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Abstract: The aim of this work is to provide a new *Linearization Principle* approach particularly suited for problems in fluid-structure stability. The complexity here, and the main difference with respect to the classical approach, comes from the fact that the full non-linear fluid equations are written in a moving (i.e. time dependent) domain. The underlying idea of our approach comes from the linearization-transpiration method developed in [11, 12]. This allows us to obtain a new coupled spectral problem involving the linearized Navier-Stokes equations and those of a reduced linear structure. The coupling is realized through specific transpiration conditions on a fixed interface, while keeping a fixed fluid domain. Finally, we provide a first rigorous mathematical treatment of this eigenproblem.

Key-words: Fluid-structure interaction, linear stability, linearization, transpiration, spectral analysis.

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Stabilité linéaire en interaction fluide-structure avec transpiration. Partie I: formulation et analyse mathématique

Résumé : Le but de ce travail est d'introduire nu nouveau *Principe de Linéarisation* adapté aux problèmes de stabilité en interaction fluide-structure. La complexité ici, et différence principale par rapport à l'approche classique, vient du fait que les équations non-linéaires du fluide sont écrites dans un domaine mobile. L'idée sous-jacente de notre approche réside dans la méthode de linéarisation-transpiration développée dans [11, 12]. Ceci nous permet d'obtenir un nouveau problème spectral couplant les équations de Navier-Stokes linéarisées et celles d'une structure réduite. Le couplage est réalisé par le moyen de conditions de type transpiration sur une interface fixe. Finalement, ce problème spectral est analysé mathématiquement.

Mots-clés : Interaction fluide-structure, stabilité linéaire, linéarisation, transpiration, analyse spectrale.

1 Introduction

An important question in the research of experimentalists and applied mathematicians, is the *stability* of an equilibrium state in a mechanical system. That is, if the equilibrium is slightly disturbed, do the perturbations grow or decay?. This question has an important role in the design of complex industrial systems.

The linear stability theory deals with perturbations of small size acting at the initial time, by supposing that the generated fluctuations remain small for all time $t > 0$. In this case, a *Linearization Principle* approach, see for instance [15], enable us to reduce the stability problem to the analysis of a specific spectral problem. Namely, neglecting the high order terms in the full non-linear equations, we obtain a linearized problem governing (at first order) the fluctuations around the equilibrium state. The study of these fluctuations can be generally obtained from the behavior of the harmonic solutions, called *normal modes*, which are directly characterized by the eigenvalues of the associated spectral problem, see figure 1.

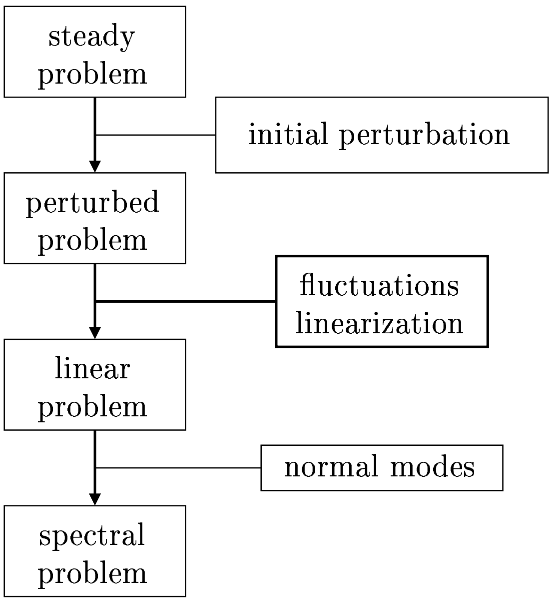

Figure 1: Linearization Principle

In this work we focus on the linear stability of a coupled fluid-structure system involving an incompressible Newtonian fluid and a reduced structure. We will generalize the classical *Linearization Principle* approach to a linear stability analysis in fluid-structure interaction. The complexity here, and the main difference with respect to the classical approach [15], comes from the fact that, due in particular to large structural displacements, the full non-linear fluid equations are written in a moving (i.e. time dependent) domain. In this sense, the new *Linearization Principle* approach requires, on the one hand, a specific definition of the state fluctuations (taking into account the fluid domain motion), and on the other hand, a new linearization method (adapted to this fluctuations definition) leading to a coupled linear problem with minimal complexity. This problem will involve the linearized incompressible Navier-Stokes equations and those of the reduced linear structure. The coupling is realized by means of specific transpiration boundary conditions. Finally, from the study of the harmonic solutions, we obtain a new spectral problem for which we provide a rigorous mathematical analysis.

The mathematical and numerical work reported here and in Part II (see [13]) is intended as the first stage of a rigorous approach to obtain, at low computational cost, reliable numerical predictions of the physical stability of the coupled system under study. The work in this paper aims, on the one hand, at defining a coupled eigenproblem of minimal complexity, overcoming all difficulties arising when dealing with moving domains, and on the other hand, at providing a rigorous mathematical analysis of its solutions.

The outline of this paper is as follows. In section 2, we introduce the fluid-structure interaction problem under study and its mathematical description. We use the classical arbitrary Euler-Lagrange (ALE) formulation for the fluid. Section 3 provides a generalized *Linearization Principle* approach particularly suited for problems involving moving boundaries. This is achieved from the linearization-transpiration method developed in [11, 12]. Finally, in section 4 we provide a mathematical study of the differential eigenproblem defined in section 3. This is done by proving that the eigenvalues of this spectral problem can be obtained from the characteristic values of a specific compact operator acting in a Hilbert space. This last result was already announced as a brief note in [14].

2 Mechanical problem

The modeling of fluid-structure interaction systems under large displacements involves, in a general way, the coupling of two formulations: the solid classically treated in lagrangian formulation, and the fluid described by an Arbitrary Lagrangian Eulerian (ALE) formulation. In this section we introduce the mechanical problem under study and its mathematical description.

2.1 Geometry

We consider a solid located at time $t \geq 0$ in a domain $\Omega^{\mathrm{s}}(t) \subset \mathbb{R}^3$ with boundary $\Gamma^{\mathrm{s}}(t)$. As in many problems of aeroelasticity at low Mach numbers, it is surrounded by a fluid in $\mathbb{R}^3$. We introduce a control volume $\Omega \subset \mathbb{R}^3$ containing the solid at each time $t \geq 0$. The notation $\partial\Omega$ stands for the boundary of Ω. Hence, the fluid evolution is restricted to the domain $\Omega^{\mathrm{f}}(t) = \Omega - \overline{\Omega}^{\mathrm{s}}(t)$. In the sequel we set $\Gamma = \partial\Omega$ and

$$\partial\Omega^{\mathrm{f}}(t) = \Gamma \cup \Gamma^{\mathrm{s}}(t),$$

stands for the fluid domain boundary, see figure 2.

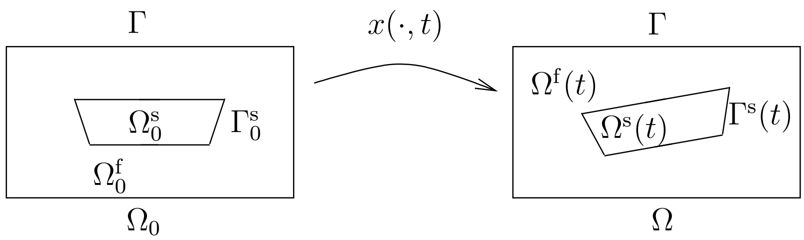

Figure 2: Geometric description

We assume the fluid to be Newtonian viscous, homogeneous and incompressible. Its behavior is described by its velocity and pressure. The elastic solid under large displacements is described by its velocity and its stress tensor. The classical conservation laws of the continuum mechanics drive the evolution of these unknowns.

2.2 The fluid: ALE formulation

The fluid state satisfies the following incompressible Navier-Stokes equations written in eulerian conservative formulation:

$$\begin{aligned} \frac{\partial u_{\mathrm{E}}}{\partial t}_{|x} + \operatorname{div}\left[u_{\mathrm{E}} \otimes u_{\mathrm{E}} - \frac{1}{\rho_{\mathrm{E}}}\sigma(u_{\mathrm{E}}, p_{\mathrm{E}})\right] &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}}(t), \\ \operatorname{div} u_{\mathrm{E}} &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}}(t), \end{aligned} \tag{1}$$

where ρ_{E}, u_{E} and p_{E} stand, respectively, for the fluid density, velocity and pressure. In addition, the fluid stress tensor is given by

$$\sigma(u_{\mathrm{E}}, p_{\mathrm{E}}) = -p_{\mathrm{E}}\,\mathrm{I} + 2\mu\varepsilon(u_{\mathrm{E}}),$$

with μ the dynamic viscosity of the fluid and

$$\varepsilon(u_{\mathrm{E}}) = \frac{1}{2}\left[\nabla u_{\mathrm{E}} + (\nabla u_{\mathrm{E}})^{\mathrm{T}}\right],$$

the strain rate tensor.

Remark 2.1 *In the sequel* $\dfrac{\partial}{\partial t}_{|a}$ *stands for the time derivative operator keeping the space variable "a" fixed. In addition, the subindex "*$_{\mathrm{E}}$*" stands for the eulerian description.*

In a fluid-structure interaction framework the evolution of the fluid domain $\Omega^{\mathrm{f}}(t)$ is induced by the structural deformation through the fluid-structure interface $\Gamma^{\mathrm{s}}(t)$. Indeed, by definition we have $\Omega^{\mathrm{f}}(t) = \Omega - \overline{\Omega}^{\mathrm{s}}(t)$. This suggests to characterize $\Omega^{\mathrm{f}}(t)$ through a map acting in a fixed reference domain. This approach is usually used for the solid domain $\Omega^{\mathrm{s}}(t)$, by means of the lagrangian formulation [2, 17].

Given a material reference configuration for the solid $\Omega_0^{\mathrm{s}} \subset \Omega$ with boundary Γ_0^{s}, we take $\Omega_0^{\mathrm{f}} = \Omega - \overline{\Omega}_0^{\mathrm{s}}$ as the reference fluid configuration. Then, the present control volume $\Omega = \Omega^{\mathrm{f}}(t) \cup \overline{\Omega}^{\mathrm{s}}(t)$ will be described by a smooth and injective map:

$$\begin{aligned} x : \quad \overline{\Omega} \times \mathbb{R}^+ &\longrightarrow \quad \overline{\Omega} \\ (x_0, t) &\longmapsto \quad x = x(x_0, t). \end{aligned}$$

We set $x^{\mathrm{f}} = x_{|\Omega_0^{\mathrm{f}}}$ and $x^{\mathrm{s}} = x_{|\Omega_0^{\mathrm{s}}}$, such that [18]:

- for $x_0 \in \Omega_0^{\rm s}$, $x^{\rm s}(x_0,t)$ represents the position at time $t \geq 0$ of the material point x_0. This corresponds to the classical lagrangian flow,
- the ALE map $x^{\rm f}$ is defined from $x^{\rm s}_{|\Gamma_0^{\rm s}}$, as an arbitrary extension over domain $\overline{\Omega}_0^{\rm f}$, which preserves $\Gamma_0^{\rm f} = \Gamma^{\rm f} = \partial\Omega$.

In short, the ALE map x is given by

$$\begin{aligned} x(x_0,t) &= \mathrm{Ext}(x^{\rm s}{}_{|\Gamma_0^{\rm s}})(x_0,t), \quad \forall x_0 \in \overline{\Omega}_0^{\rm f}, \\ x(x_0,t) &= x^{\rm s}(x_0,t), \quad \forall x_0 \in \overline{\Omega}_0^{\rm s}. \end{aligned}$$

Here, "Ext" represents an extension operator from $\Gamma_0^{\rm s}$ to $\overline{\Omega}_0^{\rm f}$ such that

$$\mathrm{Ext}(x^{\rm s}{}_{|\Gamma_0^{\rm s}})_{|\Gamma_0^{\rm f}} = I_{\Gamma_0^{\rm f}}.$$

Remark 2.2 *We set* $\Omega_0 = \Omega_0^{\rm f} \cup \overline{\Omega}_0^{\rm s}$ *as the reference domain, and* $\Omega = \Omega^{\rm f}(t) \cup \overline{\Omega}^{\rm s}(t)$ *stands for the present configuration at time* $t > 0$.

Remark 2.3 *The operator* Ext *can be defined arbitrarily inside* $\Omega_0^{\rm f}$.

This map allows us to transport the fluid equations (1) back to the reference domain $\Omega_0^{\rm f}$, leading to the classical incompressible Navier-Stokes equations written in ALE conservative formulation [18], satisfied by $u : \Omega_0^{\rm f} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$ and $p : \Omega_0^{\rm f} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$:

$$\begin{aligned} \frac{\partial Ju}{\partial t}_{|x_0} + \mathrm{div}_0 \left\{ J \left[u \otimes (u-w) - \frac{1}{\rho}\sigma(u,p) \right] F^{-T} \right\} &= 0, \quad \text{in} \quad \Omega_0^{\rm f}, \\ \mathrm{div}_0(JuF^{-T}) &= 0, \quad \text{in} \quad \Omega_0^{\rm f}, \end{aligned} \tag{2}$$

where the quantities F, J, w are defined by:

$$F = \nabla_0 x = \frac{\partial x}{\partial x_0}, \quad J = \det(F) > 0, \quad w = \frac{\partial x}{\partial t}_{|x_0}.$$

and where $\rho = \rho_{\rm E}$ (constant), and u and p are the ALE velocity and pressure defined by transport as:

$$u(x_0,t) = u_{\rm E}(x^{\rm f}(x_0,t),t), \quad p(x_0,t) = p_{\rm E}(x^{\rm f}(x_0,t),t).$$

Remark 2.4 *From the definition of* x, $w_{|\Omega_0^{\rm s}}$ *represents the solid velocity, whereas* $w_{|\Omega_0^{\rm f}}$ *stands for the fluid control volume velocity, which usually differs from the fluid velocity* u *inside* $\Omega_0^{\rm f}$.

2.3 The structure

The evolution of the structure is characterized by its motion $x^{\rm s}$. We will generally consider the case (at least for our mathematical analysis, see also [9, 10, 24]) where the structural displacement is given, around a known configuration, by a linear combination of a finite number of vibration modes $\varphi_i : \Omega_0^{\rm s} \longrightarrow \mathbb{R}^3, \quad 1 \leq i \leq n^{\rm s}$, in such a way that

$$x^{\rm s}(x_0,t) = x_0 + \sum_{i=1}^{n^{\rm s}} s_i(t)\varphi_i(x_0), \quad \forall x_0 \in \Omega_0^{\rm s},$$

with $s(t) = \{s_i(t)\}_{1\leq i\leq n^{\rm s}} \in \mathbb{R}^{n^{\rm s}}$. Hence, $x^{\rm s} = I_{\Omega_0^{\rm s}} + \Phi s$ in $\Omega_0^{\rm s}$, where $\Phi = [\varphi_1|\varphi_2|\ldots|\varphi_{n^{\rm s}}]$ is a $3\times n^{\rm s}$ matrix standing for the reduced modal basis. In this way, the structural behavior is driven by given mass and stiffness operators, $\mathcal{M}$ and $\mathcal{K}$ respectively. Thus, the equations describing the motion of the structure, around a known configuration, reduce to

$$\mathcal{M}\ddot{s} + \mathcal{K}s = f_g, \tag{3}$$

with $f_g \in \mathbb{R}^{n^{\rm s}}$ the generalized load vector, given by

$$[f_g]_i = \int_{\Gamma_0^{\rm s}} f_{\Gamma_0^{\rm s}} \cdot \varphi_i \,{\rm d}a_0, \quad 1 \leq i \leq n^{\rm s},$$

and where $f_{\Gamma_0^{\rm s}} \in \mathbb{R}^3$ stands for the surface force density applied on the structural boundary in its reference configuration.

