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N° 4610

Octobre 2002

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4610--FR+ENG

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Thème 4 — Simulation et optimisation
de systèmes complexes
Projet OPALE

Rapport de recherche n° 4610 — Octobre 2002 — 57 pages

Abstract: This report deals with recent progress in the study of shape optimization problems in case of a moving domain. We may restrict ourself to the case of newtonian viscous incompressible fluids described by the Navier-Stokes equations. We suggest three strategies in order to solve an optimal control problem involving the shape variable, respectively based on ,

1. the state derivative with respect to the shape and its associated adjoint state,
2. the Min-Max principal coupled with a function space parametrization,
3. the Min-Max principal coupled with a function space embedding.

Key-words: Navier-Stokes equations, ALE formulation, sensitivity analysis, transverse field, shape optimization, optimal control, Min-Max formulation

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Contrôle dynamique de forme pour le système de Navier-Stokes : de la dérivée de l'état au principe du Min-Max

Résumé : Dans ce rapport nous présentons de récents progrès dans l'analyse des problèmes d'optimisation de forme dans le cas où le domaine spatial est mobile. L'accent est mis sur le cas des fluides visqueux newtoniens en écoulement incompressible modélisés par le système de Navier-Stokes en domaine mobile. Nous proposons trois approches afin de résoudre un problème de contrôle optimal sur la forme qui utilisent :

1. la différentiabilité de l'état par rapport à la forme et un passage à l'adjoint,
2. le principe du Min-Max avec une paramétrisation de l'espace d'état et des multiplicateurs,
3. le principe du Min-Max avec une inclusion de domaine des espaces d'états et de multiplicateurs.

Mots-clés : équations de Navier-Stokes, formulation ALE, analyse de sensibilité, champ transverse, optimisation de forme, contrôle optimal, principe du Min-Max

Contents

1 Introduction

This article deals with the analysis of an inverse dynamical shape problem involving a fluid inside a moving domain. This type of inverse problem happens frequently in the design and the control of many industrial devices such as aircraft wings, cable-stayed bridges, automobile shapes, satellite reservoir tanks and more generally of systems involving fluid-solid interactions.
The control variable is the shape of the moving domain, and the objective is to minimize a given cost functional that may be chosen by the designer.
On the theoretical level, early works concerning optimal control problems for general parabolic equations written in non-cylindrical domains have been considered in [8], [5], [6], [25], [2]. In [24], [27], [28], the stabilization of structures using the variation of the domain has been addressed. The basic principle is to define a map sending the non-cylindrical domain into a cylindrical one. This process leads to the mathematical analysis of non-autonomous PDE's systems.
Recently, a new methodology to obtain eulerian derivative for non-cylindrical functionals has been introduced in [30], [29], [13]. This methodology was applied in [14] to perform dynamical shape control of the non-cylindrical Navier-Stokes equations where the evolution of the domain is the control variable. Hence the classical optimal shape optimization theory has been extended to deal with systems set in non-cylindrical domains.
The aim of this article is to review several results on the dynamical shape control of the Navier-Stokes system and suggest an alternative treatment using the Min-Max principle [9]. Despite its lack of rigorous mathematical justification in case where the Lagrangian functional is not convex, we shall show how this principle allows, at least formally, to bypass the tedious obtention of the state differentiability with respect to the shape of the moving domain.

2 Problem statement

Let us consider a moving domain $\Omega_t \in \mathbb{R}^d$. We introduce a diffeomorphic map sending a fix reference domain Ω_0 into the physical configuration Ω_t at time $t \geq 0$.
Without loss of generality, we choose the reference configuration to be the physical configuration at initial time $\Omega(t=0)$.
Hence we define a map $T_t \in \mathcal{C}^1(\overline{\Omega_0})$ such that

$$\begin{array}{rcl} \overline{\Omega_t} & = & T_t(\overline{\Omega_0}), \\ \overline{\Gamma_t} & = & T_t(\overline{\Gamma_0}) \end{array}$$

We set $\Sigma \equiv \bigcup_{0<t<T} (\{t\} \times \Gamma_t)$, $Q \equiv \bigcup_{0<t<T} (\{t\} \times \Omega_t)$. The map T_t can be actually defined as the flow of a particular vector field, as described in the following lemma :

Theorem 1 ([26])

$$\begin{array}{rcl}\overline{\Omega_t} & = & T_t(V)(\overline{\Omega_0}),\\ \overline{\Gamma_t} & = & T_t(V)(\overline{\Gamma_0})\end{array}$$

where $T_t(V)$ is solution of the following dynamical system :

$$\begin{array}{rrcl}T_t(V): & \Omega_0 & \longrightarrow & \Omega\\ & x_0 & \longmapsto & x(t,x_0) \equiv T_t(V)(x_0)\end{array}$$

with

$$\begin{array}{l}\dfrac{\mathrm{d}\,x}{\mathrm{d}\,\tau} = V(\tau, x(\tau)), \quad \tau \in [0,T]\\ x(\tau = 0) = x_0, \qquad \textit{in} \;\; \Omega_0\end{array} \tag{1}$$

The fluid inside Ω_t is assumed to be a viscous incompressible newtonian fluid. Its evolution is described by its velocity u and its pressure p. The couple (u,p) satisfies the classical Navier-Stokes equations written in non-conservative form,

$$\left\{\begin{array}{ll}\partial_t u + \mathrm{D}\,u \cdot u - \nu\Delta u + \nabla p = 0, & Q(V)\\ \mathrm{div}(u) = 0, & Q(V)\\ u = V, & \Sigma(V)\\ u(t=0) = u_0, & \Omega_0\end{array}\right. \tag{2}$$

where ν stands for the kinematic viscosity.
The quantity $\sigma(u,p) = -p\,\mathrm{I} + \nu(\mathrm{D}\,u + {}^*\,\mathrm{D}\,u)$ stands for the fluid stress tensor inside Ω_t, with $(\mathrm{D}\,u)_{i,j} = \partial_j u_i = u_{i,j}$.
We are interested in solving the following minimization problem :

$$\min_{V \in \mathcal{U}_{ad}} j(V) \tag{3}$$

where $j(V) = J_V(u(V), p(V))$ with $(u(V), p(V))$ is a weak solution of problem (2) and $J_V(u,p)$ is a real functional of the following form :

$$J_V(u,p) = \frac{\alpha}{2}\|\mathcal{B}\,u\|^2_{Q(V)} + \frac{\gamma}{2}\|\mathcal{K}\,V\|^2_{\Sigma(V)} \tag{4}$$

where is $\mathcal{B} \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*)$ is a general linear differential operator satisfying the following identity,

$$\langle \mathcal{B}\,u, v\rangle + \langle u, \mathcal{B}^* v\rangle = \langle \mathcal{B}_\Sigma\,u, v\rangle_{L^2(\Sigma)} \tag{5}$$

where $\mathcal{H} = \left\{v \in L^2(0,T;(H^1_0(\mathrm{div}, \Omega_t(V)))^d)\right\}$ and $\mathcal{K} \in \mathcal{L}(\mathcal{U}_{ad}, L^2(\Sigma(V)))$ is a general linear differential operator satisfying the following identity,

$$\langle \mathcal{K}\,u, v\rangle_{L^2(\Sigma)} + \langle u, \mathcal{K}^* v\rangle_{L^2(\Sigma)} = \langle \mathcal{K}_\Sigma\,u, v\rangle_{L^2(\Sigma)} \tag{6}$$

The main difficulty in dealing with such a minimization problem is related to the fact that integrals over the domain $\Omega_t(V)$ depend on the control variable V. This point will be solved by using the Arbitrary Lagrange-Euler (ALE) map $T_t(V)$ introduced previously. The purpose of this article is to prove using several methods the following result,

Main Result: For $V \in \mathcal{U}_{ad}$ and Ω_0 of class $\mathcal{C}^2$, the functional $j(V)$ possesses a gradient $\nabla j(V)$ which is supported on the moving boundary $\Gamma_t(V)$ and can be represented by the following expression,

$$\nabla j(V) = -\lambda\, n - \sigma(\varphi,\pi)\cdot n + \alpha\,\mathcal{B}_\Sigma \mathcal{B}\, u + \gamma\,\left[-\mathcal{K}^*\mathcal{K}\, V + \mathcal{K}_\Sigma \mathcal{K}\, V\right] \tag{7}$$

where (φ,π) stands for the adjoint fluid state solution of the following system,

$$\begin{cases} -\partial_t\varphi - \mathrm{D}\,\varphi\cdot u + {}^*\,\mathrm{D}\,u\cdot\varphi - \nu\Delta\varphi + \nabla\pi = -\alpha\,\mathcal{B}^*\mathcal{B}\,u, & Q(V) \\ \mathrm{div}(\varphi) = 0, & Q(V) \\ \varphi = 0, & \Sigma(V) \\ \varphi(T) = 0, & \Omega_T \end{cases} \tag{8}$$

and λ is the adjoint transverse boundary field, solution of the tangential dynamical system,

$$\begin{cases} -\partial_t\lambda - \nabla_\Gamma\,\lambda\cdot V - (\mathrm{div}\,V)\,\lambda = f, & (0,T) \\ \lambda(T) = 0, & \Gamma_T(V) \end{cases} \tag{9}$$

with

$$f = \left[-(\sigma(\varphi,\pi)\cdot n) + \alpha\,\mathcal{B}_\Sigma\mathcal{B}\,u\right]\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \frac{1}{2}\left[\alpha|\mathcal{B}\,u|^2 + \gamma\,H|\mathcal{K}\,V|^2\right] \tag{10}$$

Example 1 *We set,*

$$\begin{aligned} (\mathcal{B},\mathcal{B}^*,\mathcal{B}_\Sigma) &= (\mathrm{I},-\mathrm{I},0) \\ (\mathcal{K},\mathcal{K}^*,\mathcal{K}_\Sigma) &= (\mathrm{I},-\mathrm{I},0) \end{aligned}$$

This means, that we consider the cost functional,

$$J_V(u,p) = \frac{\alpha}{2}\|u\|^2_{L^2(Q(V))} + \frac{\gamma}{2}\|V\|^2_{L^2(\Sigma(V))} \tag{11}$$

then its gradient is given by,

$$\nabla j(V) = -\lambda\, n - \sigma(\varphi,\pi)\cdot n + \gamma\, V \tag{12}$$

where (φ, π) stands for the adjoint fluid state solution of the following system,

$$\left\{\begin{array}{ll} -\partial_t\varphi - \mathrm{D}\,\varphi\cdot u + {}^*\,\mathrm{D}\,u\cdot\varphi - \nu\Delta\varphi + \nabla\pi = \alpha\,u, & Q(V)\\ \mathrm{div}(\varphi) = 0, & Q(V)\\ \varphi = 0, & \Sigma(V)\\ \varphi(T) = 0, & \Omega_T \end{array}\right. \tag{13}$$

and λ is the adjoint transverse boundary field, solution of the tangential dynamical system,

$$\left\{\begin{array}{ll} -\partial_t\lambda - \nabla_\Gamma\,\lambda\cdot V - (\mathrm{div}\,V)\,\lambda = f, & (0,T)\\ \lambda(T) = 0, & \Gamma_T(V) \end{array}\right. \tag{14}$$

with

$$f = -\nu(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \frac{1}{2}(\alpha + \gamma\,H)|V|^2 \tag{15}$$

Example 2 *We set,*

$$\begin{array}{rcl} (\mathcal{B}, \mathcal{B}^*, \mathcal{B}_\Sigma) & = & (\mathrm{curl}, \mathrm{curl}, \wedge n)\\ (\mathcal{K}, \mathcal{K}^*, \mathcal{K}_\Sigma) & = & (\mathrm{I}, -\mathrm{I}, 0) \end{array}$$

$$J_V(u,p) = \frac{\alpha}{2}\|\,\mathrm{curl}\,u\|^2_{L^2(Q(V))} + \frac{\gamma}{2}\|V\|^2_{L^2(\Sigma(V))} \tag{16}$$

then its gradient is given by,

$$\nabla j(V) = -\lambda\,n - \sigma(\varphi,\pi)\cdot n + \alpha\,(\mathrm{curl}\,u)\wedge n + \gamma\,V \tag{17}$$

where (φ, π) stands for the adjoint fluid state solution of the following system,

$$\left\{\begin{array}{ll} -\partial_t\varphi - \mathrm{D}\,\varphi\cdot u + {}^*\,\mathrm{D}\,u\cdot\varphi - \nu\Delta\varphi + \nabla\pi = -\alpha\,\Delta u, & Q(V)\\ \mathrm{div}(\varphi) = 0, & Q(V)\\ \varphi = 0, & \Sigma(V)\\ \varphi(T) = 0, & \Omega_T \end{array}\right. \tag{18}$$

and λ is the adjoint transverse boundary field, solution of the tangential dynamical system,

$$\left\{\begin{array}{ll} -\partial_t\lambda - \nabla_\Gamma\,\lambda\cdot V - (\mathrm{div}\,V)\,\lambda = f, & (0,T)\\ \lambda(T) = 0, & \Gamma_T(V) \end{array}\right. \tag{19}$$

with

$$f = [-\nu\,\mathrm{D}\,\varphi\cdot n + \alpha\,(\mathrm{curl}\,u)\wedge n]\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \frac{1}{2}\left[\alpha|\,\mathrm{curl}\;u|^2 + \gamma\,H|V|^2\right] \tag{20}$$

In the next section, we introduce several concepts closely related to shape optimization tools for moving domain problems. We also recall elements of tangential calculus that will be used through this article. Then we treat sequentially the following points,

1. In section (4), we choose to prove the differentiability of the fluid state (u, p) with respect to the design variable V. The directional shape derivative $(u', p')(V) \cdot W$ is then used to compute the directional derivative $j'(V) \cdot W$ of the cost functional $j(V)$. Using the adjoint state $(\varphi, \pi)(V)$ associated to $(u', p')(V)$ and the adjoint field Λ associated to the transverse field Z_t introduced in section (3), we are able to furnish an expression of the gradient $\nabla j(V)$ which is a distribution supported by the moving boundary $\Gamma_t(V)$.

2. In section (5), we choose to bypass the obtention of the state shape derivative $(u', p')(V) \cdot W$, by using a Min-Max formulation of problem (3). The state and multiplier spaces are chosen in order to be independent on the perturbation parameter used in the obtention of the derivative of the lagrangian functional with respect to V. This leads to the direct obtention of the fluid state and transverse field adjoints and consequently to the gradient $\nabla j(V)$.

3. In section (6), we again use a Min-Max strategy coupled with a state and multiplier functional space embedding. This means that the state and multiplier variables live in the hold-all domain D. Hence the derivative of the lagrangian functional with respect to V only involves terms coming from the flux variation trough the moving boundary $\Gamma_t(V)$. This again leads to the direct obtention of the fluid state and transverse field adjoints and consequently to the gradient $\nabla j(V)$.

3 Elements of non-cylindrical shape and tangential calculus

This section introduce several concepts that will be intensively used through this report. It concerns the differential calculus of integrals defined on moving domains or boundaries with respect to their support and basic tools for intrinsic tangential calculus using the oriented distance function and tangential operators.

3.1 Non-cylindrical speed method

In this paragraph, we are interested in differentiability properties of integrals defined in moving configuration,

$$J_1(\Omega_t) = \int_{\Omega_t} f(\Omega_t) d\Omega$$

$$J_2(\Gamma_t) = \int_{\Gamma_t} g(\Gamma_t) d\Gamma$$

The behaviour J_1 and J_2 while perturbing their moving support highly depends on the regularity in space and time of the domains. In this work, we choose to work with domains Ω_t that are images of a fixed domain Ω_0 trough an ALE map $T_t(V)$ as introduced in the first section. Hence, the design parameter is no more the support Ω_t but rather the velocity field $V \in \mathcal{U}_{ad} \stackrel{\text{def}}{=} \mathcal{C}^0([0,T];(W^{k,\infty}(D))^d)$ that builds the support. This technique has the advantage to transform shape calculus into classical differential calculus on a vector space [30],[14]. For an other choice based on the non-cylindrical identity perturbation, the reader is referred to [4],[19]. Before stating the main result of this section, we recall the notion of transverse field.

3.1.1 Transverse applications

Definition 1 *The transverse map $\mathcal{T}_\rho^t$ associated to two vector fields $(V,W) \in \mathcal{U}_{ad}$ is defined as follows,*

$$\begin{array}{rcl} \mathcal{T}_\rho^t : \overline{\Omega_t} & \longrightarrow & \overline{\Omega_t^\rho} \stackrel{\text{def}}{=} \overline{\Omega_t(V+\rho W)} \\ x & \mapsto & T_t(V+\rho W) \circ T_t(V)^{-1} \end{array}$$

Remark 1 *The transverse map allows us to perform pertubations analysis on functions defined on the unperturbed domain $\Omega_t(V)$.*

The following result states that the transverse map $\mathcal{T}_\rho^t$ has itself the structure of a dynamical flow with respect to the perturbation variable ρ,

Theorem 2 ([29]) *The Transverse map $\mathcal{T}_\rho^t$ is the flow of a transverse field $\mathcal{Z}_\rho^t$ defined as follow :*

$$\mathcal{Z}_\rho^t \stackrel{\text{def}}{=} \mathcal{Z}^t(\rho, .) = \left(\frac{\partial \mathcal{T}_\rho^t}{\partial \rho}\right) \circ (\mathcal{T}_\rho^t)^{-1} \tag{21}$$

i.e is the solution of the following dynamical system :

$$\begin{array}{rcl} T_t^\rho(\mathcal{Z}_\rho^t) : \overline{\Omega_t} & \longrightarrow & \overline{\Omega_t^\rho} \\ x & \longmapsto & x(\rho, x) \equiv T_t^\rho(\mathcal{Z}_\rho^t)(x) \end{array}$$

with

$$\begin{array}{ll} \dfrac{\mathrm{d}\, x(\rho)}{\mathrm{d}\, \rho} = \mathcal{Z}^t(\rho, x(\rho)), & \rho \geq 0 \\ x(\rho = 0) = x, & \textit{in } \Omega_t(V) \end{array} \tag{22}$$

Since, we will mainly consider derivatives of perturbed functions at point $\rho = 0$, we set $Z_t \stackrel{\text{def}}{=} \mathcal{Z}_{\rho=0}^t$. A fundamental result lies in the fact that Z_t can be obtained as the solution of linear time dynamical system depending on the vector fields $(V,W) \in \mathcal{U}_{ad}$,

Theorem 3 ([14]) *The vector field Z_t is the unique solution of the following Cauchy problem,*

$$\begin{cases} \partial_t Z_t + [Z_t, V] = W, & D \times (0,T) \\ Z_{t=0} = 0, & D \end{cases} \tag{23}$$

where $[Z_t, V] \stackrel{\text{def}}{=} DZ_t \cdot V - DV \cdot Z_t$ stands for the Lie bracket of the pair (Z_t, V).

