq v$ and $w \leq v$, let us denote as $P(u,w)$ as the unique path of VPs of Ψ in the subchain $[1, v]$ that goes from u to w. We construct the layout Φ for the larger C_m starting from Ψ by means of the following two steps.

- *Step 1*: Replace every VP $\langle w, z\rangle$ with $z > v+1$ crossing at least one VP $\langle u, v+1\rangle$ (if any) with the pair of VPs $\langle w, v+1\rangle$, $\langle v+1, z\rangle$, and modify all the VCs of Ψ containing $\langle w, z\rangle$ accordingly. Let Υ be the resulting layout. Notice that, since no VP is modified until node v, for any two nodes $u \leq v$ and $w \leq v$, the path $P(u,w)$ of the VPs from u to v in $[1, v]$ is the same in Ψ and Υ.

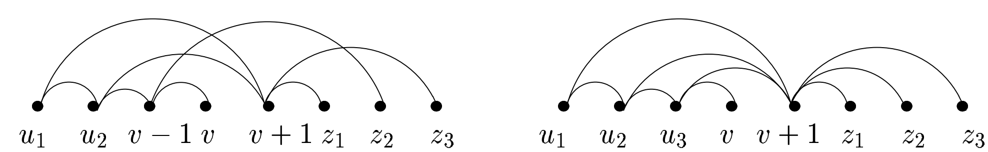

Figure 2: Crossing elimination (Step 1).

- *Step 2*: Let $\langle u_1, v+1\rangle$, $\langle u_2, v+1\rangle$, ..., $\langle u_k, v+1\rangle$ be all the VPs terminating in $v+1$ in the layout Υ resulting from Step 1. Then the chain $P(u_1, u_k)$ from u_1 to u_k steps through $u_2, \ldots, u_{k-1}$ (see Figure 2), as by hypothesis the VPs of Ψ in $[1, v]$ do not generate crossings and the same holds for Υ. In other words, $P(u_1, u_k)$ is the concatenation of the subchains $P(u_1, u_2), \ldots, P(u_{k-1}, u_k)$.

Add $k-1$ nodes w_i, $1 \le i < k$, between v and $v+1$, and replace the VPs $\langle u_i, v+1\rangle$ with $\langle u_i, w_{k-i+1}\rangle$ for $2 \le i \le k$. For simplicity we assume that $w_1, \ldots, w_{k-1}$ are rational numbers included in the interval $[v, v+1]$, so that $v+1$ and the successive nodes do not need to be renamed according to their order in the chain. In every VC of Υ containing a VP $\langle u_i, v+1\rangle$, $2 \le i \le k$, substitute the chain of VPs obtained by the concatenation of $P(u_1, u_i)$ and $\langle u_i, v+1\rangle$ with the unique VP $\langle u_1, v+1\rangle$. Let Φ be the resulting layout (see Figure 3).

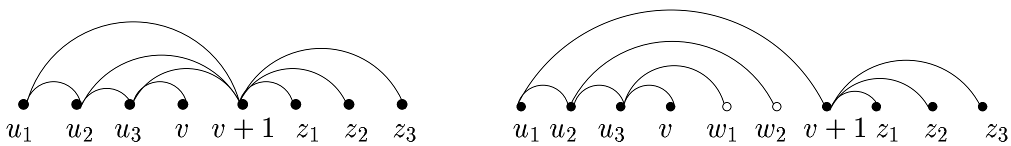

Figure 3: Cycles elimination (Step 2).

Again, since no VP is modified until node v, for any two nodes $u \le v$ and $w \le v$, $P(u, w)$ in Φ is the same of Ψ and Υ.

As a consequence of Step 1 and Step 2, we obtain a layout Φ for a chain C_m with $m = n + k - 1 > n$. In fact, $k > 1$ since by hypothesis $v+1$ is the right endpoint of more than one VP in Ψ and thus in Υ, and if such condition does not hold there exists at least one VP $\langle w, z\rangle$ in Ψ crossing a VP terminating at $v+1$. Thus, in Υ $v+1$ is again the right endpoint of more than one VP, since the VP $\langle w, v+1\rangle$ is added during Step 1. Therefore, in order to complete the proof, it remains to show that Φ is a $\langle h, l, d\rangle$-layout for C_m, that is it does not increase the three performance measures. We consider each case separately.

- Hop Count

  Since the VCs of the nodes belonging to the subchain $[1, v]$ are never modified, $h_\Phi(u) = h_\Psi(u) \le h$ for each $u \in [1, v]$.

  The virtual channel $VC_\Phi(w_i)$ of each node w_i with $1 \le i \le k-1$ added during Step 2 is obtained by the concatenation of $VC_\Psi(u_{k-i+1})$ and the VP $\langle u_{k-i+1}, w_i\rangle$. Therefore, $h_\Phi(w_i) = h_\Psi(u_{k-i+1}) + 1 \le h$, as u_{k-i+1} is the left endpoint of at least one VP in Ψ and thus has hop count $h_\Psi(u_{k-i+1}) \le h - 1$.

  Let us finally consider the subchain $[v+1, n]$. Observe first that if u is the first node of C_n having a VP $\langle u, v+1\rangle$ terminating to $v+1$ in Ψ, then $u = u_1$. In fact, in Ψ no VP $\langle w, z\rangle$ crossing a VP terminating at $v+1$ can exist for $w < u$ and $\langle u, v+1\rangle$ is also a VP in Υ. Let u be any node in $[v+1, n]$. If $VC_\Psi(u)$ contains a VP $\langle w, z\rangle$ with $w \le u_1$ and $z \ge v+1$, then by the above observation no VP of $VC_\Psi(u)$ can generate a crossing with a VP of Ψ terminating at $v+1$, thus $VC_\Psi(u)$ is not modified during

Step 1. Moreover, $VC_\Upsilon(u)$ is not modified during Step 2, as it contains $\langle w, z\rangle$ and thus no VP $\langle u_i, v+1\rangle$ with $2 \le i \le k$. Therefore, $h_\Phi(u) = h_\Psi(u) \le h$. It remains to consider the case in which $VC_\Psi(u)$ contains a VP $\langle w, z\rangle$ with $u_1 < w \le v$ and $z \ge v+1$. In this case, $VC_\Psi(u)$ is modified during Step 1 only if $z > v+1$ and in this case its VP $\langle w, z\rangle$ is substituted with the two VPs $\langle w, v+1\rangle$ and $\langle v+1, z\rangle$. Therefore, $h_\Upsilon(u) \le h_\Psi(u)+1$. Since $\langle w, v+1\rangle$ is a VP of Υ and $w > u_1$, $w = u_i$ for a given i such that $2 \le i \le k$. Thus, during Step 2, the chain of VPs obtained by the concatenation of $P(u_1, u_i)$ and $\langle u_i, v+1\rangle$ in $VC_\Upsilon(u)$ is substituted with the unique VP $\langle u_1, v+1\rangle$, so that $h_\Phi(u) \le h_\Upsilon(u) - 1$. In conclusion, $h_\Phi(u) \le h_\Upsilon(u) - 1 \le h_\Psi(u) \le h$, and therefore $\mathcal{H}_{\max}(\Phi) \le h$.

