Unfolding Of Surfaces

Jean-Marie Morvan, Boris Thibert

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Unfolding Of Surfaces

Jean-Marie Morvan — Boris Thibert

N° 4615

Novembre 2002

THÈME 2

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4615--FR+ENG

Unfolding Of Surfaces

Jean-Marie Morvan , Boris Thibert

Thème 2 — Génie logiciel
et calcul symbolique
Projet Prisme

Rapport de recherche n° 4615 — Novembre 2002 — 43 pages

Abstract: This paper deals with the approximation of the unfolding of a smooth developpable surface with a triangle mesh. First of all, we give an explicit approximation of the unfolding of a smooth developable surface with the unfolding of a developable triangle mesh close to the smooth surface. The quality of the approximation depends on the maximal angle between the normals of the two surfaces and the relative curvature distance of the smooth surface (which is linked to the curvature of the smooth surface and the Hausdorff distance between the two surfaces). We give examples of sequences of developable triangle meshes inscribed on a sphere of radius 1, with a number of vertices and edges tending to infinity.

Key-words: Developable surface, triangle mesh, Gauss curvature.

This work is partially supported by the IST Programme of the EU as a Shared-cost RTD (FET Open) Project under Contract No IST-2000-26473 (ECG - Effective Computational Geometry for Curves and Surfaces), and by Région Rhônes Alpes, contrat 99029744.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Dépliage de Surfaces

Résumé : Ce travail étudie l'approximation du dépliage dans le plan d'une surface lisse développable par le dépligage dans le plan d'une surface triangulée développable. Nous donnons dábord une approximation explicite qui dépend de l'angle maximal entre les normales des deux surfaces, de la courbure de la surface lisse et de la distance de Hausdorff entre les deux surfaces. Nous donnons ensuite des exemples de suites de triangulations developpables dont le nombre de sommets et d'arêtes tend vers l'infini, toutes inscrites sur une sphère de rayon 1.

Mots-clés : Surfaces developpables, maillages triangulés, courbure de Gauss

1 Introduction

This paper deals with the approximation of a smooth surface by a developable triangle mesh.

- In the main part of this paper, we consider the case where the smooth surface is itself developable. It is well known that a smooth surface is developable (that is, locally isometric to a plane) if and only if its Gauss curvature is identically 0. Similarly, a triangle mesh is developable if its *discrete Gauss curvature* is 0 (that is the sum of the angles incident to any interior vertex is 2π). After unfolding the smooth surface and the triangle mesh, we compare their shape by giving an upper bound of their mutual Hausdorff distance in terms of geometric invariants: the angles of the triangles of the mesh, the curvature of the smooth surface, the Hausdorff distance between both initial surfaces and the maximal angle between the normals of both surfaces. Remark that the two unfolded surfaces can be very different from one another, even if the initial surfaces are close for the Hausdorff distance. The unfolding of the *half "lampion" of Schwarz* convinces us easily. (This problem appears in many applications, such as geology, where people want to unfold strata under isometric deformations. At a certain scale, the stratum can be considered as a smooth surface, approximated by a triangle mesh (some of its vertices belong to data).)

- On the other hand, we devote a paragraph to contradict the mistaken belief that the geometry of a smooth surface is better and better approximated if the triangle mesh approximation has a bigger and bigger amount of vertices on it. We construct a sequence of developable triangle meshes whose cardinal of vertices and edges goes to infinity, and which are all incribed on a portion of a sphere of fixed radius. We end this paragraph by showing an exemple of a triangle mesh with strictly negative Gauss curvature at each interior vertex, incribed on a convex smooth surface, which has the following property: switching some edges and keeping its vertices fixed, it is still inscribed on the same smooth surface but is now positively curved at each interior vertex.

It is important to notice that our approach is different from the work of Alla Sheffer et al. [16], Mathieu Desbrun et al. [5] and Bruno Lévy et al. [11], which give explicit algorithms to unfold triangulations which are not necessarily developpable. These authors minimize an energy ([5] and [11]) related to the distorsion of the angles of the triangles. However, there is no underlying smooth surface in their work, and then no comparison betwenn smooth and discrete unfoldings is possible. On the contrary, we give a theorem which compares the unfolding of a smooth developpable surface and the unfolding of a developpable triangulation.

The notion of the *reach of a smooth surface* is one of the main tools of this paper. It allows to compare a smooth surface with a triangulated mesh "close to it". It was first introduced by H. Federer [7]. It is interesting to notice that the *reach* is in fact linked to the (more recent) notions of *medial axis* and *local feature size*, which are used in some problems of reconstructing a surface from scattered sample points. In [19], F.E. Wolter gives many interesting results related to the *medial axis* and the *cut locus*.

This paper is organized as follows. Section 2 gives classical and usual definitions. Section 3 states our results of approximations. Section 4 gives some *"bad examples"* concerning the Gauss curvature of smooth surfaces and inscribed triangle meshes. Sections 5, 6 and 7 sketch the proofs of results (in section 5 we also give results of approximation of the lengths of the curves).

2 Definitions

We recall here some classical definitions which concern smooth surfaces, triangle meshes and the relative position of two surfaces. For more details on smooth surfaces, one may refer to [1], [6] or [18]. For more details on triangle meshes, one may refer to [7], [8] or [13].

2.1 Smooth surfaces

- In the following, a smooth surface means a $\mathcal{C}^2$ surface which is regular, oriented, compact with or without boundary. Let S be a smooth surface of the (oriented) euclidean space $\mathbb{E}^3$. Let ∂S denote the boundary of S. S is endowed with the Riemannian structure induced by the standard scalar product of $\mathbb{E}^3$. We denote by da the area form on S and by ds the canonical orientation of ∂S. Let ν be the unitary normal vector field (compatible with the orientation of S) and h be the second fundamental form of S associated with ν. Its determinant at a point p of S is *the Gauss curvature* G_p, its trace is *the mean curvature* H_p. *The maximal curvature of* S *at* p is $\rho_p = max(|\lambda_p^1|, |\lambda_p^2|)$, where λ_p^1 and λ_p^2 are the principal curvatures of S at p, (that is the eigenvalues of the second fundamental form). *The maximal curvature of* S is
$$\rho_S = \sup_{p\in S} \rho_p.$$
We denote by k_p *the geodesic curvature* of ∂S at p.

- We need the following

  Proposition 1 *Let S be a smooth compact surface of $\mathbb{E}^3$. Then there exists an open set U_S of $\mathbb{E}^3$ containing S and a continuous map ξ from U_S onto S satisfying the following: if p belongs to U_S, then there exists a unique point $\xi(p)$ realizing the distance from p to S (ξ is nothing but the orthogonal projection onto S).*

  A proof of this proposition can be found in [7].
  The open set U_S depends locally and globally on the smooth surface S. Locally, the normals of S do not intersect in U_S. Globally, U_S depends on points which can be far from one another on the surface, but close in $\mathbb{E}^3$.
  We shall also need the notion of the *reach of a surface*, introduced by H. Federer in [7].

Definition 1 *The reach of a surface S is the largest $r > 0$ for which ξ is defined on the open tubular neighborhood $U_r(S)$ of radius r of S.*

Remark that the reach r_S of S is smaller than the minimal radius of curvature of S (which is $\frac{1}{\rho_S}$); (see [12] or [19] for more details). Thus, we have:

$$\rho_S \; r_S \leq 1,$$

where ρ_S is the maximal curvature of S.

2.2 Triangle meshes

2.2.1 Generalities

A triangle mesh T is a (finite and connected) union of triangles of $\mathbb{E}^3$, such that the intersection of two triangles is either empty, or equal to a vertex, or equal to an edge.
We denote by $\mathcal{T}_T$ the set of triangles of T and by Δ a generic triangle of T.

- η_Δ denotes the length of the longest edge of Δ, and $\mathcal{A}(\Delta)$ the area of Δ.
- The *area* $\mathcal{A}(T)$ is the sum of the areas of all the triangles of T.

For the following, we need to define a new geometric invariant on the mesh:

Definition 2 *Let Δ be a triangle of a triangle mesh T.*

- *The straightness of a Δ is the real number*

$$str(\Delta) = \sup_{p \text{ vertex of } \Delta} \sin(\theta_p),$$

where θ_p is the angle at p of Δ.

- *The straightness of T is:*

$$str(T) = \min_{\Delta \in \mathcal{T}_T} str(\Delta).$$

Remark 1 *In particular, if β is any of the three angles of the triangle Δ, we have:*

$$\sin \beta \leq str(\Delta).$$

2.2.2 Triangle mesh close to a smooth surface

- A triangle mesh (or a smooth surface) M is *closely near* a smooth surface S if:
  1. M lies in $U_r(S)$, where r is the reach of S,
  2. the restriction of ξ to M is one-to-one (where ξ is the map defined in Proposition 1).

- We say that a triangle mesh of $\mathbb{E}^3$ is *inscribed* in a smooth surface S if all its vertices belong to S.
- A triangle mesh T is *closely inscribed* in a smooth surface S if:
  1. T is closely near S,
  2. all the vertices of T belong to S.
- Let T be a triangle mesh closely near a smooth surface S. Let m be a point lying in the interior of a triangle Δ of T. Let N^Δ be the normal line through m to Δ. We put

  $$\alpha_m = \langle N^\Delta, \nu^S_{\xi(m)} \rangle \in \left[0, \frac{\pi}{2}\right].$$

  The real number α_m is defined almost everywhere on T. (If m is a point on an edge or a vertex, one can define α_m by taking the supremum of the angles between the triangles which contain m and the normal $\nu^S_{\xi(m)}$).

  We can define the real number

  $$\alpha = \sup_{m \in T} \alpha_m.$$

  α is called *the maximal angle between the normals of S and T.*

We introduce now an invariant which relates the triangle mesh and the smooth surface:

Definition 3 *Let T be a triangle mesh (or just a triangle) closely near a smooth surface S. The relative curvature of S to T is the real number defined by:*

$$\omega_S(T) = \sup_{m \in T \setminus \partial T} \|\xi(m) - m\| \ \rho_{\xi(m)}.$$

Remark 2
A compact triangle mesh T closely near a smooth surface S satisfies:

$$\omega_S(T) < 1.$$

Moreover, a triangle mesh T closely inscribed in a smooth surface S satisfies:

$$\omega_S(T) \le \sup_{\Delta \in \mathcal{T}_T} \eta_\Delta \ \rho_{\xi(\Delta)}.$$

(In fact, if m belongs to a triangle Δ, then $\|\xi(m) - m\|$ is smaller than the distance from m to any point of S. If s is a vertex of Δ, we have $\|\xi(m) - m\| \le ms \le \eta_\Delta$).

2.2.3 Gauss curvature of a triangle mesh

Let T denote a triangle mesh, p a vertex of T, $\mathcal{T}_T(p)$ the set of triangles of T which contain p as a vertex. Let S_T^o denote the set of interior vertices of T and $S_{\partial T}$ the set of vertices of the boundary ∂T of T.

- We call the *angle at the vertex* p the real:

$$\alpha_T(p) = \sum_{\sigma \in \mathcal{T}_T(p)} \alpha_\sigma(p),$$

  where $\alpha_\sigma(p)$ is the angle at p to the triangle σ.

- The *discrete Gauss curvature at a vertex* $p \in S_T^o$ is:

$$G_T(p) = 2\pi - \alpha_T(p).$$

- The *discrete geodesic curvature at a vertex* $p \in S_{\partial T}$ is:

$$k(p) = \pi - \alpha_T(p).$$

- The *total interior Gauss curvature* of T is:

$$G_{int}(T) = \sum_{p \in S_T^o T} G_T(p) \ = \sum_{p \in S_T^o T} (2\pi - \alpha_T(p)).$$

- The *total geodesic curvature* of ∂T is:

$$\mathcal{K}(\partial T) = \sum_{p \in S_{\partial T}} k(p) \ = \sum_{p \in S_{\partial T}} (\pi - \alpha_T(p)).$$

2.3 Flat (or developable) surfaces

A smooth surface (*resp.* a triangle mesh) M is *flat* or *developable* if its Gauss curvature is null at each interior point, (*resp.* vertex). Remark that *theorema egregium* of Gauss implies in this case that the interior of M is locally isometric to an open domain of the Euclidean plane $\mathbb{E}^2$. If M is a topological disc, then this isometry is global; we call it an *unfolding* $u(M)$ *of* M. In this case, remark that $u(M)$ is unique up to a rigid motion of $\mathbb{E}^2$.