3.1.2 Shape derivative of non-cylindrical functionals

The main theorem of this section uses the notion of non-cylindrical material derivative that we recall here,

Definition 2 *A function $f(V) \in H(\Omega_t(V))$ admits a non-cylindrical material derivative $\dot{f}(V;W)$ at point $V \in \mathcal{U}_{ad}$ in the direction $W \in \mathcal{U}_{ad}$ if the following composed function,*

$$\begin{array}{rcl} f^\rho : [0,\rho_0] & \to & H(\Omega_t(V)) \\ \rho & \mapsto & f(V+\rho W) \circ \mathcal{T}_\rho^t \end{array}$$

is differentiable at point $\rho = 0$, a.e $(t,x) \in Q(V)$ and $\dot{f}(V) \cdot W = \dot{f}(V;W) = \left. \dfrac{d}{d\rho} f^\rho \right|_{\rho=0}$.

With the above definition, we can state the differentiability properties of non-cylindrical integrals with respect to their moving support,

Theorem 4 ([14],[19]) *For a bounded measurable domain Ω_0 with boundary Γ_0, let us assume that for any direction $W \in U$ the following hypothesis holds,*

i) $f(V)$ admits a non-cylindrical material derivative $\dot{f}(V) \cdot W$

then $J_1(.)$ is Gâteaux differentiable at point $V \in \mathcal{U}_{ad}$ and its derivative is given by the following expression,

$$J_1'(V) \cdot W = \int_{\Omega_t(V)} \left[\dot{f}(V) \cdot W + f(V) \operatorname{div} Z_t \right] d\Omega \tag{24}$$

Futhermore, if

ii) $f(V)$ admits a non-cylindrical shape derivative given by the following expression,

$$f'(V) \cdot W = \dot{f}(V) \cdot W - \nabla f(V) \cdot Z_t \tag{25}$$

then

$$J_1'(V) \cdot W = \int_{\Omega_t(V)} [f'(V) \cdot W + \operatorname{div}(f(V)\, Z_t)] \, d\Omega \tag{26}$$

Furthermore, if Ω_0 is an open domain with a Lipschitzian boundary Γ_0, then

$$J_1'(V) \cdot W = \int_{\Omega_t(V)} f'(V) \cdot W d\Omega + \int_{\Gamma_t(V)} f(V) \, \langle Z_t, n \rangle d\Gamma \tag{27}$$

Remark 2 *The last identity will be of great interest while trying to prove a gradient structure result for general non-cylindrical functionals.*

It is also possible to establish a similar result for integrals over moving boundaries. For that purpose, we need to define the non-cylindrical tangential material derivative,

Definition 3 *A function $g(V) \in H(\Gamma_t(V))$ admits a non-cylindrical material derivative $\dot{g}(V;W)$ at point $V \in \mathcal{U}_{ad}$ in the direction $W \in \mathcal{U}_{ad}$ if the following composed function,*

$$\begin{array}{rcl} g^\rho : [0,\rho_0] & \to & H(\Gamma_t(V)) \\ \rho & \mapsto & g(V+\rho W) \circ \mathcal{T}_\rho^t \end{array}$$

is differentiable at point $\rho = 0$, a.e $(t,x) \in \Sigma(V)$ and $\dot{g}(V;W) = \frac{d}{d\rho} g^\rho \Big|_{\rho=0}$.

This concept is involved in the differentiability property of boundary integrals,

Theorem 5 ([19]) *For a bounded measurable domain Ω_0 with boundary Γ_0, let us assume that for any direction $W \in U$ the following hypothesis holds,*

i) $g(V)$ admits a non-cylindrical material derivative $\dot{g}(V) \cdot W$

then $J_2(.)$ is Gâteaux differentiable at point $V \in \mathcal{U}_{ad}$ and its derivative is given by the following expression,

$$J_2'(V) \cdot W = \int_{\Gamma_t(V)} [\dot{g}(V) \cdot W + g(V)\ \mathrm{div}_\Gamma Z_t]\, d\Gamma \tag{28}$$

Futhermore, if

ii) $g(V)$ admits a non-cylindrical shape derivative given by the following expression,

$$g'(V) \cdot W = \dot{g}(V) \cdot W - \nabla_\Gamma g(V) \cdot Z_t \tag{29}$$

then

$$J_2'(V) \cdot W = \int_{\Gamma_t(V)} [g'(V) \cdot W + H\, g(V) \langle Z_t, n\rangle]\, d\Gamma \tag{30}$$

where H stands for the additive curvature (Def. (4)). Furthermore, if $g(V) = \tilde{g}(V)|_{\Gamma_t(V)}$ with $\tilde{g} \in H(\Omega_t(V))$, then

$$J_2'(V) \cdot W = \int_{\Gamma_t(V)} [g'(V) \cdot W + (\nabla \tilde{g}(V) \cdot n + H\, g(V))\, \langle Z_t, n\rangle]\, d\Gamma \tag{31}$$

3.1.3 Adjoint transverse field

It is possible to define the solution of the adjoint transverse system,

Theorem 6 ([13]) *For $F \in L^2(0,T;(H^1(D))^d)$, there exists a unique field*

$$\Lambda \in \mathcal{C}^0([0,T];(L^2(D))^d)$$

solution of the backward dynamical system,

$$\begin{cases} -\partial_t \Lambda - \mathrm{D}\,\Lambda \cdot V - {}^*\mathrm{D}\,V \cdot \Lambda - (\operatorname{div} V)\Lambda = F, & (0,T) \\ \Lambda(T) = 0, \end{cases} \tag{32}$$

Remark 3 *The field Λ is the dual variable associated to the transverse field Z_t and is solution of the adjoint problem associated to the transverse dynamical system.*

In this article, we shall deal with a specific right-hand side F of the form $F(t) = {}^*\gamma_{\Gamma_t(V)}(f(t)n)$. Then the adjoint field Λ is supported on the moving boundary $\Gamma_t(V)$ and has the following structure,

Theorem 7 ([14]) *For $F(t) = {}^*\gamma_{\Gamma_t(V)}(f(t)n)$, with $f \in L^2(0,T;L^2(\Gamma_t(V))$, the unique solution Λ of problem is given by the following identity,*

$$\Lambda = (\lambda \circ p)\, \nabla\, \chi_{\Omega_t(V)} \in \mathcal{C}^0([0,T];(H^1(\Gamma_t))^d) \tag{33}$$

where $\lambda \in \mathcal{C}^0([0,T];H^1(\Gamma_t))$ is the unique solution of the following boundary dynamical system,

$$\begin{cases} -\partial_t \lambda - \nabla_\Gamma \lambda \cdot V - (\operatorname{div} V)\lambda = f, & (0,T) \\ \lambda(T) = 0, & \Gamma_t(V) \end{cases} \tag{34}$$

p is the canonical projection on $\Gamma_t(V)$ and $\chi_{\Omega_t(V)}$ is the characteristic function of $\Omega_t(V)$ inside D.

3.1.4 Gradient of non-cylindrical functionals

In the next sections, we will often deal with boundary integrals of the following forms,

$$K = \int_0^T \int_{\Gamma_t(V)} E(V)\, \langle Z_t, n\rangle$$

with $E(V) \in L^2(0,T;\Gamma_t(V))$. The following result allows us to eliminate the auxilliary variable Z_t inside the functional K,

Theorem 8 ([14],[19]) *For any $E(V) \in L^2(0,T;\Gamma_t(V))$ and $(V,W) \in \mathcal{U}_{ad}$, the following identity holds,*

$$\int_0^T \int_{\Gamma_t(V)} E(V)\, \langle Z_t, n\rangle = -\int_0^T \int_{\Gamma_t(V)} \lambda \langle W, n\rangle \tag{35}$$

where $\lambda \in \mathcal{C}^0([0,T];H^1(\Gamma_t))$ is the unique solution of problem (34) with $f = E$.

3.2 Elements of tangential calculus

In this paragraph, we review basic elements of differential calculus on a $\mathcal{C}^k$- submanifold with $k \geq 2$ of codimension one in $\mathbb{R}^d$. The following approach avoids the use of local bases and coordinates by using the intrinsic tangential derivative.

3.2.1 Oriented distance function

Let Ω be an open domain of class $\mathcal{C}^k$ in $\mathbb{R}^d$ with compact boundary Γ. We define the oriented distance function to be as follows,

$$b_\Omega(x) = \begin{cases} d_\Gamma(x), & x \in \mathbb{R}^d \setminus \overline{\Omega} \\ -d_\Gamma(x), & x \in \Omega \end{cases}$$

where $d_\Gamma(x) = \min_{y\in\Omega} |y - x|$.

Proposition 1 ([10]) *Let Ω be an open domain of class $\mathcal{C}^k$ for $k \geq 2$ in $\mathbb{R}^d$ with compact boundary Γ. There exists a neighbourhood $U(\Gamma)$ of Γ, such that $b \in \mathcal{C}^k(U(\Gamma))$. Furthermore, we have the following properties,*

i) $\nabla b|_\Gamma = n$, *where n stands for the unit exterior normal on Γ,*

ii) $\mathrm{D}^2 b : T_{p(x)}\Gamma \to T_{p(x)}\Gamma$ *coincides with the second fondamental form on Γ, where*

$$\begin{array}{rcl} p : U(\Gamma) & \to & \Gamma \\ x & \mapsto & x - b(x) \cdot \nabla b(x) \end{array}$$

stands for the projection mapping and $T_{p(x)}\Gamma$ stands for the tangent plane.

iii) $(0, \beta_1, \ldots, \beta_{d-1})$ *are the eigenvalues of $\mathrm{D}^2 b$ associated to the eigenfunctions*

$$(n, \mu_1, \ldots, \mu_{d-1})$$

where $(\beta_i, \mu_i)_{1\leq i\leq d-1}$ are the mean curvatures and principal direction of curvatures of Γ.

Proposition 2 ([10]) *For Γ of class $\mathcal{C}^2$, the projection mapping p is differentiable and its derivative has the following properties,*

$$\begin{aligned} {}^*\mathrm{D}\, p = \mathrm{D}\, p = \mathrm{I} - \nabla b \cdot {}^*\nabla b - b\, \mathrm{D}^2 b \\ \mathrm{D}\, p \cdot \tau = \tau, \quad \text{on } \Gamma, \\ \mathrm{D}\, p \cdot n = 0, \quad \text{on } \Gamma \end{aligned} \tag{36}$$

Definition 4 ([10]) *For Γ of class $\mathcal{C}^2$, the additive curvature H of Γ is defined as the trace of the second order fondamental form :*

$$H = \mathrm{Tr}\, \mathrm{D}^2 b = \Delta b = (d-1)\bar{H}, \quad \text{on } \Gamma \tag{37}$$

and $\bar{H}$ stands for the mean curvature of Γ.

3.2.2 Intrinsic tangential calculus

Using arbitrary smooth extension of functions defined on Γ to $\Omega \in \mathbb{R}^d$, is the most classical way of defining tangential operators. Hence the differential calculus on manifolds can be reduced to classical differential calculus in $\mathbb{R}^d$. In this section we recall, standard formulas for differential tangential operators using arbitrary extensions. We also emphasize the particular case where the extension is of the canonical type $(f \circ p)$. This is the basis of a simple differential calculus in the neighbourhood of Γ.

Definition 5 *For Γ of class $\mathcal{C}^2$, given any extension $F \in \mathcal{C}^1(U(\Gamma))$ of $f \in \mathcal{C}^1(\Gamma)$, the tangential gradient of f is defined as,*

$$\nabla_\Gamma f \stackrel{\text{def}}{=} \nabla F|_\Gamma - (\partial_n F)\, n \tag{38}$$

where $\partial_n F = \nabla F \cdot n$.

Proposition 3 ([10]) *Assume that Γ of class $\mathcal{C}^2$ is compact and $f \in \mathcal{C}^1(\Gamma)$, then*

i)

$$\begin{aligned} \nabla_\Gamma f &= (P \nabla F)|_\Gamma \\ n \cdot \nabla_\Gamma f &= \nabla b \cdot \nabla_\Gamma f = 0 \end{aligned} \tag{39}$$

where $P \stackrel{\text{def}}{=} \mathrm{I} - \nabla b^* \nabla b$ is the orthogonal projection operator onto the tangent plane $T_{p(x)}\Gamma$.

ii)

$$\begin{aligned} \nabla(f \circ p) &= \left[\mathrm{I} - b\, \mathrm{D}^2 b\right] \nabla_\Gamma f \circ p \\ \nabla(f \circ p)|_\Gamma &= \nabla_\Gamma f \end{aligned} \tag{40}$$

Hence $(f \circ p)$ plays the role of a canonical extension in the neigborhood $U(\Gamma)$ and its gradient is tangent to the level sets of b. Consequently, we can define in an intrinsic way the tangential gradient,

Definition 6 *For Γ of class $\mathcal{C}^2$ and $f \in \mathcal{C}^1(\Gamma)$, the tangential gradient of f is defined as,*

$$\nabla_\Gamma f = \nabla(f \circ p)|_\Gamma \tag{41}$$

In the sequel, we shall use the above definition for the tangential gradient whenever the function under derivation is intrinsically defined on Γ. We now define the other classical tangential operators,

Definition 7 *For Γ of class $\mathcal{C}^2$,* *i) for $v \in (\mathcal{C}^1(\Gamma))^d$, and $\tilde{v} \in (\mathcal{C}^1(U(\Gamma)))^d$ an arbitrary extension, the tangential jacobian is defined as follows,*

$$\begin{aligned} \mathrm{D}_\Gamma\, v &\overset{\text{def}}{=} \mathrm{D}\,\tilde{v}|_\Gamma - (\mathrm{D}\,\tilde{v}\cdot n)^* n \\ &= \mathrm{D}\,\tilde{v}|_\Gamma - \mathrm{D}\,\tilde{v}\cdot(n\otimes n) \end{aligned} \tag{42}$$

Furthermore,

$$\begin{aligned} \mathrm{D}(v\circ p) &= \mathrm{D}_\Gamma\, v\circ p\left[\mathrm{I} - b\,\mathrm{D}^2\, b\right] \\ \mathrm{D}_\Gamma\, v &= \mathrm{D}(v\circ p)|_\Gamma \end{aligned} \tag{43}$$

ii) for $v \in (\mathcal{C}^1(\Gamma))^d$, and $\tilde{v} \in (\mathcal{C}^1(U(\Gamma)))^d$ an arbitrary extension, the tangential divergence is defined as follows,

$$\mathrm{div}_\Gamma\, v \overset{\text{def}}{=} \mathrm{div}\,\tilde{v}|_\Gamma - (\mathrm{D}\,\tilde{v}\cdot n)\cdot n \tag{44}$$

Furthermore,

$$\mathrm{div}_\Gamma\, v = \mathrm{div}(v\circ p)|_\Gamma = \mathrm{Tr}(\mathrm{D}_\Gamma\, v) \tag{45}$$

iii) for $f \in \mathcal{C}^2(\Gamma)$, and $F \in \mathcal{C}^2(U(\Gamma))$ an arbitrary extension, the tangential Laplace-Beltrami operator is defined as follows,

$$\Delta_\Gamma f = \Delta F|_\Gamma - H\partial_n F - \partial_n^2 F \tag{46}$$

with $\partial_n^2 F = (\mathrm{D}^2\, F\cdot n)\cdot n$. Furthermore,

$$\Delta_\Gamma f = \mathrm{div}_\Gamma(\nabla_\Gamma\, f) = \Delta(f\circ p)|_\Gamma \tag{47}$$

In some cases, it may be interesting to use a splitting of the function v onto a normal and a tangential component,

Definition 8 *For $v \in (\mathcal{C}^1(\Gamma))^d$, we define the tangential component $v_\Gamma \in (\mathcal{C}^1(\Gamma))^d$ and the normal component $v_n \in \mathcal{C}^1(\Gamma)$ such that,*

$$v = v_\Gamma + v_n\, n \tag{48}$$

Using this definition, we obtain the following identities,

Proposition 4 *For $v \in (\mathcal{C}^1(\Gamma))^d$, we have*

$$\mathrm{D}_\Gamma\, v = \mathrm{D}_\Gamma\, v_\Gamma + v_n\cdot\mathrm{D}^2\, b + n\cdot{}^*\,\nabla_\Gamma\, v_n \tag{49}$$

$$\nabla_\Gamma\, v_n = {}^*\,\mathrm{D}_\Gamma\, v\, n + \mathrm{D}^2\, b v_\Gamma \tag{50}$$

$$\mathrm{div}_\Gamma\, v = \mathrm{div}_\Gamma\, v_\Gamma + H v_n \tag{51}$$

3.2.3 Tangential Stokes formula

In order to perform integration by parts on Γ, we will use the following tangential Stokes identity,

Proposition 5 *Let Γ be a $\mathcal{C}^2$-submanifold in $\mathbb{R}^d$, for $E \in H^1(\Gamma;\mathbb{R}^d)$ and $\psi \in H^1(\Gamma;\mathbb{R})$ the following identity holds*

$$\int_\Gamma \langle E, \nabla_\Gamma \psi\rangle_{\mathbb{R}^d} + \int_\Gamma (\mathrm{div}_\Gamma E)\psi = \int_\Gamma H\psi\langle E, n\rangle_{\mathbb{R}^d} \tag{52}$$

4 State derivative strategy

In this section, we shall prove the main theorem using an approach based on the differentiability of the solution of the Navier-Stokes system (Eq. (2)) with respect to the velocity field V. First we introduce a weak formulation for Eq. (2) and recall associated classical solvability results. Then, using the weak implicit function theorem, we will prove the existence of a weak material derivative. Finally, introducing adjoint equations associated to the linearized fluid and transverse systems, we will be able to express the gradient of the functional $j(V)$. For the sake of simplicity, we shall only prove the main theorem in the case of example (1) and with free divergence control velocity fields.

4.1 Weak formulation and solvability

In order to take into account the non-homogeneous Dirichlet boundary condition on $\Gamma_t(V)$, we use the following change of variable $\tilde{u} = u - V$, where $\tilde{u}$ satisfies the following homogeneous Dirichlet Navier-Stokes system,

$$\begin{cases} \partial_t \tilde{u} + \mathrm{D}\,\tilde{u}\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot V + \mathrm{D}\,V\cdot\tilde{u} - \nu\Delta\tilde{u} + \nabla p = F(V), & Q(V) \\ \mathrm{div}(\tilde{u}) = 0, & Q(V) \\ \tilde{u} = 0, & \Sigma(V) \\ \tilde{u}(0) = u_0 - V(0), & \Omega_0 \end{cases} \tag{53}$$

with $F(V) = -\partial_t V - \mathrm{D}\,V\cdot V + \nu\Delta V$.
We consider the following classical functional spaces [17], [23],

$$\begin{aligned} H(D) &= \left\{ v \in (L^2(D))^d, \quad \mathrm{div}\, v = 0, \text{ in } D, \quad v\cdot n = 0 \text{ on } \partial D \right\} \\ H_0^1(\mathrm{div}, D) &= \left\{ v \in (H_0^1(D))^d, \quad \mathrm{div}\, v = 0, \text{ in } D \right\} \\ \mathcal{H} &= \left\{ v \in L^2(0,T;(H_0^1(\mathrm{div},\Omega_t(V)))^d) \right\} \\ \mathcal{V} &= \left\{ v \in \mathcal{H}, \quad \partial_t v \in L^2(0,T;(H_0^1(\Omega_t(V)))^d) \right\} \end{aligned}$$

In the sequel, we shall use the notation u instead of $\tilde{u}$, keeping in mind that the original variable is obtained by translation.

