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N° 4484

Juin 2002

THÈME 2

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4484--FR+ENG

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Thème 2 — Génie logiciel
et calcul symbolique
Projet Galaad

Rapport de recherche n° 4484 — Juin 2002 — 14 pages

Abstract: This paper proposes a new method for implicitizing surfaces given by a proper rational parametrization mapping, under the assumption that the inverse mapping has been computed. The advantage of the method is that it can handle base points and it is readily generalizable to hypersurfaces in arbitrary dimension. Moreover, the computational tools required are GCD operations and taking the square-free part of a polynomial. Alternatively, one may employ factorization.

Key-words: implicitization, parametric surface, inversion

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Une méthode d'implicitisation basée sur l'inversion

Résumé : Nous proposons une méthode pour l'implicitisation des surfaces définies par une paramétrisation rationelle propre, sous l'hypothèse que l'inverse de la paramétrisation est disponible. L'avantage de la méthode est qu'elle peut traiter les points base et est généralisable directement aux hypersurfaces en dimension arbitraire. Les outils requis pour son calcul sont les opérations de PGCD et la décomposition d'un polynôme sans facteurs carrés. De manière alternative, on peut passer par une factorisation.

Mots-clés : implicitisation, surface paramétrique, inversion

1 Introduction

The implicitization of a parametric surface is an important problem in CAD and geometric modeling, especially in the presence of base points. Here we propose a new method for implicitizing surfaces given by a proper parametrization mapping, under the assumption that the inverse mapping has been computed. This is a strong assumption, although efficient methods for inversion exist (see [ChGo92, PDSS02]).

The advantage of the method is that it can handle base points and it is readily generalizable to hypersurfaces of arbitrary dimension. Moreover, the computational tools required are univariate resultants, GCD operations and computing the square-free part of polynomials in 3 variables.

This paper is organized as follows. The next section expands on related work. Section 3 describes the parametrizations and their inverses in a general setting. Section 4 details our approach to implicitization of arbitrary-dimensional hypersurfaces. Section 5 presents and analyzes our algorithm for three-dimensional surfaces, whereas the following section contains several examples studied with Maple. We conclude with extensions of our results and some open questions, in section 7.

2 Related work

Several algorithms exist for implicitization, cf. e.g. [AS01, CGZ00, Hof89, HSW97, SGD97]. The main motivation concerns 3-dimensional surfaces, which are ubiquitous in CAD and geometric modeling. Nonetheless, the implicitization problem also appears in higher dimensions, e.g. in the computation of A-discriminants; cf. [GKZ94, ch. 3.9].

Vivid interest is shown for the case of base points, where straightforward methods fail; the problem has been addressed mainly by means of resultant perturbations, the residual resultant, and Newton sums (cf. e.g. [Bus01, GV97, MC92]), or by means of Gröbner bases. More recently, there are more techniques proposed in this domain, but their complete enumeration goes beyond our scope; the interested reader may see the aforementioned works and the references thereof.

Computing the inverse of a parametrization map has been studied in [PDSS02]. Its computational steps are resultants, GCD operations and a zero test for elements in the field of rational functions of the hypersurface. This last computation can be carried out either by working modulo the implicit equation or by substituting the parametrization of the hypersurface. Obviously, since our goal is to compute the implicit equation, in this paper we consider that the zero test is performed using the second alternative. We shall use some results from this paper below.

3 The setting

We start with some known results. As usual, $\mathbb{K}[\overline{x}\,]$ and $\mathbb{K}(\overline{x}\,)$ represent, respectively, the polynomial ring and fraction field over $\overline{x} = (x_1, \ldots, x_n)$, where $\mathbb{K}$ is a field of characteristic zero.

Definition 3.1 *Let us consider the parametrization $\mathcal{P}(\overline{t}\,)$, which defines a rational map $\varphi_{\mathcal{P}} : \mathbb{K}^{n-1} \longrightarrow V$, where $\varphi_{\mathcal{P}}(\overline{t}\,) = \mathcal{P}(\overline{t}\,)$. The mapping $\mathcal{P}(\overline{t}\,)$ is said to be* proper *iff $\varphi_{\mathcal{P}}$ is* birational *or, equivalently, $\mathbb{K}(\overline{t}\,)$ and $\mathbb{K}(\mathcal{P}(\overline{t}\,))$ are isomorphic.*

When $\mathcal{P}(\overline{t}\,)$ is proper we may consider the inverse mapping of $\varphi_{\mathcal{P}}$, which is a rational mapping assigning to each $\overline{x}$ in a non–empty open Zariski subset of V the corresponding parameter vector in $\mathbb{K}^{n-1}$. We will refer to the inverse mapping of $\varphi_{\mathcal{P}}$ as the inverse of the parametrization $\mathcal{P}(\overline{t}\,)$, and we will denote it by M or $\mathcal{P}^{-1}$. Also, for simplicity, we will do not distinguish between the parametrization and the induced rational map, and we will refer to $\varphi_{\mathcal{P}}$ as the rational mapping $\mathcal{P}$.

The inverse mapping has the property that $\mathcal{P}(M(\overline{x}\,)) = \overline{x} \mod F$ for almost all $\overline{x} \in \mathbb{K}^n$ (i.e. $\mathcal{P}(M(\overline{x})) = \overline{x}$ for almost all $\overline{x} \in V$), and $M(\mathcal{P}(\overline{t}\,)) = \overline{t}$ for almost all $\overline{t} \in \mathbb{K}^{n-1}$. The next lemma shows that in fact these two properties characterize the inverse.

