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Reduced Load Equivalence under Subexponentiality

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Abstract: The stationary workload W_{A+B}^{ϕ} of a queue with capacity ϕ loaded by two independent processes A and B is investigated. When the probability of load deviation in process A decays slower than both in B and $e^{-\sqrt{x}}$, we show that W_{A+B}^{ϕ} is asymptotically equal to the reduced load queue $W_A^{\phi-b}$, where b is the mean rate of B. This complements the known result that this property does not hold when both processes have lighter than $e^{-\sqrt{x}}$ deviation decay rates. Furthermore, using the same methodology, we show that under an equivalent set of conditions the results on sampling at subexponential times hold.

Key-words: Large deviations, non-Cramér type conditions, reduced load equivalence, independent sampling, subexponential distributions

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Charge réduite équivalente dans le cas sous-exponentiel

Résumé : La charge stationnaire W^{ϕ}_{A+B} d'une file d'attente avec la capacité ϕ recevant deux processus indépendants A et B est analysée. Sous des hypothèses faibles sur A et B, nous montrons que W^{ϕ}_{A+B} est asymptotiquement égal à la charge équivalente de la file $W^{\phi-b}_{A}$, où b est le taux moyen de B. Ceci complète le résultat connu selon lequel cette propriété n'est pas vraie quand les deux processus ont une queue de distribution plus faible que $e^{-\sqrt{x}}$. De plus, en utilisant les mêmes méthodes, nous montrons que sous des hypothèses similaires, les résultats sont aussi vrais à un instant aléatoire.

Mots-clés : Grandes déviations. Charge équivalente. Distributions sous-exponentielles.

1 Introduction

Statistical resource sharing provides a mechanism for improving operating efficiency in many areas of business and engineering. This mechanism is particularly useful in communication networks, where resources, such as link transmission capacity and buffer space, are shared among different user sessions. This sharing creates potential workload backlogs that need to be addressed. A baseline model of the backlog is represented by a workload W^{ϕ}_{A+B} of a stationary queue of capacity ϕ and independent arrival processes A and B. Processes A and B can be interpreted as independent demands for a generic resource ϕ.

Thus, it is of general interest to provide analytical tools for understanding the statistical behavior of W^{ϕ}_{A+B}. In the context of heavy tails, in Theorem 4.4 of [12] (see [6] and [17] for related studies) it was shown that when process A has polynomial characteristics and B is exponentially bounded, then as $x \to \infty$

$$\mathbb{P}[W^{\phi}_{A+B} > x] = \mathbb{P}[W^{\phi-b}_{A} > x](1 + o(1)), \tag{1}$$

where b is the mean rate of process B and $W^{\phi-b}_{A}$ is the workload of a queue with reduced load A and capacity $\phi - b$. Hence, in [1] the preceding relatinship was termed *reduced load equivalence*. The results of [1] provide more general conditions under which (1) holds. For related reduced load equivalence results with polynomial tails see [18, 4, 13] and the references therein.

In this paper we further extend the results from [1]. Informally, we allow the tail of $W^{\phi-b}_{A}$ to be heavier than $e^{-\sqrt{x}}$, but lighter than $e^{-x^{1/3}}$; this range was not covered by the results of [1]. Furthermore, we allow process B to have heavier than exponential, but lighter than $W^{\phi-b}_{A}$ tails. These points will be discussed more specifically throughout the paper. Our results, in conjunction with the known result that (1) fails when both processes have lighter than $e^{-\sqrt{x}}$ characteristics [9], provide a set of conditions under which (1) is expected to hold.

Our main result on reduced load equivalence is stated in Theorem 2. The statement of this theorem significantly simplifies when the distribution of deviations in process B has lighter than $e^{-\sqrt{x}}$ tail; this is presented in Proposition 2. Informally, heuristic of our analysis can be briefly described with the following steps (for large x)

$$\mathbb{P}[W^{1}_{A+B} > x] \approx \mathbb{P}\left[\sup_{t\le lx}\{A_t - (1-b)t\} + \sup_{t\le lx}\{B_t - bt\} > x\right] \tag{2}$$

$$\approx \mathbb{P}[W^{1-b}_{A} + Z\sqrt{x} > x] \tag{3}$$

$$\approx \mathbb{P}[W^{1-b}_{A} > x], \tag{4}$$

where Z is a positive Gaussian random variable. The first step is justified by Lemma 7. Substantiating approximation (3), i.e., providing a satisfactory bound on the second supremum in (2), represents the main technical difficulty. This is facilitated by Theorem 3.2 of [11]; strengthened versions of this theorem are presented in Lemma 5 and Proposition 1. Necessary and sufficient conditions for (4) to hold are provided in Theorem 1.

