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*On the angular defect of triangulations and the  
pointwise approximation of curvatures*

V. Borrelli

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F. Cazals

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**N° 4590**

Octobre 2002

THÈME 2



*R*  
*apport*  
*de recherche*



## On the angular defect of triangulations and the pointwise approximation of curvatures

V. Borrelli \*

F. Cazals †

J.-M. Morvan ‡

Thème 2 — Génie logiciel  
et calcul symbolique  
Projets Prisme

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**Abstract:** Let  $S$  be a smooth surface of  $E^3$ ,  $p$  a point on  $S$ ,  $k_m$ ,  $k_M$ ,  $k_G$  and  $k_H$  the maximum, minimum, Gauss and mean curvatures of  $S$  at  $p$ . Consider a set  $\{p_i p_{i+1}\}_{i=1, \dots, n}$  of  $n$  Euclidean triangles forming a piecewise linear approximation of  $S$  around  $p$  —with  $p_{n+1} = p_1$ . For each triangle, let  $\gamma_i$  be the angle  $\angle p_i p p_{i+1}$ , and let the angular defect at  $p$  be  $2\pi - \sum_i \gamma_i$ . This paper establishes, when the distances  $\|p p_i\|$  go to zero, that the angular defect is asymptotically equivalent to a homogeneous polynomial of degree two in the principal curvatures.

For regular meshes, we provide closed forms expressions for the three coefficients of this polynomial. We show that vertices of valence four and six are the only ones where  $k_G$  can be inferred from the angular defect. At other vertices, we show that the principal curvatures can be derived from the angular defects of two independent triangulations.

For irregular meshes, we show that the angular defect weighted by the so-called module of the mesh estimates  $k_G$  within an error bound depending upon  $k_m$  and  $k_M$ .

Since meshes are so ubiquitous in Computer Graphics and Computer Aided Design, and Discrete Differential operators are so necessary to their processing, we believe these contributions are one step forward the intelligence of the geometry of meshes, whence one step forward more robust algorithms.

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**Key-words:** Smooth Surfaces, Meshes, Curvatures, Approximations, Differential Geometry

## Sur le déficit angulaire des triangulations et l'approximation des courbures

**Résumé :** Soit  $S$  une surface lisse de  $E^3$ ,  $p$  un point de  $S$ ,  $k_m$ ,  $k_M$ ,  $k_G$  et  $k_H$  les courbures maximale, minimale, de Gauss et moyenne de  $S$  au point  $p$ . Soit un ensemble  $\{p_i p p_{i+1}\}_{i=1, \dots, n}$  de  $n$  triangles Euclidiens formant une approximation linéaire par morceaux de  $S$  en  $p$  — avec  $p_{n+1} = p_1$ . Pour chaque triangle, soit  $\gamma_i$  l'angle  $\angle p_i p p_{i+1}$ . Le déficit angulaire en  $p$  est défini par  $2\pi - \sum_i \gamma_i$ . L'objet de ce travail est de montrer que lorsque les longueurs d'arêtes  $\|p p_i\|$  tendent vers zéro, le déficit angulaire est asymptotiquement équivalent à un polynôme homogène de degré deux en les courbures principales.

Dans le cas de maillages réguliers, on établit une forme close pour les coefficients de ce polynôme. On montre que les sommets de valence quatre et six sont les seuls où  $k_G$  peut être estimée à partir du déficit angulaire. Pour les sommets dont la valence n'est ni quatre, ni six, on établit que les courbures principales peuvent être calculées à partir du déficit de deux maillages indépendants.

Pour les maillages irréguliers, on montre que l'écart entre le déficit angulaire (normalisé par le *module* du maillage) et  $k_G$  dépend de  $k_m$  et  $k_M$ .

Ce travail est une contribution à la géométrie des maillages et à sa relation aux surfaces lisses échantillonnées. Les maillages étant omniprésents en infographie au sens large, et leur traitement nécessitant des opérateurs différentiels discrets, cette compréhension va dans le sens d'algorithmes sinon plus rapides du moins plus robustes.

**Mots-clés :** Surfaces lisses, Triangulations, Courbures, Approximations, Géométrie Différentielle

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# 1 Introduction

## 1.1 Smooth and triangulated surfaces

Meshes are ubiquitous in modern computer-related geometry. Meshes are easily obtained from physical objects through scanning and reconstruction. Meshes are commonly displayed by graphical hardware. Meshes are intuitive to deal with. They provide hierarchical representations that can be used for approximate representations. Meshes can be refined to smooth surfaces through subdivision.

In this context, and especially since meshes can be made dense enough so as to “look like” smooth surfaces, it is tempting to define a differential geometry of meshes which mimics that of smooth surfaces.

Example quantities well defined smooth surfaces that also look appealing for meshes are the surface area, the normal vector field, the curvatures, geodesics, the focal sets, the ridges, the medial axis, etc.

Interestingly, recent research in applied domains provides, for each of the notions just enumerated, several estimates adapted from classical differential geometry to the setting of piecewise linear surfaces. Several definitions of normals, principal directions and curvatures over a mesh can be found in [Tau95], [DMSB]. Ridges of polyhedral surfaces as well as cuspidal edges of the focal sets are computed in [WB01]. Geodesics and discrete versions of the Gauss-Bonnet theorem are considered in [PS98]. But none of these contributions address the question of the accuracy of these estimates or that of their convergence when the mesh is refined.

As opposed to these approaches, the literature provides a few examples of *a priori* analysis. Given a discrete set —a point cloud or a mesh— which is assumed to sample a surface in a certain way, differential operators are derived together with theoretical guarantees about the discrepancy between the discrete estimates and the true value on the underlying smooth surface.

In [AB99] it is shown that the normal to a smooth surface sampled according to a criterion involving the skeleton can be estimated accurately from the Voronoi diagram of the sample points. The surface area of a mesh and its normal vector field versus those of a smooth surface are considered in [MT01]. More closely related to the question we address is [MW00], which provides some error bounds for estimates of the normal and the Gauss curvature of a sampled surface.

## 1.2 Smooth surfaces, polyhedra, Gauss curvature and angular defect

In this section, we hi-light striking parallels between the Gauss curvature of polyhedra and smooth surfaces of  $E^3$ . More precisely, we recall:



1. the definition of the Gauss curvature for smooth surfaces and polyhedra, as well as the Gauss-Bonnet theorem in both cases.
2. how the Gauss curvature of a smooth surface can be recovered from the angular defect of geodesic triangles.

**Gauss curvature and the Gauss-Bonnet theorem for surfaces and polyhedra.**

The Gauss curvature of an (abstract) oriented smooth Riemannian surface  $M$  is a smooth function  $k_G$  on  $M$  defined by using the metric tensor. It is well known that the Gauss curvature of a domain is identically equal to zero if and only if it is locally isometric to a portion of plane. One can associate to  $k_G$  a curvature measure  $K_G$  on  $M$  by integration over any domain  $U$  of  $M$ :

$$K_G(U) = \int_U k_G da,$$

where  $da$  denotes the area form of  $M$ . Suppose now that  $M$  is isometrically embedded in  $E^3$ . One way to evaluate its Gauss curvature at a point  $p$  is to make the product of the two principal curvatures of  $M$  at  $p$ , (this is nothing but *theorema egregium* of Gauss). An equivalent way it to use the Gauss map of the embedding and to calculate the limit  $A'/A$ , where  $A$  is the surface area of a region around  $p$ ,  $A'$  the signed area of the image of  $A$  on  $S^2$  by the Gauss map, the limit being taken (*roughly speaking*) as the region  $A$  around  $p$  becomes smaller and smaller [Spi99, Vol. 2, Chapter III].

