Design of fault tolerant on board networks with priorities via selectors

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Abstract: We consider on-board networks in satellites interconnecting entering signals (inputs) to amplifiers (outputs). The connections are made via expensive switches with four links available. The paths connecting inputs to outputs should be link-disjoint. Among the input signals, some of them, called priorities, must be connected to the amplifiers which provide the best quality of service (that is to some specific outputs). In practice, amplifiers are subject to faults that cannot be repaired. Therefore we need to add extra outputs to ensure the existence of sufficiently many valid ones. Given n inputs, p priorities and k faults, the problem consists in designing a low cost network (i. e. with the minimum number of switches) where it is possible to route the p priorities to the p best quality amplifiers and the other inputs to some valid amplifiers, for any sets of k faulty and p best quality amplifiers. Let $N(n, p, k)$ be the minimum number of switches of a such a network, called *repartitor*. In [3], it was proved that $N(n, p, 0) \leq n - p + \frac{n}{2}\lceil \log_2 p \rceil$ and some exact values of $N(n, p, k)$ were given when p and k are small.

A $(n, 0, k)$-repartitor (or a (n, n, k)-repartitor) is called a $(n, n+k)$-*selector* and the minimum number of switches of a (p, n)-selector is denoted by $S(p, n)$. A selector is intrinsically easier to design than general repartitors since there exists only one type of signals to route instead of two. The approach of this paper is to construct (n, p, k)-repartitors from selectors. We show that and $N(n, p, k) \leq S(p, p+k) + S(n+k, p+k) + S(n-p, p+k)$. Then we prove that $S(p, n) \leq 33n + 4p + O(\log n)$ which implies $N(n, p, k) \leq 71n + 37p + 108k + O(\log(n+k))$. At last, we study (p, n)-selectors when p is fixed. We prove that:

if p is even then $S(p, n) \geq \dfrac{2^{p/2} - 1}{2^{p/2}} n + \theta(1)$;

if p is odd then $S(p, n) \geq \dfrac{2^{(p+3)/2} - 3}{2^{(p+3)/2}} n + \theta(1)$.

We conjecture that equality holds and show it for $p \leq 6$.

Key-words: Fault Tolerance, Network design, Concentrator, Selector

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Conception de réseaux embarqués avec prioriés tolérants aux pannes via les sélecteurs

Résumé : Nous considérons des réseaux embarqués dans les satellites interconnectant des signaux entrants (entrées) à des amplificateurs (sorties). Ces connexions sont réalisées par l'intermédiaire de commutateurs à quatre liens qui sont très onéreux. Les chemins reliant les entrées aux sorties doivent utiliser des liens différents. Parmi les signaux entrants, certains, appelés priorités, doivent être connectés aux amplificateurs qui assurent la meilleure qualité de service. En pratique, les amplificateurs peuvent tomber en panne. De ce fait, nous devons ajouter des sorties supplémentaires pour être sur qu'il y en ait suffisamment de valides. Etant donnés n entrées dont p priorités et k pannes, le problème consiste à trouver le réseau de coût minimal (i. e. avec le nombre minimum de commutateurs) pour lequel, quels que soient les k amplificateurs en panne et les p meilleurs amplificateurs, il est possible de router les p priorités vers les p amplificateurs de meilleure qualité, et les autres entrées vers des amplificateurs valides. Soit $N(n,p,k)$ le nombre minimum de commutateurs d'un tel réseau, nommé répartiteur. Dans [3], il est montré que $N(n,p,0) \leq n - p + \frac{n}{2}\lceil \log_2 p \rceil$ et des valeurs exactes de $N(n,p,k)$ sont données pour p et k petits.

Un $(n,0,k)$-repartiteur (ou un (n,n,k)-repartiteur) est appelé $(n, n+k)$-*sélecteur* et le nombre minimum de commutateurs d'un (p,n)-sélecteur est noté $S(p,n)$. Un sélecteur est intrinsèquement plus facile à concevoir qu'un répartiteur quelconque car il n'y a qu'un seul type de signaux à router au lieu de deux. Dans ce rapport, nous construisons des répartiteurs à partir de sélecteurs. Nous montrons que $N(n,p,k) \leq S(p,p+k) + S(n+k,p+k) + S(n-p,p+k)$. Nous prouvons ensuite $S(p,n) \leq 33n + 4p + O(\log n)$ ce qui implique $N(n,p,k) \leq 71n + 37p + 108k + O(\log(n+k))$. Enfin, nous étudions les (p,n)-sélecteurs avec p fixé. Nous montrons les inégalités suivantes :

si p est pair alors $S(p,n) \geq \dfrac{2^{p/2}-1}{2^{p/2}} n + \theta(1)$;

si p est impair alors $S(p,n) \geq \dfrac{2^{(p+3)/2}-3}{2^{(p+3)/2}} n + \theta(1)$.

Nous conjecturons que l'égalité est vraie et la montrons pour $p \leq 6$.

Mots-clés : Tolérance aux pannes, Conception de réseaux, Concentrateur, Sélecteur

1 Introduction

Modern telecommunications satellites are very complex to design and an important industrial issue is to provide robustness at the lowest possible cost. A key component of telecommunication satellites is an interconnection network which allows to redirect signals received by the satellite to a set of amplifiers where the signals will be retransmitted. In this paper, we consider a certain type of interconnection network as asked by Alcatel Space Industries. The network is made of expensive switches ; so we want to minimize their number subject to the following conditions : Each input and output is adjacent to exactly one link ; each switch is adjacent to exactly four links ; there are n inputs (signals) and $n + k$ outputs (amplifiers) ; among the $n + k$ outputs, k can fail permanently; among the n input signals, p of them called priorities must be connected to the amplifiers providing the best quality of service (that is to some specific outputs) and the other signals should be sent to other amplifiers. Note that the priority signals are given, but the amplifiers providing the quality of service change according the position of the satellite and so the networks should be able to route the signals for any set of k failed outputs and any set of p best quality outputs.