2.4 The coupled problem

The coupling between the solid and the fluid is realized through standard boundary conditions at the fluid-structure interface $\Gamma_0^{\rm s}$, namely, the kinematic continuity of the velocity and the kinetic continuity of the stress [18]:

$$\begin{aligned} u &= \dot{x}^{\rm s}, \quad \text{on} \quad \Gamma_0^{\rm s}, \\ f_{\Gamma_0^{\rm s}} &= J\sigma(u,p)F^{-T}n_0, \quad \text{on} \quad \Gamma_0^{\rm s}, \end{aligned} \tag{4}$$

where n_0 stands for the unit normal vector on $\Gamma_0^{\rm s}$ pointing inside $\Omega_0^{\rm s}$. Moreover, we endow the fluid equations with a Dirichlet boundary condition on $\Gamma_{\rm in}$ and an outlet condition on $\Gamma_{\rm out}$, i.e.

$$\begin{aligned} u &= u_{\Gamma_{\rm in}}, \quad \text{on} \quad \Gamma_{\rm in}, \\ \sigma n_0 &= 0, \quad \text{on} \quad \Gamma_{\rm out}, \end{aligned}$$

with $\Gamma = \Gamma_{\rm in} \cup \Gamma_{\rm out}$ and $\Gamma_{\rm in} \cap \Gamma_{\rm out} = \emptyset$.

In summary, the coupled problem, with an ALE formulation for the fluid, is given by:

$$\begin{aligned} \frac{\partial Ju}{\partial t}_{|x_0} + \operatorname{div}_0 \left\{ J \left[u \otimes (u-w) - \frac{1}{\rho}\sigma(u,p) \right] F^{-T} \right\} &= 0, \quad \text{in} \quad \Omega_0^{\rm f}, \\ \operatorname{div}_0 \left(JuF^{-T} \right) &= 0, \quad \text{in} \quad \Omega_0^{\rm f}, \\ u &= u_{\Gamma_{\rm in}}, \quad \text{on} \quad \Gamma_{\rm in}, \\ \sigma(u,p) n_0 &= 0, \quad \text{on} \quad \Gamma_{\rm out}, \\ u &= \Phi \dot{s}, \quad \text{on} \quad \Gamma_0^{\rm s}, \\ \mathcal{M}\ddot{s} + \mathcal{K}s &= -\int_{\Gamma_0^{\rm s}} J \Phi^{\rm T} \sigma(u,p) F^{-T} n_0 \, {\rm d}a_0, \\ x^{\rm s} = I_{\Omega_0^{\rm s}} + \Phi s, \quad x^{\rm f} &= {\rm Ext}(x^{\rm s}{}_{|\Gamma_0^{\rm s}}), \\ (u, s, \dot{s})_{|t=0} &= \left(u^0, s^0, s^1\right), \end{aligned} \tag{5}$$

with F, J and w as defined in (2). Here, (u^0, s^0, s^1) stands for the initial fluid velocity and the initial structural displacement and velocity, respectively.

3 Linear stability

In this section we focus on the stability, to small perturbations, of systems coupling a fluid and a structure. More precisely, our problem deals with the linear stability analysis of systems involving a reduced structure immersed in an incompressible viscous flow. We will generalize the classical *Linearization Principle* approach, see [15], using the linearization-transpiration techniques developed in [11, 12].

3.1 Equilibrium state and perturbed problem

Let (u_0, p_0, x_0) be a steady coupled fluid-structure equilibrium state, with $x_0^{\mathrm{s}} = I_{\Omega_0^{\mathrm{s}}} + \Phi s_0$ and $s_0 \in \mathbb{R}^{n^{\mathrm{s}}}$. From (5) we obtain that (u_0, p_0, x_0) satisfies the following steady coupled problem

$$\begin{aligned}
\operatorname{div}_0 \left\{ J_0 \left[u_0 \otimes u_0 - \frac{1}{\rho}\sigma(u_0, p_0) \right] F_0^{-T} \right\} &= 0, \quad \text{in} \quad \Omega_0^{\mathrm{f}},\\
\operatorname{div}_0 \left(J_0 u_0 F_0^{-T} \right) &= 0, \quad \text{in} \quad \Omega_0^{\mathrm{f}},\\
u_0 &= u_{\Gamma_{\mathrm{in}}}, \quad \text{on} \quad \Gamma_{\mathrm{in}},\\
\sigma(u_0, p_0) n_0 &= 0, \quad \text{on} \quad \Gamma_{\mathrm{out}},\\
u_0 &= 0, \quad \text{on} \quad \Gamma_0^{\mathrm{s}},\\
x_0^{\mathrm{s}} = I_{\Omega_0^{\mathrm{s}}} + \Phi s_0, \quad x_0^{\mathrm{f}} &= \operatorname{Ext}(x^{\mathrm{s}}_{0|\Gamma_0^{\mathrm{s}}}),\\
\mathcal{K} s_0 = - \int_{\Gamma_0^{\mathrm{s}}} J_0 \Phi^{\mathrm{T}} \sigma(u_0, p_0) F_0^{-T} n_0 \, \mathrm{d}a_0&,
\end{aligned} \tag{6}$$

where $F_0 = \nabla_0 x_0$ and $J_0 = \det F_0$.

We set $\Omega^{\mathrm{s}} = x^{\mathrm{s}}(\Omega_0^{\mathrm{s}})$ the configuration of the structure at equilibrium, in such a way that, after deformation of the interface, the control volume in the fluid (used in the ALE formulation) is given by $\Omega^{\mathrm{f}} = \Omega - \overline{\Omega}^{\mathrm{s}}$. We also set $\gamma = \partial \Omega^{\mathrm{s}}$, the fluid-structure interface at equilibrium. Thus, since Ω^{f} is known, by transporting (6) to this new configuration, the reference state (u_0, p_0, x_0) becomes (u_0, p_0, I) which satisfies the following uncoupled problem:

$$\begin{aligned}
\operatorname{div} \left(u_0 \otimes u_0 - \frac{1}{\rho}\sigma(u_0, p_0) \right) &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u_0 &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
u_0 &= u_{\Gamma_{\mathrm{in}}}, \quad \text{on} \quad \Gamma_{\mathrm{in}},\\
\sigma(u_0, p_0) n &= 0, \quad \text{on} \quad \Gamma_{\mathrm{out}},\\
u_0 &= 0, \quad \text{on} \quad \gamma,\\
\mathcal{K} s_0 = - \int_{\gamma} \Phi^{\mathrm{T}} \sigma(u_0, p_0) n \, \mathrm{d}a&.
\end{aligned} \tag{7}$$

Here, $n = \dfrac{F^{-T} n_0}{\|F^{-T} n_0\|}$ represents the unit normal vector on γ (pointing inside Ω^{s}).

In the same way, we can describe problem (5) taking $\Omega = \Omega^{\mathrm{f}} \cup \overline{\Omega}^{\mathrm{s}}$ as reference configuration, i.e. by defining the ALE map in Ω instead of Ω_0, see figure 3.

We denote by $\delta s \in \mathbb{R}^{n^{\mathrm{s}}}$, the degrees of freedom of the structure in this new configuration. Thus, we define the structural motion as

$$x^{\mathrm{s}} = I_{\Omega^{\mathrm{s}}} + \Phi \delta s, \quad \text{in} \quad \Omega^{\mathrm{s}},$$

and the motion of the control volume of the fluid as

$$x^{\mathrm{f}} = \mathrm{Ext}(x^{\mathrm{s}}_{|\gamma}).$$

Remark 3.1 *For simplification purposes, after transport on Ω, we have kept the same notation for the ALE map x, the deformation gradient $F = \nabla x$, the transformation jacobian J and the extension operator. We have also kept the same modal development for the structural displacement. However, we could use two different developments to describe, on the one hand, the structural motion, and on the other hand, its perturbations.*

With this choice, coupled problem (5), written in the new reference configuration Ω, becomes

$$\begin{aligned}
\frac{\partial \rho J u}{\partial t} + \mathrm{div}\left\{ J \left[\rho u \otimes (u - w) - \sigma(u,p) \right] F^{-T} \right\} = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
\mathrm{div}\left(J u F^{-T} \right) = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
u = u_{\Gamma_{\mathrm{in}}}, &\quad \text{on} \quad \Gamma_{\mathrm{in}}, \\
\sigma(u,p) n = 0, &\quad \text{on} \quad \Gamma_{\mathrm{out}}, \\
u = \Phi \dot{\delta s}, &\quad \text{on} \quad \gamma, \\
\mathcal{M} \ddot{\delta s} + \mathcal{K}(s_0 + \delta s) = -\int_\gamma J \Phi^{\mathrm{T}} \sigma(u,p) F^{-T} n \, \mathrm{d}a, & \\
x^{\mathrm{s}} = I_{\Omega^{\mathrm{s}}} + \Phi \delta s, \quad x^{\mathrm{f}} = \mathrm{Ext}(x^{\mathrm{s}}{}_{|\gamma}), \quad F = \nabla x, \quad J = \det F, & \\
\left(u, \delta s, \dot{\delta s} \right)_{|t=0} = \left(u^0, s^0, s^1 \right). &
\end{aligned} \tag{8}$$

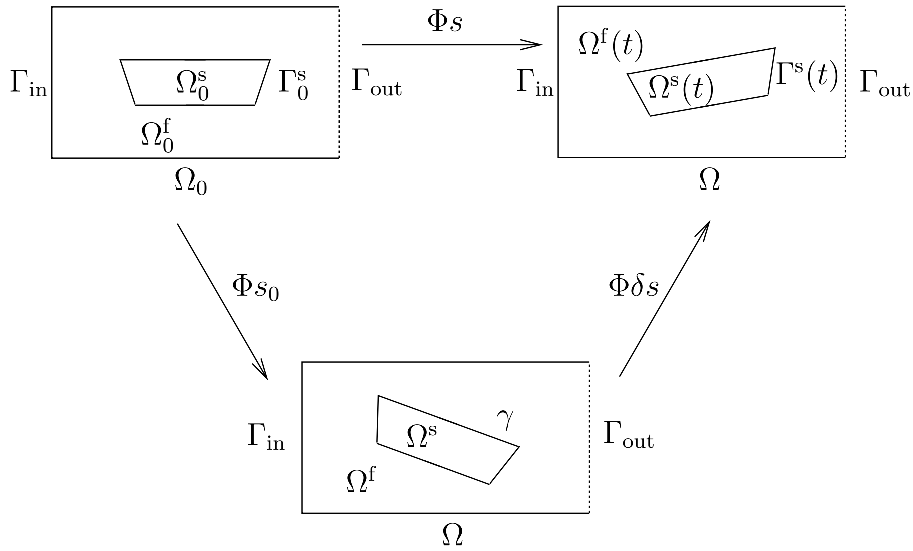

Figure 3: Change of configuration

We focus on the linear stability analysis of the steady equilibrium state (u_0, p_0, I) with respect to small perturbations acting on the initial conditions. Let us suppose (u_0, p_0, I) to be exposed to a small perturbation $(\delta u_0, \delta s_0, \delta s_1)$ at initial time. This perturbation generates a coupled unsteady state (u, p, x) which satisfies (8) with the following initial condition

$$\left(u^0, s^0, s^1\right) = (u_0 + \delta u_0, \delta s_0, \delta s_1).$$

From (8), we can obtain a non-linear coupled problem governing the fluctuations

$$(\delta u, \delta p, \delta x) = (u, p, x) - (u_0, p_0, I). \tag{9}$$

By supposing that these generated fluctuations remain small for any time $t > 0$, we can neglect the high order terms. Thus, we obtain a linearized problem driving, at first order, the fluctuations of the coupled system. At this point, we usually introduce another definition in the linear stability theory [15], by supposing that each linear fluctuation $(\delta u, \delta p, \delta x)$ can be obtained by superposition of fluctuations of type $(v, q, t)e^{-\lambda t}$, called *normal modes*. By substituting this last expression in the linearized coupled equations, we obtain that $(\lambda; v, q, t)$ is an eigenpair of the spectral problem associated to the coupled linear problem.

Definition 3.2 *The permanent state (u_0, p_0, I) is called linearly asymptotically stable, if the spectral problem does not have eigenvalues with negative real part. The permanent state is called linearly unstable, if there exists, at least, one eigenvalue with negative real part.*

In short, this general approach, known as *Linearization Principle* in [15], enables to reduce the stability problem to the problem of determine the eigenvalues of a specific spectral problem.

Nevertheless, we must point out that the complexity of the linearized problem, in particular that of eigenproblem, strongly depends on the fluctuation definition. It is straightforward to verify, see [19, 20] for instance, that taking (9) as fluctuation definition we obtain a linearized problem where the fluid equations still depend on the interface motion. Hence, in this sense, its complexity is similar to the full non-linear problem.

In the next paragraphs, we will focus on a new derivation of the *Linearization Principle*. This will be based in the linearization-transpiration method developed in [11, 12], particularly suited for problems involving moving domains. Thus, we will be able to recover a linearized coupled problem, where the linear fluid equations are independent of the fluid-structure interface motion. Indeed, the coupling with the linearized solid equations is performed via transpiration interface conditions while keeping a fixed fluid domain. In other words, we finally obtain an eigenproblem of minimal complexity. From a mathematical point of view, this can be viewed as a change of variables.

3.2 Linearization

In this paragraph we use the linearization-transpiration method introduced in [11, section 5] (see also [12, chapter 2]). First at all, we must write the coupled problem (8) in variational form. By multiplying (8) by $(\hat{v}^{\mathrm{f}}, \hat{t}) \in \mathcal{D}(\Omega)^4 \times \mathbb{R}^{n^{\mathrm{s}}}$, by integrating by parts in the fluid equations and by taking into account the boundary conditions, we obtain the following variational formulation: find

$u : \Omega^{\mathrm{f}} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$, $p : \Omega^{\mathrm{f}} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\delta s : \mathbb{R}^+ \longrightarrow \mathbb{R}^{n^{\mathrm{s}}}$ such that

$$\begin{aligned}\int_{\Omega^{\mathrm{f}}} \frac{\partial J\rho u}{\partial t} \cdot \hat{v}_1^{\mathrm{f}} \, \mathrm{d}x - \int_{\Omega^{\mathrm{f}}} J \left[\phi(u, \sigma(u,p)) - \mathrm{I}_1 \, \rho u \otimes w\right] F^{-T} : \nabla \hat{v}^{\mathrm{f}} \, \mathrm{d}x \\ + \int_{\gamma} Ju \cdot \left(F^{-T} n\right) \hat{v}_2^{\mathrm{f}} \, \mathrm{d}a + \int_{\gamma} (J\sigma(u,p)F^{-T}n) \cdot (\Phi \hat{t} - \hat{v}_1^{\mathrm{f}}) \, \mathrm{d}a \\ + (\mathcal{M}\ddot{\delta s}) \cdot \hat{t} + \big(\mathcal{K}(s_0 + \delta s)\big) \cdot \hat{t} = 0, \quad \forall (\hat{v}^{\mathrm{f}}, \hat{t}) \in \mathcal{D}(\Omega)^4 \times \mathbb{R}^{n^{\mathrm{s}}}, \quad (10)\end{aligned}$$

provided with the boundary conditions

$$\begin{aligned} u &= u_{\Gamma_{\mathrm{in}}}, &&\text{on} \quad \Gamma_{\mathrm{in}}, \\ \sigma(u,p)n &= 0, &&\text{on} \quad \Gamma_{\mathrm{out}}, \\ u &= \Phi \dot{\delta s}, &&\text{on} \quad \gamma, \end{aligned} \qquad (11)$$

and where the flux functions are given by

$$\phi(u,\sigma) = \mathrm{I}_1(\rho u \otimes u - \sigma) + \mathrm{I}_2 \otimes u, \quad \mathrm{I}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{I}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\hat{v}^{\mathrm{f}} = \begin{pmatrix} \hat{v}_1^{\mathrm{f}} \\ \hat{v}_2^{\mathrm{f}} \end{pmatrix} : \Omega \longrightarrow \mathbb{R}^4, \quad x^{\mathrm{s}} = I_{\Omega^{\mathrm{s}}} + \Phi \delta s, \quad x^{\mathrm{f}} = \mathrm{Ext}(x^{\mathrm{s}}_{|\gamma}).$$