Definition 9 *The function $u \in \mathcal{V}$ is called a weak solution of problem (53), if it satisfies the following identity,*

$$\langle e_V(u), v\rangle = \langle [e_V^1(u), e_V^2(u)], v\rangle = [0,0], \quad \forall\, v \in \mathcal{H} \tag{54}$$

with

$$\langle e_V^1(u), v\rangle = \int_0^T \int_{\Omega_t(V)} [(\partial_t u + \mathrm{D}\, u \cdot u + \mathrm{D}\, u \cdot V + \mathrm{D}\, V \cdot u) \cdot v + \nu\, \mathrm{D}\, u \cdot\cdot\, \mathrm{D}\, v - F(V) \cdot v] \tag{55}$$

$$\langle e_V^2(u), v\rangle = \int_{\Omega_0} (u(0) - \tilde{u}_0) \cdot v(0) \tag{56}$$

We set,

$$\mathcal{U}_{ad} = \left\{ V \in H^1(0,T; (H^m(D))^d), \quad \operatorname{div} V = 0 \text{ in } D, \quad V \cdot n = 0 \text{ on } \partial D \right\}$$

with $m > 5/2$.

Theorem 9 ([13]) *We assume the domain Ω_0 to be of class $\mathcal{C}^1$. For $V \in \mathcal{U}_{ad}$ and $u_0 \in H(D)$ such that $u_0|_{\Omega_0} \in H(\Omega_0)$,*

1. *it exists at least a weak solution of problem (53) with $u \in \mathcal{H} \cup L^\infty(0,T;H)$,*
2. *if $u_0 \in (H^2(D))^d \cup H_0^1(\mathrm{div}, D)$ and ν is large or u_0 is a small data, then the uniqueness of a weak solution is guaranted, and we have $\partial_t u \in \mathcal{H} \cup L^\infty(0,T;H(\Omega_t))$,*
3. *if Ω is of class $\mathcal{C}^2$, $u \in L^\infty(0,T;(H^2(\Omega_t))^d \cup H_0^1(\mathrm{div}, \Omega_t))$.*

4.2 Weak implicit function theorem and the Piola material derivative

We are interested in solving the following minimization problem :

$$\min_{V \in \mathcal{U}_{ad}} j(V) \tag{57}$$

with

$$j(V) = \frac{\alpha}{2} \int_0^T \int_{\Omega_t(V)} |u(V)|^2 + \frac{\gamma}{2} \int_0^T \int_{\Omega_t(V)} |V|^2 \tag{58}$$

Theorem 10 ([14]) *Problem (57) admits at least one solution $V^* \in \mathcal{U}_{ad}$.*

In order to derive first-order optimality conditions for problem (57), we need to analyse the derivability of the state $u(V)$ with respect to $V \in \mathcal{U}_{ad}$. There exist at least two methods in order to establish such a differentiability result :

- Limit analysis of the differential quotient,

$$\dot{u}(V;W) = \lim_{\rho\to 0} \frac{1}{\rho}(u(V+\rho W)\circ T_t(V+\rho W) - u(V)\circ T_t(V))$$

- Application of the weak implicit function theorem and deduction of the local differentiability of the solution $u(V)$ associated to the implicit equation $e(u,v)=0, \quad \forall\, v \in \mathcal{H}$.

We recall here how the second method can be applied to our problem, following the result obtained in [14].
In order to work with divergence free functions, we need to introduce the Piola transform that preserves the free divergence condition.

Lemma 1 ([4]) *The Piola transform,*

$$\begin{array}{rcl} Pt : H^1_0(\mathrm{div},\Omega_t(V)) & \longrightarrow & H^1_0(\mathrm{div},\Omega^\rho_t) \\ v & \mapsto & (\mathrm{D}\,\mathcal{T}^t_\rho \cdot v)\circ(\mathcal{T}^t_\rho)^{-1} \end{array}$$

is an isomorphism.

We consider the solution $u_\rho = u(V+\rho W)$ defined on Ω^ρ_t of the implicit equation,

$$\langle e_{(V+\rho W)}\,(u), v\rangle = 0, \quad \forall\, v \in \mathcal{H}^\rho$$

and we introduce $\hat{u}_\rho = (\mathrm{D}\,\mathcal{T}^t_\rho)^{-1}\cdot(u_\rho \circ \mathcal{T}^t_\rho)$ defined on $\Omega_t(V)$.

Lemma 2 *The element u_ρ is solution of the equation:*

$$\langle e_{(V+\rho W)}\,(u), v\rangle = 0, \quad \forall\, v \in \mathcal{H}^\rho$$

if and only if $\hat{u}_\rho$ is solution of the following equation,

$$\langle e^\rho(\hat{u}_\rho), \hat{v}\rangle = 0, \quad \forall\, \hat{v} \in \mathcal{H}$$

with

$$\langle e_1^\rho(v), w\rangle =$$

$$\int_0^T \int_{\Omega_t(V)} \big[(\partial_t(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v))\cdot(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w) - (\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1}\cdot(\partial_t\mathcal{T}_\rho^t))\cdot(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)$$

$$(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w) + (\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1}\cdot((V+\rho W)\circ\mathcal{T}_\rho^t))\cdot(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)$$

$$+(\mathrm{D}((V+\rho W)\circ\mathcal{T}_\rho^t)\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)$$

$$+\nu(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot\cdot(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})$$

$$-(F(V+\rho W)\circ\mathcal{T}_\rho^t)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)\big]$$

$$\langle e_2^\rho(v), w\rangle = \int_\Omega (v(0)-\hat{u}_0)\cdot w$$

with

$$F(V) = -\partial_t V - \mathrm{D}\,V\cdot V + \nu\Delta V$$

and

$$\partial_t\mathcal{T}_\rho^t = (V+\rho W)\circ\mathcal{T}_\rho^t - \mathrm{D}\,\mathcal{T}_\rho^t\cdot V$$

Proof :
We consider the solution u_ρ of the perturbed state equation $e_{(V+\rho W)}=0$, with

$$\langle e_{(V+\rho W)}(u), v\rangle = \int_0^T \int_{\Omega_t^\rho} [(\partial_t u + \mathrm{D}\,u\cdot u + \mathrm{D}\,u\cdot(V+\rho W) + \mathrm{D}(V+\rho W)\cdot u)\cdot v$$

$$+\nu\,\mathrm{D}\,u\cdot\cdot\,\mathrm{D}\,v - F(V+\rho W)\cdot v]$$

with $v\in\mathcal{H}^\rho$.
We introduce the variables $(\hat{u},\hat{v})$ defined in $\Omega_t(V)$ such that,

$$[u,v] = \big[(\mathrm{D}\,\mathcal{T}_\rho^t\cdot\hat{u})\circ(\mathcal{T}_\rho^t)^{-1}, (\mathrm{D}\,\mathcal{T}_\rho^t\cdot\hat{v})\circ(\mathcal{T}_\rho^t)^{-1}\big]$$

We replace this new representation inside the state equation and we use a back transport in $\Omega_t(V)$, this leads to the following identity,

$$
\begin{aligned}
\langle e_1^\rho(\hat{u}), \hat{v}\rangle = \int_{Q(V)} \big[\big(\partial_t((\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1}) + \mathrm{D}((\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1}) \cdot (\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1} \\
+ \mathrm{D}((\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1}) \cdot (V + \rho W) + \mathrm{D}(V + \rho W) \cdot (\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1}\big) \circ \mathcal{T}_\rho^t \cdot (\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{v}) \\
+\nu\, \mathrm{D}((\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t \cdot \cdot (\mathrm{D}((\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{v}) \circ (\mathcal{T}_\rho^t)^{-1})) \circ \mathcal{T}_\rho^t \\
- F(V + \rho W) \circ \mathcal{T}_\rho^t \cdot (\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{v}) \big]
\end{aligned}
$$

Lemma 3

$$\mathrm{D}((\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t = (\mathrm{D}\,\mathcal{T}_\rho^t)^{-1} \tag{59}$$

$$\partial_t((\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t = -(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1} \cdot \partial_t \mathcal{T}_\rho^t \tag{60}$$

$$\partial_t((\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t = \partial_t(\mathrm{D}\,\mathcal{T}_\rho^t \circ \hat{u}) - \mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t \circ \hat{u}) \cdot (\mathrm{D}\,\mathcal{T}_\rho^t)^{-1} \cdot \partial_t \mathcal{T}_\rho^t \tag{61}$$

Proof :
Using the identity,

$$(\mathcal{T}_\rho^t)^{-1} \circ \mathcal{T}_\rho^t = \mathrm{I}$$

we get

$$
\begin{aligned}
\mathrm{D}((\mathcal{T}_\rho^t)^{-1} \circ \mathcal{T}_\rho^t) &= \mathrm{I} \\
\mathrm{D}((\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t \cdot \mathrm{D}\,\mathcal{T}_\rho^t &= \mathrm{I}
\end{aligned}
$$

by differentiation with respect to time t, we also get,

$$
\begin{aligned}
\partial_t((\mathcal{T}_\rho^t)^{-1} \circ \mathcal{T}_\rho^t) &= 0 \\
\partial_t((\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t + \mathrm{D}((\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t \cdot \partial_t \mathcal{T}_\rho^t &= 0 \\
\partial_t((\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t + (\mathrm{D}\,\mathcal{T}_\rho^t)^{-1} \cdot \partial_t \mathcal{T}_\rho^t &= 0
\end{aligned}
$$

Using the chain rule, we deduce

$$
\begin{aligned}
[\partial_t((\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1})] \circ \mathcal{T}_\rho^t &= [\partial_t(\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1} \\
&\quad + \mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \circ (\mathcal{T}_\rho^t)^{-1} \cdot \partial_t((\mathcal{T}_\rho^t)^{-1})] \circ \mathcal{T}_\rho^t \\
&= \partial_t(\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) + \mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \cdot \partial_t((\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t \\
&= \partial_t(\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) - \mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t \cdot \hat{u}) \cdot (\mathrm{D}\,\mathcal{T}_\rho^t)^{-1} \cdot \partial_t \mathcal{T}_\rho^t
\end{aligned}
$$

□
In order to get the correct state operator, we need also the following identities,

Lemma 4

$$D(\phi \circ (\mathcal{T}_\rho^t)^{-1}) \circ \mathcal{T}_\rho^t = D(\phi) \cdot (D\,\mathcal{T}_\rho^t)^{-1} \tag{62}$$

$$D(V + \rho W) \circ \mathcal{T}_\rho^t \cdot (D\,\mathcal{T}_\rho^t) = D((V + \rho W) \circ \mathcal{T}_\rho^t) \tag{63}$$

We shall apply the first identity with $\phi = (D\,\mathcal{T}_\rho^t \cdot \hat{u})$. Finally, using all the identities proven above, we deduce the expression of $e_\rho^1(\hat{u}, \hat{v})$. Now, we simply need to prove the following lemma in order to conclude the proof,

Lemma 5

$$\partial_t \mathcal{T}_\rho^t = (V + \rho W) \circ \mathcal{T}_\rho^t - D\,\mathcal{T}_\rho^t \cdot V \tag{64}$$

Proof :
We use the definition of the Transverse map,

$$\begin{aligned}
\partial_t(\mathcal{T}_\rho^t) &= \partial_t(T_t(V + \rho W) \circ T_t(V)^{-1}) \\
&= \partial_t(T_t(V + \rho W)) \circ T_t(V)^{-1} + D(T_t(V + \rho W)) \circ T_t(V)^{-1} \cdot \partial_t(T_t(V)^{-1}) \\
&= ((V + \rho W) \circ T_t(V + \rho W)) \circ T_t(V)^{-1} \\
&\quad - D(T_t(V + \rho W)) \circ T_t(V)^{-1} \cdot (D\,T_t^{-1}(V)) \cdot \partial_t(T_t(V)) \circ T_t^{-1}(V) \\
&= (V + \rho W) \circ \mathcal{T}_\rho^t - D(T_t(V + \rho W) \circ T_t(V)^{-1}) \cdot \partial_t(T_t(V)) \circ T_t^{-1}(V) \\
&= (V + \rho W) \circ \mathcal{T}_\rho^t - D(\mathcal{T}_\rho^t) \cdot V
\end{aligned}$$

□
□
We now consider the application,

$$\begin{array}{rcl}
[0, \rho_0] \times \mathcal{V} & \rightarrow & \mathcal{H}^* \times H_0^1(\mathrm{div}, \Omega_0) \\
(\rho, v) & \mapsto & e^\rho(v)
\end{array} \tag{65}$$

and

$$\begin{array}{rcl}
[0, \rho_0] & \rightarrow & \mathcal{H} \\
\rho & \mapsto & \hat{u}_\rho = (D\,\mathcal{T}_\rho^t)^{-1} \cdot (u_\rho \circ \mathcal{T}_\rho^t)
\end{array} \tag{66}$$

where $\hat{u}_\rho \in \mathcal{V}$ is solution of the state equation,

$$\langle e^\rho(v), w \rangle = 0, \quad \forall\, w \in \mathcal{H} \tag{67}$$

Lemma 6 ([22]) *For any $F \in H^s(D)$, with $s \geq 1$,*

$$\frac{1}{\rho}(F \circ \mathcal{T}_\rho^t - F) \stackrel{\rho \to 0}{\longrightarrow} \nabla F \cdot Z_t \tag{68}$$

strongly in $H^{s-1}(D)$. In the case $s < 1$, the convergence only holds weakly in $H^{s-1}(D)$.

In order to prove the differentiability of $\hat{u}_\rho$ with respect to ρ in a neighbourhood of $\rho = 0$, we cannot use the classical implicit function theorem, since it requires strong differentiability results in H^{-1} for our application. Then we shall use the weak implicit function theorem, recalled below,

Theorem 11 ([26]) *Let X, Y^* be two Banach spaces, I an open bounded set in $\mathbb{R}$, and consider the following mapping,*

$$\begin{array}{rcl} e : I \times X & \rightarrow & Y^* \\ (\rho, x) & \mapsto & e(\rho, x) \end{array}$$

Let us assume the following hypothesis,

a) $\rho \mapsto \langle e(\rho, x), y \rangle$ *is continuously differentiable for any $y \in Y$ and* $(\rho, x) \mapsto \langle \partial_\rho e(\rho, x), y \rangle$ *is continuous.*

b) It exists $u \in X$ such that,

$$\begin{array}{l} u \in \mathcal{C}^{0,1}(I; X) \\ e(\rho, u(\rho)) = 0, \quad \forall\, \rho \in I \end{array}$$

c) $x \mapsto e(\rho, x)$ *is differentiable and* $(\rho, x) \mapsto \partial_x e(\rho, x)$ *is continuous.*

d) It exists $\rho_0 \in I$ such that $\partial_x e(\rho, x)|_{(\rho_0, x(\rho_0))} \in \mathrm{ISOM}(X, Y^*)$.

then the mapping

$$\begin{array}{rcl} u(.) : I & \rightarrow & X \\ \rho & \mapsto & u(\rho) \end{array}$$

is differentiable at point $\rho = \rho_0$ for the weak topology in X and its weak derivative $\dot{u}(\rho)$ is solution of the following linearized equation,

$$\langle \partial_x e(\rho_0, u(\rho_0)) \cdot \dot{u}(\rho_0), y \rangle + \langle \partial_\rho e(\rho_0, u(\rho_0)), y \rangle = 0, \quad \forall\, y \in Y \tag{69}$$

In order to apply the above theorem to Eq. (67), we need to state the following properties,

Lemma 7 *The mapping,*

$$\begin{array}{rcl} [0,\rho_0] & \rightarrow & \mathbb{R} \\ \rho & \mapsto & \langle e^\rho(v), w\rangle \end{array} \tag{70}$$

is $\mathcal{C}^1$ for any $(v,w) \in \mathcal{V} \times \mathcal{H}$ and its derivative is given by the following expression,

$$\langle \partial_\rho e_1^\rho(v), w\rangle =$$

$$\int_{Q(V)} \left[(\partial_t(\mathrm{D}(\mathcal{Z}_\rho^t \cdot \mathcal{T}_\rho^t)\cdot v)) + (\mathrm{D}(\mathrm{D}(\mathcal{Z}_\rho^t \cdot \mathcal{T}_\rho^t)\cdot v)\cdot V + (\mathrm{D}(\mathrm{D}(\mathcal{Z}_\rho^t \cdot \mathcal{T}_\rho^t)\cdot v)\cdot v)\right.$$

$$+\,\mathrm{D}\left[(\mathrm{D}(V+\rho W)\cdot \mathcal{Z}_\rho^t)\circ \mathcal{T}_\rho^t + W\circ \mathcal{T}_\rho^t\right]\cdot v - \partial_\rho(F(V+\rho W)\circ \mathcal{T}_\rho^t)\big]\cdot(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)$$

$$+\,\left[(\partial_t(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)) + (\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot V + (\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot v)\right.$$

$$+(\mathrm{D}((V+\rho W)\circ \mathcal{T}_\rho^t)\cdot v) - (F(V+\rho W)\circ \mathcal{T}_\rho^t\big]\cdot(\mathrm{D}(\mathcal{Z}_\rho^t\cdot \mathcal{T}_\rho^t)\cdot w)$$

$$+\,\nu(\mathrm{D}(\mathrm{D}(\mathcal{Z}_\rho^t\circ \mathcal{T}_\rho^t)\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot\cdot(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})$$

$$-\,\nu(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot \mathrm{D}(\mathcal{Z}_\rho^t\circ \mathcal{T}_\rho^t)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot\cdot(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})$$

$$+\,\nu(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot\cdot(\mathrm{D}(\mathrm{D}(\mathcal{Z}_\rho^t\circ \mathcal{T}_\rho^t)\cdot w)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})$$

$$-\,\nu(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot\cdot(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot \mathrm{D}(\mathcal{Z}_\rho^t\circ \mathcal{T}_\rho^t)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})$$

Proof :
We first simplify the expression of the weak state operator, using that

$$\partial_t \mathcal{T}_\rho^t = (V+\rho W)\circ \mathcal{T}_\rho^t - \mathrm{D}\,\mathcal{T}_\rho^t\cdot V$$

and we get,

$$\langle e_1^\rho(v), w\rangle =$$

$$\int_0^T \int_{\Omega_t(V)} \big[(\partial_t(\mathrm{D}\,\mathcal{T}_\rho^t \cdot v)) + (\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t \cdot v)\cdot V + (\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t \cdot v)\cdot v)$$
$$+(\mathrm{D}((V+\rho W)\circ \mathcal{T}_\rho^t)\cdot v) - (F(V+\rho W)\circ \mathcal{T}_\rho^t\big]\cdot(\mathrm{D}\,\mathcal{T}_\rho^t \cdot w)$$
$$+\,\nu(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot v)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})\cdot\cdot(\mathrm{D}(\mathrm{D}\,\mathcal{T}_\rho^t\cdot w)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1})$$