Remark 3 If M is flat, then there exists an homeomorphism $g_M : M \to u(M)$ which preserves distances. Furthermore, we can suppose that g_M is direct (i.e. if g_M is differentiable at a point m, then $Dg_M(m)$ is positive definite).

2.4 Hausdorff distance between two subsets of $\mathbb{E}^3$

The *Hausdorff distance* between two subsets A and B of $\mathbb{E}^3$ is:

$$\delta_{Haus}(A,B) = max(\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)).$$

Remark 4
A compact triangle mesh T closely near a smooth surface S satisfies:

$$\omega_S(T) \le \delta_{Haus}(T,S)\ \rho_S.$$

3 Unfolding of a developable surface

In this section, we consider the following situation: S is a smooth developable surface, T is a developable triangle mesh, which is closely near S. We aim at knowing whether the unfolding of the triangle mesh T is a *"good approximation"* of the unfolding of the smooth surface S. We first consider the counter-example of the *half "lampion" of Schwarz*. Then we give an explicit approximation of the unfolding of the smooth surface S in the convex case (part 3.2). Finally we give a result of convergence in the general case (part 3.3).

3.1 Half "lampion" of Schwarz

A typical example of this situation is the famous *"lampion" of Schwarz*. It is a flat triangle mesh inscribed on a cylinder. It is not simply connected, but we can "cut a piece of it" which is homeomorphic to a topological disc: we consider here a *half "lampion" of Schwarz* (which is inscribed in half a cylinder). We illustrate here two phenomena:

- In part 3.1.1, we give two half "lampions" of Schwarz whose unfoldings are very different from one another. Therefore we cannot expect to have a result of convergence without other assumptions.
- In part 3.1.2, we build two triangle meshes which have the same vertices and whose unfoldings are very different from one another. This implies that the result depends on the construction of a triangle mesh from scattered sample points.

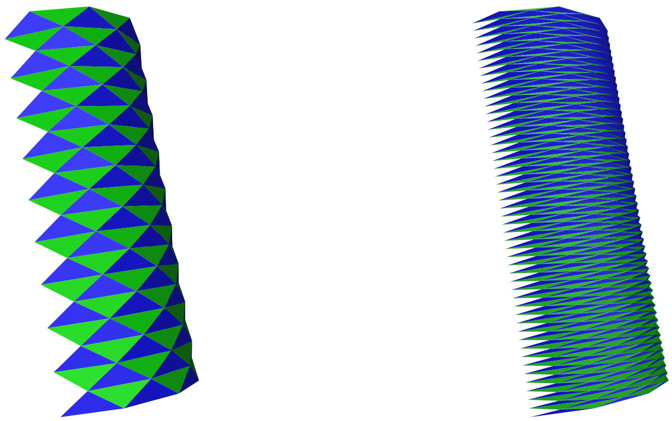
3.1.1 Comparaison of two half "lampion" of Schwarz

Let C be a half cylinder of finite height H and of radius R. It can be parametrized by:

$$\forall t \in [0,\pi]\ \forall u \in [0,H]\ f(t,u) = (R\ \cos(t), R\ \sin(t), u).$$

Let $P(n,N)$ denote the triangulated mesh whose vertices $S_{i,j}$ belong to C and are defined as follows:

$$\begin{array}{ll} \forall i \in \{0,..n-1\} & S_{i,j} = (R\ \cos(i\alpha), R\ \sin(i\alpha), jh) \text{ if } j \text{ is even,} \\ \forall j \in \{0,..N\} & S_{i,j} = (R\ \cos(i\alpha+\frac{\alpha}{2}), R\ \sin(i\alpha+\frac{\alpha}{2}), jh) \text{ if } j \text{ is odd,} \end{array}$$



(a) P(19-5) (b) P(99-5)

Figure 1: Examples of *half "lampions" of Schwarz*

and whose faces are:

$$S_{i,j}\ S_{i+1,j}\ S_{i,j+1},$$
$$S_{i,j}\ S_{i-1,j+1}\ S_{i,j+1},$$

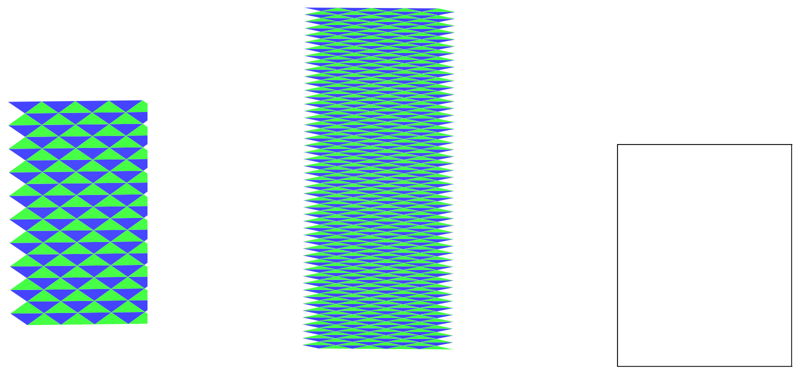
where $\alpha = \frac{\pi}{n}$ and $h = \frac{H}{N}$.

Those triangle meshes $P(n, N)$ are called *half "lampions" of Schwarz* and have the property of *being developable*. It is well-known that we can construct a *half "lampion" of Schwarz* whose area is as large as we wish (c.f. [1] or [14]). It implies that its unfolding can be "very different" from the unfolding of the half cylinder C.

Remark that the boundaries of the two unfolded *half "lampions" of Schwarz* of figure 2 are very different from one another and can be very different from the unfolding of the half cylinder C. The unfolding of C is a rectangle of height H. The height of the unfolded *half "lampions" of Schwarz* $P(99-5)$ is more than $1.5H$. In fact, the height of a *half "lampion" of Schwarz* is getting larger when it is unfolded.

Furthermore, if we consider the problem of the convergence of a sequence of triangle meshes, one may notice that the height of the unfolding of the *half "lampion" of Schwarz* $P(n, n^3)$ tends to infinity when n tends to infinity.

That is why, without other assumptions, we cannot expect the unfolding of a sequence of triangle meshes to give us a good approximation of the unfolding of the smooth surface. As we will see in part 3.3, this is linked to the fact that one of the following conditions is not satisfied:



(a) Unfolded P(19-5) (b) Unfolded P(99-5) (c) Unfolded half cylinder

Figure 2: Unfolding of C and of two *half "lampions" of Schwarz* closely inscribed in C (the scale is the same)

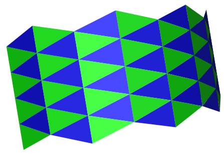
- the lengths of the edges of the sequence of triangle meshes tends to 0;
- the straightness of the sequence of triangle meshes is uniformly bounded from below by a strictly positive constant.

3.1.2 Two developable triangle mesh with the same vertices

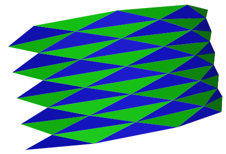
We consider a finite family of points $\mathcal{S}$ (which belong to a half cylinder) and we build two triangle meshes whose vertices are these points (figure 3(a) and 3(b)). The triangle mesh 3(b) is a half Lampion of Schwarz. The two unfoldings are different from one another. Just remark that (if we do not consider the boundary of 3(a) and 3(b)) the surface 3(a) is obtained by flipping the edges of the surface 3(b).

3.2 Approximation of the unfolding in the convex case

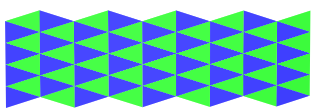
Consider now the case in which the triangle mesh T is closely near the smooth surface S (vertices of T are close to S- they are not strictly obliged to belong to S). The following result gives an explicit approximation of the unfolding of S in terms of the one of T if $u(S)$ and $u(T)$ are convex. Notice that if the normals of S are close enough to the normals of T, then the two unfoldings are quite similar.



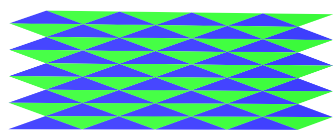
(a) A developable triangulation whose vertices are $\mathcal{S}$



(b) A Half-Lampion of Schwarz whose vertices are $\mathcal{S}$



(c) Unfolding of 3(a)



(d) Unfolding of 3(b)

Figure 3: Unfolding of two triangle meshes which have the same set of vertices $\mathcal{S}$

Theorem 1 *Let S be a smooth compact connected and developable surface of $\mathbb{E}^3$ and T be a developable triangulated mesh of $\mathbb{E}^3$ closely near S. If $u(S)$ and $u(T)$ are convex and if*

$$\epsilon = \frac{1 + \omega_S(\Delta)}{\cos(\alpha)} - 1 < 1,$$

then (up to a motion of the plane $\mathbb{E}^2$) we have:

$$\delta_{Haus}(u(S), u(T)) < diam(T)\ \sqrt{\epsilon}\ \frac{\sqrt{1+2\sqrt{\epsilon}+40\epsilon}}{(1-\epsilon)^2},$$

where $diam(T)$ is the diameter of T, α is the maximal angle between the normals and $\omega_S(T)$ is the relative curvature of S to T.

Remark 5

- We can have the same result in a more general case: if $u(T)$ is star-set in two points p and q such that $diam(u(T)) = pq$ and if $u(S)$ is star-set in two particular points (for more detail, see the proofs and especially Proposition 5).
- We could have given a result which is linear in ϵ (and not in $\sqrt{\epsilon}$). But in that case the result depends on the geometry of $u(T)$ and is more difficult to state. More precisely, if ϵ is small, we could have $\delta_{Haus}(u(S), u(T)) < diam(T)\ \tilde{K}_T\ \epsilon$. Unfortunatly, $\tilde{K}_T$ may be large (for example if there are a lot of triangles or if the angles of T are small) and the upper bound of $\delta_{Haus}(u(S), u(T))$ may be worst.

3.3 Convergence of the unfolding in the general case

Theorem 2 *Let S be a smooth compact connected developable surface of $\mathbb{E}^3$ and $(T_n)_{n\geq 0}$ a sequence of developable triangulated meshes closely near S such that:*

- *the normals of T_n tend to the normals of S,*
- *the lengths of the edges of T_n tend to 0 when n tends to infinity;*

then a sequence $(u(T_n))_{n\geq 0}$ of unfoldings of $(T_n)_{n\geq 0}$ tends in the Hausdorff sense to an unfolding $u(S)$ of S.

We know that the convergence of the normals is implied by a condition on the straightness when the vertices of triangle meshes belong to S [15]. Therefore we have the following corollary :

Corollary 1 *Let S be a smooth compact connected developable surface of $\mathbb{E}^3$ and $(T_n)_{n\geq 0}$ a sequence of developable triangulated meshes closely inscribed in S such that:*

- *the straightness of the sequence $(T_n)_{n\geq 0}$ is uniformly bounded from below by a strictly positive constant;*
- *the lengths of the edges of T_n tend to 0 when n tends to infinity;*

then a sequence $(u(T_n))_{n\geq 0}$ of unfoldings of $(T_n)_{n\geq 0}$ tends in the Hausdorff sense to an unfolding $u(S)$ of S.

Remark 6 It is possible to have an explicit result of approximation. However, the result would depend on a third surface $U_\beta \subset u(S)$ and giving further details would be tedious (if you wish to get more details, you can refer to Proposition 7).

4 Some remarks on the approximation by developable meshes

The crucial assumption in the previous theorem is that both smooth surface and triangle mesh are developable. The goal of this section is to insist on the mistaken belief that one can get a good approximation of the Gauss curvature of a smooth surface by computing the Gauss curvature of an inscribed triangle mesh closely inscribed on it, and having a big amount of vertices. *In particular, the fact that the triangle mesh is developable does not imply at all that the smooth underlying surface has a "weak Gauss curvature" at some points.*

The following theorem gives a family of examples of developable triangle meshes (the Gauss curvature is thus 0 at each interior vertex) closely inscribed in a piece of the sphere $\mathbb{S}^2(r)$ of radius $r > 0$.

Theorem 3 *Let $n \geq 3$. There exists $\alpha_0 \in]0,1]$ such that and for every $\alpha \in]0,\alpha_0]$, there exists a developable triangle mesh T^n_α satisfying:*

1. *T^n_α is closely inscribed in "an open connected portion of sphere $\mathbb{S}^2(r)$";*
2. *T^n_α contains $(3n+1)$ vertices ($(n+1)$ of them are interior) and $4n$ faces;*

Remark that the triangle mesh T^n_α depends on the parameter α. If α tends to 0, then every vertex of T^n_α tends to the same vertex. In a sense α measures "the height" of T^n_α.
On the other hand, let S^n_α denote "the open connected portion of sphere $\mathbb{S}^2(r)$". S^n_α is a smooth surface whose Gauss curvature is $\frac{1}{r^2}$ at every interior point and then it is not developable. However Theorem 3 tells us that the triangle meshes T^n_α, which are closely inscribed in S^n_α, are developable.