In the same way, the reference steady problem (7) can be written in weak form as:

$$-\int_{\Omega^{\mathrm{f}}} \phi(u_0, \sigma(u_0,p_0)) : \nabla \hat{v}^{\mathrm{f}} \, \mathrm{d}x + \int_{\gamma} \sigma(u_0,p_0)n \cdot (\Phi \hat{t} - \hat{v}_1^{\mathrm{f}}) \, \mathrm{d}a + \mathcal{K}(s_0) \cdot \hat{t} = 0, \\ \forall (\hat{v}^{\mathrm{f}}, \hat{t}) \in \mathcal{D}(\Omega)^4 \times \mathbb{R}^{n^{\mathrm{s}}}. \quad (12)$$

The following step in the linearization is the definition of the fluctuations $(\delta u, \delta p, \delta x)$, of the perturbed state (u,p,x) around the equilibrium (u_0, p_0, I). As in [11], these fluctuations are defined by

$$\begin{aligned} x &= I_{\Omega} + \delta x, &&\text{in} \quad \Omega, \\ u(I_{\Omega^{\mathrm{f}}} + \delta x) &= u_0 + \nabla u_0 \delta x + \delta u, &&\text{in} \quad \Omega^{\mathrm{f}}, \\ p(I_{\Omega^{\mathrm{f}}} + \delta x) &= p_0 + \nabla p_0 \delta x + \delta p, &&\text{in} \quad \Omega^{\mathrm{f}}, \end{aligned} \qquad (13)$$

and where

$$\delta x^{\mathrm{s}} = \Phi \delta s, \quad \delta x^{\mathrm{f}} = \mathrm{Ext}\left(x^{\mathrm{s}}{}_{|\gamma}\right) - I_{\Omega^{\mathrm{f}}}.$$

Compared to the traditional definition, used in [15, 19, 20], the non-standard definition (13) takes explicitly into account the transport of the reference state due to the fluid domain motion, and the intrinsic perturbation $(\delta u, \delta p)$ of the flow-field at the new spatial point $(x_0 + \delta x)$.

Remark 3.3 *The term* $\mathcal{K}(s_0)$ *appearing in the equation* (12) *corresponds to the residual stress introduced in [11, section 5.1]. It equilibrates the steady fluid stress field on the interface.*

As usual, the linearization could be carried out with respect to the new unknown $(\delta u, \delta p, \delta x)$, by subtracting the reference problem (12) from the perturbed problem (10), and then by neglecting the high order terms. This leads us to the following coupled linear problem [11, section 5.3]:

$$\begin{aligned}
&\int_{\Omega^{\mathrm{f}}} \rho \dot{\delta u} \cdot \hat{v}_1^{\mathrm{f}} \,\mathrm{d}x \\
&\quad - \int_{\Omega^{\mathrm{f}}} \left(\frac{\partial \phi}{\partial u}(u_0, \sigma(u_0, p_0)) \delta u + \frac{\partial \phi}{\partial \sigma}(u_0, \sigma(u_0, p_0)) \delta\sigma(\delta u, \delta p) \right) : \nabla \hat{v}^{\mathrm{f}} \,\mathrm{d}x \\
&\qquad\qquad + \int_{\gamma} (\nabla u_0 \delta x^{\mathrm{s}} + \delta u) \cdot n \hat{v}_2^{\mathrm{f}} \,\mathrm{d}a \\
&- \int_{\Omega^{\mathrm{f}}} \left\{ \phi(u_0, \sigma(u_0, p_0)) \left[\mathrm{I} \,\mathrm{div}\, \delta x - (\nabla \delta x)^{\mathrm{T}} \right] + \nabla \phi(u_0, \sigma(u_0, p_0)) \delta x \right\} : \nabla \hat{v}^{\mathrm{f}} \,\mathrm{d}x \\
&\quad + \int_{\gamma} \left[(\nabla \sigma(u_0, p_0) \delta x^{\mathrm{s}} + \delta\sigma(\delta u, \delta p)) n - \sigma(u_0, p_0) \eta(\delta x^{\mathrm{s}}) \right] \cdot (\Phi \hat{t} - \hat{v}_1^{\mathrm{f}}) \,\mathrm{d}a \\
&\qquad\qquad + (\mathrm{M}\, \ddot{\delta s} + \mathrm{K}\, \delta s) \cdot \hat{t} = 0, \quad \forall (\hat{v}^{\mathrm{f}}, \hat{t}) \in \mathcal{D}(\Omega)^4 \times \mathbb{R}^{n^{\mathrm{s}}}. \quad (14)
\end{aligned}$$

Here,

$$\delta\sigma(\delta u, \delta p) = -\delta p \,\mathrm{I} + 2\mu \varepsilon(\delta u), \quad \varepsilon(\delta u) = \frac{1}{2} \left[\nabla \delta u + (\nabla \delta u)^{\mathrm{T}} \right],$$

stand for the linearized fluid constitutive law, M and K for the linearized mass and stiffness operators around Φs_0, which we will suppose symmetric and positive definite, and $\eta(\delta x) = -\left[\mathrm{I}\,\mathrm{div}\,\delta x - (\nabla\delta x)^{\mathrm{T}}\right] n$ represents, at first order, the variation of the surface vector $-n\,\mathrm{d}a$. It only depends on the trace of δx on γ, see [24].

In addition, the boundary conditions (11), once written at first order in terms of $(\delta u, \delta p, \delta s)$, reduce to

$$\begin{aligned} \delta u = \Phi\dot{\delta s} - \nabla u \Phi \delta s, &\quad \text{on} \quad \gamma, \\ \delta u = 0, &\quad \text{on} \quad \Gamma_{\mathrm{in}}, \\ \delta\sigma n = 0, &\quad \text{on} \quad \Gamma_{\mathrm{out}}. \end{aligned} \tag{15}$$

We recover on the fixed interface γ the so called transpiration boundary conditions (see [11, 12]). Nevertheless, the second volume integral in (14) contains distributed terms in Ω^{f} depending on δx. In other words, the linear fluid equations, as for the perturbed one, still depend on the fluid domain motion. In contrast with the linearization performed in [19, 20], this difficulty can be overcome here by using a simplified version of lemma 1 in [11, section 5.2].

Lemma 3.4 (Fanion, Fernández & Le Tallec (2000)) *For each smooth displacement $\delta x \in \mathcal{C}^1(\overline{\Omega}^{\mathrm{f}})^3$ and each smooth solution, $(u_0, \sigma(u_0, p_0)) \in \mathcal{C}^1(\overline{\Omega}^{\mathrm{f}})^3 \times \mathcal{C}^1(\overline{\Omega}^{\mathrm{f}})^{3\times 3}$, of the fluid subproblem*

$$\int_{\Omega^{\mathrm{f}}} \phi(u_0, \sigma(u_0, p_0)) : \nabla \hat{v}^{\mathrm{f}}\,\mathrm{d}x = 0, \quad \forall \hat{v}^{\mathrm{f}} \in \mathcal{D}(\Omega^{\mathrm{f}})^4,$$

we obtain that

$$\begin{aligned} &-\int_{\Omega^{\mathrm{f}}} \left\{\phi(u_0, \sigma(u_0, p_0))\left[\mathrm{I}\,\mathrm{div}\,\delta x - (\nabla\delta x)^{\mathrm{T}}\right] + \nabla\phi(u_0, \sigma(u_0, p_0))\delta x\right\} : \nabla\hat{v}^{\mathrm{f}}\,\mathrm{d}x \\ &= \int_{\gamma} [\phi(u_0, \sigma(u_0, p_0))\eta(\delta x) - (\nabla\phi(u_0, \sigma(u_0, p_0))\delta x)n]\cdot\hat{v}^{\mathrm{f}}\,\mathrm{d}a, \quad \forall \hat{v}^{\mathrm{f}} \in \mathcal{D}(\Omega)^4. \end{aligned} \tag{16}$$

PROOF. See proof of lemma 1 in [11, section 5.2]. □

Subtracting now, from (14), the linearized convected problem (16), and since $\partial\phi = -\partial(\mathrm{I}_1\,\sigma) + \partial(\mathrm{I}_2 \otimes u)$ on γ, we obtain that the fluctuation $(\delta u, \delta p, \delta s)$, defined by (13), finally satisfies the following variational linear problem (independent of the extension map δx^{f}):

$$\begin{aligned}\int_{\Omega^{\mathrm{f}}} \rho \dot{\delta u} \cdot \hat{v}_1^{\mathrm{f}} \,\mathrm{d}x \\ - \int_{\Omega^{\mathrm{f}}} \left(\frac{\partial \phi}{\partial u}(u_0, \sigma(u_0, p_0))\delta u + \frac{\partial \phi}{\partial \sigma}(u_0, \sigma(u_0, p_0))\delta\sigma(\delta u, \delta p)\right) : \nabla \hat{v}^{\mathrm{f}} \,\mathrm{d}x \\ + \int_{\gamma} \left(\frac{\partial \phi}{\partial u}(u_0, \sigma(u_0, p_0))\delta u + \frac{\partial \phi}{\partial \sigma}(u_0, \sigma(u_0, p_0))\delta\sigma(\delta u, \delta p)\right) n \cdot \hat{v}^{\mathrm{f}} \,\mathrm{d}a \\ + \int_{\gamma} [(\nabla \sigma(u_0, p_0)\delta x^{\mathrm{s}} + \delta\sigma(\delta u, \delta p))n - \sigma(u_0, p_0)\eta(\delta x^{\mathrm{s}})] \cdot (\Phi \hat{t}) \,\mathrm{d}a \\ + (\mathrm{M}\, \ddot{\delta s} + \mathrm{K}\, \delta s) \cdot \hat{t} = 0, \quad \forall (\hat{v}^{\mathrm{f}}, \hat{t}) \in \mathcal{D}(\Omega)^4 \times \mathbb{R}^{n^{\mathrm{s}}}, \end{aligned} \tag{17}$$

completed with the boundary conditions (15).

In summary, as in [11, section 5.3], the linearization has been carried out by subtracting the steady reference problem, (12), and the linearized convected problem, (16), from the perturbed problem, (10), and by neglecting high order terms. This linearization provides a fluid problem written in fixed configuration Ω^{f}, and totally independent of the extension operator inside the fluid domain Ω^{f}. In addition, as we have proved and in contrast with a classical linearization, the linearization performed in a moving domain requests the introduction of a transported problem (16), that we must subtract from the perturbed one (10).

The variational formulation (17) is equivalent to two subproblems coupled along the fixed interface γ, see [11, section 5.4]. Indeed, by integrating by parts with $\hat{t} = 0$, the linearized fluid subproblem is given by

$$\begin{aligned} \rho \frac{\partial \delta u}{\partial t} + \operatorname{div}\left(\rho u_0 \otimes \delta u + \rho \delta u \otimes u_0 - \delta\sigma(\delta u, \delta p)\right) = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}, \\ \operatorname{div} \delta u = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}, \end{aligned}$$

provided with boundary conditions

$$\begin{aligned} \delta u = 0, \quad &\text{on} \quad \Gamma_{\mathrm{in}}, \\ \delta\sigma(\delta u, \delta p) n = 0, \quad &\text{on} \quad \Gamma_{\mathrm{out}}, \\ \delta u = \Phi \dot{\delta s} - \nabla u_0 \Phi \delta s, \quad &\text{on} \quad \gamma. \end{aligned}$$

Similarly, by taking $\hat{v}^{\mathrm{f}}=0$, the linearized solid subproblem is given by

$$(\mathrm{M}\,\ddot{\delta s}+\mathrm{K}\,\delta s)\cdot\hat{t}=\int_\gamma \Phi^{\mathrm{T}}\big(\sigma(u_0,p_0)\eta(\delta x^{\mathrm{s}})-\nabla\sigma(u_0,p_0)\delta x^{\mathrm{s}}n-\delta\sigma(\delta u,\delta p)n\big)\,\mathrm{d}a\cdot\hat{t},$$

for all $\hat{t}\in\mathbb{R}^{n^{\mathrm{s}}}$.

Therefore, in strong form, this linear coupled problem verified by the fluctuation $(\delta u,\delta p,\delta s)$ is given by:

$$\begin{aligned}
\rho\frac{\partial\delta u}{\partial t}+\mathrm{div}\,\big(\rho u_0\otimes\delta u+\rho\delta u\otimes u_0-\delta\sigma(\delta u,\delta p)\big)&=0, \quad \text{in}\quad \Omega^{\mathrm{f}},\\
\mathrm{div}\,\delta u&=0, \quad \text{in}\quad \Omega^{\mathrm{f}},\\
\delta u&=0, \quad \text{on}\quad \Gamma_{\mathrm{in}},\\
\delta\sigma(\delta u,\delta p)n&=0, \quad \text{on}\quad \Gamma_{\mathrm{out}},\\
\delta u&=\Phi\dot{\delta s}-\nabla u_0\Phi\delta s, \quad \text{on}\quad \gamma,\\
\mathrm{M}\,\ddot{\delta s}+\mathrm{K}\,\delta s=-\int_\gamma \Phi^{\mathrm{T}}\big(\delta\sigma(\delta u,\delta p)n&+\nabla\sigma(u_0,p_0)\Phi\delta s n-\sigma(u_0,p_0)\eta(\Phi\delta s)\big)\,\mathrm{d}a,\\
\big(\delta u,\delta s,\dot{\delta s}\big)_{|t=0}&=\big(\delta u_0,\delta s_0,\delta s_1\big).
\end{aligned} \tag{18}$$

After division by ρ in $(18)_1$, by setting $\nu=\mu/\rho$ the kinematic viscosity, by introducing the following real $n^{\mathrm{s}}\times n^{\mathrm{s}}$ matrix B^0, defining the sensitivity of the frozen stress interface vector to translation and rotation,

$$\mathrm{B}^0_{ij}=\int_\gamma\big(\nabla\sigma(u_0,p_0)\varphi_j n-\sigma(u_0,p_0)\eta(\varphi_j)\big)\cdot\varphi_i\,\mathrm{d}a,\quad 1\le i,j\le n^{\mathrm{s}}, \tag{19}$$

and since δu and u_0 are divergence free, the coupled problem (18) takes the following more compact form:

$$
\begin{aligned}
\frac{\partial \delta u}{\partial t} + \nabla u_0 \delta u + \nabla \delta u u_0 - 2\nu \operatorname{div} \varepsilon(\delta u) + \frac{1}{\rho}\nabla \delta p = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} \delta u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\delta u = 0, &\quad \text{on} \quad \Gamma_{\mathrm{in}},\\
\delta\sigma(\delta u, \delta p) n = 0, &\quad \text{on} \quad \Gamma_{\mathrm{out}},\\
\delta u = \Phi \dot{\delta s} - \nabla u_0 \Phi \delta s, &\quad \text{on} \quad \gamma,\\
\mathrm{M}\, \ddot{\delta s} + \left(\mathrm{K} + \mathrm{B}^0\right) \delta s = -\int_\gamma \Phi^{\mathrm{T}} \delta\sigma(\delta u, \delta p) n \, \mathrm{d}a. &
\end{aligned}
\tag{20}
$$

We recover in (20) the linearized Navier-Stokes equations on a fixed domain provided with a transpiration boundary condition on the fixed interface γ. For the structure, the linearization introduces a non-standard geometric term of "added stiffness" $\mathrm{B}^0\, \delta s$. Therefore, the obtained problem allows us to take into account the motion of the structure, while keeping a fixed fluid domain.