Lemma 3.2 [PDSS02, Lem. 1] *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, let*

$$\mathcal{P} : \mathbb{K}^{n-1} \to V \subset \mathbb{K}^n : \overline{t} = (t_1, \ldots, t_{n-1}) \mapsto (\mathcal{P}_1(\overline{t}\,), \ldots, \mathcal{P}_n(\overline{t}\,))$$

be a rational parametrization of a hypersurface V, and let

$$M : V \to \mathbb{K}^{n-1} : \overline{x} = (x_1, \ldots, x_n) \mapsto (M_1(\overline{x}\,), \ldots, M_{n-1}(\overline{x}\,))$$

be a rational map, where the denominators of M do not belong to the ideal of V. Then M is the inverse of $\mathcal{P} \Leftrightarrow \overline{x} = \mathcal{P}(M(\overline{x}))$ for almost all $\overline{x} \in V \Leftrightarrow \overline{t} = M(\mathcal{P}(\overline{t}\,))$, for almost all $\overline{t} \in \mathbb{K}^{n-1}$.

In addition, besides the property $\mathcal{P}(M(\overline{x}\,)) = \overline{x}$ for almost all $\overline{x} \in V$, we observe that, since $\mathcal{P}(\overline{t}\,)$ parametrizes V, for almost all $\overline{x} \in \mathbb{K}^n$ (namely those for which $\mathcal{P}(M(\overline{x}))$ is defined) one has that $\mathcal{P}(M(\overline{x}\,)) \in V$.

In the sequel we will assume that $V \subset \mathbb{K}^n$ is a rational hypersurface defined by the polynomial F, and that

$$\mathcal{P}(\overline{t}\,) = \left(\frac{p_1(\overline{t}\,)}{q_1(\overline{t}\,)}, \ldots, \frac{p_n(\overline{t}\,)}{q_n(\overline{t}\,)} \right) \in \mathbb{K}(\overline{t}\,)^n,$$

with $p_i, q_i \in \mathbb{K}[\overline{t}\,], \gcd(p_i, q_i) = 1,\ \forall i \in \{1, \ldots, n\}$, is a rational parametrization of V. Then for a polynomial $G \in \mathbb{K}[\overline{x}\,], G(\mathcal{P}(\overline{t}\,)) = 0 \Leftrightarrow G$ is in the ideal of $V \Leftrightarrow F|G$. We shall also make use of the field $\mathbb{K}(V)$ of rational functions of V; i.e. $\mathbb{K}(V)$ is the quotient field of

$\mathbb{K}[\,\overline{x}\,]/\langle F\rangle$, where $\langle\cdot\rangle$ represents a polynomial ideal. Moreover, we write the inverse mapping of $\mathcal{P}(\,\overline{t}\,)$ as

$$M(\,\overline{x}\,)=\left(\frac{A_1(\,\overline{x}\,)}{B_1(\,\overline{x}\,)},\ldots,\frac{A_{n-1}(\,\overline{x}\,)}{B_{n-1}(\,\overline{x}\,)}\right)\in\mathbb{K}(\,\overline{x}\,)^{n-1},$$

with $A_i, B_i \in \mathbb{K}[\,\overline{x}\,], B_i \neq 0$ as element in $\mathbb{K}(V)$, $\gcd(A_i, B_i) = 1$, $\forall i \in \{1,\ldots,n-1\}$ (i.e., $B_i \notin < F >$ or equivalently $B_i(\mathcal{P}(\,\overline{t}\,)) \neq 0$), and we also introduce the notation $\mathcal{P}(M(\,\overline{x}\,)) = (Q_1,\ldots,Q_n) \in \mathbb{K}(\,\overline{x}\,)^n$, where the rational functions Q_i are in reduced formed.

4 Implicitization of Hypersurface Parametrizations

In this section we present our approach to implicitization, in general dimension.

We have seen in the previous section that $Q_i(\,\overline{x}\,) = x_i \bmod F$ for $i = 1,\ldots,n$. Our analysis will depend on whether $Q_i(\,\overline{x}\,) = x_i$ or $Q_i(\,\overline{x}\,) \neq x_i$, where the equality and inequality are in $\mathbb{K}(\,\overline{x}\,)$.

Example 4.1 *The inverse of the parametrization $\mathcal{P}(t_1,t_2) = (t_1^2, t_1^3, t_2)$, of the cylinder V, can be represented as*

$$M(x_1,x_2,x_3)=\left(\frac{x_2}{x_1},x_3\right)$$

and

$$\mathcal{P}(M(x_1,x_2,x_3))=\left(\frac{x_2^2}{x_1^2},\frac{x_2^3}{x_1^3},x_3\right).$$

Hence, in this case, $Q_1 \neq x_1, Q_2 \neq x_2$, but $Q_3 = x_3$ as elements in $\mathbb{K}(x_1,x_2,x_3)$. But, $Q_1 = x_1, Q_2 = x_2, Q_3 = x_3$ as elements in $\mathbb{K}(V)$.

We denote by $I_Q \subset \{1,\ldots,n\}$ the set of all indexes i for which the corresponding component $Q_i(\,\overline{x}\,)$ of $\mathcal{P}(M(\,\overline{x}\,))$ is equal to x_i and $J_Q = \{1,\ldots,n\}\setminus I_Q$. For simplicity in the terminology, we will assume that I_Q is of the form $\{1,\ldots,r\}$. Note that this last condition does not imply a loss of generality, since one can always achieve it by a trivial linear change of coordinates. In this situation, one has the following lemmas.

Lemma 4.2 $I_Q \neq \{1,\ldots,n\}$.