This implies that without other assumptions, the knowledge of the Gauss curvature of a triangulated mesh closely inscribed in a smooth surface does not give information on the Gauss curvature of the smooth surface. In particular, the knowledge of a developable triangle mesh closely inscribed in a smooth surface does not allow to conclude whether the smooth surface is developable. It implies that the fact of building a developable triangle mesh inscribed in a smooth surface does not allow us to check an assumption of unfoldness made a priori on the smooth surface.

Remark 7 The (developable) triangle meshes T^n_α are homeomorphic to a disc. In fact, there does not exist any compact developable triangle mesh without boundary closely inscribed in the whole sphere. This is an obvious consequence Gauss Bonnet theorem (see [6]): it states that the Euler characteristic $\chi(S)$ of a smooth compact surface S (whose boundary ∂S is composed by $C_1, ..C_n$ positively oriented closed curves of class $\mathcal{C}^2$) satisfies:

$$2\pi\chi(S) = \int_S G_p \; da(p) \;+\; \sum_{i=1}^{n} \int_{C_i} k_p \; ds(p) \;+\; \sum_{i=1}^{p} \theta_i,$$

where $\{\theta_1, .., \theta_p\}$ is the set of all external angles of the curves $C_1, .., C_n$.
The discrete analogous result for the Euler characteristic $\chi(T)$ of a triangle mesh T is the following:

$$2\pi\chi(T) = G_{int}(T) + \mathcal{K}(\partial T).$$

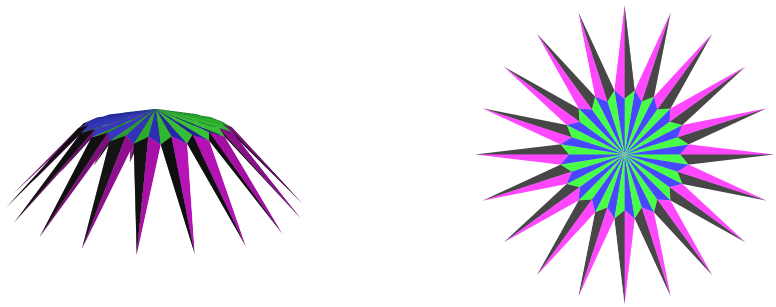
Since the Euler characteristic of a smooth surface S equals the Euler characteristic any triangle mesh T closely inscribed in it, one gets:

$$\int_S G_p \; da(p) \;+\; \sum_{i=1}^{n} \int_{C_i} k_p \; ds(p) \;+\; \sum_{i=1}^{p} \theta_i = G_{int}(T) + \mathcal{K}(\partial T).$$

In particular, if S and T *have no boundary, the previous equation becomes:*

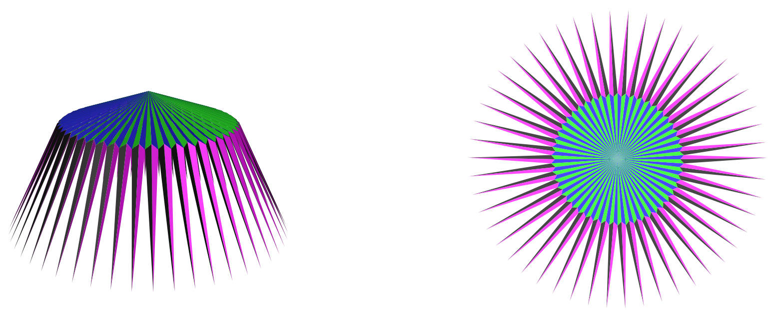
$$\int_S G_p \; da(p) \; = G_{int}(T).$$

We present here some of those triangle meshes T_α^n which are closely inscribed in a piece of sphere $\mathbb{S}^2$ and we unfold them. We use *Geomview* [9] to visualize the examples.



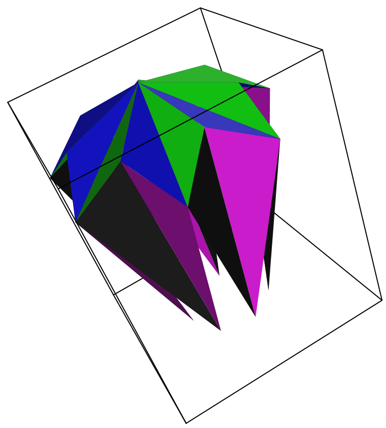
(a) Triangle mesh (b) Unfolded triangle mesh

Figure 4: case “$n = 20$ $\alpha = 0.4$”

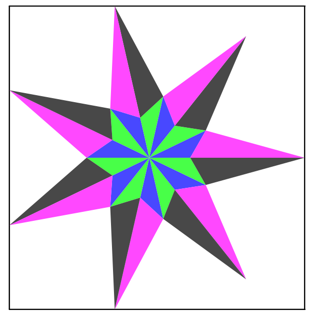


(a) Triangle mesh (b) Unfolded triangle mesh

Figure 5: case “$n = 50$ $\alpha = 0.6$”



(a) Triangle mesh



(b) Unfolded triangle mesh

Figure 6: case "$n = 7\ \alpha = 0.6$"

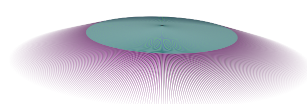


Figure 7: case "$n = 500\ \alpha = 0.2$"

The triangle mesh of figure 8 is still inscribed in sphere $\mathbb{S}^2$ and the discrete Gauss curvature at each interior vertex is strictly negative ($G_T(p) \approx -0.02$ if p is the central vertex and $G_T(p) \approx -0.04$ otherwise). Thus we have a triangle mesh with strictly negative Gauss curvature inscribed in sphere $\mathbb{S}^2$.

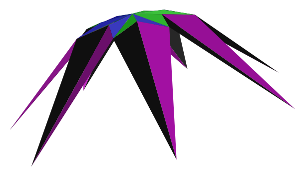
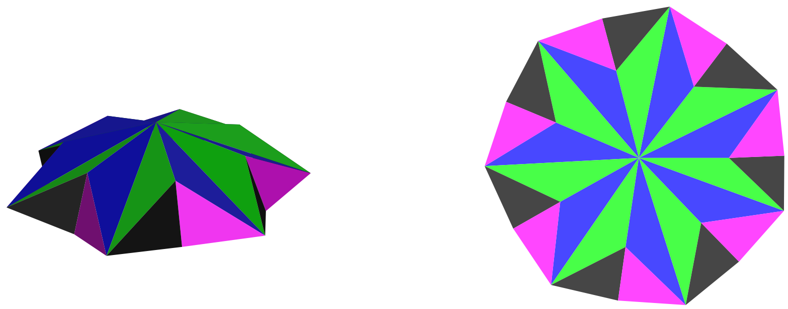


Figure 8: Triangle mesh with negative Gauss curvature inscribed in $\mathbb{S}^2$

The triangle mesh of figure 9 is developable and its boundary *"is quite regular"*, in the sense that the discrete Gauss curvature at each vertex of the boundary is not too large. This triangle mesh is not inscribed in a sphere, but in a smooth surface of revolution, whose Gauss curvature is strictly positive at each interior point.

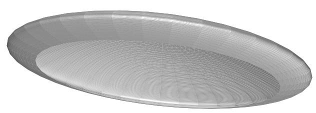


(a) Developable triangle mesh (b) Unfolded triangle mesh

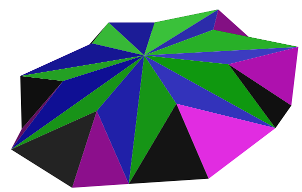
Figure 9: Developable triangle mesh inscribed in a smooth surface with strictly positive Gauss curvature

The triangle mesh of figure 10 is not developable. More precisely, the discrete Gauss curvature at each interior vertices is strictly negative (in fact $G_T(p) \leq -0.02$ at each interior vertex p). However, this triangle mesh is closely inscribed in a smooth surface of revolution, whose Gauss curvature is is strictly positive at each interior point.

Thus we have a triangle mesh with strictly negative Gauss curvature inscribed in a smooth surface with strictly positive Gauss curvature.



(a) Smooth surface with Gauss curvature larger than 0.5



(b) Triangle mesh

Figure 10: Triangle mesh of strictly negative Gauss curvature inscribed in a smooth surface with strictly positive Gauss curvature

5 Comparison of shortest paths

This part gives some intermediate results which are used in part 6 to prove the results of this paper. The main result of that part is the following proposition. Roughly speaking a triangle mesh closely near a smooth surface, whose normals are close enough to the normals of the smooth surface and which is close enough to the smooth surface is *"almost isometric"* to it.

Proposition 2 *Let S be a smooth compact connected surface of $\mathbb{E}^3$ and T be a triangle mesh closely near S. Then for every curve $\mathcal{C}$ of T:*

$$\frac{\cos(\alpha)}{1+\omega_S(T)}\ L(\mathcal{C}) \leq L(\xi(\mathcal{C})) \leq \frac{1}{1-\omega_S(T)}\ L(\mathcal{C}),$$

where $L(\mathcal{C})$ is the length of the curve $\mathcal{C}$, $L(\xi(\mathcal{C}))$ is the length of the curve $\mathcal{C}$, $\omega_S(T)$ is the relative curvature of S to T and α is the maximal angle between the normals of S and T.

This Proposition directly implies the following Corollary which compares the shortest paths of the two surfaces.

Corollary 2 *Let S be a smooth compact connected surface of $\mathbb{E}^3$ and T be a triangle mesh closely near S. Then for every points $a \in T$ and $b \in T$:*

$$\frac{\cos(\alpha)}{1+\omega_S(T)}\ L_T(a,b) \leq L_S(\xi(a),\xi(b)) \leq \frac{1}{1-\omega_S(T)}\ L_T(a,b),$$

where $L_T(a,b)$ is the distance on T between a and b, $L_S(\xi(a),\xi(b))$ is the distance on S between $\xi(a)$ and $\xi(b)$, $\omega_S(T)$ is the relative curvature of S to T and α is the maximal angle between the normals of S and T.

If the two surfaces S and T are developable, we have the same inequalities with the two unfoldings. Therefore we have:

Corollary 3 *Let S be a smooth compact connected and developable surface of $\mathbb{E}^3$ and T be a developabel triangle mesh of $\mathbb{E}^3$ closely near S. Then there exists a direct homeomorphism $f : u(S) \to u(T)$ from an unfolding $u(S)$ of S onto an unfolding $u(T)$ of T which satisfies for every curve $\mathcal{C}$ of $u(S)$:*

$$(1-\epsilon)L(\mathcal{C}) \leq L(f(\mathcal{C})) \leq (1+\epsilon)L(\mathcal{C}),$$

with

$$\epsilon = \frac{1+\omega_S(T)}{\cos(\alpha)} - 1,$$

where $\omega_S(T)$ is the relative curvature of S to T and α is the maximal angle between the normals of T and S.

Remark 8 In particular we have for every points $a \in u(T)$ and $b \in u(T)$:

$$\frac{\cos(\alpha)}{1+\omega_S(T)}\ L_{u(T)}(a,b) \leq L_{u(S)}(f(a),f(b)) \leq \frac{1}{1-\omega_S(T)}\ L_{u(T)}(a,b),$$

where $L_{u(T)}(a,b)$ is the length between a and b in $u(T)$ and $L_{u(S)}(f(a),f(b))$ is the length between $f(a)$ and $f(b)$ in $u(S)$

In the following, for every points $u \in \mathbb{E}^3$ and $v \in \mathbb{E}^3$, we denote by uv the euclidean distance between u and v.
For every compact surface M, we denote by $\mathcal{C}_M(m_1,m_2)$ the shortest path of M joining the points m_1 and m_2 of M and by $L_M(m_1,m_2)$ the distance between m_1 and m_2 on M, i.e., the length of $\mathcal{C}_M(m_1,m_2)$.
For every curve C, we denote by $L(C)$ its length.