Consider now an abstract Riemannian polyhedron  $P$ . Any point of  $P$  which is not a vertex has a neighborhood isometric to a plane, and one can define its Gauss curvature to be 0, by analogy with the smooth case. Moreover, if  $p$  is a vertex of  $P$ , one can assign to  $p$  the angular defect  $\alpha_p = 2\pi - \sum_i \gamma_i$  at  $p$ , where the  $\gamma_i$ s stand for the angles at  $p$  of the facets incident to  $p$ . We call this angular defect *the Gauss curvature  $k_G(p)$  of the vertex  $p$* , [Res93, Section 5]. Remark that it is now possible to define a curvature measure on  $P$ , as a measure concentrated on the vertices of  $P$ , by setting over any domain  $U$  of  $P$

$$K_G(U) = \sum_{p \text{ vertex in } U} \alpha(p).$$

If  $P$  is isometrically embedded in  $E^3$ , then this angular defect is exactly the signed area of the image on  $S^2$  by the Gauss map of any arbitrarily small neighborhood of  $p$ , which is a property analogous to the smooth case.

An important relationship relating the Gauss curvature to topological properties is the Gauss-Bonnet theorem. Consider a closed orientable surface  $S$  and a closed polyhedron  $P$ , the latter with vertices  $p_1, \dots, p_n$ . Let  $\chi(S)$  and  $\chi(P)$  stand for their Euler characteristic—that is  $V - E + F$  in the usual jargon. The global Gauss-Bonnet theorem respectively states for  $S$  and  $P$  that

$$2\pi\chi(S) = \int \int_S k_G d\sigma, \tag{1}$$

$$2\pi\chi(P) = \sum_{i=1,\dots,n} k_G(p_i). \quad (2)$$

These remarkable results actually state that if the topology is fixed —i.e. the genus of the surface or the polyhedron is given, the curvature distributes itself on  $S$  or  $P$  so as to comply with that topology. Notice that local versions of both theorems exist. (For the geodesic curvature involved in the local Gauss-Bonnet theorem, see [Res93, PS98].)

Remark that in this context, the Gauss curvature of a point of a polyhedron is dimensionless, while that of a surface is homogeneous to the inverse of a surface area. The topological invariant  $2\pi\chi$  is dimensionless, which is coherent with equations (1) and (2).

### Gauss curvature and geodesic triangles.

An important property of geodesic triangles also which involves the Gauss curvature is the following —see [CMS84] and [Laf86].

Let  $\tau_i$  be a geodesic triangle on  $S$ ,  $p$ ,  $p_i$  and  $p_{i+1}$  its vertices,  $l_p, l_i, l_{i+1}$  the lengths of the geodesic arcs opposite to the vertices, and  $l = \sup\{l_i, l_{i+1}\}$ . Let  $T_i$  be the Euclidean triangle whose edges have the same lengths as those of  $\tau_i$ . We call  $T_i$  a *Euclidean geodesic triangle* since it is a Euclidean triangle, but its edges' lengths are geodesic distances. Finally, let  $\beta_i$  be the angle of  $\tau_i$  at  $p$ , and  $\alpha_i$  be the angle  $\angle p_i p p_{i+1}$  of  $T_i$ . See Figure 1 for an illustration. The angles  $\beta_i$  and  $\alpha_i$  differ by a term involving the Gauss curvature  $k_G$  at  $p$ . More precisely, we have

**Proposition. 1** *The angles  $\beta_i$  and  $\alpha_i$  associated to a Euclidean geodesic triangle  $\tau_i$  satisfy*

$$\beta_i = \alpha_i + \frac{1}{6} \sin \alpha_i k_G l_i l_{i+1} + o(l^2). \quad (3)$$

Consider now a geodesic triangulation around  $p$ , that is a set of  $n$  geodesic triangles having  $p$  as common vertex and forming a topological disk around  $p$ . By looking at the tangents to the geodesics from  $p$  to the  $p_i$ s, we have  $\sum_i \beta_i = 2\pi$ . Summing Equation (3) for the  $n$  geodesic triangles immediately yields the following

**Theorem. 1** *Let  $\mathcal{T}$  be a geodesic triangulation of a smooth surface  $M$  of  $E^3$ . Let  $p$  be a vertex of  $\mathcal{T}$ , and let  $A(p)$  be the sum of the areas of the triangles  $T_i$  associated to the Euclidean geodesic triangles  $\tau_i$ s. Then*

$$2\pi - \sum_i \alpha_i = \frac{A(p)}{3} k_G + o(l^2).$$

This result is of little help from a practical standpoint since the knowledge of geodesics is required. The question addressed in this paper is actually to study the quantity estimated by the angular defect when one replaces the geodesics by the Euclidean line-segments  $pp_i$ .

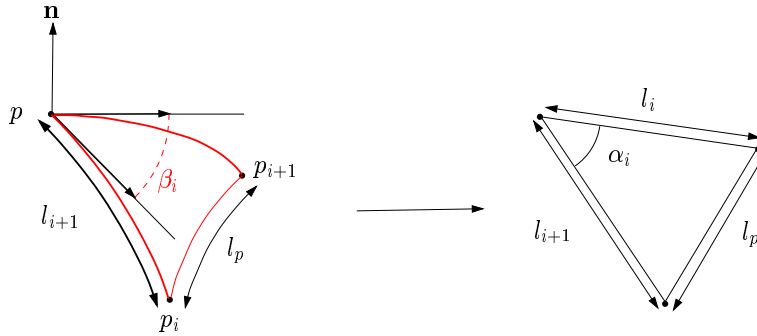


Figure 1: Flattening a geodesic triangle yields an estimate for the Gauss curvature

### 1.3 Question addressed in this paper

Let  $S$  be a surface of  $E^3$ , and let  $p$  be a point of  $S$ . Also suppose that we are given a set  $\{p_i p p_{i+1}\}_{i=1, \dots, n}$  of  $n$  Euclidean triangles forming a piecewise linear approximation of  $S$  around  $p$ . We shall refer to these triangles as the mesh, and to the  $p_i$ s as the one-ring neighbors of  $p$ . For each triangle, let  $\gamma_i$  be the angle  $\angle p_i p p_{i+1}$ , and let the angular defect at  $p$  be  $2\pi - \sum_i \gamma_i$ . Also, for a one-ring neighbor  $p_i$ , let  $\eta_i$  stand for the Euclidean distance between  $p$  and  $p_i$ .

The question we address in this paper is:

How precisely can one estimate the curvatures at a point  $p$  of a smooth surface using the angular defect of the triangles surrounding  $p$ ?

Before presenting the contributions, several comments are in order.

**Dimensionality.** In order for the previous question to make sense, we shall pay a special attention to the dimensionality of the quantities involved. As already pointed out, the Gauss curvature of a smooth surface is homogeneous to the inverse of a surface area while that of a polyhedron is dimensionless. To estimate the curvatures of a smooth surface from a polyhedron, we will therefore have to normalize by lengths (for the principal curvatures) or surface areas (Gauss curvature).

**Smooth surfaces and asymptotic estimates.** Estimating the curvatures of a smooth surface  $S$  from a single mesh is obviously an hopeless target. The folds of  $S$  may indeed occur at a resolution much lower than that of the triangulation, so that the mesh may fall short from providing accurate information on the point-wise curvatures. The problem becomes more tractable if one assumes  $S$  belongs to a restricted class of surfaces —e.g. with Lipchitz like conditions on the variation of the normal— or assumes that a sequence of meshes with edges' lengths going to zero is available. We shall in the paper follow this latter perspective and derive asymptotic results.

### Soundness of the angular defect for the Gauss curvature of smooth surfaces.

Since the angular defect over geodesic triangles provides an estimate of  $k_G$ , it is tempting to believe that when the edges lengths  $\|pp_i\|$  go to zero, one can safely replace the geodesic arcs from  $p$  to the  $p_i$ s on  $S$  by the Euclidean segments  $pp_i$ s. We shall see it is not so.