Furthermore, using the same methodology, we show that under an equivalent set of conditions the results on *independent sampling* (see [2]) at subexponential time T hold, i.e., as $x \to \infty$

$$\mathbb{P}[B_T > x] = \mathbb{P}[Tb > x](1 + o(1)),$$

where T is independent of B. Our results are stated in Theorem 3 and Proposition 3.

2 Preliminaries

2.1 Gaussian insensitivity

The purpose of this subsection is to provide a framework for rigorously justifying (4) of the introduction. Necessary and sufficient condition for this to hold is provided when W_A^{1-b} is *square-root insensitive* as in the following Definition 1. The square-root insensitivity appears in work of [2].

Throughout the paper, for any two real functions $f(x)$ and $g(x)$, we use the standard notation $f(x) \sim g(x)$ as $x \to \infty$ to denote $\lim_{x\to\infty} f(x)/g(x) = 1$.

Definition 1 *Random variable X is called* square-root insensitive *if*

$$\mathbb{P}[X > x - \sqrt{x}] \sim \mathbb{P}[X > x] \quad \textit{as } x \to \infty.$$

Remark 1 By monotonicity it follows that if X is square-root insensitive then for any k

$$\mathbb{P}[X > x - k\sqrt{x}] \sim \mathbb{P}[X > x] \quad \text{as } x \to \infty.$$

In this paper C denotes a sufficiently large positive constant, while c denotes a sufficiently small positive constant. The values of C and c are generally different in different places. For example, $C/2 = C$, $C^2 = C$, $C + 1 = C$, etc.

The hazard function $Q(x)$ of a random variable (r.v.) X is defined as $Q(x) = -\log \mathbb{P}[X > x]$.

Lemma 1 *If X is square-root insensitive, then its hazard function Q satisfies*

$$Q(x) = o(\sqrt{x}) \quad \textit{as } x \to \infty.$$

Proof: The assumption of the lemma implies that for any $\epsilon > 0$ there exists x_ϵ, such that for all $x \geq x_\epsilon \geq 1$

$$\frac{\mathbb{P}[X > x]}{\mathbb{P}[X > x + \sqrt{x}]} \leq 1 + \epsilon. \tag{5}$$

Next, define recursively $f^{(n)}(u) = f^{(n-1)}(u) + \sqrt{f^{(n-1)}(u)}$ for integers $n \geq 1$ with $f^{(0)}(u) = u$. Then,

$$\begin{aligned} f^{(2)}(x_\epsilon) &= x_\epsilon + \sqrt{x_\epsilon} + \sqrt{x_\epsilon + \sqrt{x_\epsilon}} \\ &\geq x_\epsilon + \sqrt{x_\epsilon} + 1/4 \\ &= (\sqrt{x_\epsilon} + 1/2)^2. \end{aligned}$$

From the last inequality, it is easy to show by induction that

$$f^{(2n)}(x_\epsilon) \geq (\sqrt{x_\epsilon} + n/2)^2. \tag{6}$$

Now, let n_x be the smallest integer such that $(\sqrt{x_\epsilon} + n_x/2)^2 \geq x$, i.e.

$$n_x < 2(\sqrt{x} - \sqrt{x_\epsilon}) + 1. \tag{7}$$

Next, due to (6), the choice of n_x and the monotonicity of $\mathbb{P}[X > \cdot]$

$$\frac{\mathbb{P}[X > x_\epsilon]}{\mathbb{P}[X > x]} \leq \frac{\mathbb{P}[X > x_\epsilon]}{\mathbb{P}[X > f^{(2)}(x_\epsilon)]} \frac{\mathbb{P}[X > f^{(2)}(x_\epsilon)]}{\mathbb{P}[X > f^{(4)}(x_\epsilon)]} \cdots \frac{\mathbb{P}[X > f^{(2n_x-2)}(x_\epsilon)]}{\mathbb{P}[X > f^{(2n_x)}(x_\epsilon)]}.$$

Observe that by (5) each of the ratios in the preceding inequality is upper bounded by $(1+\epsilon)^2$ and, hence, recalling (7) yields

$$\frac{\mathbb{P}[X > x_\epsilon]}{\mathbb{P}[X > x]} \le (1+\epsilon)^{2n_x} \le (1+\epsilon)^{4(\sqrt{x}-\sqrt{x_\epsilon})+2}$$

implying as $\epsilon \to 0$

$$\overline{\lim_{x\to\infty}} \frac{Q(x)}{\sqrt{x}} \le 4\log(1+\epsilon) \longrightarrow 0.$$

□

The following theorem represents the main technical result of this subsection. It provides us with a tool to justify, in Section 3, step (4) of the outline in the introduction.