5.1 One triangle case

In that part, we will prove the following Proposition:

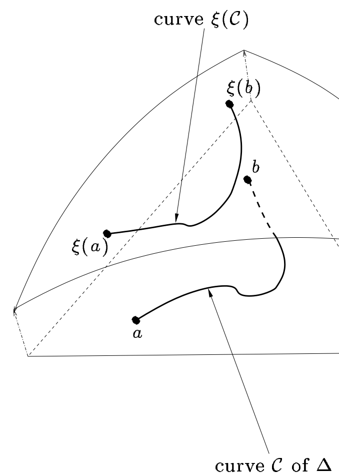


Figure 11: In that example $L(\mathcal{C})$ is the length of the curve $\mathcal{C}$ and $L(\xi(\mathcal{C}))$ is the length of the the curve $\xi(\mathcal{C})$

Proposition 3 *Let S be a smooth compact connected surface of $\mathbb{E}^3$ and Δ be a triangle closely near S. Then for every curve $\mathcal{C}$ of Δ:*

$$\frac{\cos(\alpha)}{1+\omega_S(\Delta)}\ L(\mathcal{C}) \leq L(\xi(\mathcal{C})) \leq \frac{1}{1-\omega_S(\Delta)}\ L(\mathcal{C}),$$

where $L(\mathcal{C})$ is the length of the curve $\mathcal{C}$, $L(\xi(\mathcal{C}))$ is the length of the curve $\mathcal{C}$, $\omega_S(\Delta)$ is the relative curvature of S to Δ and α is the maximal angle between the normals of S and Δ.

Remark 9 Just notice that if Δ and S are totally geodesic (that is, included in two planes and the angle between the normals is constant), then Proposition 3 leads to equalities ($\omega_S(\Delta) = 0$). Indeed, if the curve $\mathcal{C} = [a, b]$ is parallel to the two surfaces Δ and S, then we have:

$$L(\xi(\mathcal{C})) = L_S(\xi(a), \xi(b)) = \frac{1}{1-\omega_S(\Delta)} L_\Delta(a, b) = \frac{1}{1-\omega_S(\Delta)} L(\mathcal{C}).$$

Now, if we take two points c and d in Δ such that $\tilde{\mathcal{C}} = [c, d]$ is orthogonal to $[a, b]$, then we have :

$$L(\xi(\tilde{\mathcal{C}})) = L_S(\xi(c), \xi(d)) = \frac{\cos(\alpha)}{1+\omega_S(\Delta)} L_\Delta(c, d) = \frac{\cos(\alpha)}{1+\omega_S(\Delta)} L(\tilde{\mathcal{C}}).$$

5.1.1 Proof of Proposition 3

The proof of Proposition 3 is an immediate consequence of Proposition 4 and of Lemmas 1 and 2. They analyse the behavior of the differential of ξ.

Since Δ is closely near S, ξ induces a bijection between Δ and S. The restriction $\xi_{|\Delta}$ of ξ to Δ is a bijection between Δ and S. Let N^{Δ} be a normal to Δ and $\alpha_p = \langle N^{\Delta}, N^{S}_{\xi(p)} \rangle \in \left[0, \frac{\pi}{2}\right]$. For every m where ξ is differentiable, we put:

$$|D\xi(m)|_{\infty} = \sup_{X \neq 0} \frac{\|D\xi(m).X\|}{\|X\|}, \qquad |D\xi(m)|_{min} = \inf_{X \neq 0} \frac{\|D\xi(m).X\|}{\|X\|},$$

$$|D\xi|_{\infty} = \sup_{m \in \Delta} |D\xi(m)|_{\infty} \quad \text{and} \quad |D\xi|_{min} = \inf_{m \in \Delta} |D\xi(m)|_{min}.$$

Proposition 4 *Let S be a smooth compact oriented surface of $\mathbb{E}^3$ and U_S a neighborhood of S where the map ξ is defined. For every $m \in U_S$, if $\xi(m) \in S \setminus \partial S$ then ξ is differentiable at m. Furthermore,*

$$\text{if} \quad ||\xi(m) - m|| \; \rho_{\xi(m)} < 1, \quad \text{then} \quad |D\xi(m)|_{\infty} \leq \frac{1}{1 - ||\xi(m) - m|| \; \rho_{\xi(m)}},$$

and for every $X \in T_m U_S$:

$$||D\xi(m)X|| \geq \frac{1}{1 + ||\xi(m) - m|| \; \rho_{\xi(m)}} \; ||pr_{|T_{\xi(m)}M}(X)||,$$

where $pr_{|T_{\xi(m)}S}$ is the orthogonal projection onto $T_{\xi(m)}S$ and $\rho_{\xi(m)}$ is the maximal curvature of S at $\xi(m)$.

A proof of that proposition can be found in [14].

Remark 10 If the two principal curvatures of S at the the point $\xi(m)$ are opposite ($\lambda_1(\xi(m)) = -\lambda_2(\xi(m))$), then we have the inequalities :

$$|D\xi(m)|_{\infty} = \frac{1}{1 - ||\xi(m) - m|| \; \rho_{\xi(m)}}$$

and

$$||D\xi(m)X|| = \frac{1}{1 + ||\xi(m) - m|| \; \rho_{\xi(m)}} \; ||pr_{|T_{\xi(m)}M}(X)||.$$

Lemma 1

$$\begin{array}{l} |D\xi_{|\Delta}|_{\infty} \leq \frac{1}{1 - \omega_S(\Delta)}, \\ |D\xi_{|\Delta}|_{min} \geq \frac{\cos(\alpha)}{1 + \omega_S(\Delta)}, \end{array} ,$$

where $\omega_S(\Delta)$ is the relative curvature of S to Δ.

Proof of lemma 1**:**
Thanks to Proposition 4, we get for every $m \in \Delta$:

$$|D\xi_{|\Delta}(m)|_\infty \leq \frac{1}{1-\omega_S(\Delta)},$$

and for every $X \in T_m\Delta$:

$$\|D\xi_{|\Delta}(m)X\| \geq \frac{\|pr_{|T_{\xi(m)}S}(X)\|}{1+\ \omega_S(\Delta)},$$

where $pr_{|T_{\xi(m)}S}$ is the orthogonal projection onto $T_{\xi(m)}S$.
On the other hand, we have:

$$\forall X \neq 0 \quad \frac{\|pr_{|T_{\xi(m)}S}(X)\|}{\|X\|} \geq \cos(\alpha_m) \geq \cos(\alpha).$$

Thus

$$|D\xi_{|\Delta}(m)|_{min} \geq \frac{\cos(\alpha)}{1+\omega_S(\Delta)}.$$

□

Lemma 2 *Let $\mathcal{C}$ be a curve of Δ. Then*

$$L(\mathcal{C})\ |D\xi_{|\Delta}|_{min} \leq L(\xi(\mathcal{C})) \leq L(\mathcal{C})\ |D\xi_{|\Delta}|_\infty.$$

Proof of lemma 2**:**
The curve $\mathcal{C}$ is parametrized by the application $\gamma : [a,b] \to \mathcal{C}$. Since the curve $\xi(\mathcal{C})$ is parametrized by the map $\xi \circ \gamma$ the length of the curve $\xi(\mathcal{C})$ satisfies:

$$\begin{aligned} L(\xi(\mathcal{C})) &= \sup_{(t_1,\ldots,t_n)\in T(a,b)} \sum_{i=1}^{n-1} \|\xi(\gamma(t_i)) - \xi(\gamma(t_{i+1}))\| \\ &\geq |D\xi_{|\Delta}|_{min} \sup_{(t_1,\ldots,t_n)\in T(a,b)} \sum_{i=1}^{n-1} \|\gamma(t_i) - \gamma(t_{i+1})\| \\ &= |D\xi_{|\Delta}|_{min}\ L(\mathcal{C}). \end{aligned}$$

where $T(a,b)$ is the set of all the subdivisions $a = t_1 < \ldots < t_n = b$ of $[a,b]$. Similarly, we have:

$$L(\xi(\mathcal{C})) \leq |D\xi_{|\Delta}|_\infty\ L(\mathcal{C}).$$

□

5.2 Proof of Proposition 2 and its Corollaries

Proof of Proposition 2**:**
We can divide $\mathcal{C}$ into n curves $\mathcal{C}_1$, ..., $\mathcal{C}_n$ such that

- each curve $\mathcal{C}_i$ belongs to a triangle Δ_i of T;
- $\mathcal{C}_i = \cup_{i=1}^n \mathcal{C}_i$;
- $L(\mathcal{C}) = \sum_{i=1}^n L(\mathcal{C}_i)$.

Since $L(\mathcal{C}) = \sum_{i=1}^n L(\mathcal{C}_i)$ and $L(\xi(\mathcal{C})) = \sum_{i=1}^n L(\xi(\mathcal{C}_i))$, by using Proposition 3 we have the result. □

Proof of Corollary 2**:**
By using Proposition 3 with the curve $\mathcal{C} = \mathcal{C}_T(a,b)$, we have:

$$L_S(\xi(a),\xi(b)) \leq L(\xi(\mathcal{C}_T(a,b))) \leq \frac{1}{1-\omega_S(\Delta)}\ L(\mathcal{C}_T(a,b)) = \frac{1}{1-\omega_S(\Delta)}\ L_T(a,b).$$

By using Proposition 3 with the curve $\mathcal{C} = \xi^{-1}(\mathcal{C}_S(a,b))$, we have:

$$\frac{\cos(\alpha)}{1+\omega_S(\Delta)}\ L_T(a,b) \leq \frac{\cos(\alpha)}{1+\omega_S(\Delta)}\ L(\mathcal{C}) \leq L(\xi(\mathcal{C})) = L_S(\xi(a),\xi(b)).$$

□

Proof of Corollary 3**:**
Let g_T denote a direct isometry between T and $u(T)$ and g_S a direct isometry between S and $u(S)$. We define the application $f = g_T \circ \xi_{|T}^{-1} \circ g_S^{-1}$. Since g_T and g_S are two isometries, the result follows directly from Proposition 3.
Furthermore, $\xi_{|T}$ is direct (see [15]). Since g_T and g_S are direct, f is direct. □

6 Direct ϵ-isometry of the plane

Part 5 (and especially Corollary 3) tells us that if two developable surfaces S and T satisfy some assumptions, then the two unfoldings $u(S)$ and $u(T)$ are "almost isometric". In that part, we study surfaces of the plane $\mathbb{E}^2$ which are "almost isometric". In part 6.1, we define the notion of ϵ-isometry. In part 6.2, we give the Ellipse's Lemma which is well used after. In part 6.3, we compare convex surfaces of $\mathbb{E}^2$ which are ϵ-isometric and in part 6.4 we deal with the general case.

6.1 Definitions

Corollary 3 leads us to introduce the following definition :

Definition 4 *Let $\epsilon \in [0,1[$, U and V be two connected compact surfaces of $\mathbb{E}^2$.*

- *An homeomorphism $f : U \to V$ is an ϵ-isometry if it satisfies for every curve $\mathcal{C}$ of U:*

$$(1-\epsilon)L(\mathcal{C}) \leq L(f(\mathcal{C})) \leq (1+\epsilon)L(\mathcal{C}).$$

- *An ϵ-isometry $f : U \to V$ is said to be ϵ-strong if it satisfies:*

$$\forall (P,Q) \in U^2 \quad (1-\epsilon)PQ \leq f(P)f(Q) \leq (1+\epsilon)PQ.$$

Remark 11

- If U and V are convex, then every ϵ-isometry $f : U \to V$ is ϵ-strong.
- An ϵ-isometry $f : U \to V$ satisfies for every points P and Q in U:

$$(1-\epsilon)L_U(P,Q) \leq L_V(f(P), f(Q)) \leq (1+\epsilon)L_U(P,Q).$$

  (see Corollary 2).
- The map of Corollary 3 is an ϵ-isometry with $\epsilon = \frac{1+\omega_S(T)}{\cos(\alpha)} - 1$.

In the following we will denote by capitalize letters (for example P) points of U and by small letter the corresponding points in V (for example $p = f(P)$).

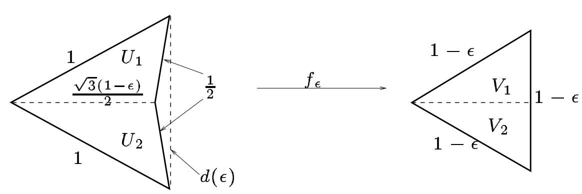


Figure 12: Example of an ϵ-isometry $f : U \to V$ which is ϵ-strong and for which U is not convex: f is affine between U_i and V_i (where $V = V_1 \cup V_2$) and we have $d(\epsilon) \leq 1-\epsilon$ for all $\epsilon \in [0,1]$.