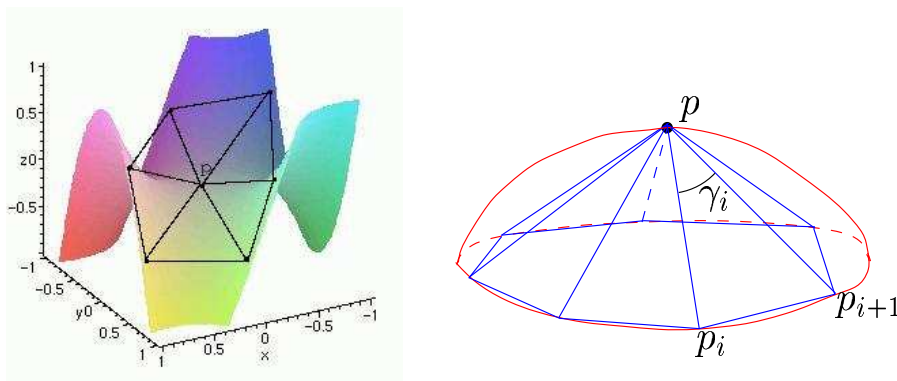


Figure 2: Can the curvatures of a smooth surface be estimated from the angular defect of a triangulation?

## 1.4 Contributions

This paper establishes, when the distances  $\|pp_i\|$  go to zero, that the angular defect is asymptotically equivalent to a homogeneous polynomial of degree two in the principal curvatures. To state the results more precisely, one needs to distinguish between regular and irregular meshes. By regular mesh, we refer to a mesh such that the  $p_i$ s lie in normal sections two consecutive of which form an angle of  $2\pi/n$ , with the additional constraint that  $\|pp_i\|$  is a constant. A mesh which is not regular is called irregular.

**Regular meshes.** We provide the closed form expression of the afore-mentioned polynomial as a function of the principal curvatures and  $2\pi/n$ . In particular, we show that  $n = 4$  is the only value of  $n$  such that  $2\pi - \sum_i \gamma_i$  depends upon the principal directions, and that  $n = 6$  is the only value such that  $2\pi - \sum_i \gamma_i$  provides an exact estimate for  $k_G$ . A corollary of these results is that the principal curvatures —whence  $k_G$  and  $k_H$ — can be computed from the angular defects of any two triangulations whose valences are not four.

**Irregular meshes.** We show that the angular defect weighted by the so-called module of the mesh estimates  $k_G$  within an error bound depending upon  $k_m$  and  $k_M$ .

From a practical standpoint, these results are of prime importance since the angular defect is widely used to estimate the Gauss curvature of triangulated surfaces —see for example [Cal86, DMSB, CW00, DHKL01].

Finally, it should be emphasized that along the derivation of these results, we prove several approximation lemmas for curves and surfaces. These results may find applications in surface meshing, surface subdivision, feature extraction, . . .

## 1.5 Paper overview

The paper is organized as follows. Section 2 provides the notations used throughout the paper. In section 3 we present approximation results for the curvature of plane curves. These results are used in section 4 to derive a formula on Euclidean triangles providing a piecewise linear approximation of a smooth surface. Using restricted hypothesis on the geometry of these triangles, we show in section 5 that the angular defect does not provide, in general, an estimate for  $k_G$ . In section 6, the hypothesis of section 5 are alleviated and a general result about the accuracy of the angular defect is proved.

## 2 Notations

Consider a point  $p$  of a smooth surface together with  $n$  Euclidean triangles  $\{p_i p p_{i+1}\}_{i=1, \dots, n}$  —with  $p_{n+1} = p_1$ — forming a piecewise linear approximation of  $S$  around  $p$ . The following notations are used throughout the paper —see Figure 3:

**Normal sections;  $\Pi_i, \varphi_i$ .** Consider the plane  $\Pi_i$  containing the normal  $\mathbf{n}$ ,  $p$  and  $p_i$ . We assume  $\Pi_i$  is defined by its angle  $\varphi_i$  with respect to some coordinate system in the tangent plane —for example the one associated with the principal directions.

**Distance to one-ring neighbor;  $\eta_i$ .** The Euclidean distance from  $p$  to its  $i$ th neighbor  $p_i$  is denoted  $\eta_i$ .

**Angle between normal sections;  $\beta_i$ .** The angle between two consecutive normal sections  $\Pi_i$  and  $\Pi_{i+1}$  is denoted  $\beta_i$ , that is  $\beta_i = \varphi_{i+1} - \varphi_i$ . Put differently,  $\beta_i$  measures, in the tangent plane, the angle between the tangents to the plane curves  $S \cap \Pi_i$  and  $S \cap \Pi_{i+1}$ .

**Polyhedral angle;  $\gamma_i$**  Consider the Euclidean triangle  $pp_i p_{i+1}$ . The angle at  $p$  i.e.  $\angle p_i p p_{i+1}$  is denoted  $\gamma_i$ . (Notice that the angle  $\gamma_i$  is different from the angle  $\alpha_i$  of proposition 1.)

**Directional curvatures;  $\lambda_i$ .** We let  $\lambda_i$  stand for the curvature of the plane curve  $S \cap \Pi_i$ . Notice that if  $\Pi_i$  contains a principal direction,  $\lambda_i$  reduces to the corresponding principal curvature  $k_M$  or  $k_m$ .

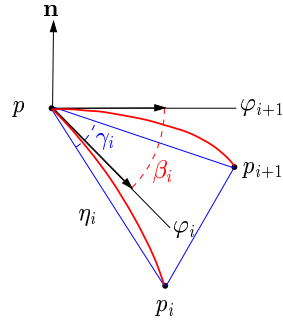
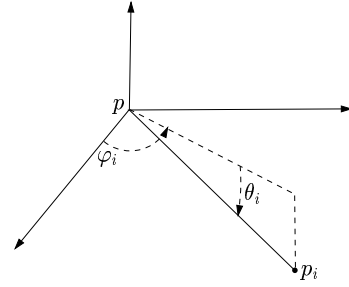


Figure 3: Notations

Figure 4:  $p_i$  in spherical coordinates

### 3 Plane curves

In this section, we provide an estimate for the curvature of a plane curve from the angular defect of an inscribed polygon.

#### 3.1 A lemma on plane curves

Let  $C$  be a  $C^\infty$  smooth regular curve. Let  $p_0$  be a point of  $C$ . It is well known that  $C$  can locally be represented by the graph  $(x, f(x))$  of a smooth function  $f$ , such that  $p_0 = (0, 0)$ , (i.e.  $f(0) = 0$ ), and such that the tangent to  $C$  at  $p_0$  is aligned with the  $x$ -axis, (i.e.  $f'(0) = 0$ ). Let  $k$  be the curvature of  $C$  at the origin (i.e.  $k = f''(0)$ ), and let  $\nu = f'''(0)$ . Near the origin we have

$$f(x) = \frac{kx^2}{2} + \frac{\nu x^3}{6} + o(x^3). \quad (4)$$

Let us now use polar coordinates:  $x = \eta \cos \theta$  and  $y = \eta \sin \theta$ . In order to approximate the curvature of  $C$  at  $p_0$ , we shall need an expression of  $\theta$  as a function of  $\eta$ . Obtaining such an expression involves the implicit function theorem, and the reader is referred to section 8 for the proof of the following

**Lemma. 1** *Let  $f(x)$  be a  $C^\infty$  smooth regular function with  $k = f''(0)$  and  $\nu = f'''(0)$ . For a point  $p = (\eta \cos \theta, \eta \sin \theta)$  on the graph of  $f$  near the origin, one has:*

$$\theta = \frac{k\eta}{2} + \frac{\nu\eta^2}{6} + o(\eta^2) \text{ if } x \geq 0, \quad (5)$$

$$\theta = \frac{k\eta}{2} - \frac{\nu\eta^2}{6} + o(\eta^2) \text{ if } x < 0. \quad (6)$$

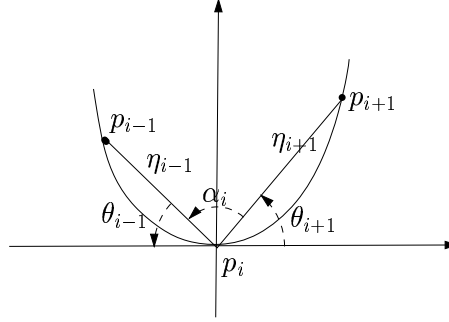


Figure 5: Smooth curve and inscribed polygon

### 3.2 Approximating the curvature of a plane curve

The previous lemma can be used to estimate the curvature of a curve from the angular defect  $\alpha_i$  of an inscribed polygon. See Figure 5 for the notations.