Proof. Let us assume that $I_Q = \{1,\ldots,n\}$, i.e. $\mathcal{P}(M(\,\overline{x}\,)) = \overline{x}$. Take two different elements $a, b \in \mathbb{K}$ such that $\mathcal{P}_1(\,\overline{t}\,) = \mathcal{P}(M(t_1,\ldots,t_{n-1},a))$ and $\mathcal{P}_2(\,\overline{t}\,) = \mathcal{P}(M(t_1,\ldots,t_{n-1},b))$ are defined. Note that this is equivalent to the non–vanishing of finitely many polynomials in $\mathbb{K}[\,\overline{x}\,]$ at $x_n = a$ and $x_n = b$; thus there exists a dense subset of $\mathbb{K}$ where a, b can be taken. Moreover, the closure of the image of $\mathcal{P}_1(\,\overline{t}\,)$ equals the closure of the image of the injection $\overline{t} \mapsto (\,\overline{t}, a)$, hence is of dimension $n-1$, and similarly for $\mathcal{P}_2(\,\overline{t}\,)$. Therefore, $\mathcal{P}_1(\,\overline{t}\,), \mathcal{P}_2(\,\overline{t}\,)$ are reparametrizations of $\mathcal{P}(\,\overline{t}\,)$, and therefore parametrize V. However, $\mathcal{P}_1(\,\overline{t}\,) = (t_1,\ldots,t_{n-1},a)$ parametrizes the hyperplane $x_n = a$, and $\mathcal{P}_2(\,\overline{t}\,) = (t_1,\ldots,t_{n-1},b)$ parametrizes the hyperplane $x_n = b$, which is impossible because $a \neq b$. □

Proof. (Alternative proof.)
Let us assume that $I_Q = \{1,\ldots,n\}$, i.e. $\mathcal{P}(M(\overline{x})) = \overline{x}$. Then, for almost all $\overline{x} \in \mathbb{K}^n$ it holds that $\mathcal{P}(M(\overline{x})) = \overline{x} \in V$. But this implies that $V = \mathbb{K}^n$ which is impossible. □

Lemma 4.3 *If* $\mathrm{card}(I_Q) = r = n-1$, *then* V *can be properly parametrized as*

$$\mathcal{P}^\star(\overline{t}) = (t_1,\ldots,t_{n-1}, Q_n(t_1,\ldots,t_{n-1},a)),$$

where a *is any element in* $\mathbb{K}$ *such that* $x_n - a$ *does not divide the denominators in* $M(\overline{x})$ *and the denominator of* Q_n.

Proof. If $\mathrm{card}(I_Q) = n-1$, then $Q_i(\overline{x}) = x_i$ for $i = 1,\ldots,n-1$. Now consider the rational parametrization defined as

$$\mathbb{K} \ni \overline{t} \mapsto \mathcal{P}(M(t_1,\ldots,t_{n-1},a)),$$

and let W be its image. Note that a is taken such that the above mapping is defined, and observe that the parametrization $\mathcal{P}(M(t_1,\ldots,t_{n-1},a))$ is equal to $\mathcal{P}^\star(\overline{t})$. Therefore $\mathcal{P}^\star(\overline{t})$ defines an irreducible subvariety W of V. Moreover, it is clear that

$$\mathbb{K}(\overline{t}) \subset \mathbb{K}(\mathcal{P}^\star(\overline{t})) \subset \mathbb{K}(\overline{t}),$$

where the first inclusion is obvious because $r = n-1$, and the second inclusion follows from $Q_n(\overline{t}, a) \in \mathbb{K}(\overline{t})$. Hence $\mathbb{K}(\overline{t}) = \mathbb{K}(\mathcal{P}^\star(\overline{t}))$. Thus, on one hand, $\dim(W) = n-1$; alternatively, this is obtained by adapting the proof of the previous lemma. Hence, $W = V$ and therefore $\mathcal{P}^\star(\overline{t})$ is a reparametrization of $\mathcal{P}(\overline{t})$. On the other hand, $\mathcal{P}^\star(\overline{t})$ is proper since it is the composition of proper mappings. This concludes the proof. □

Let $\mathrm{num}(\cdot)$ and $\mathrm{den}(\cdot)$ stand for the numerator and denominator of a rational expression respectively, and $\mathrm{SF}(\cdot)$ for the square-free part of a polynomial.

Lemma 4.4 *Let* $\mathrm{card}(I_Q) = r \geq 0$, *then it holds that:*

1. *For every* $j \in \{r+1,\ldots,n\}$ *there exists* $\ell_j \in \mathbb{N}, \ell_j > 0$, *and there exists* $H_j \in \mathbb{K}[\overline{x}]$ *such that* $\mathrm{num}(Q_j - x_j) = F^{\ell_j} H_j$ *and* $\gcd(F, H_j) = 1$.
2. *If* $r < n-1$, *then* $\gcd(H_{r+1},\ldots,H_n)$ *is either 1 or is a product of factors of the denominators* $B_1,\ldots,B_{n-1}$ *of the components of the inverse* $M(\overline{x})$.

Proof. Statement (1) follows directly from the fact that for $j \in J_Q$, $Q_j(\overline{x}) = x_j \bmod F$ but $Q_j(\overline{x}) \neq x_j$.

To prove (2), we first observe that for $j \in J_Q$, it holds that $\gcd(H_j, \mathrm{den}(Q_j)) = 1$. Indeed, let us assume that $\gcd(H_j, \mathrm{den}(Q_j)) = G \neq 1$. Then, by (1), one gets that $\mathrm{num}(Q_j) = x_j \mathrm{den}(Q_j) + F^{\ell_j} H_j$. Thus, G divides $\mathrm{num}(Q_j)$ and $\mathrm{den}(Q_j)$, which is a contradiction because Q_j was taken in reduced form. In this situation, let us assume that $\gcd(H_{r+1},\ldots,H_n) =$

$H \neq 1$ and let $H^\star$ be an irreducible factor of H not being a factor of any B_i with $i \in \{1, \ldots, n-1\}$.