6.2 The Ellipse's Lemma

Lemma 3 *Let $\epsilon \in]0,1]$ and f be an ϵ-isometry defined on a connected compact surface $U \subset \mathbb{E}^2$ onto a compact $V \subset \mathbb{E}^2$. Let P and Q be two points of U, $p = f(P)$ and $q = f(Q)$.*

1. *If $[p,q] \subset V$ (or if f is ϵ-strong), then $f(\mathcal{C}_U(P,Q))$ is included in the ellipse of foci p and q and of semi-major axis's length $a = \frac{1+\epsilon}{1-\epsilon}\frac{pq}{2}$. In particular, for every $m \in f(\mathcal{C}_U(P,Q))$ we have*
$$d(m,[p,q]) \leq \frac{pq\sqrt{\epsilon}}{1-\epsilon},$$
where $\mathcal{C}_U(P,Q)$ denotes a geodesic of U linking P and Q.

2. *If $[P,Q] \subset U$ (or if f is ϵ-strong), then $f^{-1}(\mathcal{C}_U(p,q))$ is included in the ellipse of foci P and Q and of semi-major axis's length $a = \frac{1+\epsilon}{1-\epsilon}\frac{PQ}{2}$. In particular, for every $M \in f^{-1}(\mathcal{C}_V(p,q))$ we have*
$$d(M,[P,Q]) \leq \frac{PQ\sqrt{\epsilon}}{1-\epsilon},$$
where $\mathcal{C}_V(p,q)$ denotes a geodesic of V linking p and q.

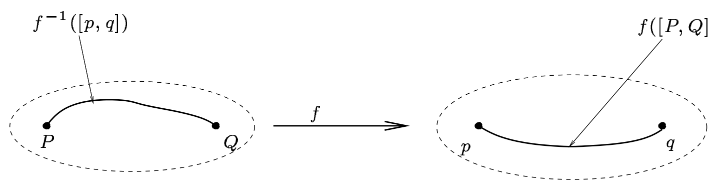


Figure 13: Illustration of the Ellipse's Lemma in the particular case where $\mathcal{C}_V(p,q) = [p,q]$ and $\mathcal{C}_U(P,Q) = [P,Q]$

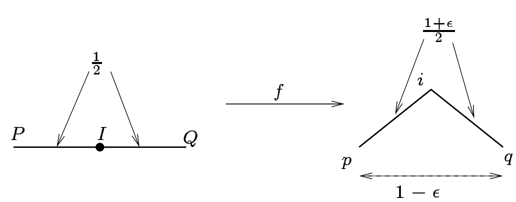


Figure 14: Exemple of the deformation of a geodesic with an ϵ-isometry f: in that case i belongs to the boundary of the ellipse and $d(i,[p,q]) = \sqrt{\epsilon} = \frac{pq\sqrt{\epsilon}}{1-\epsilon}$.

Proof of lemma 3**:**
We only prove the first part (the proof of the second part is similar). Let $M \in \mathcal{C}_U(P,Q)$ and $m = f(M)$. Then:

$$\begin{aligned} mp + mq &\leq L_V(m,p) + L_V(m,q) \\ &\leq (1+\epsilon)(L_U(M,P) + L_U(M,Q)) \\ &= (1+\epsilon)L_U(P,Q) \\ &\leq \tfrac{1+\epsilon}{1-\epsilon}L_V(p,q) \\ &= \tfrac{1+\epsilon}{1-\epsilon}pq. \end{aligned}$$

This imply that m belongs to an ellipse centered in p and q and that

$$d(m,[p,q]) \leq \frac{pq}{2}\sqrt{\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 - 1} = \frac{pq\sqrt{\epsilon}}{1-\epsilon}.$$

□

6.3 Comparison between two surfaces in the convex case

The following Proposition allows us to compare the two unfoldings in the particular case in which the ϵ-isometry is ϵ-strong.

Proposition 5 *Let $f : U \to V$ be a direct ϵ-isometry ϵ-strong (where U and V are two smooth compact connected surfaces of $\mathbb{E}^2$ and $\epsilon \in [0,1[$). If there exist two points p and q of V such that pq is equal to the diameter $diam(V)$ of V and V is star-set in both points p and q, then there exists a rigid motion d of $\mathbb{E}^2$ such that*

$$\delta_{Haus}(d(V),U) < diam(V)\ \sqrt{\epsilon}\ \frac{\sqrt{1+2\sqrt{\epsilon}+40\epsilon}}{(1-\epsilon)^2}.$$

Corollary 4 *Let $f : U \to V$ be a direct ϵ-isometry (where U and V are two smooth compact convex surfaces of $\mathbb{E}^2$ and $\epsilon \in [0,1[$). Then there exists a rigid motion d of $\mathbb{E}^2$ such that*

$$\delta_{Haus}(d(V),U) < diam(V)\ \sqrt{\epsilon}\ \frac{\sqrt{1+2\sqrt{\epsilon}+40\epsilon}}{(1-\epsilon)^2}.$$

Definition 5 *The surface $d(V)$ is said to be well-positionned with respect to U.*

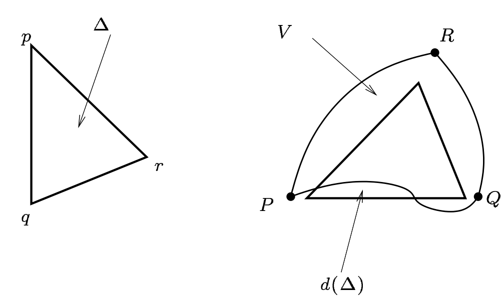


Figure 15: In this exemple, the triangle $d(\Delta)$ is well-positionned with respect to S

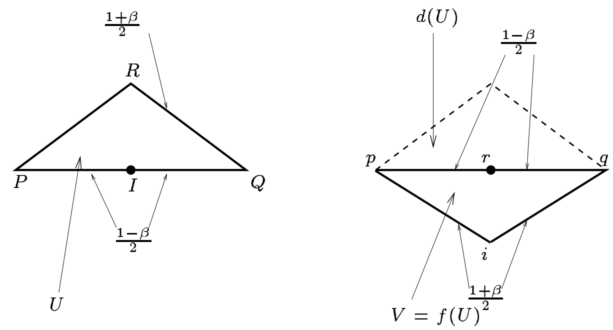


Figure 16: Exemple of an ϵ-isometry f (with $\epsilon = \frac{2\beta}{1-\beta}$) and $\delta_{Haus}(d(U), V) = f(I)d(I) = ir = \sqrt{\beta} = \sqrt{\frac{\epsilon}{2+\epsilon}}$

6.3.1 Exemple

The example of Figure 16 deals with an ϵ-isometry f between two triangles. The motion d associated to f leads to an error which is $O(\sqrt{\epsilon})$. The error of Proposition 5 is not optimal, but it is also $O(\sqrt{\epsilon})$.

6.3.2 Proof of Proposition 5

Let m be a point of V and let denote by h the orthogonal projection of m onto the line (p, q) (in fact, $h \in [p, q]$ otherwise $[p, q]$ would not be the diameter of V). Let us put $P = f^{-1}(p)$, $Q = f^{-1}(q)$, $M = f^{-1}(m)$ and $H = f^{-1}(h)$. Up to a direct rigid motion of the plane we can suppose that the two line segments $[p, q]$ and $[P, Q]$ have the same middle point and that they are parallel. Therefore, we have $pP = qQ \leq \frac{\epsilon diam(V)}{2}$ and we can have the following coordinates:

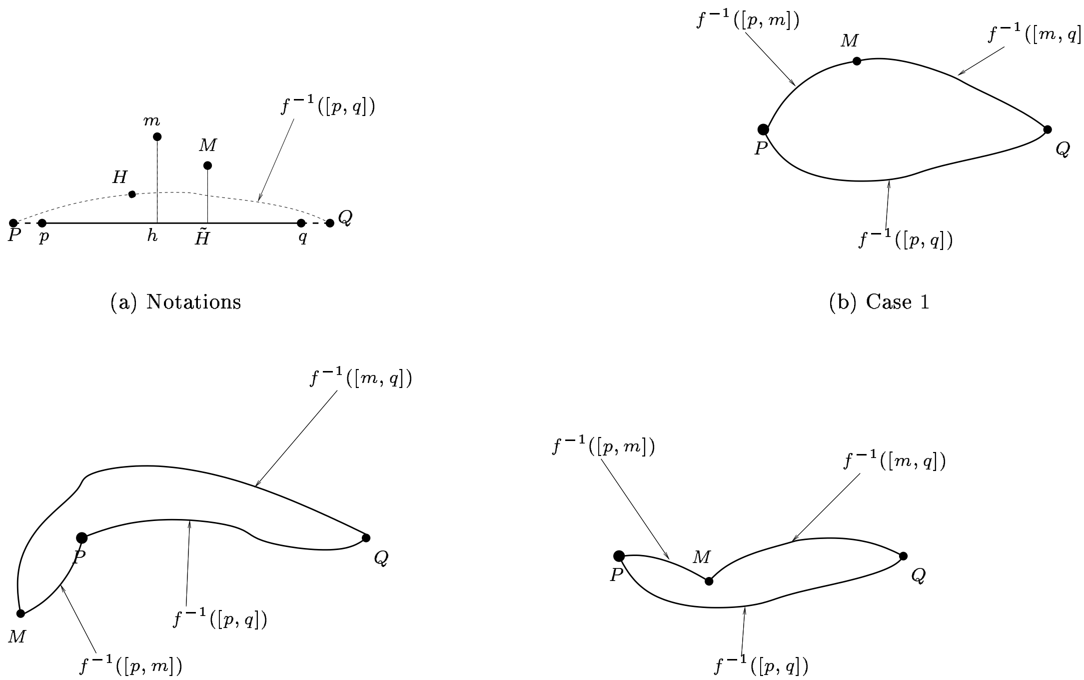
$$p = (-\frac{diam(V)}{2}, 0), \;\; q = (\frac{diam(V)}{2}, 0), \;\; m = (x_m, y_m) \text{ and } M = (X_M, Y_M).$$

We suppose here that $y_m \leq 0$.
Case 1: $Y_M \geq 0$.
From the two equations

$$\begin{cases} pm^2 = \left(x_m + \frac{diam(V)}{2}\right)^2 + y_m^2 \\ qm^2 = \left(x_m - \frac{diam(V)}{2}\right)^2 + y_m^2 \end{cases}$$

we get:

$$x_m = \frac{pm^2 - qm^2}{2\ diam(V)}.$$



(a) Notations (b) Case 1

(c) Case 2 (d) Case 3

Figure 17: Proof of Proposition 5

Similarly, we get:

$$X_M = \frac{pM^2 - qM^2}{2\ diam(V)}.$$

Furthermore, we have:

$$|pM - pm| \leq |PM - pm| + |PM - pM| \leq \epsilon pm + pP \leq \frac{3\epsilon diam(V)}{2},$$

and

$$|qM - qm| \leq \frac{3\epsilon diam(V)}{2}.$$

Therefore:

$$\begin{array}{rrl} & X_M - x_m & = \frac{(pM^2 - pm^2) - (qM^2 - qm^2)}{2diam(V)} \\ & & \leq \frac{\left[\left(pm + \frac{3\epsilon diam(V)}{2}\right)^2 - pm^2\right] - \left[\left(qm - \frac{3\epsilon diam(V)}{2}\right)^2 - qm^2\right]}{2diam(V)} \\ & & = \frac{3\epsilon(pm+qm)}{2}, \\ \text{and} & X_M - x_m & \geq -\frac{3\epsilon(pm+qm)}{2}, \\ \text{thus} & |X_M - x_m| & \leq \frac{3\epsilon(pm+qm)}{2} \leq 3\epsilon diam(V). \end{array}$$

Similarly, we get:

$$|X_H - x_h| \leq \frac{3\epsilon(ph+qh)}{2} = \frac{3\epsilon diam(V)}{2}$$

Let $\tilde{H} = (X_{\tilde{H}}, Y_{\tilde{H}})$ be the orthogonal projection of M onto (P, Q). Then we have:

$$\begin{array}{rl} |X_H - X_{\tilde{H}}| & \leq |X_H - x_h| + |x_h - X_{\tilde{H}}| \\ & = |X_H - x_h| + |x_m - X_M| \\ & \leq \frac{3\epsilon diam(V)}{2} + 3\epsilon diam(V) \\ & = \frac{9\epsilon diam(V)}{2}. \end{array}$$

Furthermore, thanks to the Ellipse's Lemma (Lemma 3), we have that:

$$|Y_H - Y_{\tilde{H}}| = |Y_H| \leq \frac{diam(V)\sqrt{\epsilon}}{1-\epsilon}.$$