- Load

  Clearly $\mathcal{L}_{\max}(\Upsilon) \le \mathcal{L}_{\max}(\Psi) \le l$, as in Step 1 some VPs $\langle w, z\rangle$ with $w < v+1$ and $z > v+1$ are split in two VPs $\langle w, v+1\rangle$ and $\langle v+1, z\rangle$. This cannot increase the load of any edge. Actually some loads can even decrease when some VPs resulting from the splits are coincident or coincide with previously existing ones in Ψ.

  Step 2 adds the new nodes $w_1, \ldots, w_k$ and modifies only the load of the edges in the subchain $[v, v+1]$, that however is always bounded by $l_\Upsilon(\{v, v+1\}) \le l$. Therefore, we can conclude that $\mathcal{L}_{\max}(\Phi) \le l$.

- Distance

  The distance in Φ between the VCs of two adjacent nodes in the subchain $[1, v]$ is the same as in Ψ, since the VCs of all these nodes are never modified.

  The distance in Φ between the VCs of v and the first added node during Step 2, w_1, is exactly one if the VP $\langle v, v+1\rangle$ was contained in Ψ, otherwise $d_\Phi(v, w_1) \le d_\Psi(v, v+1)$. In fact, while $VC_\Phi(v) \triangle VC_\Phi(w_1)$ contains the VPs contained in the chain $P(u_k, v)$ plus $\langle u_k, w_1\rangle$, $VC_\Psi(v) \triangle VC_\Psi(v+1)$ contains the VPs in the chain $P(u_i, v)$ (containing $P(u_k, v)$) for a given u_i with $i \le k$ plus $\langle u_i, v+1\rangle$. Thus, $d_\Psi(v, v+1) = |VC_\Psi(v) \triangle VC_\Psi(v+1)|$ is equal to $d_\Phi(v, w_1) = |VC_\Phi(v) \triangle VC_\Phi(w_1)|$ plus the number of VPs in the (possibly empty) subchain $P(u_i, u_k)$. Therefore, in every case $d_\Phi(v, w_1) \le d$.

  Let us now consider the subchain of the nodes in the set $\{w_1, w_2, \ldots, w_{k-1}, v+1\}$, and for simplicity let $w_k = v+1$. In order to prove that the channel distance between any two adjacent nodes is at most d, it is sufficient to show that in Υ each cycle $C(u_i, u_{i+1})$ with $1 \le i < k$ consisting of $\langle u_i, v+1\rangle$, $\langle u_{i+1}, v+1\rangle$ and the chain of VPs $P(u_i, u_{i+1})$ has length at most d. In fact, such a length after Step 2 is exactly the distance between $VC_\Phi(w_{k-i+1})$ and $VC_\Phi(w_{k-i})$ (see Figure 3). In Ψ there must necessarily exist two nodes v_1 and v_2 with $v_1 \le u_i$ and $u_{i+1} \le v_2 \le v$ such that, given $z > v+1$, $z_1 \ge v+1$ and $z_2 \ge v+1$, $\langle v_1, z_1\rangle$ belongs to $VC(z-1)$ and $\langle v_2, z_2\rangle$ belongs to $VC(z)$ or vice versa. In fact, if such condition does not hold, either the VPs starting from u_i and terminating after v or the VPs starting from u_{i+1} and terminating after v are can not be used in Ψ, thus contradicting the hypothesis that each VP belongs to at least one VC. So, after Step 1, since $D_{\triangle}(VC_{\Psi}(z-1), VC_{\Psi}(z)) \leq d$, the cycle induced by $\langle v_1, v+1\rangle$, $\langle v_2, v+1\rangle$, and $P(v_1, v_2)$ in Υ consists of at most d VPs. Therefore $C(u_i, u_{i+1})$ has length at most d, as $P(v_1, v_2)$ is contained in $P(u_i, u_{i+1})$.

We are now left with the remaining subchain $[v+1, n]$. For every $u \in [v+1, n]$, let us define $VC^1_{\Psi}(u)$ (resp. $VC^1_{\Upsilon}(u)$, $VC^1_{\Phi}(u)$) as the subset of the VPS $\langle w, z\rangle$ in $VC_{\Psi}(u)$ (resp. $VC_{\Upsilon}(u)$, $VC_{\Phi}(u)$) with $z \leq v+1$, $VC^2_{\Psi}(u)$ (resp. $VC^2_{\Upsilon}(u)$, $VC^2_{\Phi}(u)$) as the subset of the VPS $\langle w, z\rangle$ in $VC_{\Psi}(u)$ (resp. $VC_{\Upsilon}(u)$, $VC_{\Phi}(u)$) with $w \leq v+1$ and $z > v+1$, and finally $VC^3_{\Psi}(u)$ (resp. $VC^3_{\Upsilon}(u)$, $VC^3_{\Phi}(u)$) as the subset of the VPS $\langle w, z\rangle$ in $VC_{\Psi}(u)$ (resp. $VC_{\Upsilon}(u)$, $VC_{\Phi}(u)$) with $w > v+1$. Clearly, $VC^1_{\Phi}(u)$, $VC^2_{\Phi}(u)$ and $VC^3_{\Phi}(u)$ form a partition of $VC_{\Phi}(u)$. Moreover for every $u \in [v+1, n-1]$, $d_{\Psi}(u, u+1) = D_{\triangle}(VC_{\Psi}(u), VC_{\Psi}(u+1)) = |VC_{\Psi}(u) \triangle VC_{\Psi}(u+1)| = |VC^1_{\Psi}(u) \triangle VC^1_{\Psi}(u+1)| + |VC^2_{\Psi}(u) \triangle VC^2_{\Psi}(u+1)| + |VC^3_{\Psi}(u) \triangle VC^3_{\Psi}(u+1)|$. In fact, a VP in a given $VC^j_{\Psi}(u)$, $1 \leq j \leq 3$, can only be found in $VC^j_{\Psi}(u+1)$. The same considerations hold for Υ and Φ.