We shall show a contradiction. Let W be the irreducible hypersurface defined by $H^\star$. Then, for almost all $P \in W$, the expression $M(P)$ is defined because $\gcd(H^\star, B_i) = 1$ for $i = 1, \ldots, n-1$, and the expression $\mathcal{P}(M(P))$ is also defined because $\gcd(H_j, \mathrm{den}(Q_j)) = 1$ for $i = r+1, \ldots, n$ (note that for $i \leq r$, $\mathrm{den}(Q_i) = 1$). Therefore, $\mathcal{P}(M(P)) \in V$. Moreover, since $H^\star$ divides H_j one has that $\mathcal{P}(M(P)) = P$. Thus, W is dense in V. So by Bézout's theorem one concludes that $\gcd(F, H^\star) \neq 1$, which is impossible because $\gcd(F, H_j) = 1$. This concludes the proof.

□

From the above lemmas one can deduced the following theorem for computing the implicit equation F of V.

Theorem 4.5 *With the notation used before, it holds that*

1. *If* $\mathrm{card}(I_Q) = n-1$ *then the implicit equation of* V *is*

$$\mathrm{num}(x_n - Q_n(x_1, \ldots, x_{n-1}, a))$$

where a *is any element in* $\mathbb{K}$ *such that* $x_n - a$ *does not divide the denominators in* $M(\overline{x})$ *and* $\mathrm{den}(Q_n)$.

2. *If* $\mathrm{card}(I_Q) = r < n-1$ *let*

$$G(\overline{x}) = \mathrm{SF}(\gcd(\mathrm{num}(Q_{r+1} - x_{r+1}), \ldots, \mathrm{num}(Q_n - x_n))).$$

Then the implicit equation of V *is*

$$\frac{G(\overline{x})}{\gcd\left(G, \prod_{i=1}^{n-1} B_i\right)}.$$

Proof. Statement (1) follows from lemma 4.3 because the mapping $\mathcal{P}^\star(\overline{t})$ is essentially the graph of $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, Q_n(x_1, \ldots, x_n, a))$, so the implicit equation is given by the numerator of $x_n - Q_n(x_1, \ldots, x_{n-1}, a)$. Statement (2) follows from lemma 4.4. □

Remark 4.6 *Note that in general, it is very unlikely that* $\gcd(G, B_i) \neq 1$. *For this, see the examples below.*

From theorem 4.5 one can derive an inversion–based algorithm for implicitizing hypersurfaces, given $\mathcal{P}$ and M. More precisely, we first compute all the Q_i and I_Q. Depending on the latter's cardinality we follow the corresponding case of the above theorem. It is also possible to use factorization, having computed a single Q_i, provided that $i \notin I_Q$. The next two sections illustrate this discussion.

5 Three dimensional surfaces

This section applies the above results to 3-dimensional surfaces, which is the most common application. We state the precise algorithmic steps; see e.g. [Zip93] for the complexity of certain basic operations.

If we wish to use factorization, it suffices to compute a single Q_i, such that $i \notin I_Q$. Then, deciding which factor represents F is immediate by taking sample points that make F vanish and which, in general, will not make the other factors vanish. Such points are easily obtained by specializing the parameters in $\mathcal{P}(\,\bar{t}\,)$. In practice, factorization may be preferable to taking GCDs and the square-free part of multivariate polynomials; this behavior was reported by Sebastien Bis on Maple7.

In the rest of the section we examine the case that all the Q_i are available. When $r = 2$, i.e. there are two values of $k \in \{1, 2, 3\}$ such that $Q_k = x_k$, say $k = 1, 2$, then we apply statement (1) of theorem 4.5. The computation amount to finding the numerator of $x_3 - Q_3(x_1, x_2, a)$ in the notation of that theorem.

In the case that $r = |I_Q| = 1$, let us assume as before that $Q_1 = x_1, Q_2 \neq x_2, Q_3 \neq x_3$. First, we recover $G(\,\overline{x}\,)$ which is tantamount to computing $\mathrm{SF}(\gcd(\mathrm{num}(Q_2 - x_2), \mathrm{num}(Q_3 - x_3)))$ or, equivalently, $\gcd(\mathrm{SF}(\mathrm{num}(Q_2 - x_2)), \mathrm{SF}(\mathrm{num}(Q_3 - x_3)))$. Since the complexity of taking square-free parts is that of a gcd operation, both methods have the same worst-case asymptotic complexity. But the former should be faster on the average, if we compare the latter stages whose complexity is dominated. The operations required to recover F from G include computing $\gcd(G, B_i)$ and dividing out these gcd's from G. In practice, we start with the B_i of smaller degree, for $i = 1, 2$.

In the case that $r = |I_Q| = 0$, $Q_i \neq x_i,\ i = 1, 2, 3$. Recovering F requires first computing $G := \mathrm{SF}(\gcd(\mathrm{num}(Q_1 - x_1), \mathrm{num}(Q_2 - x_2), \mathrm{num}(Q_3 - x_3)))$. This requires 2 gcd operations with polynomials of 3 variables, then taking a square-free part. The operations required to recover F from G include computing $\gcd(G, B_i)$ and dividing out these gcd's from G.

We conclude this section by suggesting some further techniques, which may be useful in practice. Given polynomial $G = F^k H$, we can compute k by successive partial differentiation of F, e.g. $G_x = kF_x F^{k-1} + F^k H_x$, where x is any of the surface variables, and P_x denotes the partial derivative of any polynomial P with respect to x. By taking sample points on the surface, we can check, with high probability, whether some partial derivative of G vanishes, thus concluding whether it contains F as a factor or not. The smallest derivative that does not vanish gives us k. Moreover, the factorization of this derivative gives us H along with other factors, and a subsequent GCD computation can remove H from G. The interest of using derivatives is that it should be less costly to factorize it than factoring G.