**Theorem. 2** *Let  $p_{i-1}$ ,  $p_i$  and  $p_{i+1}$  be three points as indicated on Figure 5, with  $\eta_{i-1}$  ( $\eta_{i+1}$ ) the distance from  $p_i$  to  $p_{i-1}$  ( $p_{i+1}$ ). Also let  $\bar{\eta}_i = (\eta_{i-1} + \eta_{i+1})/2$ . The angular defect  $\alpha_i$  at  $p_i$  and the curvature  $k$  satisfy:*

- if  $\eta_{i-1} = \eta_{i+1} = \eta$ :

$$\frac{\pi - \alpha_i}{\eta} = k + o(\eta), \quad (7)$$

- if  $\eta_{i-1} \neq \eta_{i+1}$ :

$$\frac{\pi - \alpha_i}{\bar{\eta}_i} = k + o(1). \quad (8)$$

*Proof.* Since the proofs of the two statements are similar, we focus just on the first one.

Equations (5) and (6) applied to  $p_{i+1}$  and  $p_{i-1}$  yield  $\theta_{i-1} + \theta_{i+1} = k\eta + o(\eta^2)$ . But we also have  $\theta_{i-1} + \theta_{i+1} = \pi - \alpha_i$ , whence the result.  $\square$

Interestingly, the speed of convergence is faster when the two neighbors are located at the same distance from  $p_i$ .

**Remark.** The previous Theorem can be extended to space curves.

## 4 A lemma on normal sections and Euclidean triangles

This section is devoted to a general result involving Euclidean triangles and smooth surfaces. This result is the cornerstone of the next two sections.

Using the notations of section 2, we aim at finding a dependence relationship between  $\beta_i$ ,  $\gamma_i$ ,  $\eta_i$  and  $\eta_{i+1}$ . More precisely, we shall consider that the normal sections  $\Pi_i$ s are fixed—which determines the  $\varphi_i$ s,  $\beta_i$ s and  $\lambda_i$ s—and study the dependence between  $\gamma_i$ ,  $\eta_i$  and  $\eta_{i+1}$ .

**Lemma. 2** *With the above notations, let  $\eta = \max(\eta_i, \eta_{i+1})$ . Define the following sum and product functions*

$$s(\eta_i, \eta_{i+1}) = \frac{\lambda_i^2 \eta_i^2 + \lambda_{i+1}^2 \eta_{i+1}^2}{8}, \quad p(\eta_i, \eta_{i+1}) = \frac{\lambda_i \lambda_{i+1} \eta_i \eta_{i+1}}{4}.$$

The  $\beta_i$ ,  $\gamma_i$ ,  $\eta_i$  and  $\eta_{i+1}$  quantities satisfy

$$\beta_i = \gamma_i + \frac{p(\eta_i, \eta_{i+1})}{\sin \gamma_i} - s(\eta_i, \eta_{i+1}) \cot \gamma_i + o(\eta^2) \quad (9)$$

and

$$\beta_i = \gamma_i + \frac{p(\eta_i, \eta_{i+1})}{\sin \beta_i} - s(\eta_i, \eta_{i+1}) \cot \beta_i + o(\eta^2). \quad (10)$$

*Proof.* The proofs of the two claims following the same guideline, we just prove the second one. Assume point  $p$  is at the origin. Let us write the coordinates of  $p_i$  in spherical form with the conventions of Figure 4—that is  $\varphi$  measures an angle in the tangent plane of  $S$  at  $p$ . If  $X^t$  stands for the transpose of  $X$ , the coordinates of  $p_i$  are  $p_i = (\eta_i \cos \theta_i \cos \varphi_i, \eta_i \cos \theta_i \sin \varphi_i, \eta_i \sin \theta_i)^t$ , and similarly for  $p_{i+1}$ .

Since  $\beta_i = \varphi_{i+1} - \varphi_i$ , expressing the dot product  $\mathbf{p}p_i \cdot \mathbf{p}p_{i+1} = \eta_i \eta_{i+1} \cos \gamma_i$  in spherical coordinates yields

$$\cos \gamma_i = \cos \beta_i \cos \theta_i \cos \theta_{i+1} + \sin \theta_i \sin \theta_{i+1}. \quad (11)$$

Since the curve  $\Pi_i \cap S$  is a plane curve, using Equation (5) which expresses  $\theta$  as a function of  $\eta$ , we have

$$\cos \theta_i = 1 - \frac{\lambda_i^2 \eta_i^2}{8} + o(\eta_i^2), \quad \sin \theta_i = \frac{\lambda_i \eta_i}{2} + o(\eta_i).$$

Plugging these values into Equation (11) yields

$$\cos \gamma_i = (1 - s(\eta_i, \eta_{i+1})) \cos \beta_i + p(\eta_i, \eta_{i+1}) + o(\eta^2).$$

To turn the previous expression into a relationship between  $\gamma_i$  and  $\beta_i$ , we use the Taylor formula of order one to  $f(x) = \arccos(x)$  together with  $a = \cos \beta_i$  and  $b = \cos \gamma_i$ :

$$\gamma_i = \beta_i + \frac{-1}{\sin \beta_i} (-s(\eta_i, \eta_{i+1}) \cos \beta_i + p(\eta_i, \eta_{i+1}) + o(\eta^2)) + o(\eta^2).$$

Re-arranging the terms completes the proof.  $\square$



## 5 Surfaces and regular polygons

Using the lemma proved in the previous section, we are now ready to analyze the angular defect for regular meshes. We shall need the following definitions.

**Definition. 1** Let  $p$  be a point of a smooth surface  $S$  and let  $p_i, i = 1, \dots, n$  be its one ring neighbors. Point  $p$  is called a regular vertex if (i) the  $p_i$ s lie in normal sections two consecutive of which form an angle of  $\theta(n) = 2\pi/n$  (ii) the  $\eta_i$ s all take the same value  $\eta$ .

**Definition. 2** Consider the directions of maximum and minimum curvatures of  $S$  at  $p$ . Assume these directions are associated two vectors  $\mathbf{v}_M$  and  $\mathbf{v}_m$  such that  $\mathbf{v}_M \wedge \mathbf{v}_m = \mathbf{n}$  — with  $\mathbf{n}$  be the normal of  $S$  at  $p$ . The offset angle  $a$  is defined as the angle in  $[0, 2\pi[$  between the vectors  $\mathbf{v}_M$  and  $\mathbf{p}\pi(\mathbf{p}_1)$  with  $\pi(\mathbf{p}_1)$  the projection of  $p_1$  in the tangent plane.