We use the expression of the weak state operator and the following identities,

$$\partial_\rho \mathcal{T}_\rho^t = \mathcal{Z}_\rho^t \circ \mathcal{T}_\rho^t$$

$$\partial_\rho(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1} = -(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1}\cdot \mathrm{D}(\mathcal{Z}_\rho^t\circ\mathcal{T}_\rho^t)\cdot(\mathrm{D}\,\mathcal{T}_\rho^t)^{-1}$$

□

Lemma 8 *The mapping,*

$$\begin{array}{rcl} [0,\rho_0]\times\mathcal{V} & \to & \mathcal{H}^* \\ (\rho, v) & \mapsto & \partial_\rho e^\rho(v) \end{array} \tag{71}$$

is weakly continuous

Proof :
We can prove that for $(V,W)\in\mathcal{V}$, the associated flow $\mathcal{T}_\rho^t \in \mathcal{C}^1([0,\rho_0[;\mathcal{C}^2(D,\mathbb{R}^3))$, and the weak continuity follows easily.
□
In order to apply the implicit function derivative identity, we need to express the derivative $\partial_\rho e^\rho(v)$ at point $\rho = 0$,

Lemma 9

$$\langle \partial_\rho e_1^\rho|_{\rho=0}(v), w\rangle =$$

$$\int_{Q(V)} [\partial_t(\mathrm{D}\, Z_t \cdot v) + \mathrm{D}(\mathrm{D}\, Z_t \cdot v)\cdot V + \mathrm{D}(\mathrm{D}\, Z_t \cdot v)\cdot v + \mathrm{D}\,[\mathrm{D}\, V \cdot Z_t + W]\cdot v]\cdot w$$

$$+ [\partial_t v + \mathrm{D}\, v \cdot V + \mathrm{D}\, v \cdot v + \mathrm{D}\, V \cdot v]\cdot(\mathrm{D}\, Z_t \cdot w) + \nu\, \mathrm{D}(\mathrm{D}\, Z_t \cdot v)\cdot\cdot\, \mathrm{D}\, w$$

$$- \nu(\mathrm{D}\, v \cdot \mathrm{D}\, Z_t)\cdot\cdot\, \mathrm{D}\, w + \nu\, \mathrm{D}\, v \cdot\cdot\, \mathrm{D}(\mathrm{D}\, Z_t \cdot w) - \nu\, \mathrm{D}\, v \cdot\cdot(\mathrm{D}\, w \cdot \mathrm{D}\, Z_t)$$

$$+ [\partial_t W + \mathrm{D}\, W \cdot V + \mathrm{D}\, V \cdot W - \nu\Delta W]\cdot w + (\mathrm{D}\,[\partial_t V + \mathrm{D}\, V \cdot V - \nu\Delta V]\cdot Z_t)\cdot w$$

$$+ [\partial_t V + \mathrm{D}\, V \cdot V - \nu\Delta V]\cdot(\mathrm{D}\, Z_t \cdot w)$$

Proof :
We set $\rho = 0$ in the expression of $\langle \partial_\rho e^\rho(v), w\rangle$ and we use the following identities,

$$\mathcal{T}^t_{\rho=0} = \mathrm{I}$$

$$\mathcal{Z}^t_\rho|_{\rho=0} \stackrel{\text{def}}{=} Z_t$$

□

Lemma 10 *The mapping,*

$$\begin{array}{ccc} \mathcal{V} & \to & \mathcal{H}^* \\ v & \mapsto & e^\rho(v) \end{array} \tag{72}$$

is differentiable for any $\rho \in [0, \rho_0]$ and its derivative is given by the following expression,

$$\langle \partial_v e_1^\rho(v)\cdot\delta v, w\rangle =$$

$$\int_0^T \int_{\Omega_t(V)} \big[(\partial_t(\mathrm{D}\,\mathcal{T}^t_\rho \cdot \delta v)) + \mathrm{D}(\mathrm{D}\,\mathcal{T}^t_\rho \cdot \delta v)\cdot V + \mathrm{D}(\mathrm{D}\,\mathcal{T}^t_\rho \cdot \delta v)\cdot v + \mathrm{D}(\mathrm{D}\,\mathcal{T}^t_\rho \cdot v)\cdot \delta v$$

$$+(\mathrm{D}((V+\rho W)\circ\mathcal{T}^t_\rho)\cdot\delta v)\big] + \nu(\mathrm{D}(\mathrm{D}\,\mathcal{T}^t_\rho \cdot \delta v)\cdot(\mathrm{D}\,\mathcal{T}^t_\rho)^{-1})\cdot\cdot(\mathrm{D}(\mathrm{D}\,\mathcal{T}^t_\rho \cdot w)\cdot(\mathrm{D}\,\mathcal{T}^t_\rho)^{-1})$$

and the mapping,

$$\begin{array}{rcl} [0,\rho_0]\times\mathcal{V} & \rightarrow & \mathcal{L}(\mathcal{V};\mathcal{H}^*) \\ (\rho,v) & \mapsto & \partial_v e^{\rho}(v) \end{array} \tag{73}$$

is continuous.

Lemma 11 *The mapping,*

$$\begin{array}{rcl} \mathcal{V} & \rightarrow & \mathcal{F} \\ \delta v & \mapsto & \partial_v e^{\rho=0}(v)\cdot\delta v \end{array} \tag{74}$$

is an isomorphism and its expression is furnished by the following identity,

$$\begin{aligned} \langle \partial_v e_1^{\rho=0}(v)\cdot\delta v, w\rangle = \\ \int_{Q(V)} [(\partial_t \delta v)\cdot w + (\mathrm{D}\,\delta v\cdot v)\cdot w + (\mathrm{D}\,v\cdot\delta v)\cdot w + (\mathrm{D}\,\delta v\cdot V)\cdot w \\ +(\mathrm{D}\,V\cdot\delta v)\cdot w + \nu\,\mathrm{D}\,\delta v\cdot\cdot\,\mathrm{D}\,w] \end{aligned}$$

Proof :
This result follows from the uniqueness result for the Navier-Stokes system under regularity and smallness assumptions (see Th. (9) and [23]). Indeed, for u_1 and u_2 solutions of the Navier-Stokes equations, it is proven that the element $y = u_1 - u_2$ satisfying the following identity,

$$\begin{aligned} \int_{Q(V)} [(\partial_t y)\cdot w + (\mathrm{D}\,y\cdot u_1)\cdot w + (\mathrm{D}\,u_2\cdot y)\cdot w + (\mathrm{D}\,y\cdot V)\cdot w + (\mathrm{D}\,V\cdot y)\cdot w \\ +\nu\,\mathrm{D}\,y\cdot\cdot\,\mathrm{D}\,w] = 0, \quad \forall\, w\in\mathcal{H} \end{aligned}$$

exists and is identically equal to the null function. Similar a-priori estimates holds for δv and the unique solvability of the linearized system is established.
□

Lemma 12 *The solution $\hat{u}_\rho \in V$ of the implicit equation,*

$$\langle e^{\rho}(v), w\rangle = 0, \quad \forall\, w\in\mathcal{H} \tag{75}$$

is Lipschitz with respect to ρ.

Proof :
We need the identity satisfied by $\hat{u}_{\rho_1} - \hat{u}_{\rho_2}$ and we shall follow the same steps described in [12] (pp. 31).
□
Hence the hypothesis of Th. (11) are satisfied by the Eq. (67) and we can state the following differentiability result,

Theorem 12 *The Piola material derivative $\dot{u}^P = \partial_\rho(\hat{u}_\rho)|_{\rho=0}$ exists and is characterized by the linear tangent equation,*

$$\langle \partial_v e^{\rho=0}(v)|_{v=\hat{u}} \cdot \dot{u}^P, w\rangle + \langle \partial_\rho e^\rho(\hat{u})|_{\rho=0}, w\rangle = 0, \quad \forall\, w \in \mathcal{H} \tag{76}$$

which possesses the following structure,

$$\int_{Q(V)} \big[(\partial_t \dot{u}^P)\cdot w + (\mathrm{D}\,\dot{u}^P \cdot u)\cdot w + (\mathrm{D}\,u \cdot \dot{u}^P)\cdot w + (\mathrm{D}\,\dot{u}^P \cdot V)\cdot w$$
$$+(\mathrm{D}\,V \cdot \dot{u}^P)\cdot w + \nu\,\mathrm{D}\,\dot{u}^P \cdot\cdot\, \mathrm{D}\,w\big] = \langle L(u, Z_t, V, W), w\rangle$$

with

$$\langle L(u, Z_t, V, W), w\rangle =$$
$$-\int_{Q(V)} [\partial_t(\mathrm{D}\,Z_t \cdot u) + \mathrm{D}(\mathrm{D}\,Z_t \cdot u)\cdot V + \mathrm{D}(\mathrm{D}\,Z_t \cdot u)\cdot u + \mathrm{D}\,[\mathrm{D}\,V \cdot Z_t + W]\cdot u]\cdot w$$
$$-[\partial_t u + \mathrm{D}\,u\cdot V + \mathrm{D}\,u \cdot u + \mathrm{D}\,V \cdot u]\cdot(\mathrm{D}\,Z_t \cdot w) - \nu\,\mathrm{D}(\mathrm{D}\,Z_t \cdot u)\cdot\cdot\,\mathrm{D}\,w$$
$$+\nu(\mathrm{D}\,u \cdot \mathrm{D}\,Z_t)\cdot\cdot\,\mathrm{D}\,w - \nu\,\mathrm{D}\,u\cdot\cdot\,\mathrm{D}(\mathrm{D}\,Z_t\cdot w) + \nu\,\mathrm{D}\,u\cdot\cdot(\mathrm{D}\,w\cdot\mathrm{D}\,Z_t)$$
$$+[-\partial_t W - \mathrm{D}\,W\cdot V - \mathrm{D}\,V\cdot W + \nu\Delta W]\cdot w - (\mathrm{D}\,[\partial_t V + \mathrm{D}\,V\cdot V - \nu\Delta V]\cdot Z_t)\cdot w$$
$$-[\partial_t V + \mathrm{D}\,V \cdot V - \nu\Delta V]\cdot(\mathrm{D}\,Z_t\cdot w)$$

4.3 Shape derivative

In the last section, we have proven that the solution $u(V)$ of the moving Navier-Stokes system is differentiable with respect to the velocity V. We have also characterized the linearized system satisfied by the Piola material derivative $\dot{u}^P(V)\cdot W$. In this paragraph, we will identify the shape derivative $u'(V)\cdot W$ under some regularity assumptions.
Let us consider the weak solution $\tilde{u}$ of Eq. (53), i.e

$$\langle e_V(\tilde{u}), v\rangle = \langle [e_V^1(u), e_V^2(u)], v\rangle = [0,0], \quad \forall\, v \in \mathcal{H} \tag{77}$$

with

$$\langle e_V^1(\tilde{u}), v\rangle = \int_0^T \int_{\Omega_t(V)} [(\partial_t\tilde{u} + \mathrm{D}\,\tilde{u}\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot V + \mathrm{D}\,V\cdot\tilde{u})\cdot v + 2\nu\varepsilon(\tilde{u})\cdot\cdot\,\varepsilon(v) - F(V)\cdot v] \tag{78}$$

$$\langle e_V^2(\tilde{u}), v\rangle = \int_{\Omega_0} (\tilde{u}(0) - u_0)\cdot v(0) \tag{79}$$

where $\varepsilon(v) = \frac{1}{2}(\mathrm{D}\, v + {}^*\mathrm{D}\, v)$ stands for the symmetrical deviation tensor. This definition is motivated by the following lemma,

Lemma 13

$$-\nu \int_{\Omega_t} \Delta u \cdot v = 2\nu \int_{\Omega_t} \varepsilon(u)\cdot\cdot\varepsilon(v) - 2\nu \int_{\Gamma_t} \langle \varepsilon(u)\cdot n, v\rangle, \quad \forall\, v \in H^1(\mathrm{div}, \Omega_t) \tag{80}$$

Theorem 13 *For Ω_0 of class $\mathcal{C}^2$, the shape derivative $\tilde{u}' = \dot{\tilde{u}} - \mathrm{D}\,\tilde{u}\cdot Z_t$ exists and is characterized as the solution of the following linearized system,*

$$\begin{cases} \partial_t \tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot\tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot V + \mathrm{D}\, V\cdot\tilde{u}' - \nu\Delta\tilde{u}' + \nabla p' = L(V,W), & Q \\ \mathrm{div}(\tilde{u}') = 0, & Q \\ \tilde{u}' = -(\mathrm{D}\,\tilde{u}\cdot n)\langle Z_t, n\rangle, & \Sigma \\ \tilde{u}'(0) = 0, & \Omega_0 \end{cases} \tag{81}$$

with

$$L(V,W) = -\partial_t W - \mathrm{D}\, W\cdot V - \mathrm{D}\, V\cdot W + \nu\Delta W - \mathrm{D}\,\tilde{u}\cdot W - \mathrm{D}\, W\cdot\tilde{u} \tag{82}$$

Proof :
In order to state such a result, we use Th. (4) and we get,

$$\frac{d}{dV}\left(\int_{\Omega_t(V)} G(V)dx\right)\cdot W = \int_{\Omega_t(V)} G'(V)\cdot W dx + \int_{\Gamma_t(V)} G\,\langle Z_t, n\rangle \tag{83}$$

where $G'(V)\cdot W$ stands for non-cylindrical shape derivative of G and Z_t is the transverse vector field solution of the Transverse Equation (Eq. (23)) with

$$G = [(\partial_t\tilde{u} + \mathrm{D}\,\tilde{u}\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot V + \mathrm{D}\, V\cdot\tilde{u})\cdot v + \nu\varepsilon(\tilde{u})\cdot\cdot\varepsilon(v) - F(V)\cdot v]$$

We assume that v has a compact support, then $G|_{\Gamma_t(V)} = 0$.

Lemma 14

$$\begin{aligned} G'(V)\cdot W \quad = \quad & [(\partial_t\tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot\tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot V + \mathrm{D}\,\tilde{u}\cdot W \\ & + \mathrm{D}\, W\cdot\tilde{u} + \mathrm{D}\, V\cdot\tilde{u}')\cdot v + \nu\varepsilon(\tilde{u}')\cdot\cdot\varepsilon(v) - F'(V)\cdot W\cdot v] \end{aligned}$$

with

$$F'(V)\cdot W = -\partial_t W - \mathrm{D}\, W\cdot V - \mathrm{D}\, V\cdot W + \nu\Delta W$$

Finally we obtain,

$$\frac{d}{dV}\langle e_1(\tilde{u})\cdot W,\cdot v\rangle = \int_{Q(V)} [(\partial_t \tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot\tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot V + \mathrm{D}\,\tilde{u}\cdot W$$
$$+\mathrm{D}\,W\cdot\tilde{u} + \mathrm{D}\,V\cdot\tilde{u}')\cdot v + \nu\varepsilon(\tilde{u}')\cdot\cdot\varepsilon(v) - F'(V)\cdot W\cdot v]$$

for any $v \in \mathcal{H}$ with compact support.
Using integration by parts for the term $\int_{Q(V)} \nu\varepsilon(\tilde{u}')\cdot\cdot\varepsilon(v)$, we recover the correct strong formulation of the linearized equation (Eq. (81)) satisfied by the shape derivative $u'(V)\cdot W$. The boundary condition comes from the fact that the shape derivative of the condition $\tilde{u} = 0$, on $\Gamma_t(V)$ is given by

$$\tilde{u}' = -\mathrm{D}\,u\cdot Z_t, \text{ on } \Gamma_t(V)$$

Since $u = 0$ on $\Gamma_t(V)$, we have $\mathrm{D}\,u|_{\Gamma_t} = \mathrm{D}\,u\cdot(n\otimes n)$ which gives

$$\tilde{u}' = -(\mathrm{D}\,u\cdot n)\langle Z_t, n\rangle, \text{ on } \Gamma_t(V)$$

□
The shape derivative $u'(V)\cdot W$ of the solution u of the original non-homogeneous Dirichlet boundary problem (Eq. (2)) is given by the expression

$$u'(V)\cdot W = \tilde{u}'(V)\cdot W + W \tag{84}$$

Corollary 1 *The shape derivative $u'(V)\cdot W$ of the solution u of Eq. (2) exists and satisfies the following linearized problem,*

$$\begin{cases} \partial_t u' + \mathrm{D}\,u'\cdot u + \mathrm{D}\,u\cdot u' - \nu\Delta u' + \nabla p' = 0, & Q \\ \mathrm{div}(u') = 0, & Q \\ u' = W + (\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n)\langle Z_t, n\rangle, & \Sigma \\ u'(0) = 0, & \Omega_0 \end{cases} \tag{85}$$

Proof :
We simply set in Eq. (81), $\tilde{u}' = u' - W$ and $\tilde{u} = u - V$.
□

Remark 4 *If we choose $V = (V\circ p)$ the canonical extension of V in Eq. (85), then we get the simpler boundary condition,*

$$u' = W - (\mathrm{D}\,u\cdot n)\langle Z_t, n\rangle, \quad \textit{on } \Gamma_t(V) \tag{86}$$

4.4 Extractor Identity

In the last section, we have established the structure of the system satisfied by the non-cylindrical shape derivative $u'(V)\cdot W$ of the solution $u(V)$ of the Navier-Stokes problem in the moving domain $\Omega_t(V)$. This linearized system has been obtained independently of the system satisfied by the non-cylindrical material derivative $\dot{\tilde{u}}^P(V)\cdot W$. However, there exists an explicit relation between the original shape u' and the Piola material derivative $\dot{\tilde{u}}^P(V)\cdot W$ of the shift state $\tilde{u} = u - V$.

Lemma 15 *Let $u(V)$ stands for the weak solution of the non-homogeneous Navier-Stokes equations (Eq. (2)) in moving domain, $u'(V)\cdot W$ stands for its shape derivative and $\dot{\tilde{u}}^P(V)\cdot W$ stands for the Piola material derivative of the shift flow $\tilde{u} = u(V) - V$ in the direction W. Then the following identity holds,*

$$\begin{aligned}\dot{\tilde{u}}^P(V)\cdot W &= \tilde{u}'(V)\cdot W + [\tilde{u}(V), Z_t] &(87)\\ &= u'(V)\cdot W + [u(V), Z_t] - [V, Z_t] - W &(88)\end{aligned}$$

where $[X,Y] = \mathrm{D}\,X\cdot Y - \mathrm{D}\,Y\cdot X$.

This relation can be fruitful in order to obtain an identity concerning the solution $\tilde{u}(V)$ inside $\Omega_t(V)$.