Let us first determine how $VC^1_{\Psi}(u)$ and $VC^1_{\Psi}(u+1)$ are modified during Step 1 and Step 2. As already remarked for the hop count measure, $VC_{\Psi}(u)$ is modified during Step 1 and Step 2 only if it contains a VP $\langle w, z\rangle$ with $u_1 < w \leq v$ and $z \geq v+1$. In this case, if $z > v+1$, during Step 1 $\langle w, z\rangle$ in $VC_{\Psi}(u)$ is substituted with the two VPs $\langle w, v+1\rangle$ and $\langle v+1, z\rangle$. In every case, $\langle w, v+1\rangle$ is a VP of Υ and $w = u_i$ for a given i such that $2 \leq i \leq k$. Thus, during Step 2, the chain of VPs obtained by the concatenation of $P(u_1, u_i)$ and $\langle u_i, v+1\rangle$ in $VC_{\Upsilon}(u)$ is substituted with the unique VP $\langle u_1, v+1\rangle$. The same consideration holds for node $u+1$. In conclusion, $VC^1_{\Phi}(u) = VC^1_{\Phi}(u+1)$ if $VC_{\Psi}(u)$ and $VC_{\Psi}(u+1)$ are both modified in Step 2 (and eventually in Step 1), $VC^1_{\Phi}(u) = VC^1_{\Psi}(u)$ and $VC^1_{\Phi}(u+1) = VC^1_{\Psi}(u+1)$ if $VC_{\Psi}(u)$ and $VC_{\Psi}(u+1)$ are not modified. Finally, if $VC_{\Psi}(u)$ is modified and $VC_{\Psi}(u+1)$ is not modified, if $\langle w, z\rangle$ is the VP of $VC_{\Psi}(u+1)$ with $w \leq u_1$ and $z \geq v+1$ (it must exist since $VC_{\Psi}(u+1)$ is not modified), $VC^1_{\Psi}(u) \triangle VC^1_{\Psi}(u+1)$ contains the VPs of $VC_{\Psi}(u)$ in $P(w, u_1)$ and in the (not empty) chain $P(u_1, u_i)$, while $VC^1_{\Phi}(u) \triangle VC^1_{\Psi}(u+1)$ only the VPs of $VC_{\Phi}(u)$ in the chain $P(w, u_1)$ plus eventually the VP $\langle u_1, v+1\rangle$. A symmetric argument applies when $VC_{\Psi}(u)$ is not modified and $VC_{\Psi}(u+1)$ is modified, therefore in every case $|VC^1_{\Phi}(u) \triangle VC^1_{\Phi}(u+1)| \leq |VC^1_{\Psi}(u) \triangle VC^1_{\Psi}(u+1)|$.

By definition, the subsets $VC^2_{\Psi}(u)$ and $VC^2_{\Psi}(u+1)$ have both cardinality one. If $VC^2_{\Psi}(u) = VC^2_{\Psi}(u+1)$, their contained VP $\langle w, z\rangle$, is split during Step 1 in two VPs $\langle w, v+1\rangle$ and $\langle v+1, z\rangle$, and since $\langle w, v+1\rangle \in VC^1_{\Upsilon}(u)$, it follows that $VC^2_{\Upsilon}(u) = VC^2_{\Upsilon}(u+1) = \{\langle v+1, z\rangle\}$. Clearly, $VC^2_{\Upsilon}(u) = VC^2_{\Upsilon}(u+1) = \{\langle v+1, z\rangle\}$ also if $VC^2_{\Psi}(u) = VC^2_{\Psi}(u+1) = \{\langle v+1, z\rangle\}$. Since Step 2 does not modify $VC^2_{\Upsilon}(u)$ and $VC^2_{\Upsilon}(u+1)$, i.e., $VC^2_{\Upsilon}(u) = VC^2_{\Phi}(u)$ and $VC^2_{\Upsilon}(u+1) = VC^2_{\Phi}(u+1)$, $VC^2_{\Phi}(u) = VC^2_{\Phi}(u+1)$ if $VC^2_{\Psi}(u) = VC^2_{\Psi}(u+1)$, and thus $|VC^2_{\Phi}(u) \triangle VC^2_{\Phi}(u+1)| \leq |VC^2_{\Psi}(u) \triangle VC^2_{\Psi}(u+1)|$.

Finally, since during Step 1 and Step 2 all the VPs starting after $v+1$ are never modified, $VC^3_\Psi(u) = VC^3_\Upsilon(u) = VC^3_\Phi(u)$ and the same holds for $u+1$, so that $|VC^3_\Psi(u) \triangle VC^3_\Psi(u+1)| = |VC^3_\Phi(u) \triangle VC^3_\Phi(u+1)|$.

In conclusion, $d_\Phi(u,u+1) = |VC^1_\Phi(u) \triangle VC^1_\Phi(u+1)| + |VC^2_\Phi(u) \triangle VC^2_\Phi(u+1)| + |VC^3_\Phi(u) \triangle VC^3_\Phi(u+1)| \leq |VC^1_\Psi(u) \triangle VC^1_\Psi(u+1)| + |VC^2_\Psi(u) \triangle VC^2_\Psi(u+1)| + |VC^3_\Psi(u) \triangle VC^3_\Psi(u+1)| \leq d$, therefore $\mathcal{D}_{\max}(\Phi) \leq d$.

We have thus shown that Φ is a $\langle h,l,d\rangle$-layout for a chain C_m larger than C_n, thus contradicting the optimality of Ψ, hence the theorem. □

Motivated by Theorem 4.4, in the remaining part of this section we focus on canonic $\langle h,l,d\rangle$-layouts for chains, as they can be the only optimal ones. Again, since the distances D and $D_\triangle$ coincide in layouts inducing trees, all the results hold for both the two cost measures.

Let us say that a tree is ordered if it is rooted and for every internal node a total order is defined on its children. The following lemma shows that every ordered tree induces a canonic layout, by means of the following procedure from [12].

InduceVPL(T): Induces a layout according to an ordered tree T of n vertices.
1. Label vertices of T in preorder, visiting the subtrees of each internal node according to the order defined on their roots. Let $\lambda(u)$ be the label of a vertex $u \in T$, $1 \leq \lambda(u) \leq n$.
2. For every edge $(u,v) \in T$ connect a VP between $\lambda(u)$ and $\lambda(v)$.
3. Return Ψ_T, the collection of generated VPs.

Lemma 4.5 *[12] Let T be an ordered tree. Then procedure $InduceVPL(T)$ induces a canonic layout.*

Clearly, also the vice versa is true, that is every canonic layout induces an ordered tree by exploiting a reverse procedure that defines the order of the nodes according to their labels. Therefore, there exists a bijection between canonic layouts and ordered trees.

We now introduce a new class of ordered trees $\mathcal{T}(h,l,d)$ that allows to completely define the structure of an optimal $\langle h,l,d\rangle$-layout. Informally, denoted as $\mathcal{T}(h,l)$ the ordered tree corresponding to optimal layouts with maximum hop count h and load l without considering the distance measure [11], $\mathcal{T}(h,l,d)$ is a maximal subtree of $\mathcal{T}(h,l)$ with the additional property that the distance between two adjacent nodes in the preorder labelling of the ordered tree, and thus between two adjacent nodes in the induced layout, is always at most d. Moreover, the containment of $\mathcal{T}(h,l,d)$ in $\mathcal{T}(h,l)$ guarantees that the hop count h and the load l are not exceeded in the induced layout.

The definition of $\mathcal{T}(h,l,d)$ is recursive and the solution of the associated recurrence gives the exact number of the nodes reached by an optimal $\langle h,l,d\rangle$-layout. Before introducing $\mathcal{T}(h,l,d)$, let us define another ordered tree that is exploited in its definition.

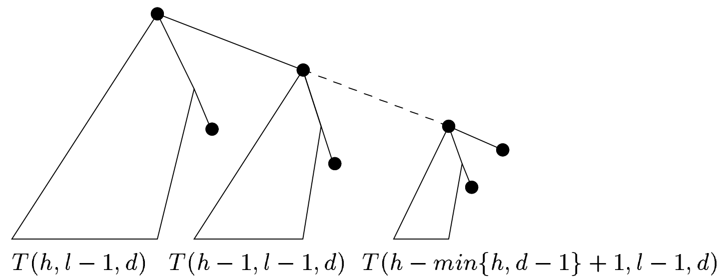

Figure 4: The recursive decomposition of $T(h,l,d)$.