As pointed out in [11], the fundamental idea of this linearization comes from the fluctuation definition (13), which leads to the transpiration interface condition $(20)_5$, and from the transported problem (16), which enables us to transport the distributed fluid equation dependencies on δx^{f} to the fluid-structure fixed interface γ.

3.3 Spectral problem definition

Finally, the last step in the *Linearization Principle*, see figure 1, is the analysis of the harmonic solutions of (20), i.e. solutions in the form of normal modes,

$$
\delta u(x,t) = u(x) e^{-\lambda t}, \quad \delta p(x,t) = p(x) e^{-\lambda t}, \quad \delta s(t) = s e^{-\lambda t}, \tag{21}
$$

with $\lambda \in \mathbb{C}$, $u : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}^3$, $p : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}$ and $s \in \mathbb{C}^{n^{\mathrm{s}}}$.

By transferring the expressions (21) in (20) we obtain the following quadratic spectral problem:

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p = \lambda u, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
u = 0, &\quad \text{on} \quad \Gamma_{\mathrm{in}}, \\
\sigma(u,p)n = 0, &\quad \text{on} \quad \Gamma_{\mathrm{out}}, \\
u = -\lambda \Phi s - \nabla u_0 \Phi s, &\quad \text{on} \quad \gamma, \\
\lambda^2 \, \mathrm{M} \, s + \left(\mathrm{K} + \mathrm{B}^0\right) s = -\int_\gamma \Phi^{\mathrm{T}} \sigma(u,p) n \, \mathrm{d}a, &
\end{aligned}
\tag{22}
$$

with unknown u, p, s and λ and where

$$
\sigma(u,p) = -p \, \mathrm{I} + 2\mu\varepsilon(u), \quad \varepsilon(u) = \frac{1}{2}\left[\nabla u + (\nabla u)^{\mathrm{T}}\right],
$$

and B^0 is given by (19).

Therefore, as in definition 3.2, the stability problem of the coupled equilibrium state (u_0, p_0, I) can be formulated from the behavior of the solutions of the new coupled spectral problem (22). In this way, we state the following definition.

Definition 3.5 *We will say that the coupled steady state* (u_0, p_0, I) *is linearly asymptotically stable, if eigenproblem* (22) *does not have eigenvalues with negative real part. However, this state will be called unstable if there exists, at least, one eigenvalue with negative real part.*

From a mathematical point of view, the difficulty now is to prove that results of the linear theory remain valid for the non-linear theory. In others words, we should prove that a linearly asymptotically stable coupled state is stable (in the sense of Liapunov), and that a linearly unstable coupled state is unstable. Generally, this proof corresponds to the reciprocal part of the *Linearization Principle*, analogous to the Liapunov's theorem for ordinary differential equations, see [15, 23, 25] for a version of this theorem in hydrodynamic stability. A rigorous mathematical justification for the sequel of our approach would require a generalization of these theorems to the fluid-structure interaction case. This could be a forthcoming topic of our work, and will not be treated in this paper.

Remark 3.6 *We must point out that the linearization method developed in [11] and, in particular, the approach presented here, does not depend on the structure of the equations. Namely, we could use another model for the fluid or the structure. Nevertheless, our choice is motivated by the mathematical analysis provided in the next section, and the numerical experiments performed in part II (see [13]).*

4 Mathematical analysis: spectrum characterization

This section provides a first rigorous mathematical study of the new spectral problem obtained in the preceding section, arising from the linear stability analysis of fluid-structure coupled systems. The eigenproblem couples the linearized Navier-Stokes equations and those of a reduced linear structure via transpiration boundary conditions on a fixed interface.

4.1 Mathematical framework

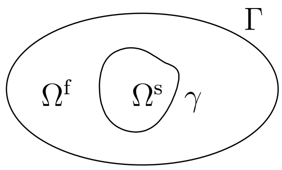

Figure 4: The computational domain, Ω, defined by the system in its equilibrium configuration

In the sequel we consider functions defined in bounded subsets of $\mathbb{R}^3$, and taking values in the complex field $\mathbb{C}$. Thus, all L^p and Sobolev spaces appearing in this section are taken as complex vector spaces of functions with complex values, see [6]. Let Ω to be an open bounded subset of $\mathbb{R}^3$, with locally Lipschitz continuous boundary Γ, see [21]. We assume that there exists a non-empty connected open subset Ω^{s} of Ω, with locally Lipschitz continuous boundary γ, and such that $\overline{\Omega}^{\mathrm{s}} \subset \Omega$. With this notations we set $\Omega^{\mathrm{f}} = \Omega - \overline{\Omega}^{\mathrm{s}}$, see figure 4.

On the one hand, we introduce a velocity field $u_0 : \Omega^{\mathrm{f}} \longrightarrow \mathbb{R}^3$ and a pressure field $p_0 : \Omega^{\mathrm{f}} \longrightarrow \mathbb{R}$, which are solution of the following steady Navier-Stokes problem:

$$
\begin{aligned}
\nabla u_0 u_0 - 2\nu \operatorname{div} \varepsilon(u_0) + \frac{1}{\rho}\nabla p_0 = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u_0 = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
u_0 = u_\Gamma, &\quad \text{on} \quad \Gamma,\\
u_0 = 0, &\quad \text{on} \quad \gamma,
\end{aligned} \tag{23}
$$

with

$$
\varepsilon(u_0) = \frac{1}{2}\left[\nabla u_0 + (\nabla u_0)^{\mathrm{T}}\right],
$$

$\rho > 0$ stands for the volume fluid density, $\nu > 0$ for the kinematic viscosity, and u_Γ for the prescribed velocity on Γ. We will assume that u_0 and p_0 are smooth functions, let us say for instance $u_0 \in \mathcal{C}^2\big(\overline{\Omega}^{\mathrm{f}}\big)^3$ and $p_0 \in \mathcal{C}^1\big(\overline{\Omega}^{\mathrm{f}}\big)$. And on the other hand, we introduce a finite number, n^{s}, of deformed modal shapes,

$$
\varphi_i : \Omega^{\mathrm{s}} \longrightarrow \mathbb{R}^3, \quad i = 1, \dots, n^{\mathrm{s}},
$$

that we will suppose volume preserving, i.e.,

$$
\int_\gamma \varphi_i \cdot n \,\mathrm{d}a = 0, \quad i = 1, \dots, n^{\mathrm{s}}. \tag{24}
$$

Here n stands for the unit normal vector on γ (pointing inside Ω^{s}). In the sequel, the $3 \times n^{\mathrm{s}}$ matrix,

$$
\Phi = \left[\varphi_1 | \varphi_2 | \dots | \varphi_{n^{\mathrm{s}}}\right],
$$

stands for the reduced modal basis. The behavior of the structure is assumed to be characterized by tangential mass and stiffness $n^{\mathrm{s}} \times n^{\mathrm{s}}$ matrices, M and K respectively, which are symmetric and positive definite.

Remark 4.1 *The regularity imposed on the reference flow (u_0, p_0) is not optimal. We only have chosen, for simplicity, spaces where the computations that we will carry out make sense.*

With this data (22) reduces to the following spectral problem: find $\lambda \in \mathbb{C}$, $u : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}^3$, $p : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}$ and $s \in \mathbb{C}^{n^{\mathrm{s}}}$, with $\displaystyle\int_{\Omega^{\mathrm{f}}} p \,\mathrm{d}x = 0$ and $(u, p, s) \neq 0$, such that

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho} \nabla p = \lambda u, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
u = 0, &\quad \text{on} \quad \Gamma, \\
u = -\lambda \Phi s - \nabla u_0 \Phi s, &\quad \text{on} \quad \gamma, \\
\lambda^2 \,\mathrm{M}\, s + \left(\mathrm{K} + \mathrm{B}^0\right) s = - \int_\gamma \Phi^{\mathrm{T}} \sigma(u, p) n \,\mathrm{d}a, &
\end{aligned}
\tag{25}
$$

where $\sigma(u, p) = -p\,\mathrm{I} + 2\mu\varepsilon(u)$, $\mu = \rho\nu$ represents the kinematic viscosity of the fluid, and B^0 is a real $n^{\mathrm{s}} \times n^{\mathrm{s}}$ matrix, given by (19):

$$
\mathrm{B}^0_{ij} = \int_\gamma \Big\{ \nabla \sigma(u_0, p_0) \varphi_j n + \sigma(u_0, p_0) \left[\mathrm{I} \operatorname{div} \varphi_j - (\nabla \varphi_j)^{\mathrm{T}} \right] n \Big\} \cdot \varphi_i \,\mathrm{d}a,
$$

for $1 \leq i, j \leq n^{\mathrm{s}}$.

Remark 4.2 *Compared to problem* (22)*, we have assumed here that the external boundary* Γ *is sufficiently far away from the structure, in order to impose a Dirichlet condition on the totality of* Γ*. This simplifies the analysis enabling us to apply the classical theory for Stokes problems [16, 8].*

4.1.1 Analysis of two particular examples

Let us consider now the particular case of problem (25) where the fluid is at rest at the equilibrium $(u_0, p_0) = 0$, the structure is rigid in translation, $n^{\mathrm{s}} = 3$ and $\Phi = \mathrm{I}$, with diagonal matrices M and K, $\mathrm{M} = m\,\mathrm{I}$, $\mathrm{K} = k\,\mathrm{I}$. Here, $m, k > 0$ are real given data. In this case we have $\mathrm{B}^0 = 0$. Hence, problem (25) yields

$$
\begin{aligned}
-2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p = \lambda u, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
u = 0, &\quad \text{on} \quad \Gamma,\\
u = -\lambda s, &\quad \text{on} \quad \gamma,\\
\lambda^2 m s + k s = -\int_{\gamma} \sigma(u,p) n \,\mathrm{d}a, &
\end{aligned}
\tag{26}
$$

which involves the Stokes equations. This kind of problem has been already proposed and completely studied in [4], for the determination of the vibration frequencies of a tube rack immersed in a viscous fluid at rest, see also [5]. Condition $(26)_4$ allows to eliminate the displacement s, leading to a purely fluid problem with non-local boundary conditions on γ. In [4] it is shown that problem (26) has eigenvalues, and those can be calculated from the characteristic values of a certain compact operator. Explicit estimates on the eigenvalues are also obtained, yielding that there is a finite number of eigenvalues with non-zero imaginary part. Numerical computations on (26) have been carried out in [3], corroborating the theory developed in [4].

Consider now a next example where the structure is rigid immersed in a permanent flow $(u_0, p_0) \neq 0$, with $n^{\mathrm{s}} = 3$, $\Phi = \mathrm{I}$, $\mathrm{M} = m\,\mathrm{I}$, $\mathrm{K} = k\,\mathrm{I}$ and where we neglect B^0 and the gradient term ∇u_0 in the transpiration condition. The quadratic eigenvalue problem (25) becomes

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p = \lambda u, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
u = 0, &\quad \text{on} \quad \Gamma,\\
u = -\lambda s, &\quad \text{on} \quad \gamma,\\
\lambda^2 m s + k s = -\int_{\gamma} \sigma(u,p) n \,\mathrm{d}a. &
\end{aligned}
\tag{27}
$$

This problem, involving the linearized Navier-Stokes equations, has been proposed in [22, 5] generalizing (26) to the case of a tube rack placed in a cross-flow. To our knowledge, no rigorous analysis on this problem has been carried out until the present.

4.2 Eigenvalues characterization

We are now tackling the mathematical analysis of problem (25). The main principle consists in defining a compact operator which characterizes the solutions of (25), see [4, 5, 7]. The classical theory of Stokes problems [16, 8] as well as fluid condensation techniques [10] will be used in this paragraph.

4.2.1 Linearization and shift

In order to linearize the quadratic term in (25) we introduce the modal velocity $z = -\lambda s \in \mathbb{C}^{n^{\mathrm{s}}}$ as new unknown, so that eigenproblem (25) can be rewritten as: find $\lambda \in \mathbb{C}$, $u : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}^3$, $p : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}$ and $s, z \in \mathbb{C}^{n^{\mathrm{s}}}$, with $\int_{\Omega^{\mathrm{f}}} p \,\mathrm{d}x = 0$ and $(u, p, s, z) \neq 0$, such that

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p = \lambda u, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
u = 0, &\quad \text{on} \quad \Gamma,\\
u = \Phi z - \nabla u_0 \Phi s, &\quad \text{on} \quad \gamma,\\
-z = \lambda s, &\\
\left(\mathrm{K} + \mathrm{B}^0\right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u,p) n \,\mathrm{d}a = \lambda \,\mathrm{M}\, z. &
\end{aligned}
\tag{28}
$$

We want to characterize the eigenvalues of (28) from the eigenvalues of a specific compact operator. By rewriting (28) in the following way: find $\lambda \in \mathbb{C}$, $u : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}^3$, $p : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}$ and $s, z \in \mathbb{C}^{n^{\mathrm{s}}}$, with $\int_{\Omega^{\mathrm{f}}} p \,\mathrm{d}x = 0$ and $(u, p, s, z) \neq 0$, such that

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p = \lambda u, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u = 0, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
u = 0, \quad \text{on} \quad \Gamma,\\
u = \Phi z - \nabla u_0 \Phi s, \quad \text{on} \quad \gamma,\\
-z = \lambda s,\\
\mathrm{M}^{-1}\left[\left(\mathrm{K}+\mathrm{B}^0\right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u,p) n \,\mathrm{d}a\right] = \lambda z.
\end{aligned}
\tag{29}
$$

it is clear that (at least formally) problem (29) can be formulated in the classical form,

$$
A\begin{pmatrix} u\\ s\\ z\end{pmatrix} = \lambda \begin{pmatrix} u\\ s\\ z\end{pmatrix}.
$$

In fact, we will focus on the "inverse" problem, i.e., on finding the eigenvalues of A^{-1}:

$$
A^{-1}\begin{pmatrix} u\\ s\\ z\end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} u\\ s\\ z\end{pmatrix},
$$

because A^{-1} will be a compact operator. Both problems are clearly equivalent. Indeed, if $\lambda \neq 0$ is solution of (29) then $1/\lambda$ is an eigenvalue of A^{-1}, and conversely, if $\omega \neq 0$ is an eigenvalue of A^{-1} then $\lambda = 1/\omega$ is an eigenvalue in (29), always with the same associated eigenfunction. Nevertheless, this approach requires the inversion of a certain operator A, which could have no inverse (λ can be zero). In order to overcome this difficulty, see [25], we introduce the following change of variable:

$$
\lambda = \omega - r, \tag{30}
$$

with $r > 0$ a real shift, to be fixed at a sufficiently large value, and $\omega \in \mathbb{C}$ standing for the new unknown. In an intuitive way, we can say that r is chosen rather large so that the fluid and solid subproblems are well posed, operator A^{-1} being thus well defined. By taking into account the change of variable (30) in (29), we obtain that $(\lambda; u, p, s, z)$ is a solution of (29) if and only if $(\omega; u, p, s, z)$ is a solution of the following eigenproblem: find $\omega \in \mathbb{C}$, $u : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}^3$, $p : \Omega^{\mathrm{f}} \longrightarrow \mathbb{C}$ and $s, z \in \mathbb{C}^{n^{\mathrm{s}}}$, with $\displaystyle\int_{\Omega^{\mathrm{f}}} p \,\mathrm{d}x = 0$ and $(u, p, s, z) \neq 0$, such that

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho} \nabla p + r u = \omega u, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u = 0, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
u = 0, \quad \text{on} \quad \Gamma,\\
u = \Phi z - \nabla u_0 \Phi s, \quad \text{on} \quad \gamma,\\
-z + r s = \omega s\\
\mathrm{M}^{-1} \left[\left(\mathrm{K} + \mathrm{B}^0 \right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u, p) n \,\mathrm{d}a \right] + r z = \omega z.
\end{aligned}
\tag{31}
$$

This eigenproblem motivates the definition of a certain operator T, which we will introduce in the following paragraph.