Proposition 6 *We consider Ω_0 of class $\mathcal{C}^2$, for all $(V,W)\in\mathcal{U}_{ad}$, $\tilde{u}$ solution of the homogeneous Navier-Stokes equations (Eq. (53)) and Z_t solution of Eq. (23), the following identity holds,*

$$\int_{Q(V)} [\partial_t(\mathrm{D}\,\tilde{u}\cdot Z_t) + \mathrm{D}(\mathrm{D}\,\tilde{u}\cdot Z_t)\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot(\mathrm{D}\,\tilde{u}\cdot Z_t) + \mathrm{D}(\mathrm{D}\,\tilde{u}\cdot Z_t)\cdot V$$

$$+\mathrm{D}\,V\cdot(\mathrm{D}\,\tilde{u}\cdot Z_t)]\cdot w - \int_{Q(V)} [\mathrm{D}\,\tilde{u}\cdot(\mathrm{D}\,Z_t\cdot\tilde{u}) + \mathrm{D}\,V\cdot(\mathrm{D}\,Z_t\cdot\tilde{u}) - \mathrm{D}(\mathrm{D}\,V\cdot Z_t)\cdot\tilde{u}]\cdot w$$

$$+[\partial_t\tilde{u} + \mathrm{D}\,\tilde{u}\cdot V + \mathrm{D}\,\tilde{u}\cdot\tilde{u} + \mathrm{D}\,V\cdot\tilde{u}]\cdot(\mathrm{D}\,Z_t\cdot w) - \mathrm{D}\,\tilde{u}\cdot W\cdot w$$

$$+\nu\,\mathrm{D}(\mathrm{D}\,\tilde{u}\cdot Z_t)\cdot\cdot\,\mathrm{D}\,w - \nu(\mathrm{D}\,\tilde{u}\cdot\mathrm{D}\,Z_t)\cdot\cdot\,\mathrm{D}\,w + \nu\,\mathrm{D}\,\tilde{u}\cdot\cdot\,\mathrm{D}(\mathrm{D}\,Z_t\cdot w) - \nu\,\mathrm{D}\,\tilde{u}\cdot\cdot(\mathrm{D}\,w\cdot\mathrm{D}\,Z_t)$$

$$+(\mathrm{D}\,[\partial_t V + \mathrm{D}\,V\cdot V - \nu\Delta V]\cdot Z_t)\cdot w + [\partial_t V + \mathrm{D}\,V\cdot V - \nu\Delta V]\cdot(\mathrm{D}\,Z_t\cdot w) = 0, \quad \forall\, w\in\mathcal{H}$$

Proof :
We recall that the shape derivative $\tilde{u}'$ satisfies the following identity,

$$\int_{Q(V)} [\partial_t\tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot\tilde{u} + \mathrm{D}\,\tilde{u}\cdot\tilde{u}' + \mathrm{D}\,\tilde{u}'\cdot V + \mathrm{D}\,V\cdot\tilde{u}']\cdot w + \nu\,\mathrm{D}\,\tilde{u}'\cdot\cdot\,\mathrm{D}\,w = \langle \ell_1, w\rangle$$

with

$$\langle \ell_1, w\rangle = \int_{Q(V)} [-\partial_t W - \mathrm{D}\, W\cdot V - \mathrm{D}\, V\cdot W + \nu\Delta W - \mathrm{D}\,\tilde u\cdot W - \mathrm{D}\, W\cdot \tilde u]\cdot w$$

Then we set $\tilde u'(V) = \dot{\tilde u}^P - [\tilde u, Z_t] = \dot{\tilde u}^P - \mathrm{D}\,\tilde u\cdot Z_t + \mathrm{D}\, Z_t\cdot\tilde u$. This leads to the following identity,

$$\int_{Q(V)} \big[(\partial_t \dot u^P)\cdot w + (\mathrm{D}\,\dot u^P\cdot u)\cdot w + (\mathrm{D}\, u\cdot \dot u^P)\cdot w + (\mathrm{D}\,\dot u^P\cdot V)\cdot w \\ +(\mathrm{D}\, V\cdot\dot u^P)\cdot w + \nu\,\mathrm{D}\,\dot u^P\cdot\cdot\,\mathrm{D}\, w\big] = \langle \ell_2, w\rangle$$

with

$$\langle \ell_2, w\rangle = \\ \int_{Q(V)} [\partial_t(\mathrm{D}\,\tilde u\cdot Z_t) + \mathrm{D}(\mathrm{D}\,\tilde u\cdot Z_t)\cdot\tilde u + \mathrm{D}\,\tilde u\cdot(\mathrm{D}\,\tilde u\cdot Z_t) + \mathrm{D}(\mathrm{D}\,\tilde u\cdot Z_t)\cdot V \\ + \mathrm{D}\, V\cdot(\mathrm{D}\,\tilde u\cdot Z_t)]\cdot w - \int_{Q(V)} [\partial_t(\mathrm{D}\, Z_t\cdot\tilde u) + \mathrm{D}(\mathrm{D}\, Z_t\cdot\tilde u)\cdot\tilde u + \mathrm{D}\,\tilde u\cdot(\mathrm{D}\, Z_t\cdot\tilde u) \\ + \mathrm{D}(\mathrm{D}\, Z_t\cdot\tilde u)\cdot V + \mathrm{D}\, V\cdot(\mathrm{D}\, Z_t\cdot\tilde u)]\cdot w + \int_{Q(V)} [\nu\,\mathrm{D}(\mathrm{D}\,\tilde u\cdot Z_t)\cdot\cdot\,\mathrm{D}\, w - \nu\,\mathrm{D}(\mathrm{D}\, Z_t\cdot\tilde u)\cdot\cdot\,\mathrm{D}\, w] \\ + \int_{Q(V)} [-\partial_t W - \mathrm{D}\, W\cdot V - \mathrm{D}\, V\cdot W + \nu\Delta W - \mathrm{D}\,\tilde u\cdot W - \mathrm{D}\, W\cdot\tilde u]\cdot w$$

Using Theorem (12), we deduce that,

$$\langle \ell_2, w\rangle = \langle L, w\rangle, \quad \forall\, w\in\mathcal{H} \tag{89}$$

with,

$$\langle L, w\rangle = \\ -\int_{Q(V)} [\partial_t(\mathrm{D}\, Z_t\cdot\tilde u) + \mathrm{D}(\mathrm{D}\, Z_t\cdot\tilde u)\cdot\tilde u + \mathrm{D}(\mathrm{D}\, Z_t\cdot\tilde u)\cdot V + \mathrm{D}(\mathrm{D}\, V\cdot Z_t)\cdot\tilde u + \mathrm{D}\, W\cdot\tilde u]\cdot w \\ - [\partial_t\tilde u + \mathrm{D}\,\tilde u\cdot V + \mathrm{D}\,\tilde u\cdot\tilde u + \mathrm{D}\, V\cdot\tilde u]\cdot(\mathrm{D}\, Z_t\cdot w) - \nu\,\mathrm{D}(\mathrm{D}\, Z_t\cdot\tilde u)\cdot\cdot\,\mathrm{D}\, w \\ + \nu(\mathrm{D}\,\tilde u\cdot\mathrm{D}\, Z_t)\cdot\cdot\,\mathrm{D}\, w - \nu\,\mathrm{D}\,\tilde u\cdot\cdot\,\mathrm{D}(\mathrm{D}\, Z_t\cdot w) + \nu\,\mathrm{D}\,\tilde u\cdot\cdot(\mathrm{D}\, w\cdot\mathrm{D}\, Z_t) \\ + [-\partial_t W - \mathrm{D}\, W\cdot V - \mathrm{D}\, V\cdot W + \nu\Delta W]\cdot w - (\mathrm{D}\,[\partial_t V + \mathrm{D}\, V\cdot V - \nu\Delta V]\cdot Z_t)\cdot w \\ - [\partial_t V + \mathrm{D}\, V\cdot V - \nu\Delta V]\cdot(\mathrm{D}\, Z_t\cdot w)$$

The sequence

$$[-\partial_t(\mathrm{D}\,Z_t\cdot\tilde{u})-\mathrm{D}(\mathrm{D}\,Z_t\cdot\tilde{u})\cdot\tilde{u}-\mathrm{D}(\mathrm{D}\,Z_t\cdot\tilde{u})\cdot V-\mathrm{D}\,W\cdot\tilde{u}]\cdot w-\nu\,\mathrm{D}(\mathrm{D}\,Z_t\cdot\tilde{u})\cdot\cdot\,\mathrm{D}\,w$$
$$+[-\partial_t W-\mathrm{D}\,W\cdot V-\mathrm{D}\,V\cdot W+\nu\Delta W]\cdot w$$

cancels and it remains the following terms,

$$\int_{Q(V)}[\partial_t(\mathrm{D}\,\tilde{u}\cdot Z_t)+\mathrm{D}(\mathrm{D}\,\tilde{u}\cdot Z_t)\cdot\tilde{u}+\mathrm{D}\,\tilde{u}\cdot(\mathrm{D}\,\tilde{u}\cdot Z_t)+\mathrm{D}(\mathrm{D}\,\tilde{u}\cdot Z_t)\cdot V$$

$$+\mathrm{D}\,V\cdot(\mathrm{D}\,\tilde{u}\cdot Z_t)]\cdot w-\int_{Q(V)}[\mathrm{D}\,\tilde{u}\cdot(\mathrm{D}\,Z_t\cdot\tilde{u})+\mathrm{D}\,V\cdot(\mathrm{D}\,Z_t\cdot\tilde{u})-\mathrm{D}(\mathrm{D}\,V\cdot Z_t)\cdot\tilde{u}]\cdot w$$

$$+[\partial_t\tilde{u}+\mathrm{D}\,\tilde{u}\cdot V+\mathrm{D}\,\tilde{u}\cdot\tilde{u}+\mathrm{D}\,V\cdot\tilde{u}]\cdot(\mathrm{D}\,Z_t\cdot w)-\mathrm{D}\,\tilde{u}\cdot W\cdot w$$

$$+\nu\,\mathrm{D}(\mathrm{D}\,\tilde{u}\cdot Z_t)\cdot\cdot\,\mathrm{D}\,w-\nu(\mathrm{D}\,\tilde{u}\cdot\mathrm{D}\,Z_t)\cdot\cdot\,\mathrm{D}\,w+\nu\,\mathrm{D}\,\tilde{u}\cdot\cdot\,\mathrm{D}(\mathrm{D}\,Z_t\cdot w)-\nu\,\mathrm{D}\,\tilde{u}\cdot\cdot(\mathrm{D}\,w\cdot\mathrm{D}\,Z_t)$$

$$+(\mathrm{D}\,[\partial_t V+\mathrm{D}\,V\cdot V-\nu\Delta V]\cdot Z_t)\cdot w+[\partial_t V+\mathrm{D}\,V\cdot V-\nu\Delta V]\cdot(\mathrm{D}\,Z_t\cdot w)=0$$

□

Remark 5 *If we set* $\tilde{u}=u-V$*, we can obtain an identity only involving* (u,Z_t,V,W)*.*

4.5 Adjoint system and cost function shape derivative

We are now coming back to the original problem of computing the gradient of the cost function $j(V)$. Let us first state a differentiability property,

Proposition 7 *For* Ω_0 *of class* $\mathcal{C}^2$*, the functional* $j(V)$ *is Gâteaux differentiable at point* $V\in\mathcal{U}_{ad}$ *and its directional derivative has the following expression,*

$$\langle j'(V),W\rangle=\int_{Q(V)}\alpha u(V)\cdot u'(V)\cdot W+\int_{\Sigma(V)}\left[\gamma V\cdot W+\frac{1}{2}(\alpha+\gamma\,H)|V|^2\langle Z_t,n\rangle\right],\quad\forall\,W\in\mathcal{U}_{ad}\tag{90}$$

where $u'(V)\cdot W$ *is solution of the shape derivative system (Eq.* (85)*).*

Proof :
We recall that,

$$j(V)=\frac{\alpha}{2}\int_0^T\int_{\Omega_t(V)}|u(V)|^2+\frac{\gamma}{2}\int_0^T\int_{\Gamma_t(V)}|V|^2\tag{91}$$

The differentiability property is an easy consequence of the differentiability of $J_V(u)$ with respect to (u, V) and the shape differentiability of $u(V)$ with respect to V. The expression of the directional derivative is a direct consequence of Th. (4) and Th. (5).
□
Using the fluid adjoint state and the adjoint transverse field, it is possible to identify the gradient distribution associated to the functional $j(V)$,

Theorem 14 *For $V \in \mathcal{U}_{ad}$ and Ω_0 of class $\mathcal{C}^2$, the functional $j(V)$ possesses a gradient $\nabla j(V)$ which is supported on the moving boundary $\Gamma_t(V)$ and can be represented by the following expression,*

$$\nabla j(V) = -\lambda\, n - \sigma(\varphi, \pi) \cdot n + \gamma\, V \tag{92}$$

where (φ, π) stands for the adjoint fluid state solution of the following system,

$$\begin{cases} -\partial_t \varphi - \mathrm{D}\,\varphi \cdot u + {}^*\,\mathrm{D}\,u \cdot \varphi - \nu \Delta \varphi + \nabla \pi = \alpha\, u, & Q(V) \\ \mathrm{div}(\varphi) = 0, & Q(V) \\ \varphi = 0, & \Sigma(V) \\ \varphi(T) = 0, & \Omega_T \end{cases} \tag{93}$$

and λ is the adjoint transverse boundary field, solution of the tangential dynamical system,

$$\begin{cases} -\partial_t \lambda - \nabla_\Gamma\, \lambda \cdot V = f, & (0, T) \\ \lambda(T) = 0, & \Gamma_T(V) \end{cases} \tag{94}$$

with $f = -(\sigma(\varphi, \pi) \cdot n) \cdot (\mathrm{D}\,V \cdot n - \mathrm{D}\,u \cdot n) + \frac{1}{2}(\alpha + \gamma\, H)|V|^2$.

Proof :
We need the following identity,

$$\begin{aligned} &\int_0^T \int_{\Omega_t(V)} [\partial_t u' + \mathrm{D}\,u' \cdot u + \mathrm{D}\,u \cdot u' - \nu \Delta u' + \nabla\, p']\, v - \int_0^T \int_{\Omega_t(V)} q\, \mathrm{div}\, u' \\ &= \int_0^T \int_{\Omega_t(V)} [-\partial_t v - \mathrm{D}\,v \cdot u + {}^*\,\mathrm{D}\,u \cdot v - \nu \Delta v + \nabla\, q]\, u' - \int_0^T \int_{\Omega_t(V)} p'\, \mathrm{div}\, v \\ &\qquad + \int_0^T \int_{\Gamma_t(V)} [p'\, n \cdot v - \nu v \cdot \partial_n u' + \nu u' \cdot \partial_n v - u' \cdot q\, n] \end{aligned} \tag{95}$$

We define (φ, π), to be the solution of the adjoint system (Eq. (93)), and we set $(v, q) = (\varphi, \pi)$ in Eq. (95), we get

$$\int_0^T \int_{\Omega_t(V)} \alpha\, u \cdot u' = -\int_0^T \int_{\Gamma_t(V)} \langle \sigma(\varphi, \pi) \cdot n, u' \rangle \tag{96}$$

We use the boundary condition on $\Gamma_t(V)$ for the linearized state u', i.e

$$u' = W + (\mathrm{D}\,V \cdot n - \mathrm{D}\,u \cdot n)\langle Z_t, n \rangle, \qquad \text{on } \Gamma_t(V) \tag{97}$$

Thus,

$$\langle j'(V), W\rangle = \int_0^T \int_{\Gamma_t(V)} \left[-(\sigma(\varphi,\pi)\cdot n)\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \frac{1}{2}(\alpha+\gamma\,H)|V|^2\right]\langle Z_t, n\rangle$$
$$+\int_0^T \int_{\Gamma_t(V)} [-\sigma(\varphi,\pi)\cdot n + \gamma V]\cdot W$$

Then we use Th. (8) with $E = -(\sigma(\varphi,\pi)\cdot n)\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \frac{1}{2}(\alpha+\gamma\,H)|V|^2$, and we get the correct result.
□

Remark 6 *Actually, we have* $\pi(\mathrm{D}(V-u)\cdot n)\cdot n = \pi\,\mathrm{div}\,V|_{\Gamma_t}$ *using the formula,*

$$(\mathrm{D}(V-u)\cdot n)\cdot n = \mathrm{div}(V-u)|_{\Gamma_t} - \mathrm{div}_\Gamma(V-u)$$

and the fact that $V-u=0$ *on* Γ_t. *Furthermore, we have considered free divergence field* V, *then this term is null and we get that*

$$f = -\nu(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \frac{1}{2}(\alpha+\gamma\,H)|V|^2$$

5 Min-Max and function space parametrization

In the previous section, we have been using the differentiability of the fluid state with respect to the eulerian velocity V as a sufficient condition in order to derive first-order optimality conditions, involving the adjoint of the linearized state. Actually, the tedious obtention of the state differentiability is not necessary in many cases, and even if the state is not differentiable, it can happen that first-order optimality conditions still hold. This is a consequence of a fundamental result in optimal control theory, the so-called Maximum Principle.
Avoiding the differentiation of the state equations with respect to the design variable V, is of great interest for shape optimization problems, especially if we deal with a moving domain system.
In this section, we are concerned with the function space parametrization, which consists in transporting the different quantities defined in the perturbed moving domain back into the reference moving domain that does not depend on the perturbation parameter. Thus, differential calculus can be performed since the functions involved are defined in a fix domain with respect to the pertubations.
In the first part, we define the saddle point formulation of the fluid state equations and the Lagrangian functional associated to the cost functional. Then, we perform a sensitivity analysis of the Lagrangian thanks to the transverse field and the fundamental Min-Max principle. This allows us to derive the expression of the cost function gradient involving the fluid and transverse field adjoints.

5.1 Saddle point formulation of the fluid state system

In the next paragraphs, we shall describe how to build an appropriate Lagrangian functional that can take into account all the constraints imposed by the mechanical problem, such as the divergence free condition or the non-homogeneous Dirichlet boundary conditions.

5.1.1 Null divergence condition

The divergence free condition coming from the fact that the fluid has an homogeneous density and evolves as an incompressible flow is difficult to impose on the mathematical and numerical point of view. We suggest at least 3 possible choices to handle this condition in our Min-Max formulation,

1. It can be taken into account in the state and multipliers spaces. In this case, the divergence free condition must be invariant with respect to the use of transport map during the derivation of optimality condition for the Lagrangian functional. This reduces the choice of appropriate maps and indeed the ALE map T_t does not satisfy this invariance condition.
   It is well known that the Piola transform does preserve the divergence quantity. Indeed we have, the following property :

   Lemma 16 ([4]) *The Piola transform*

   $$\begin{array}{rrcl} P_t: & H^1_0(\mathrm{div}, \Omega_0) & \longrightarrow & H^1_0(\mathrm{div}, \Omega_t) \\ & \varphi & \longmapsto & ((J_t)^{-1}\,\mathrm{D}\,T_t \cdot \varphi) \circ T_t^{-1} \end{array} \tag{98}$$

   is an isomorphism.

   This new transform introduces additional mathematical and computational efforts, but it seems to be the best approach in order to get rigorous mathematical justifications of the Lagrangian framework in the context of non-cylindrical and free boundary problems.

2. One way to avoid the use of this transform is the penalization of the divergence free condition inside the Navier-Stokes system. Let $\varepsilon > 0$ be a small parameter, we may consider the new penalized system :

   $$\left\{ \begin{array}{ll} \partial_t u + \mathrm{D}\, u \cdot u - \nu \Delta u - \dfrac{1}{\varepsilon} \nabla(\mathrm{div}\, u) = 0, & Q \\ u = V, & \Sigma \\ u(t=0) = u_0, & \Omega_0 \times \mathbb{R}^2 \end{array} \right. \tag{99}$$

   with $\sigma^\varepsilon(u) = \frac{1}{\varepsilon}\,\mathrm{div}(u)\,\mathrm{I} + \nu(\mathrm{D}\, u + {}^*\mathrm{D}\, u)$.
   We may work with such a modified system, derive the optimality conditions of the penalized Lagrangian functional and finally perform an asymptotic analysis on the

adjoint and primal system. For the time being, it is not clear if such a procedure may actually work, since even for non-moving Navier-Stokes problem, the convergence of the penalized adjoint is not established.