Definition 4.6 *Given any h,l,d, $T(h,l,d)$ is an ordered tree defined recursively as follows. $T(h,l,d)$ is obtained by joining the roots of $min\{h,d-1\}$ subtrees $T(i,l-1,d)$ with $h-min\{h,d-1\}+1 < i \leq h$ in such a way that the root of $T(i-1,l-1,d)$ is the rightmost child of the root of $T(i,l-1,d)$. A last node is finally added as the rightmost child of $T(h-min\{h,d-1\}+1,l-1,d)$. Trees $T(0,l,d)$ and $T(h,0,d)$ consist of a unique node.*

An example of $T(h,l,d)$ can be seen in Figure 4. Informally speaking, $T(h,l,d)$ is an ordered tree with the above stated property that the distance between two adjacent nodes in a preorder labelling of the ordered tree is at most d. Moreover, $T(h,l,d)$ has the further constraint that its rightmost leaf, the only node of $T(h,l,d)$ not having a successive one in the tree, is always at distance $min\{h,d-1\} \leq d-1$ from the root. This makes sure that, when $T(h,l,d)$ is used as a subtree in other trees and the leaf has a successive node outside the subtree, the distance between such nodes remains bounded by d.

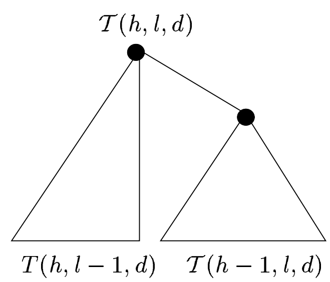

Figure 5: $\mathcal{T}(h,l,d)$ in terms of $T(h,l-1,d)$ and $\mathcal{T}(h-1,l,d)$.

Definition 4.7 *The ordered tree $\mathcal{T}(h,l,d)$ is defined recursively as the join of the roots of the tree $\mathcal{T}(h-1,l,d)$ and the tree $T(h,l-1,d)$ in such a way that the root of $\mathcal{T}(h-1,l,d)$ is the rightmost child of the root of $T(h,l-1,d)$ (see Figure 5). Trees $\mathcal{T}(0,l,d)$ and $\mathcal{T}(h,0,d)$ consist of a unique node.*

Expanding the above recursive definition, it is also possible to view $\mathcal{T}(h,l,d)$ as given by the join of the roots of $h+1$ subtrees $T(i,l-1,d)$ for $0 \le i \le h$ in such a way that for $i > 0$ the root of a $T(i-1,l-1,d)$ is the rightmost child of the root of a $T(i,l-1,d)$ (see Figure 6).

The following lemma establishes that $\mathcal{T}(h,l,d)$ is the ordered tree induced by an optimal $\langle h,l,d\rangle$-layout.

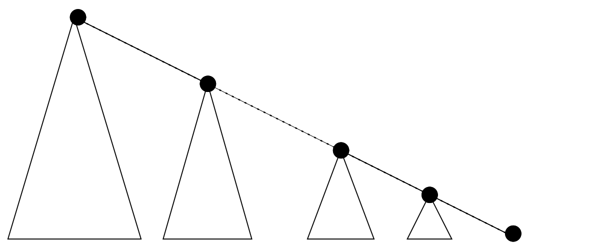

Figure 6: $\mathcal{T}(h,l,d)$ in terms of trees of type T.

Lemma 4.8 *The layout Ψ induced by $\mathcal{T}(h,l,d)$ is a $\langle h,l,d\rangle$-layout. Moreover, every canonic $\langle h,l,d\rangle$-layout Ψ induces an ordered tree T contained in $\mathcal{T}(h,l,d)$.*

Proof. Let us first prove that the layout Ψ induced by $\mathcal{T}(h,l,d)$ is a $\langle h,l,d\rangle$-layout. It is easy to see that $\mathcal{H}_{\max}(\Psi)$ is bounded by h and $\mathcal{D}_{\max}(d)$ by d, since by Definition 4.7 the height of $\mathcal{T}(h,l,d)$ is h and the distance between two consecutive nodes in the preorder labelling associated to $\mathcal{T}(h,l,d)$ is at most d. In order to show that $\mathcal{L}_{\max}(\Psi) \le l$, we first prove by induction on l that every $T(h,l,d)$ induces a layout with maximum load l. In fact, by Definition 4.7, every $\mathcal{T}(h,l,d)$ is obtained by joining the roots of $h+1$ trees $T(i,l-1,d)$ with $0 \le i \le h$, and since each $T(i,l-1,d)$ induces a layout Ψ_i such that $\mathcal{L}_{\max}(\Psi_i) \le l-1$, $\mathcal{L}_{\max}(\Psi) = 1 + \max_{0\le i\le h} \mathcal{L}_{\max}(\Psi_i) \le l$.

The claim trivially holds for $l \le 1$ as $T(h,0,d)$ consists of a single node thus yielding a layout of maximum load equal to 0, while $T(h,1,d)$ is a chain of $min\{h,d-1\}+1$ nodes and therefore it induces a layout of maximum load equal to 1. Given any $l > 1$ and assuming by induction that the claim holds for every $T(h,l-1,d)$, let us prove that every $T(h,l,d)$ induces a layout with maximum load equal to l. By Definition 4.6, $T(h,l,d)$ is obtained by joining the roots of $min\{h,d-1\}$ trees $T(i,l-1,d)$ with $h - min\{h,d-1\}+1 \le i \le h$, plus a last node attached as rightmost leaf of $T(h-min\{h,d-1\}+1,l,d)$. Again, since each $T(i,l-1,d)$ induces a layout Ψ_i such that $\mathcal{L}_{\max}(\Psi_i) \le l-1$, the maximum load of the layout induced by $T(h,l,d)$ is at most $1 + \max_{0\le i\le h} \mathcal{L}_{\max}(\Psi_i) \le l$.

We prove the second part of the claim by showing an iterative procedure that embeds the tree T induced by any canonic $\langle h,l,d\rangle$-layout Ψ into $\mathcal{T}(h,l,d)$. Let us say that an edge of T is of rank i, $1 \le i \le l$, if it corresponds to a VP $\langle u,v\rangle$ of Ψ such that $l-i$ other VPs