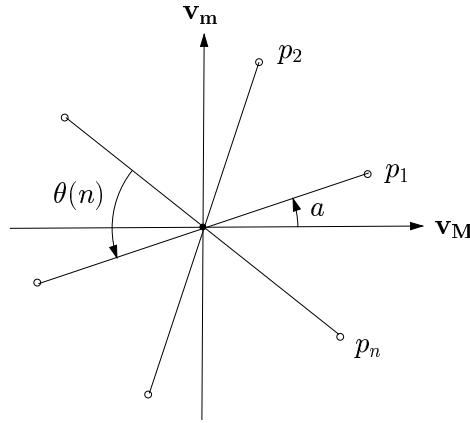


Figure 6: Regular triangulation around  $p$ : tangent plane seen from above

Let us now get back to the notations of section 2 at a regular vertex. Angle  $a$  is the angle from  $\mathbf{v}_M$  to the normal section of the first normal section. For a regular vertex, the  $\beta_i$ s are constant and equal to  $\theta(n)$ , so that  $\varphi_i = a + (i - 1)\theta(n)$ . Under these hypothesis, we provide a closed form expression for the angular defect. The reader is referred to the appendix —section 9— for the proof of the following Theorem.

**Theorem. 3** Consider a regular vertex of valence  $n$ . The following holds:

1. There exists two functions  $A(a, n)$  and  $B(a, n)$  such that

$$2\pi - \sum_{i=1}^n \gamma_i = [A(a, n) k_G + B(a, n) (k_M^2 + k_m^2)] \eta^2 + o(\eta^2). \quad (12)$$

2. The only value of  $n$  such that the functions  $A(a, n)$  and  $B(a, n)$  depend upon  $a$  is  $n = 4$ , and then

$$2\pi - \sum_{i=1}^n \gamma_i = [(1 - 2 \cos^2 a \sin^2 a) k_G + \cos^2 a \sin^2 a (k_M^2 + k_m^2)] \eta^2 + o(\eta^2). \quad (13)$$

3. If  $n \neq 4$ :

$$2\pi - \sum_{i=1}^n \gamma_i = \frac{n}{16 \sin 2\pi/n} \left[ \left( 2 - \cos \frac{4\pi}{n} - \cos \frac{2\pi}{n} \right) k_G + \left( 1 + \frac{1}{2} \cos \frac{4\pi}{n} - \frac{3}{2} \cos \frac{2\pi}{n} \right) (k_M^2 + k_m^2) \right] \eta^2 + o(\eta^2). \quad (14)$$

In particular, the only value of  $n$  such that  $B(a, n) = 0$  is  $n = 6$ , and then  $A(a, 6) = \sqrt{3}/2$ , that is

$$2\pi - \sum_i \gamma_i = \frac{\sqrt{3}}{2} k_G \eta^2 + o(\eta^2). \quad (15)$$

The graphs of the functions  $A(a, 4)$  and  $B(a, 4)$ , as well as  $A(a, n)$  and  $B(a, n)$  for  $n \neq 4$  are presented on Figures 7 and 8. For the angular defect to provide an estimate for  $k_G$ , one needs  $B(a, n)$  to vanish, which only happens for  $n = 6$  whatever the angle  $a$  is, and for  $n = 4$  when the one-ring neighbors are aligned with the principal directions.

Interestingly, although the Gauss curvature is not estimated by the angular defect, the principal curvatures can be recovered from two different meshes:

**Corollary. 1** *The principal curvatures of a smooth surface can be computed from the angular defects of any two meshes of valences  $n_1$  and  $n_2$  such that  $n_1 \neq 4, n_2 \neq 4, n_1 \neq n_2$ .*

*Proof.* Assume we are given two meshes with valences  $n_1$  and  $n_2$  different from four. Let  $E(n_i)$  be the limit value of  $(2\pi - \sum_i \gamma_i)/\eta^2$ , and  $A(n_i), B(n_i)$  be the coefficients of  $k_G$  and  $k_m^2 + k_M^2$  in Equation (12) have:

$$\begin{cases} E(n_1) = A(n_1)k_G + (k_m^2 + k_M^2)B(n_1) \\ E(n_2) = A(n_2)k_G + (k_m^2 + k_M^2)B(n_2) \end{cases}$$

Since  $n_1 \neq n_2 \Rightarrow A(n_1)B(n_2) - A(n_2)B(n_1) \neq 0$ , we have

$$k_G = \frac{E(n_1)B(n_2) - E(n_2)B(n_1)}{A(n_1)B(n_2) - A(n_2)B(n_1)}. \quad (16)$$

If  $E$  refers to one of the  $E(n_i)$  and similarly for  $A$  and  $B$ , once the Gauss curvature is known, we are left with the system

$$\begin{cases} k_m k_M = k_G \\ (k_m^2 + k_M^2)B = E - Ak_G. \end{cases}$$

Letting  $D = E - Ak_G$  and observing that  $B \neq 0$  since  $n \neq 4$  yields

$$B k_m^4 - D k_m^2 + B k_G^2 = 0, \quad (17)$$

which solves to

$$k_m^2 = \frac{E - Ak_G - \sqrt{(E - Ak_G)^2 - 4B^2 k_G^2}}{2B}. \quad (18)$$

Notice that the sign of the square root chosen in the previous expression does not matter since the two solutions corresponding to  $\pm\sqrt{\phantom{x}}$  are actually conjugated with respect to  $k_m k_M = k_G$ , that is

$$k_M^2 = \frac{k_G^2}{k_m^2} = \frac{E - Ak_G + \sqrt{(E - Ak_G)^2 - 4B^2 k_G^2}}{2B}. \quad (19)$$

Once  $k_m^2$  and  $k_M^2$  have been computed,  $k_m$  and  $k_M$  are determined observing that the product of their signs is the sign of  $k_G$ , and that  $k_m \leq k_M$ . Once  $k_m$  and  $k_M$  are known, the mean curvature  $k_H$  is  $k_H = (k_m + k_M)/2$ .  $\square$

In the same spirit, we also have:

**Corollary. 2** *Umbilics can be detected from the two angular defects without computing the principal curvatures.*

*Proof.* At an umbilic point, we have  $k_m = k_M$ . Using the notations of the previous proof, it is easily checked that  $(E - Ak_G)^2 - 4B^2 k_G^2 = 0$  and  $\text{sign}(k_G) \geq 0$ .  $\square$

Since two meshes are enough to infer the principal curvatures, it is tempting to infer the position of the principal directions using the valence four mesh and equation (13). This equation yields the value of  $\cos^2 a \sin^2 a$ , from which one is unable to distinguish between  $a$  and  $2\pi - a$ .

**Rmk.** It is important to notice that corollary 1 compares the Gauss curvature and the squares of the principal curvatures of  $S$  against the angular defect normalized by  $\eta^2$ —see section 1.2 for a discussion of the dimensionality issues.

**Rmk.** In section 1.2 we recalled the Gauss-Bonnet theorem for a polyhedra  $P$  and a compact orientable surface  $S$ . Assume  $P$  and  $S$  are homeomorphic. An interesting question in the global convergence of the sum of the angular defects over  $P$  when  $P$  is refined so as to converge to  $S$ —in the Hausdorff sense. This question is especially interesting since the angular defect usually involves the squares of the principal curvatures. See [Fu93] for similar concerns.