3. A third choice is to include the divergence free condition directly into the Lagrangian functional thanks to a multiplier that may play the role of the adjoint variable associated to the primal pressure variable. This leads in a certain sense to a saddle point formulation or mixed formulation of the Navier-Stokes system. It is well known that the well-posedness of such formulations is only established for the Stokes system, and that the Navier-Stokes suffers from a lack of convexity while taken into account in the Lagrangian functional. But still, it seems to be the easiest way, at least on the mathematical computation point of view, to deal with divergence free condition in a sensitivity analysis of the moving system. In the sequel, we adopt such a strategy, keeping in mind, its lack of rigorous mathematical justification.

5.1.2 Non-homogeneous boundary conditions

The Navier-Stokes system (Eq. (2)) involves an essential non-homogeneous Dirichlet boundary condition,

$$u = V, \quad \text{on } \Gamma_t(V) \tag{100}$$

Again, there exists different methods to take into account this boundary condition in a Min-Max formulation,

1. We can use a lifting of the boundary conditions inside the fluid domain and define a change of variable inside the coupled system, as done in Section (4). It has the drawback to put additionnal terms inside the Lagrangian functionals and to impose more regularity on the boundary conditions.

2. We can use a very weak formulation of the state equation, consisting in totally transposing the laplacian operator,

$$\int_{\Omega_t} -\nu\Delta u \cdot \phi = \int_{\Omega_t} -\nu\Delta\phi \cdot u + \int_{\Gamma_t} \nu \left[u \cdot \partial_n \phi - \phi \cdot \partial_n u\right] d\Gamma \tag{101}$$

   Then we shall substitute inside this identity the desirable boundary conditions. We recover the boundary constraints in performing an integration by parts in the optimality conditions corresponding to the sensitivity with respect to the multipliers. This procedure has been already used in [22] to perform shape optimization problems for elliptic equations using Min-Max principles.

   Remark 7 *This method has been popularized in [18] as a systematic way to study non-homogeneous linear partial differential equations. These formulations are usually called very weak formulations or transposed formulations. We shall notice that these methods are still valid in the non-linear case to obtain regularity or existence results. We refer to [3] for a recent applications to the Navier-Stokes system.*

5.1.3 Fluid state operator

In this section we shall summarize the different options that we have chosen for the Lagrangian framework and define the variationnal state operator constraint. In the sequel, we will need to define precise state and multiplier spaces in order to endow our problem with a Lagrangian functional framework.
Following the existence result stated previously, we introduce the fluid state spaces:

$$X(\Omega_t) \stackrel{\text{def}}{=} \left\{u \in H^2(0,T;(H^2(\Omega_t))^d \cap (H^1(\Omega_t))^d)\right\}$$

$$Z \stackrel{\text{def}}{=} \left\{p \in H^1(0,T;(H^1(D))^d)\right\}$$

we also need test function spaces that will be useful to define Lagrange multipliers:

$$\begin{aligned} Y(\Omega_t) &\stackrel{\text{def}}{=} \left\{v \in L^2(0,T;\,(H^2(\Omega_t))^2 \cap (H^1_0(\Omega_t))^d)\right\} \\ Q &\stackrel{\text{def}}{=} \left\{q \in H^1(0,T;(H^1(D))^2)\right\} \end{aligned}$$

We define the fluid weak state operator,

$$e_V : X \times Z \longrightarrow (Y \times Q)^*$$

whose action is defined by :

$$\begin{aligned} \langle e_V(u,p),(v,q)\rangle = &\int_0^T \int_{\Omega_t(V)} [-u \cdot \partial_t v + (\mathrm{D}\, u \cdot u) \cdot v - \nu u \cdot \Delta v + u \cdot \nabla\, q - p\ \mathrm{div}\, v] \\ &+ \int_0^T \int_{\Gamma_t(V)} V \cdot (\sigma(v,q) \cdot n) + \int_{\Omega_T} u(T) \cdot v(T) - \int_{\Omega_0} u_0 \cdot v(t=0) \\ &\qquad \forall\,(v,q) \in\ Y \times Q \end{aligned}$$

5.1.4 Min-Max problem

In this section, we introduce the lagrangian functional associated with Eq. (2) and Eq. (3) :

$$\mathcal{L}_V(u,p;v,q) \stackrel{\text{def}}{=} J_V(u,p) - \langle e_V(u,p),(v,q)\rangle \tag{102}$$

with

$$J_V(u,p) = \frac{\alpha}{2}\int_0^T \int_{\Omega_t(V)} |u|^2 + \frac{\gamma}{2}\int_0^T \int_{\Gamma_t(V)} |V|^2 \tag{103}$$

Using this functional, the optimal control problem (Eq. (3)) can be put in the following form:

$$\min_{V \in \mathcal{U}_{ad}} \quad \min_{(u,p) \in X(\Omega_t(V)) \times Z} \quad \max_{(v,q) \in Y(\Omega_t(V)) \times Q} \quad \mathcal{L}_V(u,p;v,q) \tag{104}$$

By using the Min-Max framework, we avoid the computation of the state derivative with respect to V. First-order optimality conditions will furnish the gradient of the original cost functional using the solution of an adjoint problem.
Let us first study the saddle point problem,

$$\min_{(u,p)\in X\times Z} \quad \max_{(v,q)\in Y\times Q} \quad \mathcal{L}_V(u,p;v,q) \tag{105}$$

5.1.5 Optimality Conditions

In this section, we are interested in establishing the first order optimality condition for problem ((105)), better known as Karusch-Kuhn-Tucker optimality conditions. This step is crucial, because it leads to the formulation of the adjoint problem satisfied by the Lagrange multipliers $(\varphi(V), \pi(V))$. The KKT system will have the following structure :

$$\begin{aligned}
\partial_{(v,q)}\mathcal{L}_V(u,p;v,q)\cdot(\delta v,\delta q) &= 0,\\
\forall\,(\delta v,\delta q)\in\, Y\times Q \rightarrow &\quad \text{State Equations}\\
\partial_{(u,p)}\mathcal{L}_V(u,p;v,q)\cdot(\delta u,\delta p) &= 0,\\
\forall\,(\delta u,\delta p)\in X\times Z \rightarrow &\quad \text{Adjoint Equations}
\end{aligned}$$

Lemma 17 *For $V \in \mathcal{U}_{ad}$, $(p,v,q) \in Z\times Y\times Q$, $\mathcal{L}_V(u,p;v,q)$ is differentiable with respect to $u \in X$ and we have*

$$\begin{aligned}
&\langle \partial_u \mathcal{L}_V(u,p;v,q),\, \delta u\rangle = \\
&\quad \int_0^T \int_{\Omega_t(V)} [\alpha\, u\cdot\delta u + \delta u\cdot\partial_t v - [\mathrm{D}\,\delta u\cdot u + \mathrm{D}\,u\cdot\delta u]\cdot v + \nu\delta u\cdot\Delta v - \delta u\cdot\nabla q] \\
&\qquad\qquad + \int_{\Omega_T} \delta u(T)\cdot v(T), \quad \forall\,\delta u \in\, X
\end{aligned}$$

In order to obtain a strong formulation of the fluid adjoint problem, we perform some integration by parts :

Lemma 18

$$\int_{Q(V)} (\mathrm{D}\,\delta u\cdot u)\cdot v \quad = \quad -\int_{Q(V)} [\mathrm{D}\,v\cdot u + \mathrm{div}(u)\cdot v]\cdot\delta u + \int_{\Sigma(V)} (\delta u\cdot v)(u\cdot n)$$

It leads to the following identity :

$$\langle \partial_{\hat{u}} \mathcal{L}_V(u,p;\varphi,\pi),\, \delta u\rangle = \\ -\int_{Q(V)} [-\partial_t\varphi + ({}^*Du)\cdot\varphi - (D\varphi)\cdot u - \mathrm{div}(u)\cdot\varphi - \nu\Delta\varphi + \nabla\pi - \alpha u]\cdot\delta u \\ -\nu\int_{\Sigma(V)} (\partial_n \delta u)\cdot\varphi - \int_{\Omega_T} \varphi(T)\cdot\delta u(T)$$

Lemma 19 *For $V \in \mathcal{U}_{ad}$, $(u,v,q) \in X\times Y\times Q$, $\mathcal{L}_V(u,p;v,q)$ is differentiable with respect to $p \in Z$ and we have*

$$\langle \partial_p \mathcal{L}_V(u,p;\varphi,\pi),\, \delta p\rangle = \int_0^T \int_{\Omega_t} (\delta p)\,\mathrm{div}\,\varphi, \quad \forall\, \delta p \in Z \tag{106}$$

This leads to the following fluid adjoint strong formulation,

$$\begin{cases} -\partial_t\varphi - \mathrm{D}\,\varphi\cdot u + ({}^*Du)\cdot\varphi - \nu\Delta\varphi + \nabla q = \alpha u, & Q(V) \\ \mathrm{div}(\varphi) = 0, & Q(V) \\ \varphi = 0, & \Sigma(V) \\ \varphi(T) = 0, & \Omega_T \end{cases} \tag{107}$$

Remark 8 *Existence and regularity results for the linearized Navier-Stokes adjoint problem can be found in [1, 16] for the 2D case. These results can be easily adapted for the moving domain case. There is a lack of results for the 3D case.*

5.2 Function space parametrization

To compute the first-order derivative of $j(V)$, we perturbe the moving domain $\Omega_t(V)$ by a velocity field W which generates the family of transformation $T_t^\rho \overset{\text{def}}{=} T_t(V+\rho W)$, with $\rho \geq 0$ and the family of domains and their boundaries,

$$\Omega_t^\rho \overset{\text{def}}{=} T_t(V+\rho W)(\Omega_0)$$
$$\Gamma_t^\rho \overset{\text{def}}{=} T_t(V+\rho W)(\Gamma_0)$$

We set,

$$g(\rho) = j(V+\rho W) = \min_{(u,p)\in X(\Omega_t^\rho)\times Z} \;\max_{(v,q)\in Y(\Omega_t^\rho)\times Q} \;\mathcal{L}_{(V+\rho W)}(u,p;v,q) \tag{108}$$

The objective of this section is to compute the following derivative :

$$\lim_{\rho\searrow 0} \frac{1}{\rho}(g(\rho) - g(0)) \tag{109}$$

We need a theorem that would give the derivative of a Min-Max function with respect to a real parameter $\rho \geq 0$. In our case, it is not trivial since the state and multiplier spaces $X(\Omega_t^\rho) \times Y(\Omega_t^\rho)$ depend on the perturbation parameter ρ. This point can be solved using particular parametrization of the functional spaces. To this aim, we use the transverse map introduced in Section (3),

$$\begin{array}{rcl} \mathcal{T}_\rho^t : \overline{\Omega_t} & \longrightarrow & \overline{\Omega_t^\rho} \\ x & \mapsto & T_t(V+\rho W) \circ T_t(V)^{-1} \end{array}$$

and we define the following parametrization,

$$X(\Omega_t^\rho) = \left\{ u \circ (T_t^\rho)^{-1}, \quad u \in X(\Omega_t(V)) \right\} \tag{110}$$

$$Y(\Omega_t^\rho) = \left\{ v \circ (T_t^\rho)^{-1}, \quad u \in Y(\Omega_t(V)) \right\} \tag{111}$$

This parametrization does not affect the value of the saddle point functional $g(\rho)$, but changes the parametrization of the Lagrangian functional,

$$g(\rho) = j(V+\rho W) = \min_{(u,p) \in X(\Omega_t) \times Z} \ \max_{(v,q) \in Y(\Omega_t) \times Q} \ \mathcal{L}_{(V+\rho W)}(u \circ R_\rho^t, p; v \circ R_\rho^t, q) \tag{112}$$

with $R_\rho^t \stackrel{\text{def}}{=} (T_t^\rho)^{-1}$.
We set,

$$\begin{aligned} \mathcal{L}_{V,W}^\rho(u,p;v,q) &= J_{V+\rho W}(u \circ R_\rho^t, p) \\ &- \int_0^T \int_{\Omega_t^\rho} \left[-u \circ R_\rho^t \cdot \partial_t (v \circ R_\rho^t) + (\mathrm{D}\, u \circ R_\rho^t \cdot u \circ R_\rho^t) \cdot v \circ R_\rho^t - \nu u \circ R_\rho^t \cdot \Delta(v \circ R_\rho^t) \right. \\ &\qquad \left. + u \circ R_\rho^t \cdot \nabla q - p \, \mathrm{div}(v \circ R_\rho^t) \right] - \int_0^T \int_{\Gamma_t^\rho} (V+\rho W) \cdot (\sigma(v \circ R_\rho^t, q) \cdot n^\rho) \\ &\qquad - \int_{\Omega_T} u(T) \cdot v(T) + \int_{\Omega_0} u_0 \cdot v(t=0) \\ &\qquad\qquad \forall\, (v,q) \in \ Y(\Omega_t(V)) \times Q \end{aligned}$$

where n^ρ stands for unit exterior normal of the perturbed boundary Γ_t^ρ.

5.3 Differentiability of the saddle point problem

In this section, we first state a general theorem concerning the differentiability of a Min-Max problem with respect to a scalar parameter. Then we apply it to our case of study. Finally, using a fundamental identity, we are able to express the gradient $\nabla j(V)$ as stated in the main theorem of this article.

5.3.1 General theorem

We a consider a functional,

$$G : [0, \rho_0] \times X \times Y \quad \to \quad \mathbb{R} \tag{113}$$

with $\rho_0 \geq 0$ and two topological spaces (X, Y). For each $\rho \in I \stackrel{\text{def}}{=} [0, \rho_0]$, we define

$$g(\rho) = \inf_{x \in X} \quad \sup_{y \in Y} \quad G(\rho, x, y) \tag{114}$$

and the sets,

$$X(\rho) = \left\{ x^\rho \in X, \quad \sup_{y \in Y} \quad G(\rho, x^\rho, y) = g(\rho) \right\} \tag{115}$$

$$Y(\rho, x) = \left\{ y^\rho \in Y, \quad G(\rho, x, y^\rho) = \sup_{y \in Y} \quad G(\rho, x, y) \right\} \tag{116}$$

In a similar way, we define dual functions and sets,

$$h(\rho) = \sup_{y \in Y} \quad \inf_{x \in X} \quad G(\rho, x, y) \tag{117}$$

and the sets,

$$Y(\rho) = \left\{ y^\rho \in Y, \quad \inf_{x \in X} \quad G(\rho, x, y^\rho) = h(\rho) \right\} \tag{118}$$

$$X(\rho, y) = \left\{ x^\rho \in X, \quad G(\rho, x^\rho, y) = \inf_{x \in X} \quad G(\rho, x, y) \right\} \tag{119}$$

Finally we define the sets of saddle points,

$$S(\rho) = \{(x, y) \in X \times Y, \quad g(\rho) = G(\rho, x, y) = h(\rho)\} \tag{120}$$

Theorem 15 ([7]) *Assume that the following hypothesis hold,*

(H1) The set $S(\rho) \neq \emptyset$, $\rho \in I$.

(H2) The partial derivative $\partial_\rho G(\rho, x, y)$ exists in I for all

$$(x, y) \in \left[\bigcup_{\rho \in I} X(\rho) \times Y(0) \right] \bigcup \left[X(0) \times \bigcup_{\rho \in I} Y(\rho) \right]$$

(H3) There exists a topology $\mathcal{T}_X$ *on* X *such that,*
for any sequence $(\rho_n)_{n\geq 0} \in I$ *with* $\lim_{n\nearrow\infty} \rho_n = 0$, *there exists* $x^0 \in X(0)$ *and a subsequence* ρ_{n_k} *and for each* $k \geq 1$, *there exists* $x_{n_k} \in X(\rho_{n_k})$ *such that,*

i) $\lim_{n\nearrow\infty} x_{n_k} = x^0$ *for the* $\mathcal{T}_X$ *topology,*

ii)

$$\liminf_{(\rho,k)\searrow\nearrow(0,\infty)} \partial_\rho G(\rho, x_{n_k}, y) \geq \partial_\rho G(0, x^0, y)$$

$\forall\, y \in Y(0)$.

(H4) There exists a topology $\mathcal{T}_Y$ *on* Y *such that,*
for any sequence $(\rho_n)_{n\geq 0} \in I$ *with* $\lim_{n\nearrow\infty} \rho_n = 0$, *there exists* $y^0 \in Y(0)$ *and a subsequence* ρ_{n_k} *and for each* $k \geq 1$, *there exists* $y_{n_k} \in Y(\rho_{n_k})$ *such that,*

i) $\lim_{n\nearrow\infty} y_{n_k} = y^0$ *for the* $\mathcal{T}_Y$ *topology,*

ii)

$$\liminf_{(\rho,k)\searrow\nearrow(0,\infty)} \partial_\rho G(\rho, x, y_{n_k}) \leq \partial_\rho G(0, x, y^0)$$

$\forall\, x \in X(0)$.

Then there exists $(x^0, y^0) \in X(0) \times Y(0)$ *such that*

$$\begin{aligned} dg(0) = \lim_{\rho\searrow 0} \frac{g(\rho) - g(0)}{\rho} = \inf_{x\in X(0)} \sup_{y\in Y(0)} \partial_\rho G(0, x, y) &= \partial_\rho G(0, x^0, y^0) \\ &= \sup_{y\in Y(0)} \inf_{x\in X(0)} \partial_\rho G(0, x, y) \end{aligned} \tag{121}$$

This means that $(x^0, y^0) \in X(0) \times Y(0)$ *is a saddle point of* $\partial_\rho G(0, x, y)$.