This problem can be formally restated as follows:

Definition 1 A (n,p,k)*-network* G is a graph (V,E) where the vertex set V is partitioned into four subsets P, I, O and S called respectively the *priorities*, the *ordinary inputs*, the *outputs* and the *switches*, satisfying the following constraints:
- there are p priorities, $n - p$ ordinary inputs and $n + k$ outputs;
- each priority, each ordinary input and each output is connected to exactly one switch;
- switches have degree at most 4.

A (n,p,k)-network is a *repartitor* if for any disjoint subsets F and B of O with $|F| = k$ and $|B| = p$, there exist in G, n edge-disjoint paths, p of them from P to B and the $n - p$ others joining I to $O \setminus (B \cup F)$. The set F corresponds to set of failures and B to the set of amplifiers providing the best quality of service. We denote $N(n,p,k)$ the minimum number of switches (i.e. cardinality of S) of a valid (n,p,k)-repartitor. A (n,p,k)-repartitor with $N(n,p,k)$ switches will be called minimum.

Problem 1 Determine $N(n,p,k)$ and construct a minimum (or almost minimum) repartitor.

In [2] and [4], the particular case of $(n,0,k)$-repartitors (or (n,n,k)-repartitors) also called $(n,n+k)$*-selectors*, with k fixed, were studied. Let us denote $S(p,n)$ the minimum number of switches (i.e. cardinality of $\mathcal{S}$) of a (p,n)-selector. In [2], it is shown that $S(n,n+2) = N(n,0,2) = n$. In [4], it is proved that $\frac{3n}{2} - O(\frac{n}{k}) \leq S(n,n+k) = N(n,0,k) \leq \frac{3n}{2} + O(k)$. The following values for small k are also given : $S(n,n+4) = N(n,0,4) = n + \lceil \frac{n}{4} \rceil$; $S(n,n+6) = N(n,0,6) = n + \frac{n}{4} + \sqrt{\frac{n}{8}} + O(1)$; $S(n,n+8) = N(n,0,8) = n + \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(\sqrt[4]{n})$ and $S(n,n+12) = N(n,0,12) = n + \frac{3n}{7} + O(\sqrt{n})$.

In [3], it is shown that $N(n,p,0) \leq n - p + \frac{n}{2}\lceil \log_2 p \rceil$. Some exact values of $N(n,p,k)$ were given when p and k are small.

In this paper, we study repartitors by mean of selectors. In the first section, we show how to construct a repartitor from three selectors and derive the upper bound $S(p, p+k) + S(p+k, n+k) + S(n-p, n+k)$. We then study selectors. In section 2, we establish the following upper bounds: for any $p \leq n$, $S(p,n) \leq 33n + 4p + O(\log n)$. Thus $N(n,p,k) \leq 71n + 37p + 108k + O(\log(n+k))$.

In the last section, we study (p,n)-selectors when p is fixed. We prove that:
if p is even then $S(p,n) \geq \frac{2^{p/2}-1}{2^{p/2}}n + \theta(1)$;
if p is odd then $S(p,n) = \frac{2^{(p+1)/2}-3}{2^{(p+1)/2}}n + \theta(1)$.
We conjecture that equality holds. We establish it for $p \leq 6$.

2 Constructing Repartitors from Selectors

Lemma 1 $N(n,p,0) \leq S(p,n) + S(n-p,n) + n$

Proof. Let S be a (p,n)-selector with output-set $\{o_1, o_2, \ldots, o_n\}$ and S' an $(n-p,n)$-selector with output-set $\{o'_1, o'_2, \ldots, o'_n\}$. Let H be the $(n,p,0)$-network constructed from S and S' by replacing the pair $\{o_i, o'_i\}$ by a switch s_i adjacent to an output q_i and the neighbours of o_i and o'_i. See Figure 1. The priorities of H are the inputs of S and its ordinary inputs the

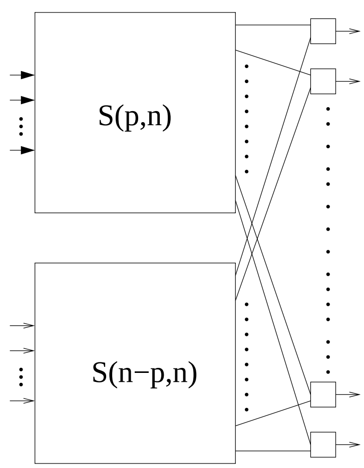

Figure 1: Construction of a $(n,p,0)$-repartitor from a (p,n)- and a $(n-p,n)$-selector

inputs of S'. It is easy to check that H is a $(n,p,0)$-repartitor. Indeed the priorities are routed through S, the ordinary inputs through S' and the switches s_i allow us to select a priority path or an ordinary one. ∎

Lemma 2 *For* $p \leq n$,

$$N(n,p,k) \leq S(p,p+k) + N(n+k,p+k,0) \leq S(p,p+k) + S(p+k,n+k) + S(n-p,n+k)$$

Proof. Let S be a $(p,p+k)$-selector with output set $O^1 = \{o^1_1, o^1_2, \ldots, o^1_{p+k}\}$ and R a $(n+k,p+k,0)$-repartitor with priority set $I^2 = \{i^2_1, i^2_2, \ldots, i^2_{p+k}\}$. Let G be the network obtained from the union of S and R by replacing each pair $\{o^1_j, i^2_j\}$ by an edge joining their neighbours. (See Figure 2). It is simple matter to check that G is a (n,p,k)-repartitor with

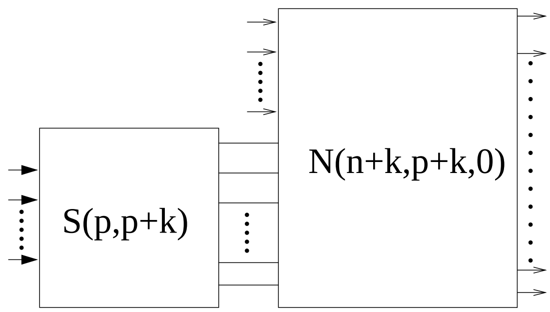

Figure 2: Construction of a (n,p,k)-repartitor from a $(p,p+k)$-selector and a $(n+k,p+k,0)$-repartitor.

priority set of G the input set of S and ordinary input set the ordinary input set of R. ■

From a (n,p,k)-repartitor, one can easily construct a $(p,p+k)$-selector a $(p+k,n+k)$-selector or a $(n-p,n+k)$-selector by removing inputs and outputs.