6 Examples

Example 6.1 *A plane may be parametrized by*

$$\mathcal{P}(t_1, t_2) := (t_1, t_2, at_1 + bt_2 + c)\,.$$

Its inverse is $M(x_1, x_2, x_3) = (x_1, x_2)$ *and* $Q(x_1, x_2, x_3) = (x_1, x_2, ax_1 + bx_2 + c)$*. Thus,* $\mathrm{card}(I_Q) = 2 = n - 1$*. Therefore, applying theorem 4.5 one has that the implicit equation is* $x_3 = ax_1 + bx_2 + c$*.*

By using Maple, we have examined the following examples.

Example 4.1 (cont'd) *A cylinder, parallel to the* x_3*-axis, may be defined as*

$$\mathcal{P}(t_1, t_2) := \left(t_1^2, t_1^3, t_2\right), t_1, t_2 \in [0, 1],$$

where an inverse mapping would be

$$M(x_1, x_2, x_3) = \left(\frac{x_2}{x_1}, x_3\right) \Rightarrow \mathcal{P}(M(x_1, x_2, x_3)) = \left(\frac{x_2^2}{x_1^2}, \frac{x_2^3}{x_1^3}, x_3\right),$$

and we obtain

$$F = \gcd(\mathrm{num}(x_1 - Q_1), \mathrm{num}(x_2 - Q_2)) = x_2^2 - x_1^3$$

by computing $\gcd(F, Fx_2)$*. Note that no factor of the* B_i *may appear above.*

Example 6.2 *One parametrization for the 3-dimensional sphere is*

$$\mathcal{P}(t_1, t_2) := \left(\frac{t_1^2 - t_2^2 - 1}{t_1^2 + t_2^2 + 1}, \frac{2t_1}{t_1^2 + t_2^2 + 1}, \frac{2t_1 t_2}{t_1^2 + t_2^2 + 1}\right),\ t_1, t_2 \in [0, 1].$$

An inversion mapping is

$$M(x_1, x_2, x_3) := \left(\frac{x_1 + 1}{x_2}, \frac{x_3}{x_2}\right).$$

Then

$$Q = \left(\frac{{x_1}^2 + 2\,x_1 + 1 - {x_3}^2 - {x_2}^2}{{x_1}^2 + 2\,x_1 + 1 + {x_3}^2 + {x_2}^2}, \frac{2\,(x_1 + 1)\,x_2}{{x_1}^2 + 2\,x_1 + 1 + {x_3}^2 + {x_2}^2}, \frac{2x_3\ (x_1 + 1)}{{x_1}^2 + 2\,x_1 + 1 + {x_3}^2 + {x_2}^2}\right)$$

yields

$$\overline{x} - Q = \left((x_1 + 1)\left({x_1}^2 - 1 + {x_3}^2 + {x_2}^2\right),\ x_2\ \left({x_1}^2 - 1 + {x_3}^2 + {x_2}^2\right), x_3\ \left({x_1}^2 - 1 + {x_3}^2 + {x_2}^2\right)\right)$$

and the gcd *of any these three expressions yields correctly* $F = {x_1}^2 + {x_3}^2 + {x_2}^2 - 1$*.*

Example 6.3 *An example with base points:*

$$\mathcal{P} := \left(t_1 + {t_2}^2, t_2 + t_1 t_2, t_2 + {t_1}^2\right).$$

Then,

$M_1 = -6\,{x_1}^2 x_2 - 6\,x_3 x_1 {x_2}^2 + 8\,x_1 x_2 x_3 - 2\,{x_1}^2 x_2 x_3 - 2\,{x_3}^2 x_2 + 2\,x_3 {x_1}^2 {x_2}^2 - {x_1}^3 + 2\,{x_3}^2 {x_1}^2 +$

$x_2{}^2x_3{}^2 - 4\,x_1{}^3x_2 + x_1{}^4 + 2\,x_3x_1{}^3 + 2\,x_1{}^2x_2{}^2 - x_1x_2{}^4 + 2\,x_3x_2{}^2 - 2\,x_2x_3 + x_2{}^2 + x_3{}^2 - 4\,x_3x_1{}^2 - 8\,x_1x_2{}^2 - x_1x_3{}^2 + 4\,x_1x_2{}^3 - x_3{}^2x_1{}^3 \;/\; \left(2\,x_2 - x_2{}^2 + x_1x_3 - x_1 - x_3\right)^2$, *and*

$$M_2 = \frac{x_2x_3 - 2\,x_1x_2 - x_3 + x_1{}^2 + x_2}{2\,x_2 - x_2{}^2 + x_1x_3 - x_1 - x_3}.$$

Also,

$$Q_1 = x_1$$

and

$Q_2 = -\left(x_2x_3 - 2\,x_1x_2 - x_3 + x_1{}^2 + x_2\right)(6\,x_1{}^2x_2 - 4\,x_3x_1x_2{}^2 + 4\,x_1x_2x_3 - 2\,x_1x_3 - 2\,x_1{}^2x_2x_3 - 2\,x_3{}^2x_2 + 2\,x_3x_1{}^2x_2{}^2 + 4\,x_2{}^3 - x_1{}^3 + x_3{}^2x_1{}^2 + x_2{}^2x_3{}^2 - 4\,x_1{}^3x_2 + x_1{}^4 + 2\,x_3x_1{}^3 + 2\,x_1{}^2x_2{}^2 - x_1x_2{}^4 - x_1{}^2 + 4\,x_1x_2 + 2\,x_2x_3 - 3\,x_2{}^2 - 2\,x_3x_1{}^2 - 10\,x_1x_2{}^2 - x_2{}^4 + x_1x_3{}^2 + 4\,x_1x_2{}^3 - x_3{}^2x_1{}^3) \;/\; \left(2\,x_2 - x_2{}^2 + x_1x_3 - x_1 - x_3\right)^3$,