$\langle w, z\rangle$ in Ψ exist with $w \leq u$ and $z \geq v$. Therefore, an edge of rank i is associated to a VP having $i-1$ VPs above it in Ψ. Let us define the rank of each edge of $\mathcal{T}(h,l,d)$ accordingly. The procedure is divided into l phases such that in each phase i all the edges of rank i in T are embedded in $\mathcal{T}(h,l,d)$. The edges of rank 1 in T correspond to the path $\langle u_0, \ldots, u_{k-1}\rangle$ of $k \leq h$ VPs from the root u_0 to the rightmost leaf u_{k-1} of T, and in phase 1 its edges are matched with the first k ones belonging to the path of length h from the root to the rightmost leaf of $\mathcal{T}(h,l,d)$. Since T is induced by a canonic $\langle h,l,d\rangle$-layout, the first endpoint u_j of each such a matched edge $\{u_j, u_{j+1}\}$, $0 \leq j < k-1$, is the starting node of a path P_j of length $k_j \leq min\{h-j, d-1\}$ containing only edges of rank 2 and terminating to the leaf of T whose successive node in the preorder labelling associated to T is the other endpoint u_{j+1}. In fact, since u_j is at level j in T and T has height h, P_j has length at most $h-j$. Moreover, P_j can not be longer than $d-1$, otherwise the distance between its last leaf node and u_{j+1} would be greater than d. Clearly, each rank 2 edge of T belongs to one of such paths P_j. During phase 2, each P_j is matched with the first k_j edges of the rank 2 path of length $min\{h-j, d-1\}$ in $\mathcal{T}(h,l,d)$ that goes from the node matched to u_j to the leaf whose successive node in the preorder labelling associated to $\mathcal{T}(h,l,d)$ is the node matched with u_{j+1}. The same steps are performed in each phase i for the rank i paths of T starting at the first endpoint of each edge of rank $i-1$ matched during the previous phase. Since T is induced by Ψ and $\mathcal{H}_{\max}(\Psi) \leq l$, l phases are sufficient to embed all T in $\mathcal{T}(h,l,d)$. Furthermore, by definition of $\mathcal{T}(h,l,d)$, all the edges can be matched until phase l included. This concludes the proof. □

Let $\mathcal{T}_n(h,l,d)$ and $T_n(h,l,d)$ denote the number of nodes in $\mathcal{T}(h,l,d)$ and in $T(h,l,d)$, respectively. Directly from Definition 4.6 and 4.7, it follows that $\mathcal{T}_n(h,l,d) = T_n(h,l-1,d) + \mathcal{T}_n(h-1,l,d) = \sum_{k=0}^{h} T_n(k,l-1,d)$, where the value of every $T_n(k,l-1,d)$ for $0 \leq k \leq h$ is obtained by the following recursive equation:

$$T_n(h,l,d) = \begin{cases} 1 & \text{if } l = 0 \text{ or } h = 0, \\ 1 + \sum_{j=0}^{min\{h,d-1\}-1} T_n(h-j, l-1, d) & \text{otherwise.} \end{cases}$$

Before solving the above recurrence, we recall that given $n+1$ positive integers $m, k_1, \ldots, k_n$ such that $m = k_1 + \cdots + k_n$, the multinomial coefficient $\binom{m}{k_1,\ldots,k_n}$ is defined as $\frac{m!}{k_1! \cdot k_2! \cdot \cdots \cdot k_n!}$ (see for instance [14]).

Lemma 4.9 *For every h, l, d, $T_n(h,l,d) =$*

$$\sum_{i=0}^{l} \sum_{j=0}^{h-1} \sum_{\substack{0 \leq k_{d-2} \leq k_{d-3} \leq \ldots \leq k_2 \leq k_1 \leq i \\ k_1 + k_2 + \ldots + k_{d-2} = j}} \binom{i}{i-k_1, k_1-k_2, \ldots, k_{d-3}-k_{d-2}, k_{d-2}}.$$

Proof. Let A be the matrix defined as follow:

$$A_{i,j} = \begin{cases} 1 & \text{if } i = 0 \text{ and } j = 0, \\ 0 & \text{if } i = 0 \text{ and } j > 0, \\ \sum_{t=\max\{0,j-d+2\}}^{j} A_{i-1,t} & \text{otherwise.} \end{cases}$$

Notice that by definition $A_{i,j} = 0$ for $j > i(d-2)$.

It is easy to see that a generic element $A_{i,j}$ represents the number of subtrees $T(h-j, l-i, d)$ that are in $T(h,l,d)$ or analogously in the expansion of the recursive definition of $T(h,l,d)$ until obtaining only trees of load $l-i$. Moreover, by the recurrence of T_n, it results that $\sum_{i=0}^{l}\sum_{j=0}^{h-1} A_{i,j}$ is exactly the number of nodes in $T(h,l,d)$, that is the value $T_n(h,l,d)$.

In order to determine the sum of the first $l+1$ rows and h columns of A, we observe that each row i of A corresponds to the coefficients of the i-th power of the polynomial $x^{d-2}+x^{d-3}+\cdots+x^2+x+1$. More precisely, a generic element $A_{i,j}$ is equal to the coefficient of x^j in the expansion of the polynomial $(x^{d-2}+x^{d-3}+\cdots+x^2+x+1)^i$. By applying $d-2$ times the well known equality $(a+b)^i = \sum_{k=0}^{i}\binom{i}{k}a^k b^{i-k}$ to $(x^{d-2}+x^{d-3}+\cdots+x^2+x+1)^i$ with $a = x^{d-2}+x^{d-3}+\cdots+x^2+x$ and $b=1$ and iterating the same argument, we obtain

$$(x^{d-2}+x^{d-3}+\cdots+x^2+x+1)^i =$$

$$= \sum_{k_1=0}^{i}\binom{i}{k_1}(x^{d-2}+x^{d-3}+\cdots+x^2+x)^{k_1} =$$

$$= \sum_{k_1=0}^{i}\binom{i}{k_1}(x^{d-3}+\cdots+x^2+x+1)^{k_1}x^{k_1} =$$

$$= \sum_{k_1=0}^{i}\binom{i}{k_1}\sum_{k_2=0}^{k_1}\binom{k_1}{k_2}(x^{d-4}+\cdots+x^2+x+1)^{k_2}x^{k_1+k_2} = \ldots =$$

$$= \sum_{k_1=0}^{i}\binom{i}{k_1}\sum_{k_2=0}^{k_1}\binom{k_1}{k_2}\cdots\sum_{k_{d-2}=0}^{k_{d-3}}\binom{k_{d-3}}{k_{d-2}}x^{k_1+k_2+\cdots+k_{d-2}} =$$

$$= \sum_{k_1=0}^{i}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{d-2}=0}^{k_{d-3}}\binom{i}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{d-3}}{k_{d-2}}x^{k_1+k_2+\cdots+k_{d-2}},$$

that can be rewritten as $\sum_{0\le k_{d-2}\le k_{d-3}\le\ldots\le k_2\le k_1\le i}\binom{i}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{d-3}}{k_{d-2}}x^{k_1+k_2+\cdots+k_{d-2}} = \sum_{j=0}^{i(d-2)}\sum_{\substack{0\le k_{d-2}\le k_{d-3}\le\ldots\le k_2\le k_1\le i\\ k_1+k_2+\ldots+k_{d-2}=j}}\binom{i}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{d-3}}{k_{d-2}}x^j$.

Therefore, recalling the definition of multinomial coefficient and that $A_{i,j}$ is the coefficient of x^j in $(x^{d-2}+x^{d-3}+\cdots+x^2+x+1)^i$,

$$A_{i,j} = \sum_{\substack{0\le k_{d-2}\le k_{d-3}\le\ldots\le k_2\le k_1\le i\\ k_1+k_2+\ldots+k_{d-2}=j}}\binom{i}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{d-3}}{k_{d-2}} =$$

$$= \sum_{\substack{0 \leq k_{d-2} \leq k_{d-3} \leq \ldots \leq k_2 \leq k_1 \leq i \\ k_1 + k_2 + \ldots + k_{d-2} = j}} \frac{i!}{(i-k_1)!(k_1-k_2)!\cdots(k_{d-3}-k_{d-2})!k_{d-2}!} =$$

$$= \sum_{\substack{0 \leq k_{d-2} \leq k_{d-3} \leq \ldots \leq k_2 \leq k_1 \leq i \\ k_1 + k_2 + \ldots + k_{d-2} = j}} \binom{i}{i-k_1, k_1-k_2, \ldots, k_{d-3}-k_{d-2}, k_{d-2}}.$$

The lemma follows by observing that $T_n(h,l,d) = \sum_{i=0}^{l} \sum_{j=0}^{h-1} A_{i,j}$. □

The following theorem is a direct consequence of Lemma 4.8, Lemma 4.9 and Definition 4.7.