Proposition 1 $\max\{S(p,n); S(n-p,n); S(p,p+k)\} \leq N(n,p,k)$

Hence, if a (n,p,k)-repartitor R is constructed from three optimum selectors using the two above constructions, it has at most $3N(n,p,k)+n+k$ switches. So finding minimum (or almost minimum) selectors will give us fairly small repartitors.

3 General Upper Bounds

Definition 2 A (n,n)-network is an *n-superselector* if for any subsets $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{O}' \subset \mathcal{O}$ with $|\mathcal{O}'| = |\mathcal{I}'|$, there exist in G, $|\mathcal{O}'|$ edge-disjoint paths joining $\mathcal{I}'$ to $\mathcal{O}'$.

We will denote $S^+(n)$ the minimum number of switches (i.e. cardinality of $\mathcal{S}$) of a n-superselector. A n-superselector with $S^+(n)$ switches will be called minimum.

From every superselector one can construct another one such that each switch has degree four by adding edges. Therefore, we will now consider that every switch has degree 4 in a superselector.

Proposition 2 *For any $k \leq n$, $S(k,n) \leq S^+(n)$.*

Proof. Let S be a n-superselector. It is easy to see that the network obtained from S by deleting any set of $n-k$ inputs is a (k,n)-selector. ■

Definition 3 Let $\theta(n) = 4\left\lceil \frac{n}{6} \right\rceil$. An $(n, 0, \theta(n))$-network is a n-*concentrator* if for any subset $\mathcal{I}' \subset \mathcal{I}$ with $|\mathcal{I}'| = k \leq \left\lceil \frac{n}{2} \right\rceil$, there exist in G, k edge-disjoint paths joining $\mathcal{I}'$ to $\mathcal{O}$. Let $C(n)$ be the minimum size of an n-concentrator.

Lemma 3

$$S^+(n) \leq 2C(n) + n + S^+(\theta(n)) \tag{1}$$

Proof. Let S_θ be a $\theta(n)$-superselector with input set $\{a_j, 1 \leq j \leq \theta(n)\}$ and output set $\{\bar{a}_j, 1 \leq j \leq \theta(n)\}$. Let C (resp. $\overline{C}$) be a n-concentrator with input set $\{c_j, 1 \leq j \leq n\}$ (resp. $\{\bar{c}_j, 1 \leq j \leq n\}$) and output set $\{d_j, 1 \leq j \leq \theta(n)\}$ (resp. $\{\bar{d}_j, 1 \leq j \leq \theta(n)\}$).

Let N be the network constructed from S_θ, C and $\overline{C}$ by the following:
For $1 \leq j \leq \theta(n)$, replace each pair $\{d_j, a_j\}$ by an edge e_j joining their neighbours;
For $1 \leq j \leq \theta(n)$, replace each pair $\{\bar{d}_j, \bar{a}_j\}$ by an edge $\bar{e}_j$ joining their neighbours;
For $1 \leq j \leq n$, replace each pair $\{c_j, \bar{c}_j\}$ by a switch v_j adjacent to their neighbours and an input i_j and an output o_j. See Figure 3.

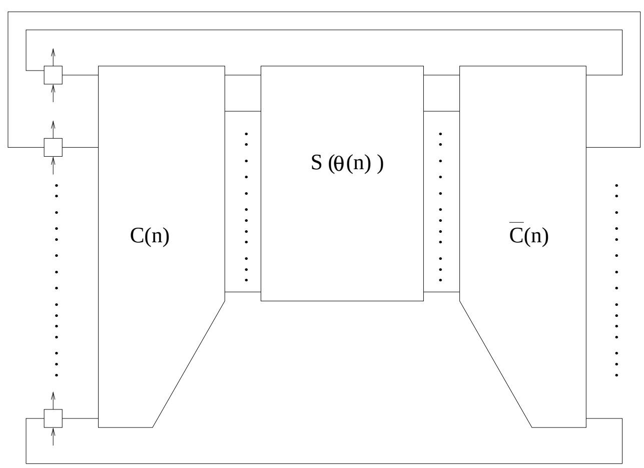

Figure 3: Construction of a n-superselector from a $\theta(n)$-superselector and two n-concentrators

Let us prove that N is an n-superselector. Let $\mathcal{I}'$ be a subset of inputs and $\mathcal{O}'$ a subset of outputs such that $|\mathcal{I}'| = |\mathcal{O}'|$. Set $I = \{j, i_j \in \mathcal{I}'$ and $o_j \in \mathcal{O}'\}$ and $J_i = \{j \notin I, i_j \in \mathcal{I}'\}$, $J_o = \{j \notin, o_j \in \mathcal{O}'\}$. Obviously, J_i and J_o have the same cardinal $q \leq n/2$. For any $j \in I$, let P_j be the path (i_j, v_j, o_j). It remains to find q paths from the inputs i_j, $j \in J_i$ to the outputs o_j, $j \in J_o$. Since C is a concentrator, there are q edge-disjoint paths Q_j, $j \in J_i$, joining i_j via v_j to an edge $e_{f(j)}$. Analogously, there are q edge-disjoint paths Q_j, $j \in J_o$, joining o_j to an edge $\bar{e}_{f(j)}$. Now, since S_θ is a $\theta(n)$-superselector, there exists q edge-disjoint paths R_j, $j \in J_i$, joining $e_{f(j)}$ to an element $\bar{e}_{g(j)} \in \{\bar{e}_{f(j)}, j \in J_o\}$. For $j \in J_i$, let P_j be the concatenation of the paths Q_j, R_j and $Q_{f^{-1}(g(j))}$. It joins the input i_j to an output $o_{f^{-1}(g(j))}$.