4.2.2 Operator definition

We introduce the Hilbert space $\mathbb{H} = L^2(\Omega^{\mathrm{f}})^3 \times \mathbb{C}^{n^{\mathrm{s}}} \times \mathbb{C}^{n^{\mathrm{s}}}$, and the operator

$$
T : (f, h, g) \in \mathbb{H} \longrightarrow T(f, g, h) = (u, z, s) \in H^1(\Omega^{\mathrm{f}})^3 \times \mathbb{C}^{n^{\mathrm{s}}} \times \mathbb{C}^{n^{\mathrm{s}}},
$$

where (u, z, s) is defined as the "solution" (for r sufficiently large) of the following coupled problem (obtained from (31)):

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p + r u = f, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
u = 0, &\quad \text{on} \quad \Gamma,\\
u = \Phi z - \nabla u_0 \Phi s, &\quad \text{on} \quad \gamma,\\
-z + r s = g, &\\
\mathrm{M}^{-1}\left[\left(\mathrm{K} + \mathrm{B}^0\right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u,p) n \,\mathrm{d}a\right] + r z = h. &
\end{aligned}
\tag{32}
$$

Remark 4.3 *The underlying idea in the definition of operator T comes from the fact that if $(\lambda; u, p, z, s)$ is a solution of (29), then $\omega = \lambda + r$ satisfies $\omega T(u,z,s) = (u,z,s)$, thus $\omega \neq 0$ and $(1/\omega; u, z, s)$ is an eigenpair of T, and conversely, if $\omega \neq 0$ is an eigenvalue of T then $1/\omega - r$ is an eigenvalue in (29). The proof of this equivalence will be detailed later (theorem 4.13).*

From $(32)_5$ $z = rs - g$, so that we can eliminate z in (32). Thus, problem (32) defining T can be reformulated as: for each $(f,h,g) \in \mathbb{H}$ find $(u,p,s) \in H^1(\Omega^{\mathrm{f}})^3 \times L^2_0(\Omega^{\mathrm{f}}) \times \mathbb{C}^{n^{\mathrm{s}}}$ such that

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p + r u = f, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
u = 0, &\quad \text{on} \quad \Gamma,\\
u = -\Phi g - \nabla u_0 \Phi s + r \Phi s, &\quad \text{on} \quad \gamma,\\
\left(\mathrm{K} + \mathrm{B}^0\right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u,p) n \,\mathrm{d}a + r^2\, \mathrm{M}\, s = \mathrm{M}(h + r g), &
\end{aligned}
\tag{33}
$$

where

$$
L^2_0(\Omega^{\mathrm{f}}) = \left\{ q \in L^2(\Omega^{\mathrm{f}}) \,\middle|\, \int_{\Omega^{\mathrm{f}}} q \,\mathrm{d}x = 0 \right\}.
$$

Remark 4.4 *To show that operator T is well defined, we have to prove that problem (33) has one and only one solution.*

4.2.3 Technical lemmas

The proof of existence and uniqueness for problem (33) requires the introduction of some preliminary results. We start decomposing (u, p) in the following way:

$$(u, p) = (u_1, p_1) + (u_2, p_2) + (u_3, p_3), \tag{34}$$

where (u_1, p_1) is solution of the purely fluid problem

$$\begin{aligned} \nabla u_0 u_1 + \nabla u_1 u_0 - 2\nu \operatorname{div} \varepsilon(u_1) + \frac{1}{\rho}\nabla p_1 + r u_1 = f, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ \operatorname{div} u_1 = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ u_1 = 0, &\quad \text{on} \quad \Gamma, \\ u_1 = -\Phi g, &\quad \text{on} \quad \gamma, \end{aligned} \tag{35}$$

(u_2, p_2) is solution of

$$\begin{aligned} \nabla u_0 u_2 + \nabla u_2 u_0 - 2\nu \operatorname{div} \varepsilon(u_2) + \frac{1}{\rho}\nabla p_2 + r u_2 = 0 &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ \operatorname{div} u_2 = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ u_2 = 0, &\quad \text{on} \quad \Gamma, \\ u_2 = -\nabla u_0 \Phi s, &\quad \text{on} \quad \gamma, \end{aligned} \tag{36}$$

and (u_3, p_3, s) is solution of

$$
\begin{aligned}
\nabla u_0 u_3 + \nabla u_3 u_0 - 2\nu \operatorname{div} \varepsilon(u_3) + \frac{1}{\rho}\nabla p_3 + r u_3 &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} u_3 &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}},\\
u_3 &= 0, \quad \text{on} \quad \Gamma,\\
u_3 &= r\Phi s, \quad \text{on} \quad \gamma,\\
\left(\mathrm{K} + r^2 \,\mathrm{M} + \mathrm{B}^0\right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u_1, p_1) n \,\mathrm{d}a + \int_\gamma \Phi^{\mathrm{T}} \sigma(u_2, p_2) n \,\mathrm{d}a &\\
+ \int_\gamma \Phi^{\mathrm{T}} \sigma(u_3, p_3) n \,\mathrm{d}a &= \mathrm{M}(h + rg).
\end{aligned}
\tag{37}
$$

Clearly, if (u_1, p_1), (u_2, p_2), and (u_3, p_3, s) are solutions of (35), (36) and (37) respectively, the triplet (u, p, s), defined by (34), is solution of (33).

In the sequel we will use the following result.

Lemma 4.5 *Let $v \in \mathcal{C}^1\left(\overline{\Omega}^{\mathrm{f}}\right)^3$ to be a smooth vector function which satisfies*

$$
\begin{aligned}
\operatorname{div} v &= 0, \quad \textit{in} \quad \Omega^{\mathrm{f}},\\
v &= 0, \quad \textit{on} \quad \gamma.
\end{aligned}
$$

and $x \in H^{1/2}(\gamma)^3$ such that

$$
\int_\gamma x \cdot n \,\mathrm{d}a = 0. \tag{38}
$$

Then we have

$$
\int_\gamma (\nabla v x) \cdot n \,\mathrm{d}a = 0. \tag{39}
$$

PROOF. We first lift x with a function $\tilde{x} \in H^1(\Omega^{\mathrm{f}})^3$ satisfying

$$\begin{aligned} \operatorname{div} \tilde{x} = 0, \quad &\text{in} \quad \Omega^{\mathrm{f}}, \\ \tilde{x} = x, \quad &\text{on} \quad \gamma, \\ \int_\Gamma \tilde{x} \cdot n \, \mathrm{d}a = 0. & \end{aligned} \tag{40}$$

We will clarify later what kind of lift to chose and how to obtain it. Since $\operatorname{div} v = 0$ and $\operatorname{div} \tilde{x} = 0$ in Ω^{f}, it is straightforward to verify that

$$\operatorname{div}(\nabla \tilde{x} v - \nabla v \tilde{x}) = (\tilde{x}_{i,j} v_j - v_{i,j} \tilde{x}_j)_{,i} = \nabla \tilde{x} : (\nabla v)^{\mathrm{T}} - \nabla v : (\nabla \tilde{x})^{\mathrm{T}} = 0,$$

in $\mathcal{D}'(\Omega^{\mathrm{f}})$. Hence, we get that $(\nabla \tilde{x} v - \nabla v \tilde{x}) \in H(\operatorname{div}; \Omega^{\mathrm{f}})$. Therefore, the Green formula yields

$$\int_{\Gamma \cup \gamma} (\nabla \tilde{x} v - \nabla v \tilde{x}) \cdot n \, \mathrm{d}a = \int_{\Omega^{\mathrm{f}}} \operatorname{div}(\nabla \tilde{x} v - \nabla v \tilde{x}) = 0,$$

see [16]. In addition, since $v = 0$ and $\tilde{x} = x$ on γ, the preceding identity implies that

$$\int_\gamma (\nabla v x) \cdot n \, \mathrm{d}a = \int_\Gamma (\nabla \tilde{x} v - \nabla v \tilde{x}) \cdot n \, \mathrm{d}a. \tag{41}$$

In summary, we have shown that, each lift $\tilde{x} \in H^1(\Omega^{\mathrm{f}})^3$ of x in Ω^{f} verifying (40) satisfies (41). To obtain (39), we have just to construct a lift $\tilde{x}$, verifying (40) and for which the left handside in (41) cancels. We proceed in the following way: since $\Gamma \cap \gamma = \emptyset$ we can build a neighborhood, ω_ϵ, of Γ in Ω^{f} such that $\overline{\omega}_\epsilon \cap \gamma \neq \emptyset$, see figure 5.

We set $\tilde{x} = v$ in ω_ϵ. In $\Omega^{\mathrm{f}} - \omega_\epsilon$ we take $\tilde{x}$ such that

$$\begin{aligned} \operatorname{div} \tilde{x} = 0, \quad &\text{in} \quad \Omega^{\mathrm{f}} - \omega_\epsilon, \\ \tilde{x} = x, \quad &\text{on} \quad \gamma, \\ \tilde{x} = v, \quad &\text{on} \quad \Gamma_\epsilon, \end{aligned}$$

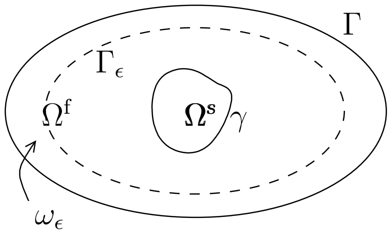

Figure 5: Neighborhood of Γ in Ω^{f}

which can be obtained by solving a traditional Stokes problem with Dirichlet boundary conditions, for instance

$$\begin{aligned} -\Delta \tilde{x} + \nabla \tilde{q} = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}} - \omega_\epsilon, \\ \operatorname{div} \tilde{x} = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}} - \omega_\epsilon, \\ \tilde{x} = x, &\quad \text{on} \quad \gamma, \\ \tilde{x} = v, &\quad \text{on} \quad \Gamma_\epsilon. \end{aligned}$$

Indeed, since $\operatorname{div} v = 0$, in Ω^{f}, and $v = 0$, on γ, we get

$$\int_{\Gamma_\epsilon} v \cdot n_\epsilon \, \mathrm{d}a_\epsilon = 0,$$

where n_ϵ stands for the unitary normal on Γ_ϵ pointing outside ω_ε. Thus, from (38) we deduce (see [16, 8]) that the above Stokes problem has an unique solution (x, q) in $H^1(\Omega^{\mathrm{f}} - \omega_\epsilon) \times L^2_0(\Omega^{\mathrm{f}} - \omega_\epsilon)$. Thus, the obtained lift $\tilde{x} \in H^1(\Omega^{\mathrm{f}})^3$ satisfies (40). Moreover, since $\tilde{x} = v$ and $\nabla \tilde{x} = \nabla v$ in ω_ϵ, the left handside in (41) cancels, which completes the proof. $\square$

Concerning the solution of problems (35) and (36),

$$\begin{aligned} \nabla u_0 u_1 + \nabla u_1 u_0 - 2\nu \operatorname{div} \varepsilon(u_1) + \frac{1}{\rho} \nabla p_1 + r u_1 = f, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ \operatorname{div} u_1 = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ u_1 = 0, &\quad \text{on} \quad \Gamma, \\ u_1 = -\Phi g, &\quad \text{on} \quad \gamma, \end{aligned}$$

$$
\begin{aligned}
\nabla u_0 u_2 + \nabla u_2 u_0 - 2\nu \operatorname{div} \varepsilon(u_2) + \frac{1}{\rho}\nabla p_2 + r u_2 = 0 \quad & \text{in} \quad \Omega^{\mathrm{f}}, \\
\operatorname{div} u_2 = 0, \quad & \text{in} \quad \Omega^{\mathrm{f}}, \\
u_2 = 0, \quad & \text{on} \quad \Gamma, \\
u_2 = -\nabla u_0 \Phi s, \quad & \text{on} \quad \gamma,
\end{aligned}
$$

we have the following result:

Theorem 4.6 *Let $l \in L^2(\Omega^{\mathrm{f}})^3$ and $\xi \in H^{\frac{1}{2}}(\gamma)^3$ with*

$$
\int_\gamma \xi \cdot n \, \mathrm{d}a = 0. \tag{42}
$$

Then for any $r \geq 3\|\varepsilon(u_0)\|_{0,\infty,\Omega^{\mathrm{f}}}$ the following problem

$$
\begin{aligned}
\nabla u_0 w + \nabla w u_0 - 2\nu \operatorname{div} \varepsilon(w) + \frac{1}{\rho}\nabla q + r w = l, \quad & \text{in} \quad \Omega^{\mathrm{f}}, \\
\operatorname{div} w = 0, \quad & \text{in} \quad \Omega^{\mathrm{f}}, \\
w = 0, \quad & \text{on} \quad \Gamma, \\
w = \xi, \quad & \text{on} \quad \gamma,
\end{aligned}
\tag{43}
$$

has an unique solution (w, q) in $H^1(\Omega^{\mathrm{f}})^3 \times L^2_0(\Omega^{\mathrm{f}})$, provided with the classic estimate

$$
\|q\|_{0,\Omega^{\mathrm{f}}} \leq C_1 \left(\|l\|_{0,\Omega^{\mathrm{f}}} + |w|_{1,\Omega^{\mathrm{f}}} + r\|w\|_{0,\Omega^{\mathrm{f}}} \right), \tag{44}
$$

where $C_1 > 0$ represents a constant which only depends on ρ, ν, u_0 and Ω^{f}. Moreover, fixing $\tilde{r} > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}}$, for any $r \geq \tilde{r}$ we get the following estimates:

$$
\begin{aligned}
\|w\|_{0,\Omega^{\mathrm{f}}} &\leq C_2 \left(1 + \frac{1}{\sqrt{r}}\right) \left(\|\xi\|_{\frac{1}{2},\Gamma} + \frac{1}{r}\|l\|_{0,\Omega^{\mathrm{f}}} \right), \\
|w|_{1,\Omega^{\mathrm{f}}} &\leq C_3 (1 + \sqrt{r}) \left(\|\xi\|_{\frac{1}{2},\Gamma} + \frac{1}{r}\|l\|_{0,\Omega^{\mathrm{f}}} \right),
\end{aligned}
\tag{45}
$$

where C_2 and C_3 are positive constants which only depend on ν, u_0, $\tilde{r}$ and Ω^{f}.