Theorem 1 *Let Z be the absolute value of a standard normal random variable. Then X is square-root insensitive if and only if*

$$\mathbb{P}[X > x - Z\sqrt{x}] \sim \mathbb{P}[X > x] \quad \textit{as } x \to \infty.$$

Remark 2 By monotonicity it follows that if X is square-root insensitive then for any k

$$\mathbb{P}[X > x - kZ\sqrt{x}] \sim \mathbb{P}[X > x] \quad \text{as } x \to \infty.$$

Proof: (*only if part*) For $k > 0$ conditioning on Z results in

$$\mathbb{P}[X > x - Z\sqrt{x}] \le \mathbb{P}[X > x - k\sqrt{x}] + C\int_k^{\sqrt{x}/2} e^{-\frac{z^2}{2}}\mathbb{P}[X > x - z\sqrt{x}]dz + \mathbb{P}[Z > \sqrt{x}/2]. \tag{8}$$

In order to estimate the integral term of (8) we choose $x_\epsilon \ge 1$ such that (5) holds for all $x \ge x_\epsilon$. Observe that $x - z\sqrt{x} \ge x_\epsilon$ for all $x \ge 2x_\epsilon$ and $z \le \sqrt{x}/2$. Now, let n_z be the smallest integer such that $(\sqrt{x - z\sqrt{x}} + n_z/2)^2 \ge x$, i.e.

$$\begin{aligned} n_z &< 2\left(\sqrt{x} - \sqrt{x - z\sqrt{x}}\right) + 1 \\ &\le 2z + 1. \end{aligned} \tag{9}$$

Next, recall the difinition of $f^{(n)}$ from the proof of Lemma 1. Due to (6), the choice of n_z and the monotonicity of $\mathbb{P}[X > \cdot]$ for all $z \le \sqrt{x}/2$ and $x \ge 2x_\epsilon$

$$\begin{aligned} \frac{\mathbb{P}[X > x - z\sqrt{x}]}{\mathbb{P}[X > x]} &\le \frac{\mathbb{P}[X > f^{(0)}(x - z\sqrt{x})]}{\mathbb{P}[X > f^{(2)}(x - z\sqrt{x})]]} \cdots \frac{\mathbb{P}[X > f^{(2n_z-2)}(x - z\sqrt{x})]}{\mathbb{P}[X > f^{(2n_z)}(x - z\sqrt{x})]]} \\ &\le (1+\epsilon)^{2n_z} \\ &\le (1+\epsilon)^{4z+2}, \end{aligned}$$

where the last inequality follows from (9). Therefore, the upper bound for the second term in (8) is as follows

$$\begin{aligned} \int_k^{\sqrt{x}/2} e^{-\frac{z^2}{2}}\mathbb{P}[X > x - z\sqrt{x}]dz &\le C\mathbb{P}[X > x]\int_k^{\sqrt{x}/2} e^{-\frac{z^2}{2}}e^{zC}dz \\ &\le C\mathbb{P}[X > x]\int_k^{\infty} e^{-\frac{(z-C)^2}{2}}dz. \end{aligned}$$

Combining the the preceding bound and Lemma 1 with (8) easily yields the "only if" part of the theorem.

(*if part*) Note that

$$\mathbb{P}[X > x - Z\sqrt{x}] \geq \mathbb{P}[Z > 1]\mathbb{P}[X > x - \sqrt{x}] + \mathbb{P}[Z \leq 1]\mathbb{P}[X > x],$$

which, in conjunction with the assumption, implies $\mathbb{P}[X > x - \sqrt{x}] \sim \mathbb{P}[X > x]$ as $x \to \infty$. □

2.2 Large deviation results

This subsection presents several large deviation bounds that will be used in proving our main results. In this paper, we use the following two classes of heavy-tailed distributions:

Definition 2 ([7]) *A nonnegative r.v. X is called subexponential ($X \in \mathcal{S}$)*

$$\lim_{x\to\infty} \frac{1 - F^{2*}(x)}{1 - F(x)} = 2,$$

where F^{2*} denotes the 2-fold convolution of F with itself, i.e., $F^{2*}(x) = \int_{[0,\infty)} F(x-y)F(dy)$.