Hence $\{P_j, j \in I \cup J_i\}$ is a set of edge-disjoint paths joining $\mathcal{I}'$ to $\mathcal{O}'$. ■

Lemma 4 (Pippenger [5]) *For every m, there is a bipartite graph $Bip(m) = (A, B)$ with $|A| = 6m$ inputs and $|B| = 4m$ outputs, in which every vertex of A has outdegree 6, every vertex of B has indegree 9, and, for every $k \leq 3m$ and every set S of k inputs, there exists a matching from S into some k-subset of the outputs.*

Theorem 1

$$C(n) \leq \frac{17}{3}n + \frac{85}{3}$$

Proof. Suppose first that $n = 6m$. Let P_A be the network consisting of three switches a_1, a_2 and a_3 such that : (a_1, a_2, a_3) is a path; each a_i has two outlinks and a_1 is adjacent to an input. Let P_B be the network consisting of four switches b_1, b_2, b_3 and b_4 such that : (b_1, b_2, b_3, b_4); each b_i, $2 \leq i \leq 4$ has two outlinks; b_1 has three outlinks and b_4 is adjacent to an output. (See Figure 4)

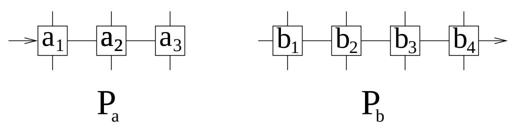

Figure 4: The networks P_a and P_b

Let C_n be the network obtained from $Bip(m)$ by replacing each vertex of A by P_A and each vertex of B by P_B. It follows from the definition of $Bip(m)$ that $C(n)$ is an n-concentrator. And C_n has $3 \times n + 4 \times \frac{2n}{3} = \frac{17n}{3}$ vertices. For $6m - 6 < n < 6m$, let C_n be the network obtained from C_{6m} by removing $6m - n$ inputs. It is easy to check that $C(n)$ is an n-concentrator. ■

Theorem 2

$$S^+(n) \leq 37n + O(\log n)$$

Proof. By Lemma 3, we have: $S^+(n) \leq 2C(n) + n + S^+(\theta(n))$. If n is sufficiently small, say $n \leq N$, it is easy to check that $S^+(n) \leq 37n$. (In particular, Waksman networks that realize any permutation from the inputs to the outputs (see [1]) are superselectors.)

For larger value of n, define $\theta^0(n) = n$ and $\theta^{t+1}(n) = \theta(\theta^t(n))$. Pick t such that $\theta^t(n) > N \geq \theta^{t+1}(n)$. Now applying Equation 1, $t+1$ times, setting $D(n) = 2C(n) + n$, we get

$$\begin{aligned} S^+(n) &\geq D(\theta^0(n)) + D(\theta^1(n)) + \cdots + D(\theta^t(n)) + S^+(\theta^{t+1}(n)) \\ &\geq \frac{37}{3}(\theta^0(n) + \theta^1(n) + \cdots + \theta^t(n)) + \frac{170}{3}(t+1) + S^+(\theta^{t+1}(n)) \end{aligned} \quad (2)$$

It is easy to show by induction on t that $\theta^t(n) \leq \left(\frac{2}{3}\right)^t n + 8$, which implies that

$$S^+(n) \leq 37n + \frac{442}{3}(t+1)$$

since $S^+(\theta^{t+1}(n)) \leq 34\theta^{t+1}(n)$.

Now $C < \theta^t(n) \leq \left(\frac{2}{3}\right)^t n + 8$, hence $t = O(\log n)$ since $\frac{2}{3} < 1$. So,

$$S^+(n) \leq 37n + O(\log(n))$$

■

Theorem 2 and Proposition 2 yield $S(p,n) \leq 37n + O(\log(n))$. However, we can get a slightly better upper bound :

Theorem 3 *$S(p,n) \leq 33n + 4p + O(\log(n))$.*

Proof. Let $S1$ be the n-superselector obtained from the construction in the proof of Theorem 2. $S1$ is constructed from n extra switches $s_1, s_2, \dots, s_n$ a concentrator C_1, a concentrator C_2 and a $\theta(n)$-superselector S_2. For $1 \leq j \leq n$, let i_j (resp. o_j, c_j and P_j) be the input (resp. output, switch of C_2, path P_a of length 3 in C_1) linked to s_j. Let S be the network obtained from S by removing i_j, s_j and P_j for $j > p$ and connecting o_j to c_j for $j > p$.

Since $S1$ is a superselector, obviously S is a (p,n)-selector. And S has $4(n-p)$ switches less than $S1$, so it has $33n + 4p + O(\log(n))$ switches. ■

Theorem 3 and Lemmas 1 and 2 yield an upper bound for $N(n,p,k)$:

Corollary 1

$$\begin{aligned} N(n,p,0) &\leq 71n + O(\log n) \\ N(n,p,k) &\leq 71n + 37p + 108k + O(\log(n+k)) \end{aligned}$$

4 Minimum (p,n)-selectors for p fixed

Let W be a set of vertices of a network. We denote by $in(W)$ (resp. $out(W)$, $sw(W)$) the number of inputs (resp. outputs, switches) in W. An edge connecting W and $\bar{W} = V \setminus W$ is said to be *cutting*. The set of cutting edges is denoted by $\Delta(W)$ and its cardinality is denoted by $deg(W)$.