and

$Q_3 = 41\,x_2{}^3x_1{}^2 + 24\,x_3x_1x_2{}^2 - 9\,x_1{}^2x_2x_3 + 55\,x_2x_3{}^2x_1{}^2 - 68\,x_2{}^2x_1x_3{}^2 + 20\,x_3{}^3x_1{}^4x_2{}^2 + 12\,x_3x_1{}^2x_2{}^6 + 8\,x_3x_1{}^5x_2{}^2 - 12\,x_3{}^3x_1x_2{}^4 - 28x_3^3x_1{}^3x_2{}^2 - 28\,x_3{}^2x_1{}^3x_2{}^4 + 50\,x_3{}^2x_1{}^2x_2{}^4 - 36\,x_1{}^5x_2x_3{}^2 - 71\,x_3x_1{}^2x_2{}^2 + 80\,x_1{}^4x_2{}^3x_3 - 13\,x_1{}^4x_2{}^2x_3{}^2 - 61\,x_1{}^3x_2{}^4x_3 - 114\,x_1{}^4x_3x_2{}^2 - 26\,x_1{}^4x_2x_3{}^3 + 88\,x_1{}^4x_2x_3{}^2 + 70\,x_1{}^3x_2{}^3x_3{}^2 + 54\,x_3x_1x_2{}^3 - 86\,x_1{}^4x_2x_3 - 70\,x_1{}^2x_2{}^5x_3 + 193\,x_1{}^2x_2{}^4x_3 - 80\,x_1{}^3x_2{}^3x_3 - 81\,x_2{}^4x_3x_1 - 6\,x_1{}^3x_2{}^2x_3{}^2 + 206\,x_1{}^3x_2{}^2x_3 - 11\,x_2x_3{}^4x_1{}^2 + 7\,x_2x_3{}^4x_1 + 5\,x_2x_3{}^4x_1{}^3 + 36\,x_2x_3{}^3x_1{}^3 + 34\,x_2x_3{}^3x_1 - 15\,x_2{}^3x_3{}^3x_1{}^2 + 26\,x_2{}^5x_3x_1 - 113\,x_2{}^3x_3x_1{}^2 - 112\,x_2x_3{}^2x_1{}^3 - 21\,x_2x_3{}^3x_1{}^2 + 24\,x_2x_3x_1{}^3 - 66\,x_2{}^2x_3{}^3x_1 + 15\,x_2{}^5x_3{}^2x_1 - 73\,x_2{}^4x_3{}^2x_1 - 125\,x_2{}^3x_3{}^2x_1{}^2 + 46\,x_2{}^3x_3{}^3x_1 + 33\,x_2{}^2x_3{}^3x_1{}^2 + 126\,x_2{}^3x_3{}^2x_1 + 20\,x_1{}^3x_2{}^5x_3 - 20\,x_1{}^6x_2x_3 - 16\,x_1{}^4x_2{}^3x_3{}^2 - 15\,x_1x_2x_3{}^2 + 4\,x_1{}^5x_2x_3{}^3 - 16\,x_3x_1{}^5x_2{}^3 + 4\,x_3{}^3x_1{}^2x_2{}^4 + 3\,x_3{}^2x_1{}^2 + 6\,x_3{}^2x_1{}^4x_2{}^4 + 24\,x_2{}^2x_3{}^2 - x_1{}^3x_2 + x_3x_1{}^3 + 6\,x_1{}^2x_2{}^2 - 20\,x_1x_2{}^4 - 4\,x_3x_1{}^3x_2{}^6 + 4\,x_3x_1{}^4x_2{}^4 + 4\,x_3{}^2x_1{}^5x_2{}^2 + 4\,x_3x_1{}^6x_2{}^2 - 4\,x_3{}^3x_1{}^5x_2{}^2 + 65\,x_2{}^2x_3{}^2x_1{}^2 - 24\,x_3x_2{}^3 + 4\,x_3{}^4x_1{}^2x_2{}^2 + 9\,x_2{}^4 - 12\,x_1x_2{}^3 - 8\,x_3{}^2x_1{}^3 + x_3{}^4x_1{}^6 + 10\,x_1{}^6x_3{}^2 + 6\,x_3{}^4x_1{}^4 - 4\,x_3{}^4x_1{}^5 + x_2{}^4x_3{}^4 + 20\,x_1{}^6x_2{}^2 - 8\,x_1{}^7x_2 - 8\,x_1{}^5x_2{}^3 + 8\,x_1{}^4x_2{}^5 + 4\,x_1{}^7x_3 - 2\,x_1{}^5x_2{}^4 - 2\,x_1{}^7x_3{}^2 - 4\,x_3{}^3x_1{}^6 - 4\,x_1{}^3x_2{}^6 + x_1{}^2x_2{}^8 - 8\,x_1{}^2x_2{}^7 - 2\,x_2{}^6x_3{}^2x_1 - 2\,x_2{}^2x_3{}^4x_1 - 2\,x_2{}^2x_3{}^4x_1{}^3 + 8\,x_1{}^6x_2x_3{}^2 - 4\,x_1x_2{}^6x_3 + 4\,x_2{}^4x_3{}^3 + 21\,x_2{}^2x_3{}^3 - 15\,x_2{}^3x_3{}^3 - 39\,x_2{}^3x_3{}^2 - 5\,x_2x_3{}^4 + 21\,x_2{}^4x_3{}^2 - 21\,x_2{}^5x_3 - 3\,x_2{}^5x_3{}^2 + 7\,x_2{}^6x_3 - x_2{}^7x_3 - 26\,x_1{}^3x_2{}^2 + 8\,x_1{}^4x_2 - 86\,x_1{}^3x_2{}^3 - 52\,x_1{}^2x_2{}^5 + 29\,x_1x_2{}^5 + 32\,x_1{}^2x_2{}^4 - 14\,x_1x_2{}^6 + 2\,x_1x_2{}^7 - 11\,x_3{}^3x_2 - 7\,x_1{}^3x_3{}^4 - 5\,x_1x_3{}^4 + 8\,x_1{}^2x_3{}^4 + 26\,x_1{}^4x_3{}^2 + 11\,x_1{}^5x_3 + 51\,x_1{}^4x_2{}^2 + 13\,x_1{}^3x_2{}^4 - 3\,x_1{}^4x_3 - 25\,x_1{}^5x_3{}^2 + 17\,x_1{}^5x_3{}^3 - 23\,x_1{}^4x_3{}^3 + 31\,x_1{}^2x_2{}^6 - 12\,x_1{}^5x_2 + 18\,x_1{}^3x_3{}^3 + 3\,x_1x_3{}^3 - 15\,x_1{}^2x_3{}^3 + 36\,x_2{}^4x_3 - 12\,x_2{}^5 + 6\,x_2{}^6 + 2\,x_3{}^4 - x_1{}^5 - x_2{}^7 + x_1{}^6 - 2\,x_1{}^7 + x_1{}^8 - 68\,x_1{}^5x_2{}^2 + 20\,x_1{}^6x_2 + 80\,x_1{}^4x_2{}^3 + 4\,x_1{}^3x_2{}^5 + 6\,x_3{}^4x_2{}^2 - 4\,x_3{}^4x_2{}^3 - 12\,x_1{}^6x_3 - 26\,x_1{}^4x_2{}^4 + 76\,x_1{}^5x_2x_3 \;/\; \left(2\,x_2 - x_2{}^2 + x_1x_3 - x_1 - x_3\right)^4$.