Then

$$H\tilde{H} \leq diam(V)\sqrt{\left(\frac{9\epsilon}{2}\right)^2 + \frac{\epsilon}{(1-\epsilon)^2}},$$

Furthermore

$$|Y_M - y_m| = \|mh - M\tilde{H}\| \leq \|mh - MH\| + \|MH - M\tilde{H}\| \leq \epsilon diam(V) + H\tilde{H}.$$

Therefore

$$\begin{array}{rl} mM & \leq \sqrt{(X_M - x_m)^2 + (Y_M - y_m)^2} \\ & \leq diam(V)\sqrt{9\epsilon^2 + \left(\epsilon + \sqrt{\left(\frac{9\epsilon}{2}\right)^2 + \frac{\epsilon}{(1-\epsilon)^2}}\right)^2} \\ & = diam(V)\sqrt{\epsilon}\sqrt{\frac{1}{(1-\epsilon)^2} + \frac{121}{4}\epsilon + 2\sqrt{\frac{81\epsilon^2}{4} + \frac{\epsilon}{(1-\epsilon)^2}}} \\ & \leq diam(V)\sqrt{\epsilon}\,\frac{\sqrt{1+2\sqrt{\epsilon}+40\epsilon}}{(1-\epsilon)^2}. \end{array}$$

Case 2: $Y_M < 0$ and $\langle\overrightarrow{PQ}, \overrightarrow{PM}\rangle \geq \pi$.

$$\begin{array}{rrl} & l(f^{-1}([m,q])) & \geq MP + PQ \geq MP + \frac{pq}{1+\epsilon} = MP + \frac{diam(V)}{1+\epsilon} \\ \text{and} & l(f^{-1}([m,q])) & \leq \frac{mq}{1-\epsilon} \leq \frac{diam(V)}{1-\epsilon}, \\ \text{thus} & MP & \leq \frac{2\epsilon\ diam(V)}{1-\epsilon^2} \\ \text{and} & mp & \leq (1+\epsilon)MP \leq \frac{2\epsilon diam(V)}{1-\epsilon}. \end{array}$$

That is why we get:

$$mM \leq mp + pP + PM \leq \frac{2\epsilon diam(V)}{1-\epsilon} + \frac{2\epsilon\ diam(V)}{1-\epsilon^2} + \frac{\epsilon\ diam(V)}{2},$$

and

$$mM \leq \frac{4.5\epsilon diam(V)}{1-\epsilon^2} \leq diam(V)\sqrt{\epsilon}\ \frac{\sqrt{1+2\sqrt{\epsilon}+40\epsilon}}{(1-\epsilon)^2}.$$

Case 3: $Y_M < 0$ and $\langle \overrightarrow{PQ}, \overrightarrow{PM} \rangle \leq 0$.
Since f is direct, then M is included in the part bounded by $[P,Q]$ and $f^{-1}([p,q])$. Thus lemma 3 implies that M belongs to the ellipse associated to foci P and Q and of semi-major axis's length $a = \frac{1+\epsilon}{1-\epsilon}PQ$ and that

$$|Y_M| \leq \frac{PQ\sqrt{\epsilon}}{1-\epsilon} \leq \frac{diam(V)\sqrt{\epsilon}}{(1-\epsilon)^2}.$$

Since f is direct and $Y_M < 0$ then m is included in the part bounded by $[p,q]$ and $f([P,Q])$. Thus lemma 3 implies that

$$|y_m| \leq \frac{pq\sqrt{\epsilon}}{1-\epsilon} = \frac{diam(V)\sqrt{\epsilon}}{1-\epsilon}.$$

In fact we have:

$$\begin{array}{rrl} & d(M, f^{-1}([p,q])) & \geq \frac{1}{1+\epsilon} d(m,[p,q]) = \frac{y_m}{1+\epsilon}, \\ \text{and} & d(M, f^{-1}([p,q])) & \leq \frac{PQ\sqrt{\epsilon}}{1-\epsilon} \leq \frac{diam(V)\sqrt{\epsilon}}{(1-\epsilon)^2}, \\ \text{thus} & |Y_M| + \frac{y_m}{1+\epsilon} & \leq \frac{diam(V)\sqrt{\epsilon}}{1-\epsilon}, \\ \text{and} & |Y_M - y_m| & = |Y_M| + y_m \\ & & = |Y_M| + \frac{y_m}{1+\epsilon} + \frac{y_m\epsilon}{1+\epsilon} \\ & & \leq \frac{diam(V)\sqrt{\epsilon}}{(1-\epsilon)^2} + \frac{diam(V)\epsilon\sqrt{\epsilon}}{(1-\epsilon)(1+\epsilon)} \\ & & = \frac{diam(V)\sqrt{\epsilon}(1+2\epsilon-\epsilon^2)}{(1-\epsilon)^2(1+\epsilon)}. \end{array}$$

Therefore

$$\begin{array}{rl} mM & \leq \sqrt{(X_M - x_m)^2 + (Y_M - y_m)^2} \\ & \leq diam(V)\sqrt{9\epsilon^2 + \left(\frac{\sqrt{\epsilon}(1+2\epsilon-\epsilon^2)}{(1-\epsilon)^2(1+\epsilon)}\right)^2} \\ & = \frac{diam(V)\sqrt{\epsilon}}{(1-\epsilon)^2}\sqrt{9\epsilon(1-\epsilon)^4 + \left(\frac{1+2\epsilon-\epsilon^2}{1+\epsilon}\right)^2} \\ & \leq \frac{diam(V)\sqrt{\epsilon}}{(1-\epsilon)^2}\sqrt{1+13\epsilon+4\epsilon^4} \\ & \leq diam(V)\sqrt{\epsilon}\ \frac{\sqrt{1+2\sqrt{\epsilon}+40\epsilon}}{(1-\epsilon)^2}. \end{array}$$

6.3.3 Proof of Theorem 3.2

By using Corollary 3, we know that there exists an ϵ-isometry $f : u(S) \to (T)$ between the two unfoldings. Since $u(T)$ and $u(S)$ are convex, f is ϵ-strong. Therefore we conclude by using Corollary 4.

6.4 Comparison between two surfaces in the general case

We want to compare two surfaces of the plane U and V when there exists an ϵ-isometry $f : U \to V$. Unfortunatly, f is not ϵ-strong in general. However, in the particular case where $U = \cup_{i=1}^{n} U_i$, $V = \cup_{i=1}^{n} V_i$ and $f_{\epsilon|U_i} : U_i \to V_i$ is ϵ-strong, Proposition 6 tells us that we can compare U and V. First we need a definition and a notation:

Definition 6 *Let V be a triangle mesh.*

- *The fatness of a triangle Δ of V is the real number:*

$$\theta_\Delta = \frac{\mathcal{A}(\Delta)}{{\eta_\Delta}^2}.$$

- *The fatness of V is:*

$$\theta_V = \min_{\Delta \text{ triangle of } V} \theta_\Delta.$$

Notation 1 *Let V be a triangle mesh of the plane $\mathbb{E}^2$, $f : U \to V$ be a strong ϵ-isometry (where U is a surface of $\mathbb{E}^2$ and $\epsilon \in [0, 1[$) and Δ_1 a triangle V.*
For every point $m \in V$, let denote by $N_m > 0$ the smallest integer such that there exist $(N_m - 1)$ triangles Δ_2,...,Δ_{N_m} such that $m \in \Delta_{N_m}$ and for every $i \in \{1, ..N_m - 1\}$ Δ_i and Δ_{i+1} have a common edge. We put:

$$N_V(\Delta_1) = \sup_{m \in V} N_m.$$

Proposition 6 *Let $f : U \to V$ be an ϵ-isometry, V be a triangle mesh. If for every triangle Δ of V $f : f^{-1}(\Delta) \to \Delta$ is ϵ-strong, then there exists a rigid motion d of the plane such that:*

$$\delta_{Haus}(d(U), V) \leq N_V(\Delta_1) \frac{K(\epsilon) diam(V) \sqrt{\epsilon}}{1 - \epsilon} \left(2 + \frac{4 + K(\epsilon)}{\theta_V} \right),$$

where $diam(V)$ is the diameter of V, θ_V its fatness and $K(\epsilon) = \frac{\sqrt{1+2\sqrt{\epsilon}+40\epsilon}}{(1-\epsilon)^2}$.

6.4.1 Proof of Proposition 6

Let $m \in V$, Δ_1,...,Δ_{N_m} be triangles as in Notation 1. We put $\mathcal{T}_i = f^{-1}(\Delta_i)$. By using Proposition 5, we suppose that Δ_1 is well-positionned with respect to $\mathcal{T}_1$ and that there exist rigid motions d_i $(i \in \{2, .., N_m\})$ such that $d_i(\Delta_i)$ is well-positionned with respect to

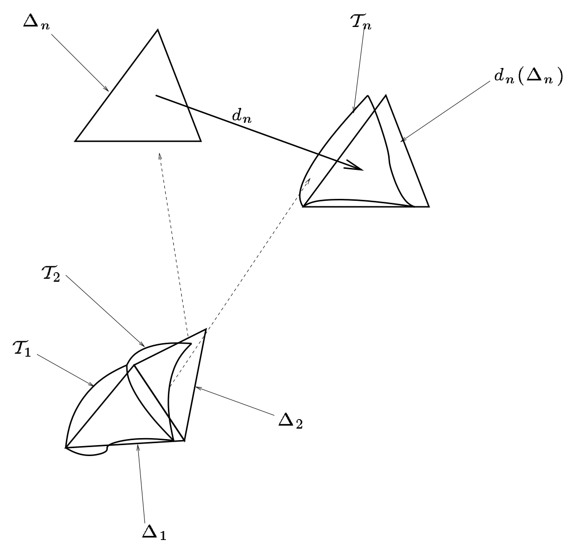


Figure 18: Visualisation of the rigid motion d_n (here we put $n = N_m$)

$\mathcal{T}_i$. We put $d_{i,i+1} = d_{i+1} \circ d_i^{-1}$ and denote by θ_{i+1} the angle of the rigid motion $d_{i,i+1}$. For every $i \in \{2, ..N_m\}$ we know that $d_{i-1,i} \ \circ ... \circ d_{1,2}(\Delta_i)$ is well-positionned with respect to $\mathcal{T}_i$. We put

$$d = d_{N_m} = d_{N_m-1,N_m} \ \circ ... \circ d_{1,2}.$$

We know that $d(\Delta_{N_m})$ is well-positionned with respect to $\mathcal{T}_{N_m}$. It means that we have :

$$d(m)f^{-1}(m) \leq K(\epsilon) \ diam(\Delta_{N_m}) \ \sqrt{\epsilon} \leq K(\epsilon) \ diam(V) \ \sqrt{\epsilon}.$$

The idea of the proof is the following : we fix a point of the plane. Therefore each rigid motion $d_{i-1,i}$ is linked to a unique translation b_i. We bound each θ_i and b_i and then will be able to bound $md(m)$.
We need the two following Lemmas to conclude :

Lemma 4 *For every $i \in \{2, .., N_m\}$, we get*

$$\sin \frac{|\theta_i|}{2} \leq \frac{K(\epsilon) \ \sqrt{\epsilon}}{(1-\epsilon) \ \theta_V}.$$

Lemma 5

$$md(m) \leq (N_m - 1)\frac{K(\epsilon)diam(V)\sqrt{\epsilon}}{1-\epsilon}\left(2 + \frac{4 + K(\epsilon)}{\theta_V}\right).$$

End of proof of Proposition 6 :
By using lemma 5, we have :

$$\begin{aligned} mf^{-1}(m) &\leq md(m) + d(m)f^{-1}(m) \\ &\leq (N_m - 1)\tfrac{K(\epsilon)diam(V)\sqrt{\epsilon}}{1-\epsilon}\left(2 + \tfrac{4+K(\epsilon)}{\theta_V}\right) + K(\epsilon)diam(V)\sqrt{\epsilon} \\ &\leq N_m \tfrac{K(\epsilon)diam(V)\sqrt{\epsilon}}{1-\epsilon}\left(2 + \tfrac{4+K(\epsilon)}{\theta_V}\right). \end{aligned}$$