Proposition 3 *A $(p, 0, n-p)$-network is a (p, n)-selector if and only if for every subset W:*

$$deg(W) \geq \min\{p, out(W)\} - in(W)$$

Proof. Let $\mathcal{O}'$ be a fixed set of p outputs and let $out'(W) = |W \cap \mathcal{O}'|$. A variant of the Ford-Fulkerson Theorem states that the problem is feasible if and only if

$$\forall\ W \subset V : deg(W) \geq demand(W) = out'(W) - in(W).$$

It follows that a (p, n)-network is (p, n)-selector if and only if:

$$\forall\ W \subset V : deg(W) \geq \max\{out'(W) | \mathcal{O}' \text{ set of } p \text{ outputs}\} - in(W).$$

Now $\max\{out'(W) | \mathcal{O}' \text{ set of } p \text{ outputs}\}$ is the maximum number of outputs of W in $\mathcal{O}'$. This maximum is attained either by choosing all the outputs in W to be in $\mathcal{O}'$ if $out(W) \leq p$, or by choosing p outputs in W to be in $\mathcal{O}'$ if $out(W) \geq p$. Hence, $\max\{out'(W) | \mathcal{O}' \text{ set of } p \text{ outputs}\} = \min\{p, out(W)\}$. ∎

Let $\mathcal{S}_0$ (resp. $\mathcal{S}_1$, $\mathcal{S}_2$) be the set of switches adjacent to no output (resp. one output, two outputs) and s_0 (resp. s_1, s_2) its cardinality.

Let $\mathcal{S}_0^0$ (resp. $\mathcal{S}_0^1$, $\mathcal{S}_0^2$) be the set of switches of $\mathcal{S}_0$ adjacent to no vertex (resp. one vertex, two vertices) of $\mathcal{S}_2$ and s_0^0 (resp. s_0^1, s_0^2) its cardinality.

Let $\mathcal{S}_1^0$ (resp. $\mathcal{S}_1^1$) be the set of switches of $\mathcal{S}_1$ adjacent to no vertex (resp. one vertex) of $\mathcal{S}_2$ and s_1^0 (resp. s_1^1) its cardinality.

Let us define the sets $\mathcal{U}_i$ and $\mathcal{T}_i$ inductively by: $\mathcal{U}_0 = \mathcal{S}_1^1$ and $\mathcal{T}_0 = \mathcal{S}_1^0$. $\mathcal{U}_{i+1}$ (resp. $\mathcal{T}_{i+1}$) is the set of switches of $\mathcal{T}_i$ having exactly one (resp. no) neighbour in $\mathcal{U}_i$.

Let us denote by k_1^0 (resp. k_1^1, k_2) the number of inputs adjacent to $\mathcal{S}_1^0$ (resp. $\mathcal{S}_1^1$,, $\mathcal{S}_2$).

From Proposition 3, one can easily prove the following:

Proposition 4 1. *If $p \geq 3$, a switch is adjacent to at most two outputs and two switches of $\mathcal{S}_2$ are not adjacent.*

2. *If $p \geq 4$, a switch of $\mathcal{S}_1$ is adjacent to at most one switch of $\mathcal{S}_2$.*
3. *If $p \geq 5$, a switch of $\mathcal{S}_0$ is adjacent to at most two switches of $\mathcal{S}_2$.*
4. *If $p \geq 2i + 4$, then $(\mathcal{U}_i; \mathcal{T}_i)$ is a partition of $\mathcal{U}_{i-1}$.*
5. *If $p \geq 2i + 5$, then any two elements of $\mathcal{U}_i$ are not adjacent.*
6. *If $p \geq i + 6$, then any element of $\mathcal{U}_i$ is not adjacent to any element of $\mathcal{S}_0^2$.*

From this Proposition, we deduce the following equations:

Corollary 2 *If $p \geq 3$,*

$$n = 2s_2 + s_1 \tag{3}$$

$$2s_2 \leq 3s_1 + 4s_0 + p \tag{4}$$

If $p \geq 4$,

$$2s_2 \leq s_1 + 4s_0 + p \tag{5}$$

If $p \geq 5$,

$$2s_2 = s_1^1 + 2s_0^2 + s_0^1 + k_2 \tag{6}$$

If $p \geq 2i+5$,

$$2s_1^1 + \sum_{j=1}^{i} u_j \leq 3t_i + 3s_0^1 + 4s_0^0 + k_1^1 + k_1^0 \tag{7}$$

If $p \geq 2i+6$,

$$2s_1^1 + \sum_{j=1}^{i} u_j \leq u_{i+1} + 3s_0^1 + 4s_0^0 + k_1^1 + k_1^0 \tag{8}$$

Theorem 4 *1) If* $p \geq 2p'-1$, $S(p,n) \geq \frac{2^{p'+1}-3}{2^{p'+1}}n - \frac{2^{p'}-3}{2^{p'+1}}p$.

2) If $p \geq 2p'$, $S(p,n) \geq \frac{2^{p'}-1}{2^{p'}}n - \frac{2^{p'-1}-1}{2^{p'}}p$.

Proof. Since a minimum (p,n)-selector must be connected, it follows that $S(p,n) \geq 1/2(p+n-2)$, hence $S(1,n) \geq \lfloor \frac{n}{2} \rfloor$ and $S(2,n) \geq \lceil \frac{n}{2} \rceil$.
If $p \geq 3$, Eq. 3 + 1/5 Eq. 4 gives $8/5s_2 + 8/5s_1 + 4/5s_0 \geq n - \frac{3}{5}$. Thus $S(3,n) \geq \frac{5n}{8} - \frac{3}{8}$.
If $p \geq 4$, Eq. 3 + $\frac{1}{3}$ Eq. 5 gives $4/3s_0 + 4/3s_1 + 4/3s_2 \geq n - 4/3$. Thus $S(4,n) \geq \frac{3n}{4} - 1$.
Suppose now that $p \geq 5$.