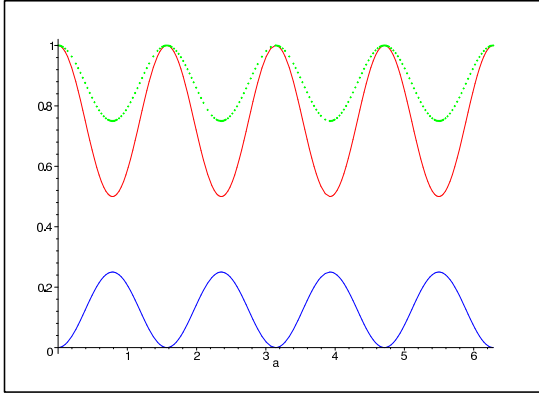


Figure 7: The coefficients  $A(a,4)$ ,  $B(a,4)$  and  $A(a,4) + B(a,4)$  — respectively (solid curve, top), (solid curve, bottom), (dotted curve)

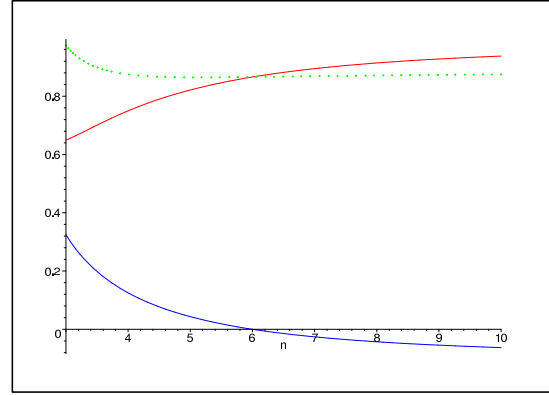


Figure 8: The coefficients  $A(a,n)$ ,  $B(a,n)$  and  $A(a,n) + B(a,n)$  — respectively (solid curve, top), (solid curve, bottom), (dotted curve)

## 6 Surfaces: the general case

This section generalizes the analysis carried out in the previous section. We relax the hypothesis on the edges' lengths  $\eta_i$ s as well as on the angles  $\varphi_i$ s, and provide an expression of the angular defect as a Taylor expansion in the edges' lengths.

### 6.1 Angular defect and $k_G$

We shall need the following definition:

**Definition. 3** Consider the one-ring neighbors of  $p$ . For the sake of conciseness, let  $c_i = \cos \varphi_i$ ,  $s_i = \sin \varphi_i$ , and define the following quantities:

$$A_i = \frac{1}{4 \sin \gamma_i} [\eta_i \eta_{i+1} (c_i^2 s_{i+1}^2 + s_i^2 c_{i+1}^2) - \frac{\cos \gamma_i}{2} (\eta_i^2 (2c_i^2 s_i^2) + \eta_{i+1}^2 (2c_{i+1}^2 s_{i+1}^2))], \quad (20)$$

$$B_i = \frac{1}{4 \sin \gamma_i} [\eta_i \eta_{i+1} (c_i^2 c_{i+1}^2) - \frac{\cos \gamma_i}{2} (\eta_i^2 c_i^4 + \eta_{i+1}^2 c_{i+1}^4)], \quad (21)$$

$$C_i = \frac{1}{4 \sin \gamma_i} [\eta_i \eta_{i+1} (s_i^2 s_{i+1}^2) - \frac{\cos \gamma_i}{2} (\eta_i^2 s_i^4 + \eta_{i+1}^2 s_{i+1}^4)], \quad (22)$$

$$S_p = A + B + C \text{ with } A = \sum_i A_i, \quad B = \sum_i B_i, \quad C = \sum_i C_i. \quad (23)$$

The quantity  $S_p$  is called the module of the mesh at  $p$ .

The first lemma we shall need is about the independence of the module wrt the positions of the normal sections:

**Lemma. 3** *The module of the mesh at  $p$  is independent from the angles  $\varphi_1, \dots, \varphi_n$ .*

*Proof.* Simply observe that

$$A_i + B_i + C_i = \frac{1}{4 \sin \gamma_i} \left[ \eta_i \eta_{i+1} - \frac{\cos \gamma_i}{2} (\eta_i^2 + \eta_{i+1}^2) \right].$$

□

The second lemma states, as in the regular case, that the angular defect is a homogeneous polynomial of degree two in the principal curvatures:

**Lemma. 4** *Let  $\eta = \sup_i \eta_i$ . The angular defect and the principal curvatures satisfy*

$$2\pi - \sum_{i=1}^n \gamma_i = (A k_G + B k_M^2 + C k_m^2) + o(\eta^2) \quad (24)$$

*Proof.* We express the directional curvature  $\lambda_i$  using Euler's relation. Plugging the values of  $\lambda_i$  and  $\lambda_{i+1}$  into Equation (9), summing over the  $n$  one-ring neighbors and grouping terms in  $k_G$ ,  $k_M^2$  and  $k_m^2$  yields Equation (24). □

The main result, at last, provides an upper bound for the discrepancy between the normalized angular defect and the Gauss curvature:

**Theorem. 4** *Let  $T_m$  be a sequence of meshes on a surface having  $p$  as a common vertex. Consider the one-ring around  $p$ . Let  $\bar{\eta}_m = \sup_i \eta_{m_i}$ ,  $\underline{\eta}_m = \inf_i \eta_{m_i}$ . Suppose that*

1. *there exist two positive constants  $\gamma_{min}, \gamma_{max}$  such that  $\forall i, \forall m, 0 < \gamma_{min} \leq \gamma_i \leq \gamma_{max}$ .*
2. *there exist two positive constants  $\eta_1, \eta_2$  such that  $\forall m, \eta_1 \leq \bar{\eta}_m / \underline{\eta}_m \leq \eta_2$ .*

*Then, there exists a positive constant  $C$  such that*

$$\limsup_m \left| \frac{2\pi - \sum_i \gamma_{i_m}}{S_{p_m}} - k_G \right| \leq \frac{nC}{2 \sin \gamma_{min}} [(k_M - k_m)^2 + |k_M^2 - k_m^2|] \quad (25)$$

*Proof.* Let us consider a particular mesh in the sequence, and for the sake of clarity, let us omit its index  $m$ . We have  $B_i \leq \eta^2 / (2 \sin \gamma_i)$  whence  $B \leq n\eta^2 / (2 \sin \gamma_i)$ . The same inequalities hold for  $C_i$  and  $C$ . From Equation (24), we get

$$\begin{aligned} |2\pi - \sum_i \gamma_i - (A + B + C)k_G| &= |B(k_M^2 - k_m^2) + C(k_m^2 - k_G)| + o(\eta^2) \\ &= \left| \frac{B+C}{2}(k_M - k_m)^2 + \frac{B-C}{2}(k_M^2 - k_m^2) \right| + o(\eta^2) \\ &\leq \frac{B+C}{2}(k_M - k_m)^2 + \left| \frac{B-C}{2} \right| |k_M^2 - k_m^2| + o(\eta^2) \end{aligned} \quad (26)$$

Using the upper bounds on  $B$  and  $C$  and since  $S_p$  does not depend on the angles  $\varphi_1, \dots, \varphi_n$ , we get:

$$\left| \frac{2\pi - \sum_i \gamma_i}{S_p} - k_G \right| \leq \frac{\eta^2}{|S_p|} \frac{n}{2 \sin \gamma_{min}} ((k_M - k_m)^2 + |k_M^2 - k_m^2|) + o(1). \quad (27)$$

Let us now get back to the sequence of meshes i.e. consider that the previous equation is indexed by  $m$  —that is  $\eta = \bar{\eta}_m$ ,  $\gamma_i = \gamma_{m_i}$ ,  $S_p = S_{p,m}$ . The assumptions on the  $\gamma_{m_i}$  angles and those on  $\bar{\eta}_m/\eta_m$  imply that there exists a constant  $\mathcal{C}$  such that  $\eta^2/|S_p| \leq \mathcal{C}$ , whence the result.  $\square$

The previous theorem deserves several comments:

- Theorem 4 refines the  $O(1)$  upper bound provided in [MW00].
- The lim sup accounts for the fact that the limit of the discrepancy between the normalized angular defect and  $k_G$  may not exist. The hypothesis used make it bounded, but one may have a sequence of alternating triangulations (e.g. indexed by odd and even integers) with different properties.
- Theorem 4 shows that the error incurred when approximating the Gauss curvature by  $(2\pi - \sum_i \gamma_i)/S_p$  depends upon  $k_M$  and  $k_m$ . In particular, this error is minimum if  $k_M = k_m$  if  $k_G > 0$  —or  $k_M = -k_m$  if  $k_G < 0$ . Although the Gauss curvature is intrinsic i.e. invariant upon isometric transformations of the surface, the accuracy of the estimate depends upon the particular embedding considered.
- In using geodesic triangulations to estimate  $k_G$ , the natural quantity to divide the angular defect by is the area of the triangles surrounding  $p$ . As the previous analysis shows, using Euclidean triangles induces the module of the mesh rather than its area. (Notice however that in both cases, the denominator is homogeneous to a surface area.)
- It should be observed that the error term may vanish under very special circumstances. We have encountered two of them in section 5, namely for a valence six triangulation, or a valence four triangulation with  $a = 0 \pmod{\pi/2}$ . Other cancellations can certainly be obtained exploiting the independence of the  $\eta_i$ s and the  $\varphi_i$ s.