Hence $J_Q = \{2, 3\}$ *and therefore only the second and third component have to been taken into account. This yields to*

$F = -4\,x_1{}^2x_2 + 2\,x_3x_1x_2{}^2 - x_1x_2x_3 - x_1x_3 + 3\,x_2{}^3 + x_1{}^3 - x_3{}^2x_1{}^2 - 4\,x_3x_2{}^2 + x_3{}^3 + x_1x_2 + 5\,x_2x_3 - 3\,x_2{}^2 - 2\,x_3{}^2 + 2\,x_3x_1{}^2 + 2\,x_1x_2{}^2 - x_2{}^4$.

It is interesting to note that $H_2 = -\left(x_1 - x_2\right)\left(x_1{}^2 - x_1x_2 - x_1 - 1 + 3\,x_2 - x_2{}^2\right)$ *and* $H_3 = -3\,x_2{}^2 - 17\,x_1x_3x_2 + 3\,x_2x_3 - 6\,x_2x_1{}^2 - x_1x_3 + 8\,x_1x_2{}^2 + 10\,x_3x_1x_2{}^2 - x_3{}^2 + 7\,x_3x_1{}^2 + x_1{}^3 - 6\,x_3{}^2x_1{}^2 + 2\,x_3x_2{}^2 + x_2{}^3 + 13\,x_1{}^2x_3x_2 - 4\,x_3x_2{}^3 - 2\,x_2{}^3x_1 - 3\,x_2x_3x_1{}^3 - 8\,x_3x_1{}^2x_2{}^2 -$

$x_1{}^2 + 2\,x_1{}^3 x_2{}^2 x_3 + 3\,x_1 x_2 + 4\,x_1 x_3{}^2 - 2\,x_1{}^4 - 13\,x_1{}^2 x_2{}^2 - 7\,x_3 x_1{}^3 + 4\,x_3{}^2 x_1{}^3 + 11\,x_1{}^3 x_2 + x_2{}^4 x_3 + 2\,x_1{}^3 x_2{}^2 + 5\,x_1{}^2 x_2{}^3 - 4\,x_1{}^4 x_2 - x_1{}^2 x_2{}^4 + 2\,x_1{}^4 x_3 - x_1{}^4 x_3{}^2 + x_1{}^5$, *hence their* gcd *is* $H = 1$.

Example 6.4 *The following example shows that in general one may need to use all of the components corresponding to* J_Q *unless it suffices, for a particular application, to obtain a nontrivial multiple of the implicit equations. Consider the surface parametrization*

$$\mathcal{P}(t_1, t_2) = \left(\frac{t_1 + 1}{t_2 + t_1}, -\frac{t_1}{-t_2 + t_1}, \frac{3\,t_1{}^3 + 2\,t_1{}^2 - t_2\,t_1{}^2 + t_1\,t_2{}^2 + t_2{}^2 - 2\,t_2\,t_1}{(t_2 + t_1)\,(-t_2 + t_1)^2} \right)$$

An expression for the inverse of $\mathcal{P}$ *is* $M(x_1, x_2, x_3) = (M_1, M_2)$ *where*

$$M_1 = -\frac{4\,x_3{}^2 - 4\,x_2{}^2 x_1\,x_3 - 18\,x_3\,x_2{}^2 + 9\,x_2\,x_3 - 4\,x_1\,x_3 + 7\,x_2 - 4\,x_2{}^2 + 12\,x_2{}^2 x_1 - 8\,x_2{}^3 + 12\,x_2{}^4 x_1 - 8\,x_2{}^3 x_1 - 8\,x_1\,x_2 + 12\,x_2{}^4}{-10\,x_1\,x_2 + 3\,x_2{}^2 - 7\,x_3 + 7\,x_2 + 15\,x_2{}^2 x_1 - 6\,x_3\,x_2{}^2 + x_2\,x_3 - 5\,x_1\,x_3 + 4\,x_3{}^2}$$

$$M_2 = -\frac{7 - 7\,x_1 + 3\,x_2 + 8\,x_3 + 4\,x_1\,x_3 - x_2\,x_3 - 11\,x_2{}^2 - 4\,x_3{}^2 + 2\,x_3\,x_2{}^2 - 7\,x_2{}^2 x_1 + 4\,x_2{}^2 x_1\,x_3}{-10\,x_1\,x_2 + 3\,x_2{}^2 - 7\,x_3 + 7\,x_2 + 15\,x_2{}^2 x_1 - 6\,x_3\,x_2{}^2 + x_2\,x_3 - 5\,x_1\,x_3 + 4\,x_3{}^2}.$$