Theorem 4.10 *For every h, l, d, the maximum number of nodes reachable on a chain network by a $\langle h, l, d\rangle$-layout is $\mathcal{T}_n(h,l,d) =$*

$$1 + \sum_{k=1}^{h} \sum_{i=0}^{l-1} \sum_{j=0}^{k-1} \sum_{\substack{0 \leq k_{d-2} \leq k_{d-3} \leq \ldots \leq k_2 \leq k_1 \leq i \\ k_1 + k_2 + \ldots + k_{d-2} = j}} \binom{i}{i-k_1, k_1-k_2, \ldots, k_{d-3}-k_{d-2}, k_{d-2}}.$$

Unfortunately $\mathcal{T}_n(h,l,d)$ in general cannot be expressed by means of a more compact closed formula. However, there are a few cases in which it can be significantly simplified. Some of them are listed below.

- $d = 2$: $\mathcal{T}_n(h,l,2) = h \cdot l + 1$.

  In fact, by the definition of the matrix A in the proof of Lemma 4.9, the only non null elements of A belong to the first column and their value is always equal to one. Hence, the number of the nodes of every $T(k,l,2)$ is $l+1$ and

  $$\mathcal{T}_n(h,l,2) = 1 + \sum_{k=1}^{h} T_n(k, l-1, 2) = 1 + h \cdot l.$$

  Such a number of nodes can be directly inferred also by exploiting the equation stated in the claim of Theorem 4.10.

- $d > h$: $\mathcal{T}_n(h,l,d) = \binom{h+l}{l}$.

In fact, the generic element $A_{i,j}$ of A for $1 \le i \le l$ and $0 \le j \le h-1$ can be simplified as $A_{i,j} = \binom{i-1+j}{j}$. Therefore, the sum of the elements of the submatrix given by the first $l+1$ rows and h columns of A is

$$1 + \sum_{i=1}^{l} \sum_{j=0}^{h-1} \binom{j+i-1}{i-1} = 1 + \sum_{i=1}^{l} \binom{h-1+i}{i} = 1 + \sum_{i=1}^{l} \binom{i+h-1}{h-1} =$$

$$= \sum_{i=0}^{l} \binom{i+h-1}{h-1} = \binom{h+l}{h} = \binom{h+l}{l}.$$

Therefore,

$$\mathcal{T}_n(h,l,d) = 1 + \sum_{k=1}^{h} \binom{k+l-1}{l-1} = \sum_{k=0}^{h} \binom{k+l-1}{l-1} = \binom{h+l}{l}.$$

Notice that in this case $\mathcal{T}_n(h,l,d)$ coincides with the number of nodes $\mathcal{T}_n(h,l)$ in the ordered tree $\mathcal{T}(h,l)$ defined in [11], that is with the maximum size of a chain admitting a $\langle h,l\rangle$-layout for standard ATM networks (i.e., without the distance measure). In fact, clearly $\mathcal{T}_n(h,l,d) \le \mathcal{T}_n(h,l)$ for every h,l,d. Moreover, the ordered tree $\mathcal{T}(h,l)$ of [11] induces a layout with distance $d = h+1$, so that $\mathcal{T}_n(h,l,d) \ge \mathcal{T}_n(h,l)$ when $d > h$. Therefore, in this case $\mathcal{T}(h,l)$ and $\mathcal{T}(h,l,d)$ coincide and $\mathcal{T}_n(h,l) = \mathcal{T}_n(h,l,d) = \binom{h+1}{l}$.

- $d = h$: $\mathcal{T}_n(h,l,d) = \mathcal{T}_n(h,l) - \frac{l(l-1)}{2} = \binom{h+l}{l} - \frac{l(l-1)}{2}$.

  In fact, it is easy to see that the number of nodes removed from $\mathcal{T}(h,l)$ to get $\mathcal{T}(h,l,d)$ can be suitably bounded as $\frac{l(l-1)}{2}$.

- $(l-1)(d-2) < d \le h$: $\mathcal{T}_n(h,l,d) = (h-d)\frac{(d-1)^l}{d-2} + \binom{d+1}{l} - \frac{l(l-1)}{2}$.

  In fact, since all the elements of the row of index i in A are null starting from the column of index $l(d-2)+2$, the sum of all the elements of the submatrix given by the first $l+1$ rows and h columns of A coincides with the sum of all the elements in the first $l+1$ rows of A. Therefore, since by definition of A the sum of the elements in the row of index i is $(d-1)^i$, it results $T_n(h,l,d) = \sum_{i=0}^{l}(d-1)^i = \frac{(d-1)^{l+1}}{d-2}$,

  By Definition 4.7, we can decompose $\mathcal{T}(h,l,d)$ in $(h-d)$ trees $T(k,l-1,d)$ for $h-d \le k \le h$ and a remaining tail of trees in the recursive decomposition that coincides with $\mathcal{T}(d,l,d)$. Hence, by the previous considerations for the case $h = d$, we have $\mathcal{T}_n(h,l,d) = (h-d)\frac{(d-1)^l}{d-2} + \binom{d+l}{l} - \frac{l(l-1)}{2}$.

5 Optimal layouts for ring networks

In this section we provide optimal layouts for ring networks R_n with $V = \{0, 1, \ldots, n-1\}$ and $E = \{\{i, (i+1)\, mod\, n\} | 0 \leq i \leq n-1\}$. Again we assume that the adjacency graph A coincides with R_n and without loss of generality we take $s = 0$ as the source node. Moreover, we let $d > 1$, since as remarked in Section 3, no layout with maximum distance 1 exists for cyclic adjacency graphs.

Notice that in any $\langle h, l, d \rangle$-layout Ψ for R_n, by the shortest path property, if n is odd the nodes in the subring $[1, \lfloor \frac{n}{2} \rfloor]$ are reached in one direction from the source, say clockwise, while all the remaining ones anti-clockwise. This means that Ψ can be divided into two separated sublayouts Ψ_c and Ψ_a respectively for the subchains of the nodes reached clockwise in Ψ, that is $[0, \lfloor \frac{n}{2} \rfloor]$, and anticlockwise, that is from $\lceil \frac{n}{2} \rceil$ to 0 in clockwise direction, extremes included. However, the results of the previous section for chains do not extend in a trivial way, as a further constraint exists for the final nodes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$, that are adjacent in A and thus must be at distance at most d in Ψ. A similar observation holds when n is even.