Proof of Lemma 4**:**
Let Δ_{i-1} and Δ_i be two triangles of V with a common edge $[p,q]$. We put $P = f^{-1}(p)$ and $Q = f^{-1}(q)$. Suppose that $\eta_{\Delta_i} \geq \eta_{\Delta_{i-1}}$. If $K(\epsilon)\eta_{\Delta_i}\sqrt{\epsilon} \leq \frac{PQ}{2}$, the worst case is obtained for an angle θ_i satisfying :

$$\sin\frac{|\theta_i|}{2} = \frac{K(\epsilon)\eta_{\Delta_i}\sqrt{\epsilon}}{\frac{PQ}{2}} \leq \frac{2K(\epsilon)\eta_{\Delta_i}\sqrt{\epsilon}}{pq(1-\epsilon)} \leq \frac{K(\epsilon)\sqrt{\epsilon}}{\theta_{\Delta_i}(1-\epsilon)} \leq \frac{K(\epsilon)\sqrt{\epsilon}}{\theta_V(1-\epsilon)}.$$

□

Proof of Lemma 5**:**
We suppose that the origin is in Δ_1. At each rigid motion $d_{i-1,i}$ we associate the complex fonction $g_i(z) = e^{i\theta_i}z + b_i$. We associate the fonction $g(z) = e^{i\theta}z + b$ to the rigid motion d. We have

$$md(m) = |m - g(m)| = |e^{i\theta}m - m + b| \leq |m||e^{i\theta} - 1| + |b| = 2|m|\sin\frac{|\theta|}{2} + |b|.$$

Since $|m| \leq diam(V)$, we just have to bound $|b|$ and $\sin\frac{|\theta|}{2}$.
Since

$$\theta = \sum_{i=2}^{N_m}\theta_i,$$

we have :

$$\left|\sin\frac{\theta}{2}\right| \leq \sum_{i=2}^{N_m}\sin\frac{|\theta_i|}{2} \leq (N_m - 1)\frac{K(\epsilon)\ \sqrt{\epsilon}}{(1-\epsilon)\ \theta_V}.$$

We first notice that :

$$|b| \leq \sum_{i=2}^{N_m}|b_i|.$$

For every i, we denote by p^i a point which is a vertex of both triangles Δ_{i-1} and Δ_i. We put $P^i = f^{-1}(p^i)$, $p_1^i = d_{i-2,i-1}\circ \ldots \circ d_{2,1}(p^i)$ and $p_2^i = d_{i-1,i}(p_1^i)$. We have :

$$|b_i| = |p_2^i - e^{i\theta_i}p_1^i| \leq |p_2^i - p_1^i| + |p_1^i||e^{i\theta_i} - 1| = |p_2^i - p_1^i| + |p_1^i|\sin\frac{|\theta_i|}{2}.$$

However

$$\begin{array}{rl} & |p_2^i - p_1^i| \leq 2K(\epsilon)diam(V)\sqrt{\epsilon}, \\ \text{and} & |p_1^i| \leq |P^i - 0| + |p_1^i - P^i| \leq diam(U) + K(\epsilon)\eta_V\sqrt{\epsilon}. \end{array}$$

Therefore

$$\begin{aligned} |b_i| &\leq 2K(\epsilon)diam(V)\sqrt{\epsilon} + (diam(U) + K(\epsilon)\eta_V\sqrt{\epsilon})\tfrac{K(\epsilon)\sqrt{\epsilon}}{\theta_V(1-\epsilon)} \\ &\leq \tfrac{K(\epsilon)diam(V)\sqrt{\epsilon}}{1-\epsilon}\left(2 + \tfrac{2+K(\epsilon)}{\theta_V}\right), \end{aligned}$$

and

$$|b| \leq (N_m - 1)\frac{K(\epsilon)diam(V)\sqrt{\epsilon}}{1-\epsilon}\left(2 + \frac{2+K(\epsilon)}{\theta_V}\right)$$

Consequently, we have :

$$md(m) \leq (N_m - 1)\frac{K(\epsilon)diam(V)\sqrt{\epsilon}}{1-\epsilon}\left(2 + \frac{4+K(\epsilon)}{\theta_V}\right).$$

□

6.4.2 End of the proof of theorem 2

Thanks to Corollary 3, we know that there exists an ϵ_n-isometry $f_n : u(U) \to u(T_n)$ between the unfoldings of two developable surfaces T_n and S. Unfortunatly we need f_n to be strong (so as to be able to compare both unfoldings) and we want the Hausdorff distance between both unfoldings to be bounded by quantities which do not depend on $u(T_n)$.
Therefore, the idea of this part is to build a triangle mesh $U_\beta \subset u(S)$ such that for every triangle Δ of U_β $(f_{n|\Delta})^{-1} : \mathcal{T} \to \Delta$ is a strong $\tilde{\epsilon_n}$-isometry (where U_β is close to $u(S)$ if β is small, $\tilde{\epsilon}$ is small and $\mathcal{T} = f_n(\Delta)$) .
Just notice that the triangle mesh U_β does not depend on the triangles of the triangle mesh $u(T_n)$ (if not, we would not have the result of convergence we expect, because the error between two unfoldings grows up with the number of triangles, see Proposition 6).

Proposition 7 *Let $f : U \to V$ be an ϵ-isometry (where U and V are two connected compact surfaces of $\mathbb{E}^2$ and $\epsilon \in]0, 0.2]$). Let U_β be a triangle mesh included in U such that*

- $\delta_{Haus}(U, U_\beta) \leq 2\beta$,
- $d(U_\beta, \partial U) \geq \beta$,
- *its height η_β is strictly less than $\frac{\beta(1-\epsilon)}{\sqrt{\epsilon}}$.*

Then

$$\delta_{Haus}(U, V) \leq \frac{K_\beta\sqrt{\epsilon} + 2\beta}{(1-2\epsilon)^4}.$$

where K_β is a constant which only depends on U_β.

Remark 12

1. If β is small enough, there always exists a triangular mesh U_β satisfying the assumptions of Proposition 7. Indeed, if we put
$$\widetilde{U_\beta} = \{m \in U, d(m, \partial U) \geq \beta\} \text{ and } \widetilde{U_{2\beta}} = \{m \in U, d(m, \partial U) \geq 2\beta\},$$
we notice that (if β is small enough) $\widetilde{U_\beta}$, $\widetilde{U_{2\beta}}$ and $\widetilde{U_\beta} \setminus \widetilde{U_{2\beta}}$ are connected surfaces. Therefore, we can build a closed polygonal line $\Gamma_\beta \subset \widetilde{U_\beta} \setminus \widetilde{U_{2\beta}}$. We triangulate the compact surface of $\mathbb{E}^2$ whose boundary is Γ_β such that the height η_β of V_β is strictly less than $\frac{\beta(1-\epsilon)}{\sqrt{\epsilon}}$. This triangular mesh is U_β.

2. The triangular mesh U_β can be built so as to have the angles of the triangles larger than 26.56 degres. Therefore the fatness θ_{U_β} of U_β is larger than 0.2. *(the construction can be as follows : we can build Γ_β such that the angles at every vertex is more than 60 degres. Then, by using Chew's algorithm [17] we can triangulate U_β such that its minimal angle is greater than 26.56 degres).*

Proof of Proposition 7
We put $V_\beta = f(U_\beta)$.
The idea of the proof is the following : we are going to compare V_β and U_β thanks to Proposition 6. Then we will be able to compare U and V.

Lemma 6 *Let Δ be triangle of U_β.*
Then $(f_{|\Delta})^{-1} : f(\Delta) \to \Delta$ is an $\left(\frac{\epsilon}{1-\epsilon}\right)$-strong isometry.

Proof of Lemma 6**:**
The fact that $f : U_\beta \to V_\beta$ is an ϵ-isometry implies that $f^{-1} : V_\beta \to U_\beta$ is an $\left(\frac{\epsilon}{1-\epsilon}\right)$-isometry. Let $[a,b]$ be an edge of U_β. Using the last assumption of Proposition 7, we have:
$$d([a,b], \partial U) \geq \beta > \frac{\eta\sqrt{\epsilon}}{1-\epsilon} \geq \frac{ab\sqrt{\epsilon}}{1-\epsilon},$$
Just suppose that the shortest path $\mathcal{C}_V(f(a), f(b))$ intersets the boundary ∂V. Then it means that $f^{-1}(\mathcal{C}_V(f(a), f(b)))$ intersects the boundary ∂U. By using the Ellipse's Lemma (Lemma 3), we have
$$d([a,b], \partial U) \leq \frac{ab\sqrt{\epsilon}}{1-\epsilon},$$
which is not possible. Therefore the shortest path $\mathcal{C}_V(f(a), f(b))$ is the line $[f(a), f(b)]$. $\square$

Lemma 7
$$\delta_{Haus}(U_\beta, V_\beta) \leq \frac{K_\beta}{(1-2\epsilon)^3}\sqrt{\frac{\epsilon}{1-\epsilon}},$$
where K_β only depends on the diameter and the fatness of U_β.

Proof of Lemma 7**:**
Thanks to Lemma 6, we can apply Proposition 6. Therefore, we get:

$$\delta_{Haus}(U_\beta, V_\beta) \leq \frac{K_\beta}{(1-2\epsilon)^3}\sqrt{\frac{\epsilon}{1-\epsilon}}.$$

□

Proof of Proposition 7**:**
Let v be a point of V. There exists a point $v_\beta \in V_\beta$ such that $f^{-1}(v)f^{-1}(v_\beta) \leq 2\beta$. Therefore

$$vv_\beta \leq \frac{1}{1-\epsilon} f^{-1}(v)f^{-1}(v_\beta) \leq \frac{2\beta}{1-\epsilon}.$$

By using lemma 7, we have there exists a point $u_\beta in U_\beta$ such that

$$uu_\beta \leq \frac{K_\beta}{(1-2\epsilon)^3}\sqrt{\frac{\epsilon}{1-\epsilon}}.$$

Therefore

$$d(v,U) \leq vv_\beta + v_\beta u_\beta \leq \frac{2\beta}{1-\epsilon} + \frac{K_\beta}{(1-2\epsilon)^3}\sqrt{\frac{\epsilon}{1-\epsilon}} \leq \frac{K_\beta\sqrt{\epsilon}+2\beta}{(1-2\epsilon)^4}.$$

Similarly, for every point $u \in U$, we have

$$d(u,V) \leq 2\beta + \frac{K_\beta}{(1-\epsilon)^3}\sqrt{\frac{\epsilon}{1-\epsilon}} \leq \frac{K_\beta\sqrt{\epsilon}+2\beta}{(1-2\epsilon)^4}.$$

□

Proof of Theorem 2**:**
Using Corollary 3, we know that for every n, there exist an ϵ_n-isometry $f_n : u(S) \rightarrow u(T_n)$. Since the normals of T_n tend to the normals of S and the height of T_n tends to 0, we have that $\lim_{n\rightarrow\infty} \epsilon_n = 0$.
Let $\gamma > 0$. Let $\beta \in]0, \frac{\gamma}{4}]$ and U_β be a triangle mesh which satisfies the two first assumptions of Proposition 7 (U_β exists, see Remark 12). Then there exists $n_0 \geq 0$ such that for every $n \geq n_0$, the height of T_n is less than $\frac{\beta(1-\epsilon_n)}{\sqrt{\epsilon_n}}$, $\frac{K_\beta\sqrt{\epsilon_n}}{(1-\epsilon_n)^4} \leq \frac{\gamma}{2}$ and $\frac{1}{2(1-\epsilon_n)^4} \leq \frac{\gamma}{2}$. Therefore, by using Proposition 7:

$$\delta_{Haus}(d_n(u(T_n)), u(S)) \leq \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma.$$

□

7 Proof of Theorem 3

Let $\alpha \in]0,1]$. We denote by z_0 the point of $\mathbb{S}^2$ of coordinates $(0,0,1)$. We define the points z_1^α,..,z_n^α on sphere $\mathbb{S}^2$ by:

$$\forall i \in \{1,..,n+1\}\ z_i^\alpha = \left(\alpha \cos\frac{\pi(2i-3)}{n}, \alpha \sin\frac{\pi(2i-3)}{n}, \sqrt{1-\alpha^2}\right).$$

Remark that $z_{n+1}^\alpha = z_1^\alpha$.
<u>Step 1:</u>
We are going to build points w_1^α,..,w_n^α on $\mathbb{S}^2$ so as to get:

$$\forall i \in \{1,..,n\} \quad \widehat{z_i^\alpha z_0 w_i^\alpha} = \frac{\pi}{n} \quad \text{and} \quad \widehat{w_i^\alpha z_0 z_{i+1}^\alpha} = \frac{\pi}{n}.$$