PROOF. Let us consider the following Hilbert space

$$H = \left\{ v \in H^1(\Omega^{\mathrm{f}})^3 \;\middle|\; \begin{array}{rcc} \operatorname{div} v = 0 & \text{in} & \Omega^{\mathrm{f}} \\ v = 0 & \text{on} & \Gamma \cup \gamma \end{array} \right\},$$

provided with the $H^1(\Omega^{\mathrm{f}})^3$-seminorm (from the Poincaré's inequality [2] this seminorm becomes a norm), a divergence free continuous linear lift operator (see [16]),

$$R : H^{\frac{1}{2}}(\gamma)^3 \longrightarrow \left\{ v \in H^1(\Omega^{\mathrm{f}})^3 \;\middle|\; \begin{array}{rcc} \operatorname{div} v = 0 & \text{in} & \Omega^{\mathrm{f}} \\ v = 0 & \text{on} & \Gamma \end{array} \right\},$$

defined for any $\xi \in H^{\frac{1}{2}}(\gamma)^3$ such that $\displaystyle\int_\gamma \xi \cdot n \,\mathrm{d}a = 0$, and the following continuous sesquilinear form:

$$a : (w, v) \in H \times H \longrightarrow a(w, v) = a_0(w, v) + r(w, \bar{v})_{0,\Omega^{\mathrm{f}}} \in \mathbb{C},$$

with

$$a_0(w, v) = (\nabla u_0 w + \nabla w u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + 2\nu(\varepsilon(w), \varepsilon(\bar{v}))_{0,\Omega^{\mathrm{f}}}.$$

With this notation, problem (43) can be reformulated in the following variational framework: find $w \in H^1(\Omega^{\mathrm{f}})^3$ such that

$$\begin{aligned} w - R(\xi) \in H, \\ a_0(w, v) + r(w, \bar{v})_{0,\Omega^{\mathrm{f}}} = (l, \bar{v})_{0,\Omega^{\mathrm{f}}}, \quad \forall v \in H. \end{aligned} \tag{46}$$

Thus, we obtain that $\hat{w} = w - R(\xi) \in H$ solves the following internal variational problem: find $\hat{w} \in H$ such as

$$a_0(\hat{w}, v) + r(\hat{w}, v)_{0,\Omega^{\mathrm{f}}} = (l, \bar{v})_{0,\Omega^{\mathrm{f}}} - a_0(R(\xi), v) - r(R(\xi), \bar{v})_{0,\Omega^{\mathrm{f}}}, \quad \forall v \in H. \tag{47}$$

At this point, we introduce a continuous antilinear form

$$L : v \in H \longrightarrow L(v) = (l, \bar{v})_{0,\Omega^{\mathrm{f}}} - a_0(R(\xi), v) - r(R(\xi), \bar{v})_{0,\Omega^{\mathrm{f}}},$$

so that problem (47) takes the following classic formalism: find $\hat{w} \in H$ such as

$$a(\hat{w}, v) = L(v), \quad \forall v \in H. \tag{48}$$

By construction we have

$$\operatorname{Re} a(v,v) = 2\nu \|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} + \operatorname{Re}(\nabla u_0 v, \bar{v}) + \operatorname{Re}(\nabla v u_0, \bar{v}) + r\|v\|^2_{0,\Omega^{\mathrm{f}}}. \tag{49}$$

By using the fact that u_0 is divergence free and that v cancels on $\Gamma \cup \gamma$, we obtain, on the one hand,

$$\begin{aligned}
2\operatorname{Re}(\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} &= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + \overline{(\nabla v u_0, \bar{v})}_{0,\Omega^{\mathrm{f}}} \\
&= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + (v, \nabla \bar{v} u_0)_{0,\Omega^{\mathrm{f}}} \\
&= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + \int_{\Omega^{\mathrm{f}}} (v \otimes u_0) : \nabla \bar{v} \,\mathrm{d}x \\
&= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} - \int_{\Omega^{\mathrm{f}}} (\nabla v u_0 + v \operatorname{div} u_0) \cdot \bar{v} \,\mathrm{d}x \\
&\quad + \int_{\Gamma\cup\gamma} \|v\|^2 u_0 \cdot n \,\mathrm{d}a \\
&= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} - (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} \\
&= 0,
\end{aligned} \tag{50}$$

and on the other hand,

$$\begin{aligned}
\operatorname{Re}(\nabla u_0 v, \bar{v})_{0,\Omega^{\mathrm{f}}} &= \frac{1}{2}\left[(\nabla u_0 v, \bar{v})_{0,\Omega^{\mathrm{f}}} + \overline{(\nabla u_0 v, \bar{v})}_{0,\Omega^{\mathrm{f}}}\right] \\
&= \frac{1}{2}\left[(\nabla u_0 v, \bar{v})_{0,\Omega^{\mathrm{f}}} + (v, \nabla u_0 \bar{v})_{0,\Omega^{\mathrm{f}}}\right] \\
&= \frac{1}{2}\left[(\nabla u_0 v, \bar{v})_{0,\Omega^{\mathrm{f}}} + \left((\nabla u_0)^{\mathrm{T}} v, \bar{v}\right)_{0,\Omega^{\mathrm{f}}}\right] \\
&= (\varepsilon(u_0) v, \bar{v})_{0,\Omega^{\mathrm{f}}}.
\end{aligned} \tag{51}$$

Hence, from (49), (50) and (51), we get

$$\operatorname{Re} a(v,v) \geq 2\nu\|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} - 3\|\varepsilon(u_0)\|_{0,\infty,\Omega^{\mathrm{f}}}\|v\|^2_{0,\Omega^{\mathrm{f}}} + r\|v\|^2_{0,\Omega^{\mathrm{f}}}.$$

Therefore, by taking $r \geq 3\|\varepsilon(u_0)\|_{0,\infty,\Omega^{\mathrm{f}}}$ and from Korn's inequality [2], we deduce that the sesquilinear form a is H-elliptic, i.e.,

$$\operatorname{Re} a(v,v) \geq 2\nu\|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} \geq \alpha|v|^2_{1,\Omega^{\mathrm{f}}}, \quad \forall v \in H,$$

where $\alpha > 0$ stands for a constant which only depends on ν and Ω^{f}. Lax-Milgram's theorem [16] ensures the existence and uniqueness of $\hat{w}$ as solution of (48). We also obtain that $w = \hat{w} + R(\xi)$ is a solution of (46). The uniqueness of w comes from (46) and from Lax-Milgram's theorem. In addition, since v cancels on $\Gamma \cup \gamma$, integrating by parts in $(46)_2$ yields

$$(\nabla u_0 w + \nabla w u_0 - 2\nu \operatorname{div}\varepsilon(w) + rw - l, \bar{v})_{0,\Omega^{\mathrm{f}}} = 0, \quad \forall v \in H.$$

Consequently, there exists (see [8], propositions 1 and 2 of chapter 9) an unique distribution $q \in L^2_0(\Omega)$ such that

$$\begin{aligned} &\nabla u_0 w + \nabla w u_0 - 2\nu \operatorname{div}\varepsilon(w) + rw - l = -\frac{1}{\rho}\nabla q, \quad \text{in} \quad D'(\Omega^{\mathrm{f}})^3, \\ &\|q\|_{0,\Omega^{\mathrm{f}}} \leq c\|\nabla u_0 w + \nabla w u_0 - 2\nu \operatorname{div}\varepsilon(w) + rw - l\|_{-1,\Omega^{\mathrm{f}}}, \end{aligned} \tag{52}$$

where c represents a positive constant which only depends on ρ and Ω^{f}. On the one hand, equality $(52)_1$ allows us to complete the proof of existence and uniqueness of solution for (43) and, on the other hand, from $(52)_2$ we directly obtain the estimate (44). Obtaining the inequalities $(45)_{1,2}$ requests now a finer analysis.

By considering the real part of the expression (47) with $v = \hat{w}$, and from (50) and (51) we obtain

$$\begin{aligned} &(\varepsilon(u_0)\hat{w}, \hat{w})_{0,\Omega^{\mathrm{f}}} + 2\nu\|\varepsilon(\hat{w})\|^2_{0,\Omega^{\mathrm{f}}} + r\|\hat{w}\|^2_{0,\Omega^{\mathrm{f}}} \\ &\qquad = \operatorname{Re}\big((l,\hat{w})_{0,\Omega^{\mathrm{f}}} - a_0(R(\xi),\hat{w}) - r(R(\xi),\hat{w})_{0,\Omega^{\mathrm{f}}}\big) \end{aligned}$$

which implies

$$
\begin{aligned}
&-3\|\varepsilon(u_0)\|_{0,\infty,\Omega^{\mathrm f}}\|\hat w\|^2_{0,\Omega^{\mathrm f}} + 2\nu\|\varepsilon(\hat w)\|^2_{0,\Omega^{\mathrm f}} + r\|\hat w\|^2_{0,\Omega^{\mathrm f}}\\
&\quad \le \left|(l,\hat w)_{0,\Omega^{\mathrm f}} - a_0(R(\xi),\hat w) - r(R(\xi),\hat w)_{0,\Omega^{\mathrm f}}\right|\\
&\quad \le \big(\|l\|_{0,\Omega^{\mathrm f}} + r\|R(\xi)\|_{0,\Omega^{\mathrm f}} + \|\nabla u_0 R(\xi)\|_{0,\Omega^{\mathrm f}} + \|\nabla R(\xi) u_0\|_{0,\Omega^{\mathrm f}}\big)\,\|\hat w\|_{0,\Omega^{\mathrm f}}\\
&\qquad + 2\nu\|\varepsilon(R(\xi))\|_{0,\Omega^{\mathrm f}}\|\varepsilon(\hat w)\|_{0,\Omega^{\mathrm f}}\\
&\quad \le \big(\|l\|_{0,\Omega^{\mathrm f}} + r\|R(\xi)\|_{0,\Omega^{\mathrm f}} + 3\|\nabla u_0\|_{0,\infty,\Omega^{\mathrm f}}\|R(\xi)\|_{0,\Omega^{\mathrm f}}\\
&\qquad + \sqrt{3}\|u_0\|_{0,\infty,\Omega^{\mathrm f}}\|\nabla R(\xi)\|_{0,\Omega^{\mathrm f}}\big)\|\hat w\|_{0,\Omega^{\mathrm f}} + 2\nu\|R(\xi)\|_{1,\Omega^{\mathrm f}}\|\varepsilon(\hat w)\|_{0,\Omega^{\mathrm f}}\\
&\quad \le \Big[\|l\|_{0,\Omega^{\mathrm f}} + r\|R(\xi)\|_{1,\Omega^{\mathrm f}} + \big(3+\sqrt{3}\big)\|u_0\|_{1,\infty,\Omega^{\mathrm f}}\|R(\xi)\|_{1,\Omega^{\mathrm f}}\Big]\,\|\hat w\|_{0,\Omega^{\mathrm f}}\\
&\qquad + 2\nu\|R(\xi)\|_{1,\Omega^{\mathrm f}}\|\varepsilon(\hat w)\|_{0,\Omega^{\mathrm f}}.
\end{aligned}
\tag{53}
$$

Thus, from the continuity of R, from $H^{\frac{1}{2}}(\gamma)^3$ in $H^1(\Omega^{\mathrm f})^3$ and by supposing that $r \ge \|u_0\|_{1,\infty,\Omega^{\mathrm f}}$, the last inequality of (53) yields

$$
\begin{aligned}
&-\underbrace{3\|\varepsilon(u_0)\|_{0,\infty,\Omega^{\mathrm f}}}_{r_0}\|\hat w\|^2_{0,\Omega^{\mathrm f}} + 2\nu\|\varepsilon(\hat w)\|^2_{0,\Omega^{\mathrm f}} + r\|\hat w\|^2_{0,\Omega^{\mathrm f}}\\
&\qquad \le 2\nu c_1\|\xi\|_{\frac12,\Gamma}\|\varepsilon(\hat w)\|_{0,\Omega^{\mathrm f}} + c_2\left(\|l\|_{0,\Omega^{\mathrm f}} + r\|\xi\|_{\frac12,\Gamma}\right)\|\hat w\|_{0,\Omega^{\mathrm f}}\\
&\qquad = 2\nu\underbrace{c_1\|\xi\|_{\frac12,\Gamma}}_{2a}\underbrace{\|\varepsilon(\hat w)\|_{0,\Omega^{\mathrm f}}}_{x} + r\underbrace{c_2\left(\frac{1}{r}\|l\|_{0,\Omega^{\mathrm f}} + \|\xi\|_{\frac12,\Gamma}\right)}_{2b}\underbrace{\|\hat w\|_{0,\Omega^{\mathrm f}}}_{y},
\end{aligned}
\tag{54}
$$

where c_1 and c_2 are positive constants that only depend on u_0 and $\Omega^{\mathrm f}$.

By taking into account this new notation, and from (54), we obtain the following inequality

$$-r_0y^2 + 2\nu x^2 + ry^2 \le (2\nu)(2a)x + r2by.$$

We can complete the squares here, and then we get

$$2\nu(x-a)^2 + r(y-b)^2 \le 2\nu a^2 + rb^2 + r_0y^2. \tag{55}$$

In the sequel we will take $r > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}}$, which implies $r > r_0$. On the one hand, from inequality (55), we get

$$r(y-b)^2 \leq 2\nu a^2 + rb^2 + r_0 y^2,$$

i.e., after division by r

$$(y-b)^2 \leq \frac{2\nu}{r} a^2 + b^2 + \frac{r_0}{r} y^2.$$

By taking the square root, and since $\alpha^2 + \beta^2 + \varsigma^2 \leq (\alpha + \beta + \varsigma)^2$ for each $\alpha, \beta, \varsigma \geq 0$, we deduce

$$|y-b| \leq \frac{\sqrt{2\nu}}{\sqrt{r}} a + b + \sqrt{\frac{r_0}{r}} y,$$

which implies

$$y \leq \frac{\sqrt{2\nu}}{\sqrt{r}} a + 2b + \sqrt{\frac{r_0}{r}} y,$$

that is

$$y \left(1 - \sqrt{\frac{r_0}{r}}\right) \leq \frac{\sqrt{2\nu}}{\sqrt{r}} a + 2b,$$

i.e., finally

$$y \leq \frac{1}{1 - \sqrt{\frac{r_0}{r}}} \left(\frac{\sqrt{2\nu}}{\sqrt{r}} a + 2b \right). \tag{56}$$

Moreover, since

$$r \in (r_0, +\infty) \longmapsto 1 - \sqrt{\frac{r_0}{r}},$$

is an increasing function, for each fixed $\tilde{r} > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}} \geq r_0$ we have

$$1 - \sqrt{\frac{r_0}{r}} \geq 1 - \sqrt{\frac{r_0}{\tilde{r}}}, \quad \forall r \geq \tilde{r}.$$

From (56) and the preceding bound we arrive to the following estimate

$$y \leq c_3 \left(\frac{1}{\sqrt{r}} a + b \right), \quad \forall r \geq \tilde{r}. \tag{57}$$

where $c_3 > 0$ is a constant which only depends on r_0, $\tilde{r}$ and ν.

In the same way, from (55), we have

$$2\nu(x-a)^2 \leq 2\nu a^2 + rb^2 + r_0 y^2,$$

that is

$$(x-a)^2 \leq a^2 + \frac{r}{2\nu} b^2 + \frac{r_0}{2\nu} y^2.$$

By taking the square root, we obtain

$$|x-a| \leq a + \frac{\sqrt{r}}{\sqrt{2\nu}} b + \sqrt{\frac{r_0}{2\nu}} y,$$

which yields

$$x \leq 2a + \frac{\sqrt{r}}{\sqrt{2\nu}} b + \sqrt{\frac{r_0}{2\nu}} y.$$

Thus, with (57) and the preceding inequality we obtain that for $r \geq \tilde{r}$

$$x \leq 2a + \frac{\sqrt{r}}{\sqrt{2\nu}} b + c_3 \sqrt{\frac{r_0}{2\nu}} \left(\frac{1}{\sqrt{r}} a + b \right),$$

from where we deduce

$$x \leq \left(2 + c_3 \sqrt{\frac{r_0}{2\nu}} \frac{1}{\sqrt{r}} \right) a + \left(\frac{\sqrt{r}}{\sqrt{2\nu}} + c_3 \sqrt{\frac{r_0}{2\nu}} \right) b.$$

This can be written as

$$x \leq c_4 \left[a + (\sqrt{r} + 1) b \right], \quad \forall r \geq \tilde{r}, \tag{58}$$

where $c_4 > 0$ represents a constant which only depends on r_0, $\tilde{r}$ and ν.