1) Set $l = p'-3$.
Eq. 3 + $\frac{1}{2^{l+4}-3}\left\{(2^{l+3}-3)\text{Eq. } 6 + \sum_{i=0}^{l-1} 2^{l+1-i}\text{Eq. } 8[i] + \text{Eq. } 7[l]\right\}$ yields:

$$\begin{aligned} n \quad \leq \quad & \frac{2^{l+4}}{2^{l+4}-3}\left(s_2 + s_1^1 + \sum_{i=1}^{l} u_i + t_l\right) + \frac{2^{l+4}-6}{2^{l+4}-3}s_0^2 + \frac{7 \times 2^{l+1}-12}{2^{l+4}-3}s_0^1 + \frac{2^{l+4}-12}{2^{l+4}-3}s_0^0 \\ & + \frac{2^{l+3}-3}{2^{l+4}-3}k_2 + \frac{2^{l+2}-4}{2^{l+4}-3}k_1^1 + \frac{2^{l+1}-4}{2^{l+4}-3}k_1^0 \end{aligned}$$

Thus $n \leq \frac{2^{l+4}}{2^{l+4}-3}s + \frac{2^{l+3}-3}{2^{l+4}-3}p$.

2) Set $l = p'-3$.
Eq. 3 + $\frac{1}{2^{l+3}-1}\left\{(2^{l+2}-1)\text{Eq. } 6 + \sum_{i=0}^{l} 2^{l-i}\text{Eq. } 8[i]\right\}$ yields:

$$n \quad \leq \quad \frac{2^{l+3}}{2^{l+3}-1}\left(s_2 + s_1^1 + \sum_{i=1}^{l+1} u_i\right) + t_{l+1} + \frac{2^{l+3}-2}{2^{l+3}-1}s_0^2 + \frac{7 \times 2^{l}-2}{2^{l+3}-1}s_0^1 + \frac{2^{l+3}-4}{2^{l+3}-1}s_0^0$$

$$+\frac{2^{l+2}-1}{2^{l+3}-1}k_2+\frac{2^{l+1}-1}{2^{l+3}-1}k_1^1+\frac{2^{l}-1}{2^{l+3}-1}k_1^0$$

Thus $n \leq \frac{2^{l+3}}{2^{l+3}-1}s + \frac{2^{l+2}-1}{2^{l+3}-1}p$.

■

We conjecture that the inequalities obtained in the above corollary are tight:

Conjecture 1 *For any fixed p,*
if p is even then $S(p,n) = \dfrac{2^{p/2}-1}{2^{p/2}}n + \theta(1)$*;*
if p is odd then $S(p,n) = \dfrac{2^{(p+3)/2}-3}{2^{(p+3)/2}}n + \theta(1)$*.*

We now show that Conjecture 1 holds for $p \leq 6$.

Therefore we prove a reinforcement of Proposition 3, which allows us to check the cut criterion only for a certain kind of subsets called *suitable*. A subset is *suitable* if it is connected, with no inputs and containing all the outputs adjacent to its switches.

Proposition 5 *A* (k,n)*-network is a* (k,n)*-selector if and only if* $deg(W) \geq \min\{k, out(W)\}$ *for any suitable subset* W*.*

Proof. Suppose that $deg(W) \geq \min\{k, out(W)\}$ for any suitable subset W.

Let us prove that for any subset X connected with no input then $deg(X) \geq \min\{k, out(X)\}$. Let W be the set obtained from X by adding all the outputs adjacent to a switch of X. Then $deg(W) \leq deg(X)$ and $out(W) \geq out(X)$. So $deg(X) \geq deg(W) \geq \min\{k, out(W)\} \geq \min\{k, out(X)\}$.

Let us prove that for any subset Y with no input then $deg(Y) \geq \min\{k, out(Y)\}$, by induction on the number c of connected component. The result is true if $c = 1$. Suppose now that it is true for c, and suppose Y has $c+1$ connected components. Let C be one of it and $X = Y \setminus C$. We have $deg(Y) = deg(C) + deg(X) \geq \min\{k, out(C)\} + \min\{k, out(X)\}$. Since $out(Y) = out(C) + out(X)$, we obtain $deg(Y) \geq \min\{k, out(Y)\}$.

Let us now prove that for any subset Z, $deg(Z) \geq \min\{k, out(Z)\} - in(Z)$. Let Y be the set obtained from Z by removing all the inputs. We have $deg(Y) \geq deg(Z) - in(Z)$, and $out(Y) = out(Z)$. Now $deg(Y) \geq \min\{k, out(Y)\}$, so $deg(Z) \geq \min\{k, out(Z)\} - in(Z)$. ■

Theorem 5

$$\begin{aligned} S(1,n) &= \left\lfloor \frac{n}{2} \right\rfloor \\ S(2,n) &= \left\lceil \frac{n}{2} \right\rceil \end{aligned} \tag{9}$$

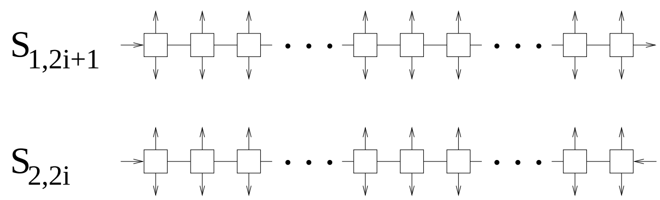

Figure 5: Minimum $(1,2i+1)$- and $(2,2i)$-selectors.