Computing (Q_1, Q_2, Q_3) *one checks that* $I_Q = \emptyset$ *and*

$$\gcd(\mathrm{num}(x_1 - Q_1), \mathrm{num}(x_2 - Q_2), \mathrm{num}(x_3 - Q_3)) = x_1 + x_2^2 + x_2^2 x_1 - x_3$$

that is the implicit equation, but

$$\gcd(\mathrm{num}(x_2 - Q_2), \mathrm{num}(x_3 - Q_3)) = (-4x_3 - x_2 + 6x_2^2)(x_1 + x_2^2 + x_2^2 x_1 - x_3).$$

Example 6.5 *We consider the parametrization*

$$\mathcal{P}(\,\bar{t}\,) = (t_1^3, t_2^3, t_1 + t_2^2).$$

Its inverse can be expressed as

$$M = \left(\frac{x_1{}^2 x_3{}^3 - 8\,x_3{}^3 x_2{}^2 x_1 - 2\,x_3{}^6 x_1 + x_3{}^9 - x_2{}^2 x_1{}^2 - 2\,x_2{}^4 x_1 - x_2{}^6}{x_3{}^2\,(-x_1 + 2\,x_2{}^2 + x_3{}^3)^2}, \frac{x_2\,\left(x_1 + x_2{}^2 + 2\,x_3{}^3\right)}{x_3\,\left(-x_1 + 2\,x_2{}^2 + x_3{}^3\right)} \right)$$

and (Q_1, Q_2, Q_3) *is*

$$\left(-\frac{\left(-x_1{}^2 x_3{}^3 + 8\,x_3{}^3 x_2{}^2 x_1 + 2\,x_3{}^6 x_1 - x_3{}^9 + x_2{}^2 x_1{}^2 + 2\,x_2{}^4 x_1 + x_2{}^6\right)^3}{x_3{}^6\,(x_1 - 2\,x_2{}^2 - x_3{}^3)^6}, -\frac{x_2{}^3\,\left(x_1 + x_2{}^2 + 2\,x_3{}^3\right)^3}{x_3{}^3\,(x_1 - 2\,x_2{}^2 - x_3{}^3)^3}, x_3 \right).$$

Finally computing $\gcd(\mathrm{num}(x_1 - Q_1), \mathrm{num}(x_2 - Q_2))$ *one gets the implicit equation*

$$x_1{}^3 + 3\,x_2{}^2 x_1{}^2 + 3\,x_2{}^4 x_1 + x_2{}^6 - 3\,x_1{}^2 x_3{}^3 + 21\,x_3{}^3 x_2{}^2 x_1 - 3\,x_3{}^3 x_2{}^4 + 3\,x_3{}^6 x_1 + 3\,x_3{}^6 x_2{}^2 - x_3{}^9$$

7 Conclusion

The method deals with implicitization in arbitrary dimension and handles the case of base points without any modification. We plan to fully compare the practical complexity of this algorithm to that of other implicitization methods.

Let us focus on $n = 3$ and $r = 1$. We have considered another means for computing F, by applying a univariate resultant, but with no formal proof yet. Let us use $Q_i(x_1, t_2, a) = \mathcal{P}_i(M(x_1, t_2)) = x_i$ for $i = 2, 3$ to eliminate t_2 by taking the resultant of two polynomials:

$$\mathrm{Res}_{t_2}\left(x_2 \mathrm{den}(Q_2(t_2)) - \mathrm{num}(Q_2(t_2)), x_3 \mathrm{den}(Q_3(t_2)) - \mathrm{num}(Q_3(t_2))\right),$$

with respect to t_2. This yields F once we prove $\mathcal{P}^\star$ is proper. One may use Bézout's matrix for computing this resultant, thus reducing the question to interpolating the determinant in 2 variables. Any knowledge on the degrees of F in these variables may help in order to accelerate the interpolation, by avoiding a completely dense interpolation.

Example 4.1 (cont'd) *Let us apply the above idea. Then,*

$$\mathcal{P}^\star(s_1, s_3) = (\frac{a^2}{s_1^2}, \frac{a^3}{s_1^3}, s_3),$$

and we can take the resultant

$$Res_{s_1}(x_1 s_1^2 - a^2, x_2 s_1^3 - a^3) = a^6(x_1^3 - x_2^2),$$

which defines the same cylindrical variety. Yet another possibility is to consider

$$\mathcal{P}^\star(s_2, s_3) = (\frac{s_2^2}{a^2}, \frac{s_2^3}{a^3}, s_3), \qquad Res_{s_2}(x_1 a^2 - s_2^2, x_2 a^3 - s_2^3) = a^6(x_1^3 - x_2^2).$$

Acknowledgments

The first author had fruitful discussions with Sebastien Bis, and was partially supported by the INRIA Sophia-Antipolis COLOR project SIMPLES and FET Open European Project IST-2001-35512 (GAIA-II). The second author was partially supported by DGES PB98-0713-C02-01. Work on this paper was partially conducted while the second author was visiting the GALAAD group at INRIA Sophia-Antipolis.

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http://www.inria.fr
ISSN 0249-6399