Definition 3 ([14]) *A nonnegative r.v. X (or its hazard function) belongs to class $\mathcal{SC}$ if its hazard function $Q(x)$ is* eventually concave*, such that, as $x \to \infty$*

$$Q(x)/\log x \to \infty$$

and for $x \geq x_0$, $\beta x \leq u \leq x$,

$$\frac{Q(x) - Q(u)}{Q(x)} \leq \alpha \frac{x-u}{x},$$

where $0 < \alpha < 1$, $0 < \beta < 1$.

As pointed out in [11], r.v.s with hazard functions $(\log x)^\alpha$, $1 < \alpha$, and x^α, $0 < \alpha < 1$, belong to $\mathcal{SC}$; see [11] for additional discussion on the class $\mathcal{SC}$.

In analyzing renewal processes, residual random variables and distribution functions play an important role. For a nonnegative random variable X with distribution F and finite mean $\mathbb{E}X$, the residual distribution F_r is defined by $F_r(x) = (\mathbb{E}X)^{-1}\int_0^x (1 - F(u))\,du$, $x \geq 0$. A random variable X_r with distribution F_r is called the residual variable of X.

We now state two basic lemmas.

Lemma 2 *If $X \in \mathcal{S}$ and $\mathbb{P}[U > x] = o(\mathbb{P}[X > x])$, then as $x \to \infty$*

$$\mathbb{P}[X + U > x,\ X \leq x] = o(\mathbb{P}[X > x]).$$

Proof: Corollary 2 of [16] states $\mathbb{P}[X + U > x] \sim \mathbb{P}[X > x]$ as $x \to \infty$, which in conjunction with $\mathbb{P}[X + U > x] = \mathbb{P}[X > x] + \mathbb{P}[X + U > x,\ X \leq x]$ concludes the proof. □

Lemma 3 ([11]) (a) *If $Q \in \mathcal{SC}$, then $Q(x) \leq Q(u)(x/u)^\alpha$ for all $x_0 \leq u \leq x$.* (b) *If $X \in \mathcal{SC}$ then $X, X_r \in \mathcal{S}$.*

In the remaining part of this subsection, we assume that $\{X, X_i, i \geq 1\}$ is a sequence of i.i.d. r.v.s. The next sequence of lemmas will be used in Section 3 to estimate the deviations of process B.

Lemma 4 *If $\mathbb{E}e^{Q(X)} < \infty$ for some $Q \in \mathcal{SC}$, then for any $\phi > \mathbb{E}X$*

$$\lim_{x\to\infty} e^{o(Q(x))}\mathbb{P}\left[\sup_{n\geq 1}\left\{\sum_{i=1}^{n} X_n - \phi n\right\} > x\right] = 0.$$

Proof: The lemma follows from stochastic dominance, Lemma 3 and Pakes' theorem [15]. □

Lemma 5 *If $\mathbb{E}e^{Q(X)} < \infty$ for some $Q \in \mathcal{SC}$, then for all x and $u \geq 0$*

$$\mathbb{P}\left[\max_{1\leq n\leq x}\left\{\sum_{i=1}^{n} X_i - n\mathbb{E}X\right\} > u\right] \leq C\left(e^{-c\frac{u^2}{x}} + xe^{-\frac{1}{2}Q(u)}\right).$$

Proof: See the appendix.

Lemma 6 *Let $X \geq 0$ a.s. and $0 < \mathbb{E}X < \infty$. If $N_x = \max\{n : \sum_{i=1}^{n} X_i < x\}$, then there exists $\delta > 0$ such that for all x and $0 \leq u \leq \delta x$*

$$\mathbb{P}\left[N_x - x/\mathbb{E}X > u\right] \leq Ce^{-cu^2/x}.$$

Proof: See the appendix.

3 Main results

In this section we present our main results on reduced load equivalence and independent sampling. As mentioned in the introduction, these results extend work of [1] and [2]. It will become clear from our analysis that the two problems are strongly related: the square-root insensitivity plays a central role in both of them.

We assume that B is a regenerative process with $B_0 = 0$. The length of the nth regenerative cycle is denoted by $\nu_n > 0$. Random variables $\{\nu_n\}_{n=1}^{\infty}$ are i.i.d. independent of a.s. finite ν_0 and have finite second moment, $\mathbb{E}\nu_1^2 < \infty$. With each ν_n, $n \geq 1$, we associate random variables γ_n and γ_n^*. If $T_n = \sum_{i=0}^{n} \nu_i$ then

$$\gamma_n = B_{T_n} - B_{T_{n-1}},$$
$$\gamma_n^* = \sup_{T_{n-1}\leq t\leq T_n} B_t - B_{T_{n-1}},$$

with $\gamma_0 = B_{\nu_0}$ and $\gamma_0^* = \sup_{0\leq t\leq \nu_0} B_t$. The random variables $\{\gamma_n\}_{n=1}^{\infty}$ and $\{\gamma_n^*\}_{n=1}^{\infty}$ are i.i.d. independent of a.s. finite γ_0 and γ_0^* and have the finite first moment. Note that, by the SLLN, the mean rate $b \triangleq \mathbb{E}\gamma_1/\mathbb{E}\nu_1 = \lim_{t\to\infty} B_t/t$ a.s.