Therefore, from (57), (58) and the considered notation, we have for each $r \geq \tilde{r}$

$$
\begin{aligned}
\|\varepsilon(\hat{w})\|_{0,\Omega^{\mathrm{f}}} &\leq c_4 \left[\frac{c_1}{2} \|\xi\|_{\frac{1}{2},\Gamma} + (\sqrt{r}+1)\frac{c_2}{2}\left(\frac{1}{r}\|l\|_{0,\Omega^{\mathrm{f}}} + \|\xi\|_{\frac{1}{2},\Gamma}\right)\right] \\
&\leq c_5(1+\sqrt{r})\left(\|\xi\|_{\frac{1}{2},\Gamma} + \frac{1}{r}\|l\|_{0,\Omega^{\mathrm{f}}}\right), \qquad (59)\\
\|\hat{w}\|_{0,\Omega^{\mathrm{f}}} &\leq c_3 \left[\frac{1}{\sqrt{r}}\frac{c_1}{2} \|\xi\|_{\frac{1}{2},\Gamma} + \frac{c_2}{2}\left(\frac{1}{r}\|l\|_{0,\Omega^{\mathrm{f}}} + \|\xi\|_{\frac{1}{2},\Gamma}\right)\right] \\
&\leq c_6\left(1+\frac{1}{\sqrt{r}}\right)\left(\|\xi\|_{\frac{1}{2},\Gamma} + \frac{1}{r}\|l\|_{0,\Omega^{\mathrm{f}}}\right), \qquad (60)
\end{aligned}
$$

with c_5 and c_6 positive constants which only depend on ν, u_0, $\tilde{r}$ and Ω^{f}. The estimates $(45)_{1,2}$ are obtained from the fact that $w = \hat{w} + R(\xi)$ and using Korn's inequality in (59), which completes the proof of this theorem. □

The following corollary can be directly obtained from the preceding theorem.

Corollary 4.7 *Let* $\tilde{r} > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}}$ *and* $s \in \mathbb{C}^{n^{\mathrm{s}}}$. *For* $r \geq \tilde{r}$ *problems* (35) *and* (36) *have an unique solution in* $H^1(\Omega^{\mathrm{f}})^3 \times L^2_0(\Omega^{\mathrm{f}})$, *and we have the following estimates:*

$$
\left.
\begin{aligned}
\|u_1\|_{0,\Omega^{\mathrm{f}}} &\leq C_4\left(1+\frac{1}{\sqrt{r}}\right)\left(\|g\| + \|f\|_{0,\Omega^{\mathrm{f}}}\right),\\
|u_1|_{1,\Omega^{\mathrm{f}}} &\leq C_5(1+\sqrt{r})\left(\|g\| + \|f\|_{0,\Omega^{\mathrm{f}}}\right),\\
\|p_1\|_{0,\Omega^{\mathrm{f}}} &\leq C_6\left(1+\sqrt{r}+r\right)\left(\|g\| + \|f\|_{0,\Omega^{\mathrm{f}}}\right),\\
\|u_2\|_{0,\Omega^{\mathrm{f}}} &\leq C_4\left(1+\frac{1}{\sqrt{r}}\right)\|s\|,\\
|u_2|_{1,\Omega^{\mathrm{f}}} &\leq C_5(1+\sqrt{r})\|s\|,\\
\|p_2\|_{0,\Omega^{\mathrm{f}}} &\leq C_6\left(1+\sqrt{r}+r\right)\|s\|,
\end{aligned}
\right\} \qquad (61)
$$

where C_4, C_5 *are* C_6 *are positive constants that only depend on* ρ, ν, u_0, $\tilde{r}$, Φ *and* Ω^{f}.

PROOF. Since u_1 and u_2 are divergence free, we have to check the compatibility condition of the trace on γ (42), i.e.

$$\int_\gamma (\Phi g) \cdot n \,\mathrm{d}a = 0, \qquad \int_\gamma (\nabla u_0 \Phi s) \cdot n \,\mathrm{d}a = 0.$$

The first identity is obvious because, from (24),

$$\int_\gamma \Phi^{\mathrm{T}} n \,\mathrm{d}a = 0,$$

and g is a constant vector. The second one can be obtained from a direct application of lemma 4.5, with $v = u_0$ and $x = \Phi_{|\gamma} s$. The corollary holds after direct application of theorem 4.6 to problems (35) and (36). $\square$

The above corollary allows us to completely determine (u_1, p_1) from the data f and g, thus we can transfer

$$\int_\gamma \Phi^{\mathrm{T}} \sigma(u_1, p_1) n \,\mathrm{d}a,$$

to the right handside in $(37)_5$, so that we only have to determine (u_2, p_2) and (u_3, p_3, s).

From the linearity of

$$\int_\gamma \Phi^{\mathrm{T}} \sigma(u_2, p_2) n \,\mathrm{d}a,$$

with respect to (u_2, p_2), and of (u_2, p_2) with respect to s, we can write

$$\int_\gamma \Phi^{\mathrm{T}} \sigma(u_2, p_2) n \,\mathrm{d}a,$$

as a linear function of s, i.e.,

$$\int_\gamma \Phi^{\mathrm{T}} \sigma(u_2, p_2) n \,\mathrm{d}a = \mathrm{F}(r) s, \tag{62}$$

with $\mathrm{F}(r)$ a $n^{\mathrm{s}} \times n^{\mathrm{s}}$ real matrix, associated to problem (36) and given by the following expression:

$$\mathrm{F}_{ij}(r) = \int_\gamma \big(\sigma(w_j, q_j)n\big) \cdot \varphi_i \, \mathrm{d}a,$$

where (w_j, q_j) is the unique solution (see corollary 4.7), of the fluid problem

$$\begin{aligned} \nabla u_0 w_j + \nabla w_j u_0 - 2\nu \operatorname{div} \varepsilon(w_j) + \frac{1}{\rho}\nabla q_j + r w_j = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ \operatorname{div} w_j = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\ w_j = 0, &\quad \text{on} \quad \Gamma, \\ w_j = -\nabla u_0 \varphi_j, &\quad \text{on} \quad \gamma, \end{aligned} \tag{63}$$

for $j = 1, \dots, n^{\mathrm{s}}$.

Remark 4.8 *The fact that we could express the fluid load on the interface as* (62)*, in other words, that we could condense the fluid effect from the computation of a finite number of elementary fluid solutions, is directly related to the assumption that the structural displacement is given as superposition of a finite number of vibration modes. Expression* (62) *is crucial in the sequel of our approach, and explains our assumption of a reduced behavior of the structure. In a more general case, for instance where the displacement of the structure satisfies the linear elastodynamic equations, our approach would not be valid.*

In the sequel we will need an estimate for $\mathrm{F}(r)$ in terms of r. This is the object of the following lemma.

Lemma 4.9 *Let $\tilde{r} > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}}$. For each $r \geq \tilde{r}$ we have*

$$\|\,\mathrm{F}(r)\| \leq C_7(1 + \sqrt{r} + r),$$

with C_7 a positive constant which only depends on ρ, ν, u_0, $\tilde{r}$, Φ and Ω^{f}.

PROOF. For $s \in \mathbb{R}^{n^{\mathrm{s}}}$, (u_2, p_2) is uniquely determined from corollary 4.7. From (62) we get, modulo some multiplicative constants independent of r and (u_2, p_2), the following estimate:

$$\begin{aligned}
\|\,\mathrm{F}(r)s\| &= \left\| \int_\gamma \Phi^{\mathrm{T}} \sigma(u_2,p_2) n \,\mathrm{d}a \right\| \\
&\le \|\sigma(u_2,p_2)\|_{H(\mathrm{div};\Omega^{\mathrm{f}})} \\
&\le \|\sigma(u_2,p_2)\|_{0,\Omega^{\mathrm{f}}} + \|\operatorname{div}\sigma(u_2,p_2)\|_{0,\Omega^{\mathrm{f}}} \\
&\le \|p_2\|_{0,\Omega^{\mathrm{f}}} + |u_2|_{1,\Omega^{\mathrm{f}}} + \|\nabla u_0 u_2 + \nabla u_2 u_0\|_{0,\Omega^{\mathrm{f}}} + r\|u_2\|_{0,\Omega^{\mathrm{f}}} \\
&\le \|p_2\|_{0,\Omega^{\mathrm{f}}} + |u_2|_{1,\Omega^{\mathrm{f}}} + r\|u_2\|_{0,\Omega^{\mathrm{f}}}. \qquad (64)
\end{aligned}$$

Corollary 4.7 provides estimates for $|u_2|_{1,\Omega^{\mathrm{f}}}$, $\|u_2\|_{0,\Omega^{\mathrm{f}}}$ and $\|p_2\|_{0,\Omega^{\mathrm{f}}}$ depending on r and s. Thus, from (64) and $(61)_{4,5,6}$, we get

$$\begin{aligned}
\|\,\mathrm{F}(r)s\| &\le \left[C_6\left(1+\sqrt{r}+r\right) + C_5(1+\sqrt{r}) + rC_4\left(1+\frac{1}{\sqrt{r}}\right)\right]\|s\| \\
&\le C_7\left(1+\sqrt{r}+r\right)\|s\|,
\end{aligned}$$

where C_7 is a constant which neither depends on r, u_2 nor s. Thus, we complete the proof of the lemma. □

Now let us take again the problem of determining (u_2,p_2) and (u_3,p_3,s). From (62) we can eliminate (u_2,p_2) in $(37)_5$ and, by multiplying by ρ equation $(37)_1$, the problem reduces to: find $(u_3,p_3,s) \in H^1(\Omega^{\mathrm{f}})^3 \times L^2_0(\Omega^{\mathrm{f}}) \times \mathbb{C}^{n^{\mathrm{s}}}$ verifying

$$\begin{aligned}
\rho\big(\nabla u_0 u_3 + \nabla u_3 u_0\big) - 2\mu \operatorname{div}\varepsilon(u_3) + \nabla p_3 + r\rho u_3 &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}}, \\
\operatorname{div} u_3 &= 0, \quad \text{in} \quad \Omega^{\mathrm{f}}, \\
u_3 &= 0, \quad \text{on} \quad \Gamma, \\
u_3 &= r\Phi s, \quad \text{on} \quad \gamma, \\
\big(\mathrm{K} + r^2\,\mathrm{M} + \mathrm{B}^0 + \mathrm{F}(r)\big)\, s + \int_\gamma \Phi^{\mathrm{T}}\sigma(u_3,p_3) n\,\mathrm{d}a & \\
= \mathrm{M}(h+rg) - \int_\gamma \Phi^{\mathrm{T}}\sigma(u_1,p_1)n\,\mathrm{d}a. &
\end{aligned} \qquad (65)$$

Lemma 4.10 *There exists* $r > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}}$ *such that the coupled problem* (65) *has an unique solution in* $H^1(\Omega^{\mathrm{f}})^3 \times L_0^2(\Omega^{\mathrm{f}}) \times \mathbb{C}^{n^{\mathrm{s}}}$. *In addition, for each* r *sufficiently large, we have the estimate*

$$\|s\| + |u_3|_{1,\Omega^{\mathrm{f}}} + \|p_3\|_{0,\Omega^{\mathrm{f}}} \leq C_8 \left(\|f\|_{0,\Omega^{\mathrm{f}}} + \|g\| + \|h\|\right), \qquad (66)$$

where $C_8 > 0$ *is a constant independent of* (u_3, p_3, s) *and* (f, g, h).

PROOF. We introduce the following Hilbert space

$$\mathbb{V} = \left\{ V = (v,t) \in H^1(\Omega^{\mathrm{f}})^3 \times \mathbb{C}^{n^{\mathrm{s}}} \;\middle|\; \begin{array}{rcl} \operatorname{div} v = 0 & \text{in} & \Omega^{\mathrm{f}} \\ v = 0 & \text{on} & \Gamma \\ v = r\Phi t & \text{on} & \gamma \end{array} \right\},$$

provided with the norm

$$\|(v,t)\|_{\mathbb{V}} = \left(|v|_{1,\Omega^{\mathrm{f}}}^2 + \|t\|^2\right)^{\frac{1}{2}}.$$

Let $V = (v,t) \in \mathbb{V}$, multiplying equation $(65)_1$ by $\bar{v}$ and integrating by parts, we get

$$\rho(\nabla u_0 u_3 + \nabla u_3 u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + 2\mu(\varepsilon(u_3), \varepsilon(\bar{v}))_{0,\Omega^{\mathrm{f}}} + r\rho(u_3, \bar{v})_{0,\Omega^{\mathrm{f}}} - r\int_\gamma \Phi^{\mathrm{T}} \sigma(u_3, p_3) n \,\mathrm{d}a \cdot \bar{t} = 0.$$

Now, multiplying equation $(65)_5$ by $r\bar{t}$ and by adding these two last expressions, we obtain the following variational formulation for the coupled problem (65): find $(u_3, s) \in \mathbb{V}$ such that

$$\rho(\nabla u_0 u_3 + \nabla u_3 u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + 2\mu(\varepsilon(u_3), \varepsilon(\bar{v}))_{0,\Omega^{\mathrm{f}}} + r\rho(u_3, \bar{v})_{0,\Omega^{\mathrm{f}}} + r\left(\mathrm{K} + \mathrm{B}^0\right) s \cdot \bar{t} + r^3\,\mathrm{M}\, s \cdot \bar{t} + r\,\mathrm{F}(r) s \cdot \bar{t} = r\,\mathrm{M}(rg + h) \cdot \bar{t} - r\int_\gamma \Phi^{\mathrm{T}} \sigma(u_1, p_1) n \,\mathrm{d}a \cdot \bar{t}, \quad \forall \in (v,t) \in \mathbb{V}. \qquad (67)$$

We introduce a sesquilinear form $A : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{C}$ and an antilinear form $L : \mathbb{V} \longrightarrow \mathbb{C}$ that are defined by

$$\begin{aligned} A\big((u,s),(v,t)\big) &= \rho(\nabla u_0 u + \nabla u u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + 2\mu(\varepsilon(u),\varepsilon(\bar{v}))_{0,\Omega^{\mathrm{f}}} + r\rho(u,\bar{v})_{0,\Omega^{\mathrm{f}}} \\ &\quad + r\left(\mathrm{K} + \mathrm{B}^0\right) s \cdot \bar{t} + r^3\,\mathrm{M}\, s \cdot \bar{t} + r\,\mathrm{F}(r) s \cdot \bar{t}, \\ L\big((v,t)\big) &= r\,\mathrm{M}(rg+h)\cdot \bar{t} - r\int_\gamma \Phi^{\mathrm{T}} \sigma(u_1,p_1) n \,\mathrm{d}a \cdot \bar{t}, \end{aligned}$$

for $U = (u,s)$ and $V = (v,t)$ in $\mathbb{V}$. In this way, problem (67) takes the following classical form: find $U = (u_3, s) \in \mathbb{V}$ such that

$$A(U,V) = L(V), \quad \forall V \in \mathbb{V}. \tag{68}$$

Let $V = (v,t) \in \mathbb{V}$. By integrating by parts in Ω^{f}, by using the fact that u_0 is divergence free and since $\|v\|^2 u_0 \cdot n$ vanishes on $\Gamma \cup \gamma$ (we remember that u_0 is solution of the reference problem (23) in Ω^{f}), we get the following expression similar to (50):

$$\begin{aligned} 2\,\mathrm{Re}(\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} &= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + \overline{(\nabla v u_0, \bar{v})}_{0,\Omega^{\mathrm{f}}} \\ &= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + (v, \nabla \bar{v} u_0)_{0,\Omega^{\mathrm{f}}} \\ &= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} + \int_{\Omega^{\mathrm{f}}} (v \otimes u_0) : \nabla \bar{v} \,\mathrm{d}x \\ &= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} - \int_{\Omega^{\mathrm{f}}} (\nabla v u_0 + v \,\mathrm{div}\, u_0)\cdot \bar{v}\,\mathrm{d}x \\ &\quad + \int_{\Gamma\cup\gamma} \|v\|^2 u_0 \cdot n \,\mathrm{d}a \\ &= (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} - (\nabla v u_0, \bar{v})_{0,\Omega^{\mathrm{f}}} \\ &= 0. \end{aligned} \tag{69}$$