## 6.2 Angular defect and the second fundamental form: illustrations

We proceed with a couple of examples illustrating the relationship between the angular defect, the Gauss curvature and the second fundamental form.

**Example 1 .** Consider the monkey saddle of Figure 9, point  $p$  being at the origin. Using a triangulation whose one-ring neighbors are located above or below the  $z = 0$  plane results in a positive angular defect —as if we were processing an elliptic vertex. Using a triangulation whose points are distributed on the two sides of the tangent plane results in a negative

angular defect —as if an hyperbolic vertex was processed. Does this mean that two different triangulations may yield Gauss curvatures with opposite signs?

Fortunately not! For a surface such as the Monkey saddle, the second fundamental form  $II_p$  is null so that the point is a planar point and any directional curvature around  $p$  is also null. No matter what triangulation is used, the angular defect will converge to zero which corresponds to a null Gauss curvature.

**Example 2.** Consider now the two surfaces of Figure 10 and assume they differ by a small “fold” located in-between two consecutive one-ring neighbors of the mesh. If  $II_p$  does not change —the fold affects third or higher order terms of the Monge form of  $S$  at  $p$ , neither the directional curvatures nor the angular defect are affected.

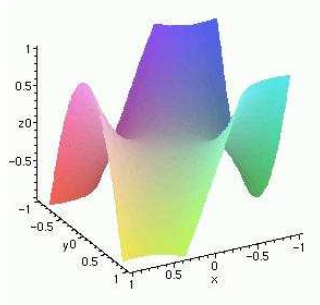


Figure 9: Monkey saddle

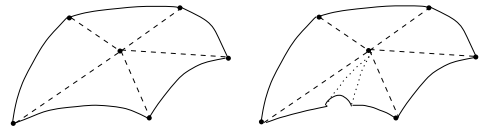


Figure 10: Local fold

## 7 Conclusion

Let  $S$  be a smooth surface,  $p$  a point of  $S$ , and consider a mesh providing a piecewise linear approximation of  $S$  around  $p$ . This paper establishes, asymptotically, several approximation results relating the curvatures of  $S$  at  $p$  and normalized angular defects of meshes at  $p$ . In particular, we show that the angular defect does not provide in general, an accurate point-wise estimate of the Gauss curvature.

From a practical standpoint, these results are especially interesting since the angular defect is a widely used estimate for the Gauss curvature of triangulated surfaces. From a theoretical perspective, we believe these contributions might find applications for the many operations involving differential operators on meshes, that is fairing, smoothing, as well as subdivision. A clear understanding of the geometry of meshes is certainly one step forward more robust algorithms.

On a broader perspective, these contributions illustrate the difficulties one has to face in order to perform differential geometry on non smooth objects. It would therefore be very interesting to generalize the analysis presented in this paper to all the methods —least-

square quadrics, gradient-based operators, etc— which are used to estimate the normal, mean curvature, principal directions, ridges, etc of triangulated surfaces.

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## 8 Appendix I: Proof of lemma 1

Using spherical coordinates ( $x = \eta \cos \theta$ ,  $y = \eta \sin \theta$ ) to express the position of a point  $p \in C$ , we get that  $C$  is implicitly represented by  $F(\eta, \theta) = 0$  with

$$F(\eta, \theta) = y - f(x) = \eta \sin \theta - f(\eta \cos \theta). \quad (28)$$

Obtaining an expression of  $\theta$  as a function of  $\eta$  involves the implicit function theorem applied to  $F$  at  $(\eta = 0, \theta = 0)$ . Unfortunately,  $\partial F / \partial \theta(0, 0) = 0$ . We get around the difficulty using an auxiliary function  $\Phi$  defined as follows:

**Lemma. 5** *Let  $\Phi(\eta, \theta)$  be defined by*

$$\eta \neq 0 : \Phi(\eta, \theta) = \frac{F(\eta, \theta)}{\eta} = \sin \theta - \frac{f(\eta \cos \theta)}{\eta}, \quad (29)$$

$$\eta = 0 : \Phi(0, \theta) = \sin \theta. \quad (30)$$

The function  $\Phi$  is  $C^2$ , and the point  $(\eta = 0, \theta = 0)$  is a regular point of  $\Phi$ . Moreover, near the origin

$$\theta = A\eta + B\eta^2 + o(\eta^2).$$

*Proof.* The proof consists of three parts.

**Part I.**  $\Phi(\eta, \theta)$  versus  $F(\eta, \theta)$ .

We first observe that working with  $F(\eta, \theta)$  or  $\Phi(\eta, \theta)$  is equivalent. When  $\eta \neq 0$ , the solutions of  $F = 0$  and  $\Phi = 0$  are the same. If  $\eta = 0$ , the only solution of  $\Phi = 0$  is  $(0, \theta = 0)$  which is also a solution of  $F = 0$ . To be more precise, any pair  $(0, \theta)$  is a solution of  $F = 0$ , and working with  $\Phi$  instead of  $F$  retains only one of these solutions, namely  $(0, 0)$ .

**Part II.**  $\Phi(\eta, \theta)$  is  $C^2$ .

To begin with, observe that

$$f(x) = kx^2/2 + \nu x^3/6 + o(x^3) \text{ and } f'(x) = kx + \nu x^2/2 + o(x^2).$$

Using these two expressions in the following calculations are straightforward.

$\Phi$  is  $C^0$ .

$$\lim_{\eta \rightarrow 0} \frac{F(\eta, \theta)}{\eta} = \lim_{\eta \rightarrow 0} \sin \theta - f(\eta \cos \theta) = \sin \theta$$

$\Phi$  is  $C^1$ . We consider  $\partial \Phi / \partial \eta(0, \theta)$  and  $\partial \Phi / \partial \theta(0, \theta)$  in the two settings —  $\eta \neq 0$  and  $\eta = 0$ .

•  $\eta \neq 0$ ,  $\partial \Phi / \partial \eta(0, \theta)$

$$\frac{\partial \Phi}{\partial \eta}(0, \theta) = \lim_{\eta \rightarrow 0} \frac{\partial \Phi}{\partial \eta}(\eta, \theta) = -\frac{k \cos^2 \theta}{2}$$

- $\eta = 0, \partial\Phi/\partial\eta(0, \theta)$

$$\frac{\partial\Phi}{\partial\eta}(0, \theta) = \lim_{\eta \rightarrow 0} \frac{\Phi(\eta, \theta) - \Phi(0, \theta)}{\eta} = \lim_{\eta \rightarrow 0} (\sin \theta - \frac{f(\eta \cos \theta)}{\eta} - \sin \theta) = -\frac{k \cos^2 \theta}{2}$$

- $\eta \neq 0, \partial\Phi/\partial\theta(0, \theta)$

$$\frac{\partial\Phi}{\partial\eta}(0, \theta) = \lim_{\eta \rightarrow 0} \frac{\partial\Phi}{\partial\theta}(\eta, \theta) = \cos \theta$$

- $\eta \neq 0, \partial\Phi/\partial\theta(0, \theta)$

$$\frac{\partial\Phi}{\partial\eta}(0, \theta) = \frac{\partial \sin \theta}{\partial \theta}(0, \theta) = \cos \theta$$

$\Phi$  is  $C^2$ . The equality of the four second order derivatives  $\partial^2\Phi/(\partial u^1 \partial u^2)$  with  $u^1 = \{\eta, \theta\}$  and  $u^2 = \{\eta, \theta\}$  in the two settings are checked similarly.