The proofs of our main results use the following proposition that generalizes Theorem 3.2 of [11] to regenerative processes.

Proposition 1 *If* $\mathbb{E}e^{Q(\gamma_i^*)} < \infty$, $i = 0,1$ *for some* $Q \in \mathcal{SC}$, *then for all* x *and* $u \geq 0$

$$\mathbb{P}\left[\sup_{0\leq t\leq x}\{B_t - bt\} > u\right] \leq C\left(e^{-c\frac{u^2}{x}} + e^{-cx} + xe^{-cQ(u)}\right).$$

Proof: Let $N_x = \max\{n : \sum_{i=1}^n \nu_i < x\}$. Since for all $t \geq 0$

$$B_t - bt \leq \gamma_0^* + \gamma^*_{N_{t-\nu_0}+1} + \sum_{i=1}^{N_{t-\nu_0}} \gamma_i - b\sum_{i=1}^{N_{t-\nu_0}} \nu_i, \tag{10}$$

one concludes

$$\mathbb{P}\left[\sup_{0\leq t\leq x}\{B_t - bt\} > u\right] \leq \mathbb{P}\left[\gamma_0^* > \frac{u}{4}\right] + \mathbb{P}\left[\gamma_1^* > \frac{u}{4}\right] + \mathbb{P}\left[\max_{1\leq n\leq N_x}\sum_{i=1}^n(\gamma_i - b\nu_i) > \frac{u}{2}\right]. \tag{11}$$

Lemma 6 provides a bound on N_x for all δ small enough

$$\mathbb{P}[N_x - x/\mathbb{E}\nu_1 > \delta x] \leq Ce^{-cx}$$

and, hence, the third term in (11) can be upper bound as follows

$$\begin{aligned}\mathbb{P}\left[\max_{0\leq n\leq N_x}\sum_{i=1}^n(\gamma_i - b\nu_i) > \frac{u}{2}\right] &\leq Ce^{-cx} + \mathbb{P}\left[\max_{0\leq n\leq(\delta+1/\mathbb{E}\nu_1)x}\sum_{i=1}^n(\gamma_i - b\nu_i) > \frac{u}{2}\right]\\ &\leq Ce^{-cx} + C\left(e^{-c\frac{u^2}{x}} + xe^{-\frac{1}{2}Q(u/2)}\right),\end{aligned}$$

where the last inequality is due to Lemma 5. Substituting the preceding bound in (11) leads to

$$\mathbb{P}\left[\sup_{0\leq t\leq x}\{B_t - bt\} > u\right] \leq Ce^{-Q(u/4)} + Ce^{-cx} + C\left(e^{-c\frac{u^2}{x}} + xe^{-\frac{1}{2}Q(u/2)}\right)$$

and the statement follows by Lemma 3. □

3.1 Reduced load equivalence

In this subsection we investigate the tail behavior of the stationary workload of a stable queue. The stationary workload W_X^ϕ of a queue with service rate ϕ fed by a process X with stationary increments, is know to satisfy

$$W_X^\phi \stackrel{d}{=} \sup_{t\geq 0}\{X_t - \phi t\},$$

where $\stackrel{d}{=}$ denotes equality in distribution and X_t represents the amount of work that arrives to the queue in $(-t, 0)$; throughout the paper X will be considered equal to A, B or $A+B$. In the queueing context, a natural assumption on B is that sample paths are a.s. increasing, i.e., in this subsection $\gamma_i = \gamma_i^*$ for $i \geq 0$. For convenience, we put $W_X \equiv W_X^1$.

The following theorem is the first main result of this paper. Let a denote the mean rate of process A.