As for chains, let us say that a $\langle h, l, d \rangle$-layout Ψ for rings is optimal if it reaches the maximum number of nodes. Moreover, let us call Ψ canonic if the clockwise and anticlockwise sublayouts Ψ_c and Ψ_a are both crossing-free and the virtual topologies induced by their VPs are trees. The following lemma is the equivalent of Theorem 4.4 for rings.

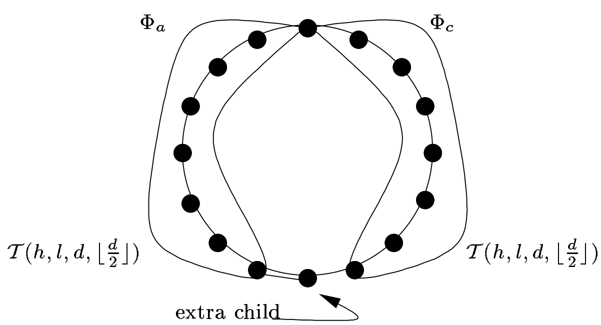

Figure 7: Optimal layout for a ring network with an odd d.

Lemma 5.1 *For every h, l, d, there exists an optimal $\langle h, l, d \rangle$-layout for rings that is canonic.*

Proof.

Let Ψ be an optimal $\langle h, l, d \rangle$-layout for a ring R_n, Ψ_c and Ψ_a be the sublayouts of Ψ induced respectively by the subchains of the nodes reached clockwise and anticlockwise from the source in Ψ, n_c and n_a be the number of nodes of Ψ_c and Ψ_a, and finally u and v be the extreme nodes of Ψ_c and Ψ_a, that is the farthest ones from the source. Clearly, the number of nodes in R_n is $n = n_c + n_a - 1$, since the source belongs both the two subchains and thus it must be counted only once.

We distinguish between the following two cases.

Case 1: $h_\Psi(u) \leq \lfloor \frac{d}{2} \rfloor$ and $h_\Psi(v) \leq \frac{d}{2}$.
Without loss of generality let us assume that the clockwise subchain is not smaller than the anticlockwise one, that is $n_c \geq n_a$. If Ψ_c is not canonic, then by performing the same steps of Theorem 4.4, it is possible to obtain a $\langle h,l,d \rangle$-layout Φ_c for a larger clockwise subchain (if Ψ_c is canonic let $\Phi_c = \Psi_c$). Then, it is possible to replace Ψ_a with the canonic layout Φ_a symmetric to Φ_c. Clearly, the layout Φ given by the union of Φ_c and Φ_a forms a canonic layout for a ring R_m with $m \geq n$. Such a ring is obtained by adding the edge $\{w,z\}$ between w and z, where w and z are the last nodes of Φ_c and Φ_a, respectively. In order to show that Φ is a $\langle h,l,d \rangle$-layout for R_m, we observe that the only violation of the performance measures can be the distance between the VCs of the two extreme nodes w, z. However, by the definition of distance and by the construction of Theorem 4.4, $d_\Phi(w,z) \leq h_\Phi(w) + h_\Phi(z) \leq h_\Psi(u) + h_\Psi(v) \leq 2\frac{d}{2} = d$.

Case 2 $h_\Psi(u) > \frac{d}{2}$ or $h_\Psi(v) > \frac{d}{2}$ (or analogously $h_\Psi(u) \geq \frac{d+1}{2}$ or $h_\Psi(v) \geq \frac{d+1}{2}$).
Since u and v are adjacent in R_n and reached respectively clockwise and anticlockwise, a handover from u to v (or vice versa) requires the replacement of all the VPs in $VC_\Psi(u)$ and the addition of all the VPs in $VC_\Psi(v)$. Therefore, $d_\Psi(u,v) = h_\Psi(u) + h_\Psi(v) \leq d$ and thus either $h_\Psi(u) \leq \frac{d-1}{2}$ or $h_\Psi(v) \leq \frac{d-1}{2}$. Without loss of generality, assume that the first case holds.

Again, if Ψ_c is not canonic, let Φ_c the canonic $\langle h,l,d \rangle$-layout for the clockwise subchain obtained like in Theorem 4.4 ($\Phi_c = \Psi_c$ otherwise). Let Φ_a the canonic $\langle h,l,d \rangle$-layout for the anticlockwise subchain symmetric to Φ_c. Before constructing the final global layout Φ, we append to the last node z of Φ_a a further node z' by means of the edge $\{z,z'\}$ (see Figure 7), we add the VP $\langle z,z' \rangle$ to Φ_a and finally fix $VC_\Phi(z')$ as the concatenation of $VC_\Phi(z)$ and $\langle z,z' \rangle$. Let Φ be the layout given by the union of Φ_c and Φ_a.

Again, Φ forms a canonic layout for a ring R_m with $m \geq n$ obtained by joining z' and the last node of Φ_c, w, with the edge $\{w,z'\}$. In fact, by the shortest path property, $|n_c - n_a| \leq 1$, and if m_c and m_a are the cardinalities of the clockwise and anticlockwise subchains of Φ, as $m_c \geq n_c$, $m_a = m_c + 1 \geq n_c + 1 \geq n_a$. Therefore, $m = m_c + m_a - 1 \geq n_c + n_a - 1 = n$.

Clearly, all the VCs of Φ correspond to shortest paths, and in order to complete the proof it remains to show that again Φ is a $\langle h,l,d \rangle$-layout for R_m. To this aim, we observe that the only violations of the performance measure can be the hop count $h_\Phi(z')$ of z', plus the distances between the VCs of the nodes w and z', and of the nodes z and z'. Since by hypothesis $h_\Psi(u) \leq h_\Psi(v) - 1$, $h_\Phi(z') = h_\Phi(z) + 1 = h_\Phi(w) + 1 \leq h_\Psi(u) + 1 \leq h_\Psi(v) \leq h$. Moreover, $d_\Phi(z,z') = 1$ and $d_\Phi(w,z')) \leq h_\Phi(w) + h_\Phi(z') = h_\Phi(w) + h_\Phi(z) + 1 \leq h_\Psi(u) + h_\Psi(v) + 1 \leq 2\frac{d-1}{2} + 1 = d$. □

Notice that Lemma 5.1 holds for the distances D and $D_\triangle$ indifferently, as the same holds for Theorem 4.4. Again, since D and $D_\triangle$ coincide in layout inducing trees, all the following results are valid under both the two cost measures.

Starting from Lemma 5.1, we generalize the ordered tree $\mathcal{T}(h,l,d)$ to $\mathcal{T}(h,l,d,t)$ by adding a further parameter $t \leq h$, which fixes the hop count of the rightmost leaf to t.

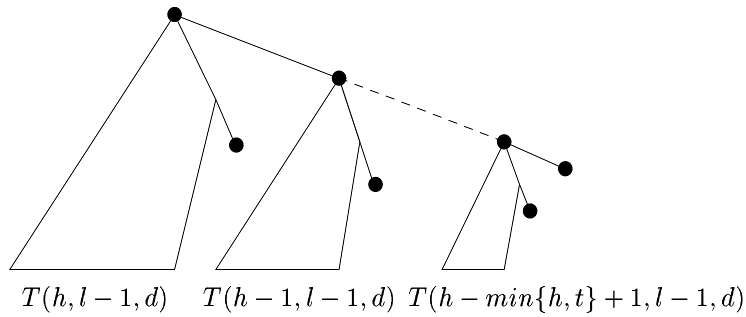

Figure 8: $\mathcal{T}(h, l, d, t)$ in terms of trees of type T.