5.3.2 Derivative of the perturbed Lagrangian

Following Th. (15), we need to differentiate the perturbed Lagrangian functional $\mathcal{L}(\rho)$. We shall successively differentiate the distributed and the boundary integrals involved in the perturbed Lagrangian:

a) Distributed terms:
We set,

$$\begin{aligned} G(\rho, .) = \big[&-u \circ \mathcal{R}^t_\rho \cdot \partial_t (v \circ \mathcal{R}^t_\rho) + D(u \circ \mathcal{R}^t_\rho) \cdot (u \circ \mathcal{R}^t_\rho) \cdot v \circ \mathcal{R}^t_\rho \\ &-\nu (u \circ \mathcal{R}^t_\rho) \cdot \Delta (v \circ \mathcal{R}^t_\rho) + (u \circ \mathcal{R}^t_\rho) \cdot \nabla q - p \operatorname{div}(v \circ \mathcal{R}^t_\rho) \big] \end{aligned}$$

with $\mathcal{R}_\rho^t \stackrel{\text{def}}{=} (\mathcal{T}_\rho^t)^{-1}$.
We shall need the following lemmas in order to derivate $G(\rho,.)$ with respect to ρ,

Lemma 20

$$\left(\frac{d\mathcal{T}_\rho^t}{d\rho}\right)\Big|_{\rho=0} = Z_t$$

$$\left(\frac{d\mathcal{R}_\rho^t}{d\rho}\right)\Big|_{\rho=0} = -Z_t$$

Lemma 21

$$\left(\frac{d\left(u\circ\mathcal{R}_\rho^t\right)}{d\rho}\right)\Big|_{\rho=0} = -\,\mathrm{D}\,u\cdot Z_t$$

Proof :
Using the chain rule we get

$$\begin{aligned}\left(\frac{\mathrm{D}\left(u\circ\mathcal{R}_\rho^t\right)}{\mathrm{D}\,\rho}\right)\Big|_{\rho=0} &= \left(\mathrm{D}\,u\circ\mathcal{R}_\rho^t\right)\cdot\left(\frac{\mathrm{D}\,\mathcal{R}_\rho^t}{\mathrm{D}\,\rho}\right)\Big|_{\rho=0}\\ &= -\left(\mathrm{D}\,u\circ\mathcal{R}_\rho^t\right)\cdot\mathcal{Z}^t(\rho,.)\big|_{\rho=0}\\ &= -\,\mathrm{D}\,u\cdot Z_t\end{aligned}$$

□

Lemma 22 *Then, we have the following result*

$$\begin{aligned}\partial_\rho G(\rho,.)|_{\rho=0} = [&(\mathrm{D}\,u\cdot Z_t)\cdot\partial_t v + u\cdot(\partial_t(\mathrm{D}\,v\cdot Z_t))\\ &-[(\mathrm{D}(\mathrm{D}\,u\cdot Z_t))\cdot u + \mathrm{D}\,u\cdot(\mathrm{D}\,u\cdot Z_t)]\cdot v - (\mathrm{D}\,u\cdot u)\cdot(\mathrm{D}\,v\cdot Z_t)\\ &+\nu(\mathrm{D}\,u\cdot Z_t)\cdot\Delta v + \nu u\cdot(\Delta(\mathrm{D}\,v\cdot Z_t)) + p\,\mathrm{div}(\mathrm{D}\,v\cdot Z_t) - (\mathrm{D}\,u\cdot Z_t)\cdot\nabla q]\end{aligned}$$

Proof :
It comes easily using definition of $G(\rho,.)$ and Lem. (20)-(21).
□
Then we have an expression of the derivative of distributed terms coming from the

Lagrangian with respect to ρ,

$$\frac{d}{d\rho}\left(\int_{\Omega_t^\rho} G(\rho,x)dx\right)\Bigg|_{\rho=0} = \int_{\Omega_t} [(\mathrm{D}\, u\cdot Z_t)\cdot\partial_t v + u\cdot(\partial_t(\mathrm{D}\, v\cdot Z_t))$$
$$-[(\mathrm{D}(\mathrm{D}\, u\cdot Z_t))\cdot u + \mathrm{D}\, u\cdot(\mathrm{D}\, u\cdot Z_t)]\cdot v - (\mathrm{D}\, u\cdot u)\cdot(\mathrm{D}\, v\cdot Z_t)$$
$$+\nu(\mathrm{D}\, u\cdot Z_t)\cdot\Delta v + \nu u\cdot(\Delta(\mathrm{D}\, v\cdot Z_t)) + p\,\mathrm{div}(\mathrm{D}\, v\cdot Z_t) - (\mathrm{D}\, u\cdot Z_t)\cdot\nabla q]$$
$$+\int_{\Gamma_t}[-u\cdot\partial_t v + (Du\cdot u)\cdot v - \nu u\cdot\Delta v + u\cdot\nabla q - p\,\mathrm{div}(v)]\,\langle Z_t, n\rangle$$

b) Boundary terms :
We must now take into account the terms coming from the moving boundary Γ_t^ρ. Then we set,

$$\begin{aligned}\phi(\rho,.) &= (V+\rho W)\cdot\left[-q\,\mathrm{I} + \nu\,\mathrm{D}(v\circ\mathcal{R}_\rho^t)\right]\cdot n^\rho \\ &= E(\rho)\cdot n^\rho\end{aligned} \tag{122}$$

Since $\phi(\rho,.)$ is defined on the boundary Γ_t^ρ, we need some extra identities corresponding to boundary shape derivates of terms involved in $\phi(\rho,.)$.

Lemma 23 ([11])

$$\partial_\rho n^\rho|_{\rho=0} = n_\Gamma^{'} = -\nabla_\Gamma(Z_t\cdot n)$$

Lemma 24

$$\begin{aligned}\frac{d}{d\rho}\left(\int_{\Gamma_t^\rho}\langle E(\rho), n^\rho\rangle d\Gamma\right)\Bigg|_{\rho=0} &= \int_{\Gamma_t}\langle E'|_{\Gamma_t}, n\rangle + (\mathrm{div}\, E)\langle Z_t, n\rangle \\ &= \int_{\Gamma_t}\langle E'_{\Gamma_t}, n\rangle + (\mathrm{div}_\Gamma\, E)\langle Z_t, n\rangle\end{aligned} \tag{123}$$

Proof :
First, we use that,

$$\int_{\Gamma_t^\rho}\langle E(\rho), n^\rho\rangle = \int_{\Omega_t^\rho}\mathrm{div}\, E(\rho)$$

then we derive this quantity using Th. (4),

$$\frac{d}{d\rho}\left(\int_{\Omega_t^\rho}\mathrm{div}\, E(\rho)\right)\Bigg|_{\rho=0} = \int_{\Omega_t}\mathrm{div}\, E' + \int_{\Gamma_t}(\mathrm{div}\, E)\langle Z_t, n\rangle$$

We conclude using $\int_{\Omega_t} \operatorname{div} E' = \int_{\Gamma_t} \langle E', n\rangle$. For the second identity, using the Th. (5), we have

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^\rho} \langle E(\rho), (n \circ \mathcal{R}_\rho^t)\rangle d\Gamma\right)\Bigg|_{\rho=0} = \int_{\Gamma_t} \langle E'_\Gamma, n\rangle + \langle E, n'_\Gamma\rangle + H\langle E, n\rangle\langle Z_t, n\rangle$$

using Lem. (23), we get

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^\rho} \langle E(\rho), (n \circ \mathcal{R}_\rho^t)\rangle d\Gamma\right)\Bigg|_{\rho=0} = \int_{\Gamma_t} \langle E'_\Gamma, n\rangle - \langle E, \nabla_\Gamma (Z_t \cdot n)\rangle + H\langle E, n\rangle\langle Z_t, n\rangle$$

then using the tangential Stokes identity from Lem. (5), we obtain the correct result.
□

Hence, we only need to compute the quantity E'_Γ. To this end, we need the following identities,

Lemma 25

$$\left(v \circ \mathcal{R}_\rho^t\right)'_\Gamma\Big|_{\rho=0} = -\operatorname{D}_\Gamma v \cdot Z_t$$

Proof : Since $\left(v \circ \mathcal{R}_\rho^t\right)^{\cdot}\big|_{\rho=0} = \partial_\rho\left(v \circ \mathcal{R}_\rho^t \circ \mathcal{T}_\rho^t\right)\big|_{\rho=0} = \partial_\rho v|_{\rho=0} = 0.$
□

Lemma 26

$$\left(\operatorname{D}\left(v \circ \mathcal{R}_\rho^t\right)\right)'_\Gamma\Big|_{\rho=0} = -\operatorname{D} v \cdot \operatorname{D} Z_t - (\operatorname{D}_\Gamma(\operatorname{D} v)) \cdot Z_t$$

Proof : By definition we have,

$$\begin{aligned}
\left(\operatorname{D}\left(v \circ \mathcal{R}_\rho^t\right)\right)'_\Gamma\Big|_{\rho=0} &= \left(\operatorname{D}\left(v \circ \mathcal{R}_\rho^t\right)\right)^{\cdot}_\Gamma\Big|_{\rho=0} - \operatorname{D}_\Gamma\left(\operatorname{D}(v \circ \mathcal{R}_\rho^t)\right) \cdot \mathcal{Z}_\rho^t\big|_{\rho=0} \\
&= \partial_\rho\left(\operatorname{D}\left(v \circ \mathcal{R}_\rho^t\right) \circ \mathcal{T}_\rho^t\right)\big|_{\rho=0} - (\operatorname{D}_\Gamma(\operatorname{D} v)) \cdot Z_t \\
&= \partial_\rho\left[\left((\operatorname{D} v) \circ \mathcal{R}_\rho^t \cdot \operatorname{D}\mathcal{R}_\rho^t\right) \circ \mathcal{T}_\rho^t\right]\big|_{\rho=0} - (\operatorname{D}_\Gamma(\operatorname{D} v)) \cdot Z_t \\
&= \partial_\rho\left[(\operatorname{D} v) \cdot \operatorname{D}\mathcal{R}_\rho^t \circ \mathcal{T}_\rho^t\right]\big|_{\rho=0} - (\operatorname{D}_\Gamma(\operatorname{D} v)) \cdot Z_t \\
&= -\operatorname{D} v \cdot \operatorname{D} Z_t + \left[\operatorname{D} v \cdot \operatorname{D}(\operatorname{D}\mathcal{R}_\rho^t) \cdot \partial_\rho(\mathcal{T}_\rho^t)\right]\big|_{\rho=0} - (\operatorname{D}_\Gamma(\operatorname{D} v)) \cdot Z_t \\
&= -\operatorname{D} v \cdot \operatorname{D} Z_t - (\operatorname{D}_\Gamma(\operatorname{D} v)) \cdot Z_t
\end{aligned}$$

Using these results, we can state the following :

Lemma 27

$$E'_\Gamma = W \cdot [-q\,\mathrm{I} + \nu\,\mathrm{D}\,v] + \nu\,V \cdot [-\,\mathrm{D}\,v \cdot \mathrm{D}\,Z_t - \mathrm{D}_\Gamma(\mathrm{D}\,v)\cdot Z_t] \tag{124}$$

This means that we have,

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^\rho} \phi(\rho,x)d\Gamma\right)\Bigg|_{\rho=0} = \int_{\Gamma_t(V)} W \cdot [-q\,n + \nu\,\mathrm{D}\,v\cdot n]$$
$$+\,\nu\,V\cdot[-(\mathrm{D}\,v\cdot\mathrm{D}\,Z_t)\cdot n - (\mathrm{D}_\Gamma(\mathrm{D}\,v)\cdot Z_t)\cdot n] + \mathrm{div}_\Gamma(V\cdot[-q\,\mathrm{I}+\nu\,\mathrm{D}\,v])\langle Z_t, n\rangle$$

We have also,

Lemma 28

$$E'|_\Gamma = W\cdot[-q\,\mathrm{I}+\nu\,\mathrm{D}\,v] - \nu\,V\cdot[\mathrm{D}(\mathrm{D}\,v)\cdot Z_t] \tag{125}$$

Hence, we have

$$\frac{d}{d\rho}\left(\int_{\Gamma_t^\rho} \phi(\rho,x)d\Gamma\right)\Bigg|_{\rho=0} = \int_{\Gamma_t(V)} W \cdot [-q\,n + \nu\,\mathrm{D}\,v\cdot n]$$
$$-\,\nu\,V\cdot[\mathrm{D}(\mathrm{D}\,v)\cdot Z_t\cdot n] + \mathrm{div}(V\cdot[-q\,\mathrm{I}+\nu^*\,\mathrm{D}\,v])\langle Z_t, n\rangle$$

Remark 9 *We recall that,*

$$\begin{aligned}\int_{\Gamma_t} V\cdot(\mathrm{D}\,v\cdot n) &= \int_{\Omega_t} \mathrm{div}({}^*\,\mathrm{D}\,v\cdot V)\\ &= \int_{\Omega_t} \mathrm{D}\,v\cdot\cdot\,\mathrm{D}\,V + V\cdot\Delta v\end{aligned} \tag{126}$$

We shall use this expression in the sequel. We recall that the perturbed lagrangian has the following form,

$$\mathcal{L}^\rho_{V,W} = J^\rho_{V,W} - \int_0^T\int_{\Omega_t^\rho} G(\rho) - \int_0^T\int_{\Gamma_t^\rho}\phi(\rho)$$
$$-\int_{\Omega_T} u(T)\cdot v(T) + \int_{\Omega_0} u_0\cdot v(t=0)$$
$$\forall\,(v,q)\in\;Y(\Omega_t)\times Q$$

Hence its derivative with respect to ρ at point $\rho = 0$ has the following expression,

$$\frac{d}{d\rho}\left(\mathcal{L}^{\rho}_{V,W}\right)\Big|_{\rho=0} = \frac{d}{d\rho}\left(J^{\rho}_{V,W}\right)\Big|_{\rho=0} - \int_0^T \frac{d}{d\rho}\left(\int_{\Omega_t^\rho} G(\rho)\right)\Big|_{\rho=0} - \int_0^T \frac{d}{d\rho}\left(\int_{\Gamma_t^\rho} \phi(\rho)\right)\Big|_{\rho=0}$$
$$\forall\,(v,q) \in\ Y(\Omega_t)\times Q$$

Furthermore we have,

Lemma 29

$$\frac{d}{d\rho}\left(J^{\rho}_{V,W}\right)\Big|_{\rho=0} = -\alpha\int_0^T\int_{\Omega_t(V)} u\cdot(\mathrm{D}\,u\cdot Z_t) + \int_0^T\int_{\Gamma_t(V)}\gamma\,V\cdot W$$
$$+\int_0^T\int_{\Gamma_t(V)}\left[\frac{\alpha}{2}|u|^2 + \frac{\gamma}{2}H|V|^2\right]\langle Z_t, n\rangle$$

Using the last identities concerning the derivative of the distributed and the boundary terms with respect to ρ, we shall get the following expression,

$$\frac{d}{d\rho}\left(\mathcal{L}^{\rho}_{V,W}\right)\Big|_{\rho=0} = -A_{Z_t} - B_{Z_t} - C_W \tag{127}$$

with

$$A_{Z_t} = \int_0^T\int_{\Omega_t(V)}\left[\alpha u\cdot(\mathrm{D}\,u\cdot Z_t) + (\mathrm{D}\,u\cdot Z_t)\cdot\partial_t v - [(\mathrm{D}(\mathrm{D}\,u\cdot Z_t))\cdot u\right.$$
$$\left. + \mathrm{D}\,u\cdot(\mathrm{D}\,u\cdot Z_t)]\cdot v + \nu(\mathrm{D}\,u\cdot Z_t)\cdot\Delta v - (\mathrm{D}\,u\cdot Z_t)\cdot\nabla q\right]$$
$$+\int_0^T\int_{\Omega_t(V)}\left[u\cdot(\partial_t(\mathrm{D}\,v\cdot Z_t)) - (\mathrm{D}\,u\cdot u)\cdot(\mathrm{D}\,v\cdot Z_t) + \nu u\cdot(\Delta(\mathrm{D}\,v\cdot Z_t)) + p\,\mathrm{div}(\mathrm{D}\,v\cdot Z_t)\right]$$

$$B_{Z_t} = \int_0^T\int_{\Gamma_t(V)}\left[-u\cdot\partial_t v + (Du\cdot u)\cdot v - \nu u\cdot\Delta v + u\cdot\nabla q - p\,\mathrm{div}(v)\right](Z_t\cdot n)$$

$$-\,\nu\,V\cdot[(\mathrm{D}(\mathrm{D}\,v)\cdot Z_t)\cdot n] + \mathrm{div}(V\cdot[-q\,\mathrm{I} + \nu\,\mathrm{D}\,v])\langle Z_t, n\rangle - \left[\frac{\alpha}{2}|u|^2 + \frac{\gamma}{2}H|V|^2\right]\langle Z_t, n\rangle$$

$$C_W = \int_0^T\int_{\Gamma_t(V)}\left[W\cdot[-q\,n + \nu\,\mathrm{D}\,v\cdot n] - \gamma\,V\cdot W\right]$$

5.3.3 The shape derivative kernel identity

We shall now, assume that $(u,p,v,q) = (u,p,\varphi,\pi)$ is a saddle point of the Lagrangian functional $\mathcal{L}_V$. This will help us to simplify several terms involved in the derivative of $\mathcal{L}_V$ with respect to V.
Indeed, we would like to express the distributed term A_{Z_t} as a boundary quantity defined on the moving boundary Γ_t.