Thus, if we define T_n^α as being the triangulated mesh

$$\begin{array}{ll} \text{whose vertices are} & \left\{ \begin{array}{l} z_i^\alpha \text{ for } 0 \le i \le n, \\ w_i^\alpha \text{ for } 1 \le i \le n, \end{array}\right. \\ \text{and whose faces are} & \left\{ \begin{array}{l} z_i^\alpha z_0 w_i^\alpha \text{ for } 1 \le i \le n, \\ w_i^\alpha z_0 z_{i+1}^\alpha \text{ for } 1 \le i \le n, \end{array}\right. \end{array}$$

we get the following property:

$$\alpha_{T_n^\alpha}(z_0) = \sum_{i=1}^{n}\left(\widehat{z_i^\alpha z_0 w_i^\alpha} + \widehat{w_i^\alpha z_0 z_{i+1}^\alpha}\right) = 2n\frac{\pi}{n} = 2\pi.$$

Let us build the point w_1^α:
let $(x,0,z)$ be the coordinates of w_1^α. We have to solve the following equation:

$$(E) = \left\{ \begin{array}{l} w_1^\alpha \in \mathbb{S}^2, \\ x \ge 0, \\ \widehat{w_1^\alpha z_0 z_2^\alpha} = \frac{\pi}{n}. \end{array}\right.$$

Let:

$$a = \alpha\cos\frac{\pi}{n}, \quad b = 1-\sqrt{1-\alpha^2}, \quad c = 2\cos\frac{\pi}{n}\sqrt{1-\sqrt{1-\alpha^2}},$$

and

$$\left\{ \begin{array}{l} z_1 = \frac{(a^2-b^2)(c^2-a^2-b^2)+2abc\sqrt{2a^2+2b^2-c^2}}{(a^2+b^2)^2}, \\ z_2 = \frac{(a^2-b^2)(c^2-a^2-b^2)-2abc\sqrt{2a^2+2b^2-c^2}}{(a^2+b^2)^2}. \end{array}\right.$$

A simple calculation leads to:

$$w_1^\alpha \text{ solution of } (E) \Leftrightarrow \left\{ \begin{array}{l} x = \sqrt{1-z^2}, \\ z = z_1 \ ou\ z = z_2. \end{array}\right.$$

There are two solutions. We take $z = z_2$, which is linked to the farthest point from z_0. Let us construct the points w_2^α,..,w_n^α. For every $i \in \{2, .., n\}$ let w_i^α be the point of coordinates

$$\left(x_{w_1}^\alpha \cos \frac{2\pi(i-1)}{n}, x_{w_1}^\alpha \sin \frac{2\pi(i-1)}{n}, z_{w_1^\alpha} \right).$$

Remark that if r is the rotation of angle $\frac{2\pi}{n}$ and of axis (Oz_0), we get

$$\forall i \in \{1, .., n-1\} \quad \begin{array}{l} r^i(z_1^\alpha z_0 w_1^\alpha) = z_{i+1}^\alpha z_0 w_{i+1}^\alpha, \\ r^i(w_1^\alpha z_0 z_2^\alpha) = w_{i+1}^\alpha z_0 z_{i+2}^\alpha. \end{array}$$

<u>Step 2:</u>
We are going to build new points u_1^α,..,u_n^α on the sphere $\mathbb{S}^2$ satisfying:

$$\forall i \in \{1, ..n\} \; \widehat{w_i^\alpha z_{i+1}^\alpha u_i^\alpha} = \pi - \widehat{w_i^\alpha z_{i+1}^\alpha z_0} \quad \text{and } \widehat{u_i^\alpha z_{i+1}^\alpha w_{i+1}^\alpha} = \pi - \widehat{w_{i+1=}^\alpha z_{i+1}^\alpha z_0}.$$

Thus, by adding to the triangulated mesh T_n^α:

$$\begin{array}{ll} \text{the points} & u_i \text{ for } 1 \le i \le n, \\ \text{and the faces} & \left\{ \begin{array}{l} w_i^\alpha z_{i+1}^\alpha u_i^\alpha \text{ for } 1 \le i \le n, \\ u_i^\alpha z_{i+1}^\alpha w_{i+1}^\alpha \text{ for } 1 \le i \le n \; (avec \; z_{n+1}^\alpha = z_1^\alpha), \end{array} \right. \end{array}$$

we obtain:

$$\forall i \in \{1, ..n\} \; \alpha_{T_\alpha^n}(z_i^\alpha) = 2\pi.$$

Let us construct the point u_1^α:
Let P denote the plane determined by the points O, z_0 and z_2. We want to show that

$$\exists u_1^\alpha \in P \cap \mathbb{S}^2, \quad \widehat{w_1^\alpha z_2^\alpha u_1^\alpha} = \pi - \widehat{z_0 z_2^\alpha w_1^\alpha}.$$

We define the application β by:

$$\begin{array}{lll} \beta : & \mathbb{S}^2 \cap P & \rightarrow \mathbb{E} \\ & z & \mapsto \widehat{w_1^\alpha z_2^\alpha z}. \end{array}$$

There exists $\tilde{z} \in P \cap \mathbb{S}^2$ close to z_2^α, such that the triangulated mesh K whose vertices are z_0, z_2^α, w_1^α, w_2^α and $\tilde{z}$ and whose faces are $z_0 z_2^\alpha w_1^\alpha$, $z_0 z_2^\alpha w_2^\alpha$, $w_1^\alpha z_2^\alpha z$ and $w_2^\alpha z_2^\alpha z$ is a subset of the boundary of a strictly convex set of $\mathbb{E}^3$.

$$\begin{array}{ll} \text{Thus} & \alpha_K(z_2) < 2\pi. \\ \text{Since} & \alpha_K(z_2) = 2\beta(\tilde{z}) + 2\widehat{z_0 z_2^\alpha w_1^\alpha}, \\ \text{we get} & \beta(\tilde{z}) < \pi - \widehat{w_1^\alpha z_2^\alpha z_0}. \end{array}$$

Let $\tilde{\tilde{z}}$ denote the point of coordinates $(0,0,-1)$.

$$
\begin{array}{ll}
 & \beta(\tilde{\tilde{z}}) > \pi - \widehat{w_1^\alpha z_2^\alpha z_0} \\
\Leftrightarrow & \cos(\widehat{w_1^\alpha z_2^\alpha \tilde{\tilde{z}}}) < -\cos(\widehat{w_1^\alpha z_2^\alpha z_0}) \\
\Leftrightarrow & -\frac{\cos(\widehat{w_1^\alpha z_2^\alpha \tilde{\tilde{z}}})}{\cos(\widehat{w_1^\alpha z_2^\alpha z_0})} < 1. \\
\text{Since} & \lim\limits_{\alpha \to 0} -\dfrac{\cos(\widehat{w_1^\alpha z_2^\alpha \tilde{\tilde{z}}})}{\cos(\widehat{w_1^\alpha z_2^\alpha z_0})} = 0, \\
\text{we have} & \exists \alpha_0 \in]0,1], \ \forall \alpha \in]0,\alpha_0] \quad \beta(\tilde{\tilde{z}}) > \pi - \widehat{w_1^\alpha z_2^\alpha z_0}.
\end{array}
$$

Since β continuous, we get:

$$\exists u_1^\alpha \in D \cap \mathbb{S}^2, \quad \widehat{w_1^\alpha z_2^\alpha u_1^\alpha} = \beta(u_1^\alpha) = \pi - \widehat{z_0 z_2^\alpha w_1^\alpha}.$$

Furthermore, we know that the abscissa and the ordinate of u_1^α are positive. Thanks to the symmetry with respect to the plane P, we get:

$$\widehat{u_1^\alpha z_2^\alpha w_2^\alpha} = \pi - \widehat{w_2^\alpha z_2^\alpha z_0^\alpha}.$$

We know that for $\alpha \in]0,\alpha_0]$, u_1^α is well-defined. We construct the points u_2^α,..,u_n^α: if r always denotes the rotation of angle $\frac{2\pi}{n}$ and of axis (Oz_0), we define those points by:

$$\forall i \in \{1,..n\} \quad r^i(u_1^\alpha) = u_{i+1}^\alpha.$$

We clearly have:

$$\forall i \in \{1,..n\} \begin{array}{ll} r^i(w_1^\alpha u_1^\alpha z_2^\alpha) = w_{i+1}^\alpha u_{i+1}^\alpha z_{i+2}^\alpha & \text{with } z_{i+2^\alpha} = z_2^\alpha, \\ r^i(u_1^\alpha z_2^\alpha w_2^\alpha) = u_{i+1}^\alpha z_{i+2}^\alpha w_{i+2}^\alpha & \text{with } w_{i+2}^\alpha = w_2^\alpha. \end{array}$$

The triangulated mesh T_n^α satisfies the conditions of Theorem 3.

8 Conclusion and perspectives

The fact that two surfaces are close from one another (in the Hausdorff sense) does not imply that we can compare their Gauss curvature. In particular, the fact of having a developable triangle mesh closely inscribed in a smooth surface does not allow to conclude on the "unfoldness" of the smooth surface.

However, in the case in which both surfaces are developable, the unfolding of the triangle mesh gives a "good approximation" of the unfolding of the smooth surface if the normals of both surfaces are close enough and if both surfaces are close enough.

In this paper, we do not care about the construction of the triangle mesh. Results do not depend on a precise construction. However, it would be interesting to construct a developable triangle mesh from a set of sample points which are on a smooth surface.

Furthermore, we can wonder whether the approximation of the total Gauss curvature of a smooth surface depends on the *fatness* of a triangle mesh inscribed in the smooth surface. An interesting question could be to find an analogous result with an assumption on the straightness of the triangle. Another interesting question concerns the convergence (or the approximation) of the discrete pointwise Gauss curvature. What other definition can we take for the discrete pointwise Gauss curvature of a triangle mesh? What assumptions do we have to make on a sequence of triangle meshes, so as to get results of convergence?

References

[1] M. Berger, B. Gostiaux, Géométrie différentielle : variétés, courbes et surfaces (Second edition, Presses Universitaires de France, Paris, 1992).

[2] U. Brehm, W. Kühnel, Smooth approximation of polyhedral surfaces regarding curvatures, Geom. Dedicata 12 no. 4 (1982) 435-461.

[3] J. Cheeger, W. Müller, R. Schrader, On the curvature of piecewise flat spaces, Communications in mathematical Physics 92 (1984) 405-454.

[4] M. Desbrun, M. Meyer, P. Schröder, A. H. Barr, Discrete differential-geometry operators in nd, submitted (2000).
http://citeseer.nj.nec.com/desbrun00discrete.html

[5] Mathieu desbrun, Mark Meyer and Pierre Alliez, Intrinsec Parameterizations of surfaces meshes, eurographics 2002.

[6] M. P. Do Carmo, Differential geometry of curves and surfaces (Translated from the Portuguese, Prentice-Hall Inc., Englewood Cliffs, N.J., 1976).

[7] H. Federer, Curvature measures, Trans. Amer. Math. Soc 93 (1959) 418-491.

[8] J. Fu, Convergence of curvatures in secant approximations, J. Differential Geometry 37 (1993) 177-190.

[9] S. Levy, T. Munzner, M. Phillips, Geomview.
http://www.geomview.org/

[10] B. Hamman, Curvature approximation for triangulated surfaces, Computing Suppl. 8 (1993) 139-153 .

[11] Bruno Lévy, Sylvain Petitjean, Nicolas Ray and Jérome Maillot, Least square conformal maps for automatic texture atlas generation.

[12] J. Milnor, Morse theory (Princeton University Press, Princeton, New Jersey, 1963).

[13] F. Morgan, Geometric measure theory (Acad. Press, INC 1987).

[14] J.M. Morvan, B. Thibert, On the Approximation of the Area Of A Surface, rapport de recherche INRIA N 4375, février 2002.

[15] J.M. Morvan, B. Thibert, On the Approximation of the Normal Vector Field of a Smooth Surface with the normals of a Triangulated Mesh, preprint.

[16] A. Sheffer and E. de Sturler, Parameterization of Faceted Surfaces for Meshing Using Angle Based Flattening, Engineering with Computers, 17 (3), 326-337, 2001.

[17] Jonathan Richard Shewchuk, Delaunay Refinement Algorithms for Triangular Mesh Generation, 2001.

[18] M. Spivak, A comprehensive introduction to differential geometry, Vol. III (Second edition, Publish or Perish Inc., Wilmington, Del., 1979).

[19] F. E. Wolter, Cut locus and medial axis in global shape interrogation and representation, MIT Design Laboratory Memorandum 92-2 and MIT Sea Grant Report, 1992.

Contents

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399