Proof. Let P_i be the network consisting of a path $(v_1, v_2, \dots, v_i)$ of switches and $2i$ outputs $1 \leq o_j \leq 2i$ such that for every $1 \leq j \leq i$, then v_j is adjacent to o_j and o_{i+j}. Let $S_{1,2i+1}$ (resp. $S_{2,2i}$) be the network obtained from P_i by adding an input adjacent to v_1 and an output (resp. an input) adjacent to v_i see Figure 5.

Let W be a suitable subset of $S_{1,2i+1}$. And let j be the smallest integer such that $v_j \in W$. Then v_i is adjacent to an element in $\bar{W}$. Thus $deg(W) \geq 1$. By Proposition 5, it follows that $S_{1,2i+1}$ is a $(1,2i+1)$-selector.

Analogously considering j and j' the smallest and biggest integer such that v_j is in a suitable subset W of $S_{2,2i}$, we obtain that $deg(W) \geq 2$ for any suitable subset of $S_{2,2i}$. Hence $S_{2,2i}$ is a $(2,2i)$-selector by Proposition 5.

The network $S_{1,2i}$ (resp. $S_{2,2i-1}$) obtained from $S_{1,2i+1}$ (resp. $S_{2,2i}$) by removing an output is obviously a $(1,2i)$-selector (resp. $(2,2i-1)$-selector). ∎

Theorem 6

$$S(3,n) \quad = \quad \left\lceil \frac{5n}{8} \right\rceil + \theta(1)$$

Proof. Let $S_{3,8i+5}$ be the network depicted Figure 6 with $5i+3$ switches, r_j, s_j, t_j, for $1 \leq j \leq i$, and v_j, w_j, for $1 \leq j \leq i+1$, and u such that:

- for $1 \leq j \leq i$, r_j is adjacent to v_j, v_{j+1}, t_j and an output;
- for $1 \leq j \leq i$, s_j is adjacent to w_j, w_{j+1}, t_j and an output;
- for $1 \leq j \leq i$, v_j and w_j are adjacent to two outputs;
- for $1 \leq j \leq i-1$, t_j is adjacent to two outputs;
- u is adjacent to v_{i+1}, w_{i+1} an input and an output;
- v_1, and w_1 are adjacent to an input.

Let W be a suitable subset of $S_{3,8i+4}$. If W contains a unique switch then $deg(W) \geq 2 \geq out(W)$. Set $u = r_{i+1} = s_{i+1}$ and let $R = \{r_j, 1 \leq j \leq i+1\}$ and $S = \{r_j, 1 \leq j \leq i+1\}$. Suppose now that W contains at least two switches. Then because W is connected, it contains an element of $R \cup S$. By symmetry, we may assume that $W \cap R$ is not empty. Let j and j' be the smallest and biggest integer such that $r_j \in W$. Then if $v_j \in \bar{W}$ then $(v_j, r_j) \in \Delta(W)$ otherwise (v_j, r_{j-1}) (with r_0 being the input adjacent to v_1) is in $\Delta(W)$.

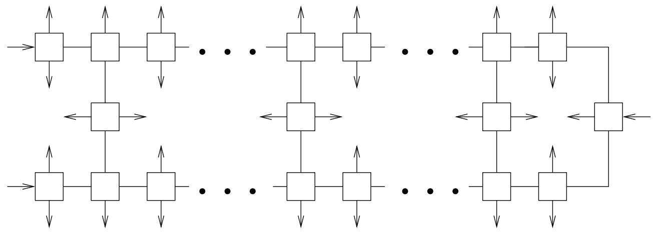

Figure 6: Minimum $(3, n)$-selector

Analogously, if $v_{j'+1} \in \bar{W}$ then $(v_{j'+1}, r_{j'}) \in \Delta(W)$ otherwise $(v_{j'+1}, r_{j'+2})$ (with $r_{i=2}$ being the input adjacent to u) is in $\Delta(W)$.

Suppose first that $W \cap S \neq \emptyset$. Let j'' be the minimum integer such that $s_{j''} \in W$. There is a cutting edge which is incident to $w_{j''}$. Hence $deg(W) \geq 3$.

Suppose now that $W \cap S = \emptyset$. Then $j \leq i$ and (r_j, t_j) or (t_j, s_j) is in $\Delta(W)$. Again $deg(W) \geq 3$.

Thus by Proposition 5, $S_{3,8i+5}$ is a $(3, 8i+5)$-selector. And obviously, for $1 \leq j \leq 7$, $S_{3,8i+5-j}$ obtained from $S_{3,8i+5}$ by removing j outputs is a $(3, 8i+5-j)$-selector. ■

Theorem 7

$$S(4,n) \quad = \quad \frac{3}{4}n + \theta(1)$$

Proof. Let $S_{4,4i}$ be the network depicted Figure 7 with $3i-1$ switches v_j, $1 \leq j \leq 2i$ and u_j, $1 \leq j \leq i-1$, such that:

- for $1 \leq j \leq i-1$, u_j is adjacent to v_j, v_{j+1}, v_{i+j} and v_{i+j+1};
- for $1 \leq j \leq 2i$, v_j is adjacent to two outputs;
- v_1, v_i, v_{i+1} and v_{2i} are adjacent to an input.

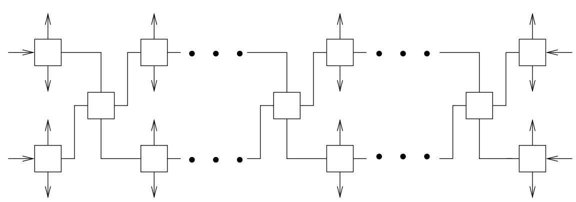

Figure 7: Minimum $(4, n)$-selector

Let W be a suitable subset of $S_{4,4i}$. If W contains a unique switch then $deg(W) \geq 2 \geq out(W)$. Suppose now that it contains at least two switches. Then because W is connected, it contains at least of of the u_j. Let j and j' be the smallest and biggest integer such that $u_j \in W$. Then one of the two edges (v_j, u_j) and (v_j, u_{j-1}) is in $\Delta(W)$ (with u_0 the input adjacent to v_1). Analogously v_{i+j}, $v_{j'+1}$ and $v_{i+j'+1}$ are incident to a cutting edge. Hence $deg(W) \geq 4$. Therefore, by Proposition 5, $S_{4,4i}$ is a $(4,4i)$-selector.