Theorem 16 *For (u,p,φ,π) saddle points of the Lagrangian functional (Eq. (102)), the following identity holds,*

$$\begin{aligned}
&\int_0^T \int_{\Omega_t(V)} [\alpha u \cdot (\mathrm{D}\, u \cdot Z_t) + (\mathrm{D}\, u \cdot Z_t) \cdot \partial_t v - [(\mathrm{D}(\mathrm{D}\, u \cdot Z_t)) \cdot u \\
&\qquad\qquad + \mathrm{D}\, u \cdot (\mathrm{D}\, u \cdot Z_t)] \cdot v + \nu(\mathrm{D}\, u \cdot Z_t) \cdot \Delta v - (\mathrm{D}\, u \cdot Z_t) \cdot \nabla q] \\
&+ \int_0^T \int_{\Omega_t(V)} [u \cdot (\partial_t(\mathrm{D}\, v \cdot Z_t)) - (\mathrm{D}\, u \cdot u) \cdot (\mathrm{D}\, v \cdot Z_t) + \nu u \cdot (\Delta(\mathrm{D}\, v \cdot Z_t)) + p \,\mathrm{div}(\mathrm{D}\, v \cdot Z_t)] \\
&- \int_0^T \int_{\Gamma_t(V)} [\nu\, V \cdot (\mathrm{D}(\mathrm{D}\,\varphi \cdot Z_t) \cdot n) - (\mathrm{D}\,\varphi \cdot Z_t) \cdot (-p\, n + \nu(\mathrm{D}\, u \cdot n))] = 0, \quad \forall\, W \in \mathcal{U}_{ad}
\end{aligned}$$

Proof :
We shall use extremal conditions associated to variations with respect to (u,v) in the Lagrangian functional where we add a boundary integral since we consider test functions v that do not vanish on the boundary $\Gamma_t(V)$, i.e

$$\begin{aligned}
\mathcal{L}_V^2(u,p;v,q) = J_V(u,p) - &\int_0^T \int_{\Omega_t(V)} [-u \cdot \partial_t v + (\mathrm{D}\, u \cdot u) \cdot v - \nu u \cdot \Delta v + u \cdot \nabla\, q - p \ \mathrm{div}\, v] \\
- \int_0^T \int_{\Gamma_t(V)} V \cdot (\sigma(v,q) \cdot n) + &\int_0^T \int_{\Gamma_t(V)} v \cdot (\sigma(u,p) \cdot n) - \int_{\Omega_T} u(T) \cdot v(T) + \int_{\Omega_0} u_0 \cdot v(t=0) \\
&\forall\, (v,q) \in \ Y \times Q
\end{aligned}$$

This leads to the following perturbation identity,

$$\begin{aligned}
\partial_{(u,v)} \mathcal{L}_V^2 \cdot (\delta u, \delta v) = -&\int_{Q(V)} [-\alpha u \cdot \delta u - \delta u \cdot \partial_t v - u \cdot \partial_t \delta v + D(\delta u \cdot u) \cdot v + D(u \cdot \delta u) \cdot v \\
+ D(u \cdot u) \cdot \delta v - \nu(\delta u \cdot \Delta v) - \nu(u \cdot \Delta \delta v) &+ \delta u \cdot \nabla q - p \,\mathrm{div}(\delta v)] - \int_0^T \int_{\Gamma_t(V)} \nu\, V \cdot (\mathrm{D}\, \delta v \cdot n) \\
+ \int_0^T \int_{\Gamma_t(V)} [\nu\, v \cdot (\mathrm{D}\, \delta u \cdot n) &+ \delta v \cdot (-p\, n + \nu(\mathrm{D}\, u \cdot n))] - \int_{\Omega_T} [\delta u(T) v(T) + u(T) \delta v(T)] \\
&\forall\, (\delta u, \delta v) \in \ X(\Omega_t) \times Y(\Omega_t)
\end{aligned}$$

We choose specific perturbation directions, i.e

$$\delta u = \mathrm{D}\, u \cdot Z_t \quad \delta v = \mathrm{D}\, v \cdot Z_t$$

with $\delta u(T) = \delta v(T) = \delta u(0) = \delta v(0) = 0$, where (u, v) are saddle points of the lagrangian, i.e solutions of respectively the primal and adjoint fluid problem. We recognize immediately the distributed and boundary terms involved in the shape derivative kernel identity.
□

5.3.4 Cost functional gradient

Now, we set $(u, v) = (u, \varphi)$ and we use the fact that $u = V$, on Γ_t and $\varphi = 0$, on Γ_t to simplify the remaining terms.

$$A_{Z_t} = \int_0^T \int_{\Gamma_t(V)} [\nu\, V \cdot (\mathrm{D}(\mathrm{D}\,\varphi \cdot Z_t) \cdot n) - \nu(\mathrm{D}\,\varphi \cdot Z_t) \cdot (\mathrm{D}\, u \cdot n)] \tag{128}$$

Remark 10 *We have used, that* $(\mathrm{D}\,\varphi \cdot Z_t) \cdot (p\, n) = (\mathrm{D}\,\varphi \cdot (n \otimes n) \cdot Z_t) \cdot (p\, n) = p\, ((\mathrm{D}\,\varphi \cdot n) \cdot n) \cdot \langle Z_t, n\rangle = (p \operatorname{div} \varphi)\langle Z_t, n\rangle = 0.$

$$B_{Z_t} = \int_0^T \int_{\Gamma_t(V)} [-\nu V \cdot \Delta\varphi + V \cdot \nabla\pi]\, \langle Z_t, n\rangle - \frac{1}{2}\, [\alpha + \gamma\, H]\, |V|^2 \langle Z_t, n\rangle$$

$$-\,\nu\, V \cdot [(\mathrm{D}(\mathrm{D}\,\varphi) \cdot Z_t) \cdot n] + [-\pi \operatorname{div} V - V \cdot \nabla\, \pi + \nu\, \mathrm{D}\,\varphi \cdot\cdot\, \mathrm{D}\, V + \nu\, V \cdot \Delta\varphi])\langle Z_t, n\rangle$$

$$C_W = \int_0^T \int_{\Gamma_t(V)} [W \cdot [-\pi\, n + \nu\, \mathrm{D}\,\varphi \cdot n] - \gamma\, V \cdot W]$$

We need to establish the following identity,

Lemma 30

$$\int_{\Gamma_t} (\mathrm{D}\,\varphi \cdot Z_t) \cdot (\mathrm{D}\, u \cdot n) = \int_{\Gamma_t} (\mathrm{D}\,\varphi \cdot n) \cdot (\mathrm{D}\, u \cdot n)\langle Z_t, n\rangle \tag{129}$$

Then

$$-\,\frac{d}{d\rho} \left(\mathcal{L}^{\rho}_{V,W}\right)\Big|_{\rho=0} = \int_{\Sigma(V)} \nu\, V \cdot (\mathrm{D}(\mathrm{D}\,\varphi \cdot Z_t) \cdot n) + [-\nu(\mathrm{D}\,\varphi \cdot n) \cdot (\mathrm{D}\, u \cdot n) - \nu V \cdot \Delta\varphi$$

$$+V \cdot \nabla\pi]\, \langle Z_t, n\rangle + [-\pi \operatorname{div} V - V \cdot \nabla\, \pi + \nu\, \mathrm{D}\,\varphi \cdot\cdot\, \mathrm{D}\, V + \nu\, V \cdot \Delta\varphi])\langle Z_t, n\rangle$$

$$-\,\nu\, V \cdot [(\mathrm{D}(\mathrm{D}\,\varphi) \cdot Z_t) \cdot n] - \frac{1}{2}\, [\alpha + \gamma\, H]\, |V|^2 \langle Z_t, n\rangle + [W \cdot [-\pi\, n + \nu\, \mathrm{D}\,\varphi \cdot n] - \gamma\, V \cdot W]$$

This allows us to derive the expression of the cost function directional derivative,

Proposition 8

$$dg(0) = \int_{\Sigma(V)} \left[-\nu(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \pi\,\mathrm{div}\,V + \frac{1}{2}(\alpha+\gamma\,H)|V|^2 \right] \langle Z_t, n\rangle + \int_{\Sigma(V)} [-\sigma(\varphi,q)\cdot n + \gamma V]\cdot W \tag{130}$$

Then we use Th. (8) with,

$$E = -\nu(\mathrm{D}\,\varphi\cdot n)\cdot(\mathrm{D}\,V\cdot n - \mathrm{D}\,u\cdot n) + \pi\,\mathrm{div}\,V + \frac{1}{2}(\alpha+\gamma\,H)|V|^2$$

and we get the correct result.

6 Min-Max and function space embedding

In the previous section, we have used a function space parametrization in order to get the gradient of a given functional related to the solution of the Navier-Stokes system in moving domain, with respect to the speed of the moving domain. In this section, we use a different method based on function space embedding particulary suited for non-homogeneous Dirichlet boundary problems. It means that the state and multiplier variables are defined in a hold-all domain D that contains the moving domain $\Omega_t(V)$ for $t \in (0,T)$ and $\forall\, V \in \mathcal{U}_{ad}$.

6.1 Saddle point formulation of the fluid state system

We recall that we are dealing with the Navier-Stokes in a moving domain $\Omega_t(V)$ which is driven by an eulerian velocity field $V \in \mathcal{U}_{ad}$,

$$\begin{cases} \partial_t u + \mathrm{D}\,u\cdot u - \nu\Delta u + \nabla p = 0, & Q(V) \\ \mathrm{div}(u) = 0, & Q(V) \\ u = V, & \Sigma(V) \\ u(t=0) = u_0, & \Omega_0 \end{cases} \tag{131}$$

and

$$\mathcal{U}_{ad} = \left\{ V \in H^1(0,T;(H^m(D))^d), \quad \mathrm{div}\,V = 0 \text{ in } D, \quad V\cdot n = 0 \text{ on } \partial D \right\} \tag{132}$$

with $m > 5/2$.
We introduce a Lagrange multiplier μ and a functional,

$$E_V(u,p;v,q,\mu) = \int_0^T \int_{\Omega_t(V)} [\partial_t u + \mathrm{D}\,u\cdot u - \nu\Delta u + \nabla\,p]\cdot v - \int_0^T \int_{\Omega_t(V)} q\,\mathrm{div}\,u - \int_0^T \int_{\Gamma_t(V)} (u - V)\cdot\mu$$

for $(u,p) \in X \times P$, $(v,q) \in Y \times Q$ and $\mu \in M$ with

$$X \stackrel{\text{def}}{=} Y \stackrel{\text{def}}{=} H^1(0,T;H^2(D))$$

$$P \stackrel{\text{def}}{=} Q \stackrel{\text{def}}{=} H^1(0,T;H^1(D))$$

$$M = H^1(0,T;H^{3/2}(\Gamma_t))$$

We are interested in the following Min-Max problem,

$$\min_{(u,p)\in X\times P} \quad \max_{(v,q,\mu)\in Y\times Q\times M} E_V(u,p;v,q,\mu) \tag{133}$$

The solution $(y,p,\varphi,\pi,\lambda)$ of this problem is characterized by the following optimality system,

- The primal state (y,p) is solution of the Navier-Stokes system,

$$\begin{cases} \partial_t y + \mathrm{D}\, y \cdot y - \nu \Delta y + \nabla p = 0, & Q(V) \\ \mathrm{div}(y) = 0, & Q(V) \\ y = V, & \Sigma(V) \\ y(t=0) = y_0, & \Omega_0 \end{cases} \tag{134}$$

- The dual state (φ,π) is solution of the fluid adjoint system,

$$\begin{cases} -\partial_t \varphi - \mathrm{D}\, \varphi \cdot u + ({}^*Du) \cdot \varphi - \nu \Delta \varphi + \nabla \pi = 0, & Q(V) \\ \mathrm{div}(\varphi) = 0, & Q(V) \\ \varphi = 0, & \Sigma(V) \\ \varphi(t=T) = 0, & \Omega_T \end{cases} \tag{135}$$

- The multiplier satisfies the following identity,

$$\mu = -q\, n + \nu(\mathrm{D}\, \varphi \cdot n), \qquad \text{on } \Gamma_t(V) \tag{136}$$

Then we can choose the above particular representation of the boundary Lagrange multiplier μ. This yields to the following functional,

$$\begin{aligned} E_V(u,p;v,q) = \int_0^T \int_{\Omega_t(V)} [\partial_t u + \mathrm{D}\, u \cdot u - \nu \Delta u + \nabla\, p] \cdot v - \int_0^T \int_{\Omega_t(V)} q \,\mathrm{div}\, u \\ - \int_0^T \int_{\Gamma_t(V)} (u - V) \cdot \sigma(v,q) \cdot n \end{aligned}$$

for $(u,p) \in X \times P$, $(v,q) \in Y \times Q$, with

$$\sigma(v,q) \cdot n = -q\, n + \nu(\mathrm{D}\, \varphi \cdot n), \qquad \text{on } \Gamma_t(V)$$

The following identities hold true,

Lemma 31

$$\begin{aligned}\int_{\Gamma_t(V)} (u-V)\cdot(\mathrm{D}\,v\cdot n) &= \int_{\Omega_t(V)} \mathrm{div}\,[{}^*\,\mathrm{D}\,v\cdot(u-V)] \\ &= \int_{\Omega_t(V)} [\mathrm{D}(u-V)\cdot\cdot\,\mathrm{D}\,v + (u-V)\cdot\Delta v]\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma_t(V)} (u-V)\cdot q\,n &= \int_{\Omega_t(V)} \mathrm{div}\,[q(u-V)] \\ &= \int_{\Omega_t(V)} [(u-V)\cdot\nabla q + q\,\mathrm{div}(u-V)]\end{aligned}$$

Using this identity, we may get the final expression of our saddle functional,

$$\begin{aligned}E_V(u,p;v,q) = \int_0^T \int_{\Omega_t(V)} [\partial_t u + \mathrm{D}\,u\cdot u - \nu\Delta u + \nabla\,p]\cdot v - \int_0^T \int_{\Omega_t(V)} q\,\mathrm{div}\,u \\ + \int_0^T \int_{\Omega_t(V)} [(u-V)\cdot\nabla q + q\,\mathrm{div}(u-V) - \nu\,\mathrm{D}(u-V)\cdot\cdot\,\mathrm{D}\,v - \nu(u-V)\cdot\Delta v]\end{aligned}$$

for $(u,p)\in X\times P$, $(v,q)\in Y\times Q$.

Remark 11 *The above expression of the Lagrange functional has the advantage to include only distributed terms. This will be useful for its differentiation with respect to V.*

6.2 Lagrange functional and non-cylindrical shape derivative

We are interested in the following minimization problem,

$$\min_{V\in\mathcal{U}_{ad}} j(V) \tag{137}$$

where $j(V) = J_V(u(V),p(V))$ with $(u(V),p(V))$ is a weak solution of problem (2) and $J_V(u,p)$ is a real functional of the following form :

$$J_V(u,p) = \frac{\alpha}{2}\int_0^T \int_{\Omega_t(V)} |u|^2 + \frac{\gamma}{2}\int_0^T \int_{\Gamma_t(V)} |V|^2 \tag{138}$$

We may solve this problem by the studying the equivalent Min-Max problem,

$$\min_{V\in\mathcal{U}_{ad}} \;\min_{(u,p)\in X\times P} \;\max_{(v,q)\in Y\times Q} \;\mathcal{L}_V(u,p;v,q) \tag{139}$$

with

$$\mathcal{L}_V(u,p;v,q) = J_V(u,p) - E_V(u,p;v,q) \tag{140}$$

Our main concern is the differentiation of the above functional with respect to $V \in \mathcal{U}_{ad}$. As in the previous section we perturbe the tubes using a vector field $W \in \mathcal{U}_{ad}$ with an increment parameter $\rho \geq 0$. Since the functions are embedded in the hold-all domain D, the perturbed Lagrangian has the following form,

$$\mathcal{L}^\rho(u,p;v,q) = J_{V+\rho W}(u,p) - E_{V+\rho W}(u,p;v,q) \tag{141}$$

The set of saddle points,

$$S(\rho) = X(\rho) \times P \times Y(\rho) \times Q \in X \times P \times Y \times Q$$

is not a singleton since,

$$X(\rho) = \left\{u \in X, u|_{\Omega_t^\rho} = y(\rho)\right\}$$

$$Y(\rho) = \left\{v \in Y, v|_{\Omega_t^\rho} = \varphi(\rho)\right\}$$

We make the conjecture that we can bypass the min-max, and state

$$\frac{d}{d\rho} j(V+\rho W)\Big|_{\rho=0} = \min_{(u,p)\in X\times P} \quad \max_{(v,q)\in Y\times Q} \quad \frac{d}{d\rho}\mathcal{L}^\rho(u,p;v,q)\Big|_{\rho=0} \tag{142}$$

Using non-cylindrical shape derivative framework, we can state

Lemma 32

$$\partial_V \mathcal{L}_V(u,p;v,q)\cdot W = -\int_0^T \int_{\Gamma_t(V)} [(\partial_t u + \mathrm{D}\, u\cdot u - \nu\Delta u + \nabla\, p)\cdot v - q\,\mathrm{div}\, u$$

$$+(u-V)\cdot\nabla q + q\,\mathrm{div}(u-V) - \nu\,\mathrm{D}(u-V)\cdot\cdot\,\mathrm{D}\, v - \nu(u-V)\cdot\Delta v - \frac{\alpha}{2}|u|^2$$

$$-H\frac{\gamma}{2}|V|^2\Big]\,\langle Z_t, n\rangle - \int_0^T \int_{\Omega_t(V)} [-W\cdot\nabla q - q\,\mathrm{div}\, W + \nu\,\mathrm{D}\, W\cdot\cdot\,\mathrm{D}\, v + \nu W\cdot\Delta v]$$

$$+\int_0^T \int_{\Gamma_t(V)} \gamma V\cdot W$$

Then we set $(u,p) = (y,p)$ and $(v,q) = (\varphi,\pi)$ with

$$\begin{cases} -\partial_t\varphi - \mathrm{D}\,\varphi\cdot u + {}^*\,\mathrm{D}\, u\cdot\varphi - \nu\Delta\varphi + \nabla\pi = \alpha\, u, & Q(V)\\ \mathrm{div}(\varphi) = 0, & Q(V)\\ \varphi = 0, & \Sigma(V)\\ \varphi(T) = 0, & \Omega_T \end{cases} \tag{143}$$

and we use that

$$(y, \varphi) = (V, 0) \text{ on } \Gamma_t(V)$$

and

$$\int_{\Omega_t(V)} [-W \cdot \nabla q - q \operatorname{div} W + \nu \operatorname{D} W \cdot \cdot \operatorname{D} v + \nu W \cdot \Delta v] = \int_{\Gamma_t(V)} W \cdot \sigma(v, q) \cdot n$$

Then,

$$\partial_V j(V) \cdot W = -\int_0^T \int_{\Gamma_t(V)} \Big[\Big(-\pi \operatorname{div} y + \pi \operatorname{div}(y - V) - \nu \operatorname{D}(y - V) \cdot \cdot \operatorname{D} \varphi - \frac{1}{2}(\alpha + H\gamma)|V|^2 \Big) \langle Z_t, n \rangle + (\sigma(\varphi, \pi) \cdot n - \gamma V) \cdot W \Big]$$

Using regularity assumptions on y and the free divergence condition on y, we may state that $\operatorname{div} y|_{\Gamma_t} = 0$.

Lemma 33

$$\operatorname{D} y \cdot \cdot \operatorname{D} \varphi|_{\Gamma_t(V)} = (\operatorname{D} y \cdot n) \cdot (\operatorname{D} \varphi \cdot n) \tag{144}$$

Proof :
Using that $\varphi = 0$ on $\Gamma_t(V)$, yields to

$$\operatorname{D} \varphi|_{\Gamma_t} = \operatorname{D} \varphi \cdot (n \otimes n)|_{\Gamma_t}$$

then, we get

$$\begin{aligned} \operatorname{D} y \cdot \cdot \operatorname{D} \varphi &= \operatorname{D} y \cdot \cdot (\operatorname{D} \varphi \cdot (n \otimes n)) \\ &= (\operatorname{D} y \cdot n) \cdot (\operatorname{D} \varphi \cdot n) \end{aligned}$$

□
Consequently we get,

$$\langle j'(V), W \rangle = \int_0^T \int_{\Gamma_t(V)} \Big[-\nu(\operatorname{D} \varphi \cdot n) \cdot (\operatorname{D} V \cdot n - \operatorname{D} u \cdot n) + \pi \operatorname{div} V + \frac{1}{2}(\alpha + \gamma H)|V|^2 \Big] \langle Z_t, n \rangle + \int_0^T \int_{\Gamma_t(V)} [-\sigma(\varphi, q) \cdot n + \gamma V] \cdot W \tag{145}$$

We then use theorem (8) with

$$E = -\nu(\operatorname{D} \varphi \cdot n) \cdot (\operatorname{D} V \cdot n - \operatorname{D} u \cdot n) + \pi \operatorname{div} V + \frac{1}{2}(\alpha + \gamma H)|V|^2$$

and we get the correct result.
□

7 Conclusion

In this article, we have been dealing with a particular shape optimization problem involving the Navier-Stokes equations. Its originality lies in the fact that the domain containing the fluid is moving. We have introduced an open loop control problem based on the velocity of the moving domain with the goal of reaching a given objective related to the behaviour of the fluid. Our main concern was to show how the gradient of the cost functional involved in the optimal control problem can be obtained by using non-cylindrical shape optimization concepts. In addition to the classical method based on the state derivative with respect to shape motions, we have introduced two different methods based on the Min-Max principle. Even if for the time being these methods lacks from a rigorous mathematical framework, they allow more flexible computations which can be very useful for pratical purpose. On the numerical point of view, an implementation of the open loop control is under study in the 2D case [15]. We believe that the concepts introduced in this article, will prove large efficiency for coupled problems involving a moving boundary, as it will be shown in [20], [21].

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Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399