**Part III.** Expression of  $\theta$  as a function of  $(\eta)$ .

To see that  $(0, 0)$  is a regular point, observe that  $\partial\Phi/\partial\theta(0, 0) = \cos 0 = 1$ . Since  $\Phi$  is  $C^2$ , applying the implicit function theorem yields

$$\theta = A\eta + B\eta^2 + o(\eta^2).$$

□

The proof of lemma is now straightforward:

*Proof.* From Equation (8), one easily derives the equivalents of  $\cos \theta$  and  $\sin \theta$  as a function of  $\eta$ . Plugging them into Equation (4) yields Equation (5). Equation (6) is proved in the same way observing that  $p_{i-1} = (-\eta_{i-1} \cos \theta_{i-1}, \eta_{i-1} \sin \theta_{i-1})$ . □

## 9 Appendix II: proof of Theorem 3

The proof of Theorem 3 consists of working out the expressions of  $A(a, n)$ ,  $B(a, n)$  and  $C(a, n)$  from Equation (10). We begin with a lemma about trigonometric sums.

**Lemma. 6** *Define the three trigonometric sums*

$$t_1(a, n) = \sum_{i=1}^n \cos(2\varphi_i), \quad t_2(a, n) = \frac{1}{2} \sum_{i=1}^n \cos(2\varphi_i + 2\varphi_{i-1}), \quad t_3(a, n) = \sum_{i=1}^n \cos(4\varphi_i). \quad (31)$$

For a regular uniform mesh of valence  $n$ , we have

$$2\pi - \sum_{i=1}^n \gamma_i = [A(a, n) k_G + B(a, n) k_M^2 + C(a, n) k_m^2] \eta^2 + o(\eta^2) \quad (32)$$

with

$$A(a, n) = \frac{1}{16 \sin \theta(n)} (2n - n \cos 2\theta(n) - n \cos \theta(n) - t_2(a, n) + t_3(a, n) \cos \theta(n)), \quad (33)$$

$$B(a, n) = \frac{1}{16 \sin \theta(n)} \left( n + \frac{n}{2} \cos 2\theta(n) - \frac{3n}{2} \cos \theta(n) + 2(1 - \cos \theta(n))t_1(a, n) + t_2(a, n) - \frac{\cos \theta(n)}{2} t_3(a, n) \right), \quad (34)$$

$$C(a, n) = \frac{1}{16 \sin \theta(n)} \left( n + \frac{n}{2} \cos 2\theta(n) - \frac{3n}{2} \cos \theta(n) - 2(1 - \cos \theta(n))t_1(a, n) + t_2(a, n) - \frac{\cos \theta(n)}{2} t_3(a, n) \right). \quad (35)$$

*Proof.* Since the angles in the tangent plane are known, we work out the angular defect using Equation (10) rather than Equation (9). Assuming that  $\eta_i = \eta$  and  $\beta_i = \theta(n)$ , we rewrite Equation (10) as follows:

$$\frac{4 \sin \theta(n)}{\eta^2} (\beta_i - \gamma_i) = \mu_i + o(1) \quad (36)$$

with

$$\mu_i = \lambda_i \lambda_{i+1} - \frac{1}{2} (\lambda_i^2 + \lambda_{i+1}^2) \cos \theta(n). \quad (37)$$

Expressing the directional curvatures  $\lambda_i$  with Euler's relation, that is

$$\lambda_i = \cos^2 \varphi_i k_M + \sin^2 \varphi_i k_m, \quad (38)$$

and substituting into  $\sum_{i=1}^n \mu_i$  yields a homogeneous polynomial of degree four in sines and cosines. We linearize this polynomial using the standard formulae  $\cos^2 a = (1 + \cos 2a)/2$ ,  $\sin^2 a = (1 - \cos 2a)/2$ ,  $\cos a \cos b = (\cos(a + b)/2 + \cos(a - b))/2$ , as well as  $\cos^2 a \sin^2 a = (1 - \cos 4a)/8$ ,  $\cos^4 a = (\cos 4a + 4 \cos 2a + 3)/8$ ,  $\sin^4 a = (\cos 4a - 4 \cos 2a + 3)/8$ . These calculations simplify to Equations (33), (34) and (35).

Notice that the expressions of  $B(a, n)$  and  $C(a, n)$  just differ by the sign of the coefficient of  $t_1(a, n)$ .  $\square$

**Lemma. 7** *Let  $\alpha$  and  $\beta$  be positive integers, and consider the sum*

$$S(\alpha, \beta, n) = \sum_{k=1}^n \cos\left(\alpha + k \frac{\beta\pi}{n}\right). \quad (39)$$

*If  $\beta\pi/n = 0 \pmod{2\pi}$ , then  $S(\alpha, \beta, n) = n \cos \alpha$ . If  $\beta\pi/n \neq 0 \pmod{2\pi}$  and  $\beta\pi = 0 \pmod{2\pi}$ , then  $S(\alpha, \beta, n) = 0$ .*

*Proof.* If  $\beta\pi/n = 0 \pmod{2\pi}$ , the result is trivial. Otherwise, consider the two terms:

$$S(\alpha, \beta, n) = \sum_{k=1}^n \cos\left(\alpha + k\frac{\beta\pi}{n}\right), \quad T(\alpha, \beta, n) = \sum_{k=1}^n \sin\left(\alpha + k\frac{\beta\pi}{n}\right). \quad (40)$$

Let  $\beta(n) = \beta\pi/n$ . Using complex numbers we have the geometric sum

$$S(\alpha, \beta, n) + iT(\alpha, \beta, n) = \sum_{k=1}^n e^{i\alpha} (e^{i\beta(n)})^k = e^{i(\alpha+\beta(n))} \frac{1 - e^{i n\beta(n)}}{1 - e^{i\beta(n)}} \quad (41)$$

whose numerator is null if  $n\beta(n) = \beta\pi = 0 \pmod{2\pi}$ .  $\square$

We are now ready to prove Theorem 3:

*Proof.* To prove **1.**, we need to show that  $B(a, n) = C(a, n)$  in lemma 6. Using the trigonometric sum of lemma 7, we have  $t_1(a, n) = S(2a, 4, n)$ ,  $t_2(a, n) = S(4a-4\pi/n, 8, n)$ ,  $t_3(a, n) = S(4a, 8, n)$ .

But  $t_1(a, n) = 0$  for all values of  $n$  since, by the same lemma, we never have  $4\pi/n = 0 \pmod{2\pi}$  for  $n \geq 3$ . Since  $B$  and  $C$  just differ by the sign of the coefficient of  $t_1$ ,  $B = C$  for all  $ns$ .

For **2.**,  $A(a, n)$  and  $B(a, n)$  depend upon  $a$  if the trigonometric sums  $t_2$  or  $t_3$  are non vanishing. According to the above lemma the condition is  $8\pi/n = 0 \pmod{2\pi}$ , which occurs for  $n = 4$  only.

For **3.**, it is easily checked that the only value of  $n$  such that  $B(a, n)$  vanishes is  $n = 6$ .  $\square$



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