And the networks $S_{4,4i-j}$, $1 \leq j \leq 3$, obtained from $S_{4,4i}$ by removing j outputs is a $(4, 4i-j)$-selectors. ■

Theorem 8

$$S(5,n) \quad = \quad \frac{13}{16}n + \theta(1) \tag{10}$$

Proof. Using Proposition 5, one can prove that the network depicted Figure 8 is a $(5,n)$-selector. ■

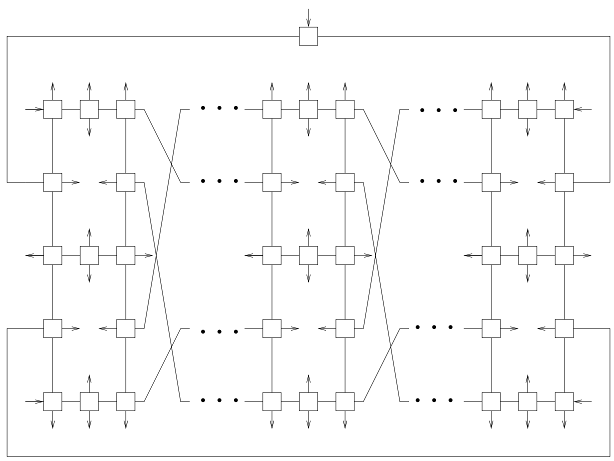

Figure 8: Minimum $(5,n)$-selector

Theorem 9

$$S(6,n) \quad = \quad \frac{7}{8}n + \theta(1)$$

Proof. Let $S_{6,8i}$ be the network depicted Figure 9 whose switch set is the partition of seven sets, $A = \{a_j, 1 \leq j \leq i\}$, $B = \{b_j, 1 \leq j \leq i\}$, $C = \{c_j, 1 \leq j \leq i\}$, $D = \{d_j, 1 \leq j \leq i\}$, $E = \{e_j, 1 \leq j \leq i\}$, $F = \{f_j, 1 \leq j \leq i\}$, and $G = \{g_j, 1 \leq j \leq i\}$ such that:

- A, B, C and D induces paths;
- every switch of $A \cup B \cup C \cup D \cup E \cup F$ is adjacent to one output;
- every switch of G is adjacent to two outputs;

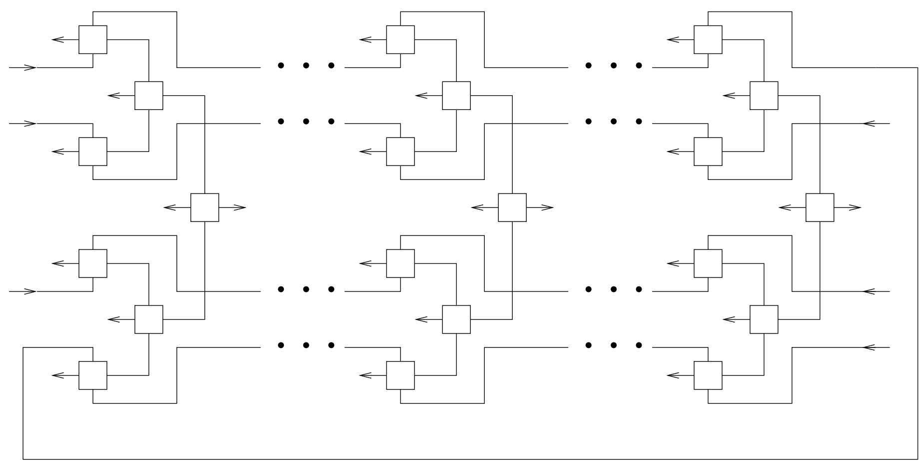

Figure 9: Minimum $(6, n)$-selector

- for $1 \leq j \leq i$, e_i is adjacent to a_i, b_i and g_i;
- for $1 \leq j \leq i$, f_i is adjacent to c_i, d_i and g_i.

Let W be a suitable set of $S_{6,8i}$.

Assume first that W has $sw < 6$ switches. Since there is no cycle of length less than 6 and the distance between to switches of G is at least 6, then W is a tree containing at most one element of G. Thus, $deg(W) \geq 2sw + 2 - out(W)$ and $out(W) \leq sw + 1$. Thus, $deg(W) \geq out(W)$.

Suppose now that W has at least 6 switches. Let us prove that $deg(W) \geq 6$.

Let us consider the paths P_1, P_2 and P_3 induces by the vertices of $A \cup D$, B and C respectively. For $1 \leq l \leq 3$, if there is a vertex on P_l then P_l contains at least two cutting edges. In particular, if there is a vertex on each path then $deg(W) \geq 6$.

Let T_j be the network induced by $\{a_j, b_j, c_j, d_j, e_j, f_j, g_j\}$. Suppose now that W intersects two paths P_l. Then each T_j containing a vertex of W contains a cutting edge. If there at least two such trees, then $deg(W) \leq 6$ because there are at least four cutting edges on the paths. If there only one, it is easy that there are six cutting edges because there are 4 on P_1, (two on $P_1 \cap A$ and two on $P_1 \cap D$). Thus $deg(W) \geq 6$.

At last, assume that W intersects one of the P_l. For any tree T_j, if $|W \cap T_j| = 1$, there is one cutting edge in T_j, if $|W \cap T_j| \in \{2, 3\}$ there are two cutting edges and if $|W \cap T_j| = 4$, there are three cutting edges. It follows easily that $deg(W) \geq 6$. ■

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