Let $m = \min \sigma(M) > 0$ and $k = \min \sigma(\mathrm{K}) > 0$, where $\sigma(\mathrm{M})$ and $\sigma(K)$ stand, respectively, for the spectrum of M and K. We fix $\tilde{r} > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}}$ and we take $r \geq \tilde{r}$. From (51) and (69), we obtain that for each $V = (v,t) \in \mathbb{V}$

$$
\begin{aligned}
\operatorname{Re} A(V,V) &= \rho \operatorname{Re}(\nabla u_0 v, \bar{v})_{0,\Omega^{\mathrm{f}}} + 2\mu \|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} + r\rho \|v\|^2_{0,\Omega^{\mathrm{f}}} \\
&\quad + r \left(\mathrm{K} + r^2 \, \mathrm{M} \right) t \cdot \bar{t} + r \operatorname{Re}\left(\mathrm{B}^0 \, t \cdot \bar{t} \right) + r \operatorname{Re}\left(\mathrm{F}(r) t \cdot \bar{t} \right) \\
&\geq \rho(\varepsilon(u_0) v, \bar{v})_{0,\Omega^{\mathrm{f}}} + 2\mu \|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} + r\rho \|v\|^2_{0,\Omega^{\mathrm{f}}} + r\left(k + r^2 m\right) \|t\|^2 \\
&\quad + r \operatorname{Re}\left(\mathrm{B}^0 \, t \cdot \bar{t} \right) + r \operatorname{Re}\left(\mathrm{F}(r) t \cdot \bar{t} \right) \\
&\geq 2\mu \|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} - \rho 3 \|\varepsilon(u_0)\|_{0,\infty,\Omega^{\mathrm{f}}} \|v\|^2_{0,\Omega^{\mathrm{f}}} + r\rho \|v\|^2_{0,\Omega^{\mathrm{f}}} \\
&\quad + r \left(k + r^2 m - \left\| \mathrm{B}^0 \right\| - \|\mathrm{F}(r)\| \right) \|t\|^2 \\
&\geq 2\mu \|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} + r[\underbrace{k + r^2 m - \left\| \mathrm{B}^0 \right\| - C_7 \left(1 + \sqrt{r} + r\right)}_{\alpha(r)}] \|t\|^2 .
\end{aligned}
\tag{70}
$$

where the last inequality is a consequence of lemma 4.9. We take $r \geq \tilde{r}$ sufficiently large such as $\alpha(r) > 0$, which is possible because the dominating term grows at infinity as $r^2 m$, with $m > 0$. Then, the last inequality in (70) yields

$$
\operatorname{Re} A(V,V) \geq 2\mu \|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} + r\alpha(r) \|t\|^2 \geq \min\left\{2\mu, r\alpha(r)\right\} \left(\|\varepsilon(v)\|^2_{0,\Omega^{\mathrm{f}}} + \|t\|^2 \right).
$$

This, with Korn's inequality, implies that A is $\mathbb{V}$-ellipticity, i.e.,

$$
\operatorname{Re} A(V,V) \geq \beta \|(v,t)\|^2_{\mathbb{V}}, \quad \forall V \in \mathbb{C},
$$

where $\beta > 0$ depends on M, K, B^0, ρ, μ, u_0, r and Ω^{f}. Consequently, Lax-Milgram's theorem gives the existence and uniqueness of $(u_3, s) \in \mathbb{V}$ as solution of (65). The pressure p_3 is obtained from theorem 4.6. We also obtain the estimate

$$
\|(v,t)\|_{\mathbb{V}} \leq \frac{\|L\|_{\mathbb{V}'}}{\beta}.
$$

This last inequality combined with the estimate (44) for p_3 and the estimates $(61)_{1,2,3}$, for u_1 and p_1, allow us to obtain estimate (66), which completes the proof. □

4.2.4 Existence of the operator

We are now ready to prove an existence and uniqueness result for problem 33. This is the object of the following theorem.

Theorem 4.11 *There exists $r > 3\|u_0\|_{1,\infty,\Omega^{\mathrm{f}}}$ such that problem* (33) *has an unique solution $(u,p,s) \in H^1(\Omega^{\mathrm{f}})^3 \times L^2_0(\Omega^{\mathrm{f}}) \times \mathbb{C}^{n^{\mathrm{s}}}$. In addition, for each r sufficiently large, we have the estimate*

$$\|s\| + |u|_{1,\Omega^{\mathrm{f}}} + \|p\|_{0,\Omega^{\mathrm{f}}} \leq C_3 \left(\|f\|_{0,\Omega^{\mathrm{f}}} + \|g\| + \|h\| \right), \tag{71}$$

where C_9 is a positive constant independent of (u,p,s) and (f,g,h).

PROOF. First, the existence is a direct consequence of decomposition (34) and of the above lemmas. Indeed, corollary 4.7 allows to determine first (u_1,p_1), then we define problem (65). From lemmas 4.9 and 4.10 we can determine (u_3,p_3,s). Finally, once s known, we can compute (u_2,p_2) in (36) from corollary 4.7, or simply from the following expression

$$(u_2,p_2) = \sum_{j=1}^{n^{\mathrm{s}}} s_j (w_j, q_j),$$

where (w_j,q_j) is the solution of (63).

By linearity in problem (33), the uniqueness of solution can be reduced to prove that problem (33) provided with zero data only admits the trivial solution. In other words, we should prove that if $(u,p,s) \in H^1(\Omega^{\mathrm{f}})^3 \times L^2_0(\Omega^{\mathrm{f}}) \times \mathbb{C}^{n^{\mathrm{s}}}$ satisfies

$$\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p + r u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
\operatorname{div} u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}}, \\
u = 0, &\quad \text{on} \quad \Gamma, \\
u = r\Phi s - \nabla u_0 \Phi s, &\quad \text{on} \quad \gamma, \\
\left(\mathrm{K} + \mathrm{B}^0\right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u,p) n \,\mathrm{d}a + r^2 \,\mathrm{M}\, s = 0, &
\end{aligned} \tag{72}$$

then $(u,p,s)=0$.

In this way, let us suppose that $(u,p,s)\in H^1(\Omega^{\mathrm{f}})^3\times L^2_0(\Omega^{\mathrm{f}})\times\mathbb{C}^{n^{\mathrm{s}}}$ is a solution of (72). Then we define $(\tilde u,\tilde p)$ as a solution of

$$\begin{aligned}
\nabla u_0\tilde u+\nabla\tilde u u_0-2\nu\,\mathrm{div}\,\varepsilon(\tilde u)+\frac{1}{\rho}\nabla\tilde p+r\tilde u=0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\mathrm{div}\,\tilde u=0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\tilde u=0, &\quad \text{on} \quad \Gamma,\\
\tilde u=-\nabla u_0\Phi s, &\quad \text{on} \quad \gamma.
\end{aligned}$$

Corollary 4.7 ensures the existence and uniqueness of $(\tilde u,\tilde p)$. In addition, from (62), we get

$$\mathrm{F}(r)s=\int_\gamma \Phi^{\mathrm{T}}\sigma(\tilde u,\tilde p)n\,\mathrm{d}a.$$

We set $(\hat u,\hat p)=(u,p)-(\tilde u,\tilde p)$. Thus from (72) we obtain that $(\hat u,\hat p,s)$ is a solution of the following problem:

$$\begin{aligned}
\nabla u_0\hat u+\nabla\hat u u_0-2\nu\,\mathrm{div}\,\varepsilon(\hat u)+\frac{1}{\rho}\nabla\hat p+r\hat u=0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\mathrm{div}\,\hat u=0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\hat u=0, &\quad \text{on} \quad \Gamma,\\
\hat u=r\Phi s, &\quad \text{on} \quad \gamma,\\
\big(\mathrm{K}+r^2\,\mathrm{M}+\mathrm{B}^0+\mathrm{F}(r)\big)\,s+\int_\gamma \Phi^{\mathrm{T}}\sigma(\hat u,\hat p)n\,\mathrm{d}a=0. &
\end{aligned}$$

This is a coupled problem (65), supplied with zero data. Then, from lemma 4.10 we get $\hat u=0$, $\hat p=0$ and $s=0$. In particular, this last identity implies that $\tilde u=0$ and $\tilde p=0$, which gives $u=0$ and $p=0$.

Finally, inequality (71), giving the continuity of operator T, comes from the estimates on (u_1,p_1) and (u_2,p_2) (given by corollary 4.7) and from inequality (66) in the preceding lemmas. $\square$

4.2.5 Operator compactness

From theorem 4.11, operator T is well defined, linear and continuous from $\mathbb{H}$ to $H^1(\Omega^{\rm f})^3 \times \mathbb{C}^{n^{\rm s}} \times \mathbb{C}^{n^{\rm s}}$. In addition, since $H^1(\Omega^{\rm f})^3 \times \mathbb{C}^{n^{\rm s}} \times \mathbb{C}^{n^{\rm s}} \hookrightarrow \mathbb{H}$ is a compact embedding, the continuity of T leads to the compactness of T as an operator from $\mathbb{H}$ to $\mathbb{H}$ [7]. The following classical result, concerning the spectrum of compact operators [1], then applies:

Theorem 4.12 *Let $\mathbb{H}$ and T be, respectively, a Hilbert space and a compact operator on $\mathbb{H}$. The non-zero elements of the spectrum of T are eigenvalues with finite multiplicity and, which can only cluster at 0. These eigenvalues are ,at most, a countable infinite set. Zero always belongs to the spectrum.*

The following theorem characterizes the eigenvalues of (28) from the non-zero eigenvalues of the operator T.

Theorem 4.13 *For r sufficiently large we have that, if $(\lambda; u, p, s, z)$ is an eigenpair of (28) then*

$$\left(\frac{1}{\lambda + r}; u, s, z\right),$$

is an eigenpair of T, with non-zero eigenvalue, and conversely.

PROOF. Let us fix r as in theorem 4.11. If $(\lambda; u, p, s, z)$ is a solution of (28), we obtain by definition of an eigenvector that $(u, p, s, z) \neq 0$. This implies that $(u, s, z) \neq 0$. Indeed, otherwise we would have, from (28), that $\nabla p = 0$ in $\Omega^{\rm f}$, implying $p = 0$, because the pressure is defined modulo a constant. We take $\omega = \lambda + r$. From (28) we get

$$\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p + r u = \omega u, \quad \text{in} \quad \Omega^{\rm f},\\
\operatorname{div} u = 0, \quad \text{in} \quad \Omega^{\rm f},\\
u = 0, \quad \text{on} \quad \Gamma,\\
u = \Phi z - \nabla u_0 \Phi s, \quad \text{on} \quad \gamma,\\
-z + rs = \omega s,\\
\mathrm{M}^{-1}\left[\left(\mathrm{K} + \mathrm{B}^0\right) s + \int_\gamma \Phi^{\rm T} \sigma(u,p) n \, \mathrm{da}\right] + rz = \omega z.
\end{aligned}$$

Hence, from definition of T (32), we get

$$\omega T(u,s,z) = (u,s,z).$$

But, ω can not be zero because, since we have just proved that $(u,s,z) \neq 0$. Therefore, we can write

$$T(u,s,z) = \frac{1}{\omega}(u,s,z).$$

Consequently, if $(\lambda; u, p, s, z)$ is an eigensolution of (28), then

$$\frac{1}{\omega} = \frac{1}{\lambda + r},$$

is an eigenvalue of T and (u,z,s) an associated eigenvector.

Conversely, if $\omega \neq 0$ is an eigenvalue of T and (u,s,z) an associated eigenvector, namely,

$$T(u,s,z) = \omega(u,s,z),$$

then from the definition of T (32), we get that $(\omega u, \omega s, \omega z)$ satisfies

$$\begin{aligned}
\omega \nabla u_0 u + \omega \nabla u u_0 - \omega 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla p + \omega r u = u, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\operatorname{div} \omega u = 0, &\quad \text{in} \quad \Omega^{\mathrm{f}},\\
\omega u = 0, &\quad \text{on} \quad \Gamma,\\
\omega u = \omega \Phi z - \omega \nabla u_0 \Phi s, &\quad \text{on} \quad \gamma,\\
-\omega z + \omega r s = s, &\\
\mathrm{M}^{-1}\left[\left(\mathrm{K} + \mathrm{B}^0\right)\omega s + \int_\gamma \Phi^{\mathrm{T}} \sigma(\omega u, p) n \,\mathrm{d}a\right] + \omega r z = z. &
\end{aligned}$$

After division by $\omega \neq 0$, and by denoting $q = p/\omega$ we obtain

$$
\begin{aligned}
\nabla u_0 u + \nabla u u_0 - 2\nu \operatorname{div} \varepsilon(u) + \frac{1}{\rho}\nabla q &= \left(\frac{1}{\omega} - r\right) u, \quad \text{in} \quad \Omega^{\mathrm{f}}, \\
\operatorname{div} u = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}, \\
u = 0, \quad \text{on} \quad \Gamma, \\
u = \Phi z - \nabla u_0 \Phi s, \quad \text{on} \quad \gamma, \\
-z &= \left(\frac{1}{\omega} - r\right) s \\
\mathrm{M}^{-1}\left[\left(\mathrm{K} + \mathrm{B}^0\right) s + \int_\gamma \Phi^{\mathrm{T}} \sigma(u, q) n \, \mathrm{d}a\right] &= \left(\frac{1}{\omega} - r\right) z,
\end{aligned}
$$

which implies that

$$
\left(\frac{1}{\omega} - r; u, \frac{p}{\omega}, s, z\right),
$$

is a solution of (28).

In summary, the eigenvalues of (28) are in the form

$$
\lambda = \frac{1}{\omega} - r,
$$

where $\omega \neq 0$ is an eigenvalue of T. □

4.2.6 Spectral analysis

Finally, we can prove the following theorem.

Theorem 4.14 *The eigenvalues of the coupled problem* (25) *are, at most, a countable sequence of complex numbers, each with finite multiplicity, which can cluster only at infinity.*

PROOF. This result can be directly obtained from the preceding theorem, from the compactness of operator T and from theorem 4.12. □

Remark 4.15 *This result agrees with that obtained in [4], for* (26)*, and that conjectured in [22, 5], for* (27)*, both being particular cases of* (25)*.*

5 Conclusion

In this paper we provided a new *Linearization Principle* approach particularly suited for fluid-structure interaction systems. Our problem involves an incompressible Newtonian fluid and a reduced structure. In our approach, the main idea comes from the linearization-transpiration method developed in [11, 12]. We obtained a new spectral problem coupling the linearized Navier-Stokes equations and those of a reduced linear structure. The coupling is realized through transpiration conditions on a fixed interface. This allow us to keep a fixed fluid domain and, from a numerical point of view, to overcome all difficulties arising when dealing with moving grids. In short, we obtain an eigenproblem of minimal complexity.

On the other hand, we have characterized the eigenvalues of this new eigenproblem. Our main result shows that the spectrum consists of a discrete set of complex eigenvalues, each of finite multiplicity, which can cluster only at infinity. This result is obtained by proving that the eigenproblem can be reduced to that of finding the characteristic values of a compact operator acting in a Hilbert space.

The real numerical issue is to obtain reliable numerical predictions of the physical stability of the coupled system, by computing the eigenvalues in (22) with smallest real part. This is the subject of Part II, see [13].

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