Theorem 2 *Let* $\mathbb{E}e^{Q(\gamma_i)} < \infty$, $i = 0,1$ *for some* $Q \in \mathcal{SC}$ *and* $\mathbb{E}\nu_1^2 < \infty$. *If* $W_A^{1-b} \in \mathcal{S}$ *is square-root insensitive,* $\mathbb{P}[W_A^{1-b} > x] = e^{-o(Q(x))}$ *and for some* $a < \phi < 1-b$

$$\lim_{k\to\infty}\overline{\lim_{x\to\infty}}\frac{\mathbb{P}[W_A^\phi > kx]}{\mathbb{P}[W_A^{1-b} > x]} = 0,$$

then as $x \to \infty$

$$\mathbb{P}[W_{A+B} > x] \sim \mathbb{P}[W_A^{1-b} > x].$$

When the regenerative increments of B do not have tails heavier than $e^{-\theta\sqrt{x}}$, $\theta > 0$, the conditions of the preceding theorem can be weakened. In particular, the assumptions $W_A^{1-b} \in \mathcal{S}$ and $\mathbb{P}[W_A^{1-b} > x] = e^{-o(Q(x))}$ are not needed.

Proposition 2 *Let* $\mathbb{E}e^{\theta\sqrt{\gamma_i}} < \infty$, $i = 0, 1$ *for some* $\theta > 0$ *and* $\mathbb{E}\nu_1^2 < \infty$. *If* W_A^{1-b} *is square-root insensitive and for some* $a < \phi < 1 - b$

$$\lim_{k\to\infty} \overline{\lim_{x\to\infty}} \frac{\mathbb{P}[W_A^{\phi} > kx]}{\mathbb{P}[W_A^{1-b} > x]} = 0,$$

then as $x \to \infty$

$$\mathbb{P}[W_{A+B} > x] \sim \mathbb{P}[W_A^{1-b} > x].$$

Remark 3 The double limit condition in the statements of Theorem 2 and Proposition 2 is implied by $x^{\epsilon}\mathbb{P}[X > x]$ being eventually monotonically decreasing in x for some $\epsilon > 0$.

These results extend Propositions 8.2 and 8.3 of [1], where A is assumed to be an On-Off process with a specific form of the distribution of On periods and B is exponentially bounded. In particular, Proposition 8.3 in [1] assumes that the tail of the residual distribution of On periods is of the form $e^{-\alpha x^{\beta}}$ with $\beta < 1/3$.

Possible choices of A include, for instance, an On-Off process with subexponential On periods and a Gaussian process exhibiting long-range dependence, cf. [5]. Next, we specialize our result to the case where A is a stationary On-Off process. For the exact construction of such a process see [10]. Let On periods be equal in distribution to τ. Denote by r and a the peak and average rate, respectively.

Corollary 1 *Let* $\mathbb{E}e^{Q(\gamma_i)} < \infty$, $i = 0, 1$ *for some* $Q \in \mathcal{SC}$ *and* $\mathbb{E}\nu_1^2 < \infty$. *If* $\tau_r \in \mathcal{S}$ *is square-root insensitive,* $\mathbb{P}[\tau_r > x] = e^{-o(Q(x))}$, $r > 1 - b > a$ *and*

$$\lim_{k\to\infty} \overline{\lim_{x\to\infty}} \frac{\mathbb{P}[\tau_r > kx]}{\mathbb{P}[\tau_r > x]} = 0,$$

then as $x \to \infty$

$$\mathbb{P}[W_{A+B} > x] \sim \mathbb{P}[W_A^{1-b} > x].$$

Proof: Follows from Theorem 2 and the asymptotics for W_A^{1-b} (see Theorem 4.3 of [12]). □

Before we turn to the proofs of Theorem 2 and Proposition 2, we state an additional preliminary result.

Lemma 7 *Let* $\mathbb{E}e^{Q(\gamma_i)} < \infty$, $i = 0, 1$ *for some* $Q \in \mathcal{SC}$. *If* $\mathbb{P}[W_A^{1-b} > x] = e^{-o(Q(x))}$ *and for some* $a < \phi < 1 - b$

$$\lim_{l\to\infty} \overline{\lim_{x\to\infty}} \frac{\mathbb{P}[W_A^{\phi} > lx]}{\mathbb{P}[W_A^{1-b} > x]} = 0,$$

then

$$\lim_{l\to\infty} \overline{\lim_{x\to\infty}} \frac{\mathbb{P}[\sup_{t>lx}\{A_t + B_t - t\} > x]}{\mathbb{P}[W_A^{1-b} > x]} = 0.$$

Proof: See the appendix. □

We are now ready to present a proof of Theorem 2.

Proof of Theorem 2: The proof consists of deriving upper and lower bounds.