Roughly speaking, $\mathcal{T}(h, l, d, h) = \mathcal{T}(h, l, d)$ and $\mathcal{T}(h, l, d, d-1) = T(h, l, d)$. More precisely, $\mathcal{T}(h, l, d, t)$ is defined recursively as the join of the roots of $min\{h, t\}$ subtrees $T(i, l-1, d)$ for $h - min\{h, t\} < i \leq h$ in such a way that for $i < h$ the root of a $T(i, l-1, d)$ is the rightmost child of the root of a $T(i+1, l-1, d)$, plus a final node as rightmost child of $T(h - min\{h, t\} + 1, l-1, d)$ (see Figure 8). Thus, $\mathcal{T}_n(h, l, d, t) = 1 + \sum_{k=h-min\{h,t\}+1}^{h} T_n(k, l-1, d)$, that is $\mathcal{T}_n(h, l, d, t) =$

$$1 + \sum_{k=h-min\{h,t\}+1}^{h} \sum_{i=0}^{l-1} \sum_{j=0}^{k-1} \sum_{\substack{0 \leq k_{d-2} \leq k_{d-3} \leq \ldots \leq k_2 \leq k_1 \leq i \\ k_1 + k_2 + \ldots + k_{d-2} = j}} \binom{i}{i-k_1, k_1-k_2, \ldots, k_{d-3}-k_{d-2}, k_{d-2}}.$$

Lemma 4.8 extends directly to $\mathcal{T}(h, l, d, t)$, that in turn corresponds to an optimal $\langle h, l, d\rangle$-layout for a chain with the further property that the rightmost node (opposite of the source) has hop count t. Therefore, it is possible to prove the following theorem.

Theorem 5.2 *The maximum number of nodes reachable on a ring network by a $\langle h, l, d\rangle$-layout is $2\mathcal{T}_n(h, l, d, \lfloor \frac{d}{2} \rfloor) - ((d+1) \bmod 2)$, with $\mathcal{T}_n(h, l, d, \lfloor \frac{d}{2} \rfloor) =$*

$$1 + \sum_{k=h-min\{h,\lfloor \frac{d}{2} \rfloor\}+1}^{h} \sum_{i=0}^{l-1} \sum_{j=0}^{k-1} \sum_{\substack{0 \leq k_{d-2} \leq k_{d-3} \leq \ldots \leq k_2 \leq k_1 \leq i \\ k_1 + k_2 + \ldots + k_{d-2} = j}} \binom{i}{i-k_1, k_1-k_2, \ldots, k_{d-3}-k_{d-2}, k_{d-2}}.$$

Proof. As already observed, any layout Ψ for a ring R_n can be split in two separated sublayouts Ψ_c and Ψ_a for the subchains of the nodes reached clockwise and anticlockwise in Ψ, respectively. If t_1 and t_2 are the hop counts of the last two nodes u and v of the two subchains, the condition $d(u, v) = h(u) + h(v) \leq d$ must hold, therefore $t_1 \leq \lfloor \frac{d}{2} \rfloor$ or $t_2 \leq \lfloor \frac{d}{2} \rfloor$. Without loss of generality let us focus on layouts in which $t_1 \leq t_2$ and let us consider first the case in which d is odd.

Since $\mathcal{T}(h,l,d,t_1)$ corresponds to an optimal $\langle h,l,d\rangle$-layout for a chain in which the last node has hop count t_1, if n_c and n_a are the number of nodes in the clockwise and anticlockwise subchains, it results $n_c \leq \mathcal{T}_n(h,l,d,\frac{d-1}{2})$ and by the shortest path property $n_a \leq n_c + 1$, hence any ring R_n admitting a $\langle h,l,d\rangle$-layout has at most $n = n_a + n_c - 1 \leq 2n_c \leq 2\mathcal{T}_n(h,l,d,\frac{d-1}{2})$ nodes.

A layout with such a number of nodes can be obtained by taking the layout Ψ_c induced by $\mathcal{T}(h,l,d,\frac{d-1}{2})$ for the clockwise subchain, and for the anticlockwise one the symmetric layout Ψ_a induced by $\mathcal{T}(h,l,d,\frac{d-1}{2})$ plus a final node attached as rightmost child of the righmost leaf of $\mathcal{T}(h,l,d,\frac{d-1}{2})$. The union of Ψ_c and Ψ_a clearly forms an optimal canonic $\langle h,l,d\rangle$-layout Ψ, that is for a ring R_n with a maximum number of nodes.

The case in which d is even is simpler, as it is immediate to see that the maximum number of nodes is obtained when Ψ_c and Ψ_a are both induced by $\mathcal{T}(h,l,d,\frac{d}{2})$, thus yielding an optimal canonic $\langle h,l,d\rangle$-layout Ψ for a ring R_n with $n = 2\mathcal{T}_n(h,l,d,\frac{d}{2}) - 1$ nodes. This concludes the proof. □

Before concluding this section, let us observe that as for chains there are cases in which the formula of Theorem 5.2 can be simplified. For instance we obtain $\mathcal{T}_n(h,l,d,\lfloor\frac{d}{2}\rfloor) = 1 + l$ for $d = 2$, $\mathcal{T}_n(h,l,d,\lfloor\frac{d}{2}\rfloor) = 1 + \binom{h+l}{l} - \binom{h-\lfloor\frac{d}{2}\rfloor+l}{l}$ for $d > h$, $\mathcal{T}_n(h,l,d,\lfloor\frac{d}{2}\rfloor) = 1 + \binom{h+l}{l} - \binom{h-\lfloor\frac{d}{2}\rfloor+l}{l} - \frac{l(l-1)}{2}$ for $d = h$ and $\mathcal{T}_n(h,l,d,\lfloor\frac{d}{2}\rfloor) = 1 + (h-d)\frac{(d-1)^l}{d-2} + \binom{d+l}{l} - \binom{d-\lfloor\frac{d}{2}\rfloor+l}{l} - \frac{l(l-1)}{2}$ for $(l-1)(d-2) < d \leq h$.

6 Conclusion

We have extended the basic ATM model presented in [12, 7] to cope with quality of service and mobility aspect in wireless ATM networks. This is obtained by adding a further measure, the VCs distance, that represents the time needed to reconstruct connecting VCs when handovers occur and must be maintained as low as possible in order to avoid the rerouting mechanism to be appreciated by the mobile users. We have shown that the problem of finding suitable trade-offs between the various performance measures is in general an intractable problem, while optimal constructions have been given for chain and ring topologies.

Among the various questions left open, we have the extension of our results to more general topologies. Moreover, it would be nice to consider the case in which the physical graph does not coincide with the adjacency graph. A typical example, is a GSM like network in which the physical network is a tree and cells correspond to its leaves. Another worth investigating issue is the determination of layouts in which the routed paths are not necessarily the shortest ones, but have a fixed stretch factor or even unbounded length. Finally, all the results should be extended to the all-to-all communication pattern, that is when connections can be established between every pair of mobile users.

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