Upper bound. Observe that for $l > 0$

$$\mathbb{P}[W_{A+B} > x] \leq \mathbb{P}\left[\sup_{0\leq t\leq lx}\{A_t + B_t - t\} > x\right] + \mathbb{P}\left[\sup_{t>lx}\{A_t + B_t - t\} > x\right]. \tag{12}$$

The second term is negligible by Lemma 7 as $x \to \infty$ for large l. To estimate the first term, we proceed as follows. By using $\sup_t\{f(t) + g(t)\} \leq \sup_t f(t) + \sup_t g(t)$ for any two functions $f(x)$ and $g(x)$ one obtains for $k > 0$

$$\begin{aligned}\mathbb{P}\left[\sup_{0\leq t\leq lx}\{A_t + B_t - t\} > x\right] &\leq \mathbb{P}\left[\sup_{0\leq t\leq lx}\{A_t - (1-b)t\} + \sup_{0\leq t\leq lx}\{B_t - bt\} > x\right]\\ &\leq \mathbb{P}[W_A^{1-b} > x - k\sqrt{x}] + \mathbb{P}[W_A^{1-b} + Y_{lx} > x,\ W_A^{1-b} \leq x - k\sqrt{x}]\\ &\triangleq f_1 + f_2,\end{aligned} \tag{13}$$

where $Y_x \triangleq \sup_{0\leq t\leq x}\{B_t - bt\}$. Proposition 1 yields an upper bound on f_2

$$\begin{aligned}f_2 &\leq \int_0^{x-k\sqrt{x}} \mathbb{P}[Y_{lx} > x - u] d\mathbb{P}[W_A^{1-b} \leq u]\\ &\leq C\int_0^{x-k\sqrt{x}} \left(e^{-c\frac{(x-u)^2}{lx}} + e^{-clx} + lxe^{-cQ(x-u)}\right) d\mathbb{P}[W_A^{1-b} \leq u]\\ &\triangleq f_{21} + f_{22} + f_{23}.\end{aligned} \tag{14}$$

Integration by parts and change of variables ($z = (x-u)/\sqrt{x}$) result in

$$\begin{aligned}f_{21} &\leq Ce^{-c\frac{x}{l}} + C\frac{k}{l\sqrt{x}}\int_0^{x-k\sqrt{x}} \frac{x-u}{\sqrt{x}} e^{-c\frac{(x-u)^2}{lx}} \mathbb{P}[W_A^{1-b} > u]\, du\\ &= Ce^{-c\frac{x}{l}} + C\frac{k}{l}\int_k^{\sqrt{x}} ze^{-c\frac{z^2}{l}} \mathbb{P}[W_A^{1-b} > x - z\sqrt{x}]\, dz\\ &\leq Ce^{-c\frac{x}{l}} + C\frac{k}{l}\mathbb{P}[W_A^{1-b} + Z\sqrt{x} > x,\ Z > k],\end{aligned}$$

where r.v. Z is the absolute value of a normal random variable. Combining the preceding bound with Lemma 3 and Theorem 1 we obtain

$$\overline{\lim_{x\to\infty}} \frac{f_{21}}{\mathbb{P}[W_A^{1-b} > x]} \leq C\frac{k}{l}\mathbb{P}[Z > k].$$

It easy follows that the upper bound for the second term in (14) is $f_{22} \leq Ce^{-clx}$. To handle f_{23}, define r.v. U such that $\mathbb{P}[U > x] = e^{-cQ(x)}$ for $x \geq x_U$. Then,

$$\begin{aligned}f_{23} &\leq Clxe^{-cQ(k\sqrt{x})}\int_0^{x-k\sqrt{x}} e^{-cQ(x-u)} d\mathbb{P}[W_A^{1-b} \leq u]\\ &= Clxe^{-cQ(k\sqrt{x})}\mathbb{P}[U + W_A^{1-b} > x,\ W_A^{1-b} \leq x - k\sqrt{x}];\end{aligned}$$

thus, by Lemmas 2 and 3 (b) one obtains $f_{23} = o(\mathbb{P}[W_A^{1-b} > x])$ as $x \to \infty$.

Combining the bounds on f_{21}, f_{22} and f_{23} with (14), (13) and square-root insensitivity of W_A^{1-b} results, after passing $k \to \infty$, in

$$\overline{\lim_{x\to\infty}} \frac{\mathbb{P}\left[\sup_{0\le t\le lx}\{A_t + B_t - t\} > x\right]}{\mathbb{P}[W_A^{1-b} > x]} = 1.$$

Therefore, the proof of the upper bound is concluded recalling (12) and Lemma 7.

Lower bound. As usual, the lower bound is somewhat easier:

$$\begin{aligned}\mathbb{P}[W_{A+B} > x] &\geq \mathbb{P}\left[\sup_{0\le t\le lx}\{A_t + B_t - t\} > x\right] \\ &\geq \mathbb{P}\left[\sup_{0\le t\le lx}\{A_t - (1-b)t\} + \inf_{0\le t\le lx}\{B_t - bt\} > x\right] \\ &= \mathbb{P}\left[\sup_{0\le t\le lx}\{A_t - (1-b)t\} - \sup_{0\le t\le lx}\{bt - B_t\} > x\right].\end{aligned}$$

Hence, for any $k, l > 0$,

$$\mathbb{P}[W_{A+B} > x] \geq \mathbb{P}\left[\sup_{0\le t\le lx}\{A_t - (1-b)t\} > x + k\sqrt{x}\right]\mathbb{P}\left[\sup_{0\le t\le lx}\{bt - B_t\} \le k\sqrt{x}\right]. \qquad (15)$$

Note that for $t \geq 0$

$$bt - B_t \le b\nu_0 + b\nu_{N_{t-\nu_0}+1} + \sum_{i=1}^{N_{t-\nu_0}} (b\nu_i - \gamma_i)$$

and, thus, by Lemma 6 and the CLT for maximums [8, Ch. 7] one obtains

$$\lim_{k\to\infty} \overline{\lim_{x\to\infty}}\, \mathbb{P}\left[\sup_{0\le t\le lx}\{bt - B_t\} \le k\sqrt{x}\right] = 1.$$

The proof is now completed by diving both sides of (15) by $\mathbb{P}[W_A^{1-b} > x + k\sqrt{x}]$, letting $x \to \infty$, using the square-root insensitivity of W_A^{1-b}, setting first $k \to \infty$, then $l \to \infty$ and applying Lemma 7. □

Proof of Proposition 2: The proof is identical to the proof of Theorem 2 with $Q(x) = \theta\sqrt{x}$ except the derivation of the upper bound for f_{23}. In the following we show that $f_{23} = o(\mathbb{P}[W_A^{1-b} > x])$ as $x \to \infty$. For any $0 < \delta < 1$ integration by parts yields

$$\begin{aligned}f_{23} &\le Clx \int_0^{x-k\sqrt{x}} e^{-c\sqrt{x-u}} d\mathbb{P}[W_A^{1-b} \le u] \\ &\le Clxe^{-c\sqrt{x}} + Clx \int_{\delta x}^{x-k\sqrt{x}} e^{-c\sqrt{x-u}} d\mathbb{P}[W_A^{1-b} \le u] \\ &\le Clxe^{-c\sqrt{x}} + Clx \int_{\delta x}^{x-k\sqrt{x}} e^{-c\sqrt{x-u}} \mathbb{P}[W_A^{1-b} > u] du. \qquad (16)\end{aligned}$$

Next, square-root insensitivity yields (see the proof of Lemma 1 for details) that for any $\epsilon > 0$ there exists $x_\epsilon \ge 1$ such that for all $x_\epsilon \le u \le x - k\sqrt{x}$

$$\frac{\mathbb{P}[W_A^{1-b} > u]}{\mathbb{P}[W_A^{1-b} > x - k\sqrt{x}]} \le Ce^{\epsilon(\sqrt{x}-\sqrt{u})}.$$

By using the preceding bound in (16) and recalling the concavity of Q one obtains for $\delta x \geq x_\epsilon$

$$\begin{aligned} f_{23} &\leq Clxe^{-c\sqrt{x}} + Clx\mathbb{P}[W_A^{1-b} > x - k\sqrt{x}] \int_{\delta x}^{x-k\sqrt{x}} e^{-c\sqrt{x-u}+\epsilon(\sqrt{x}-\sqrt{u})}\, du \\ &\leq Clxe^{-c\sqrt{x}} + Clx^2\mathbb{P}[W_A^{1-b} > x - k\sqrt{x}] \left(e^{-c\sqrt{k\sqrt{x}}+\epsilon(\sqrt{x}-\sqrt{x-k\sqrt{x}})} + e^{-\sqrt{x}(c\sqrt{1-\delta}-\epsilon(1-\sqrt{\delta}))} \right). \end{aligned}$$

Clearly, we can chose ϵ and δ in the preceding inequality to obtain

$$f_{23} \leq Clxe^{-c\sqrt{x}} + Clx^2\mathbb{P}[W_A^{1-b} > x - k\sqrt{x}]e^{-cx^{1/4}},$$

which by Lemma 1 and square-root insensitivity yields $f_{23} = o(\mathbb{P}[W_A^{1-b} > x])$ as $x \to \infty$. □

3.2 Independent sampling

Our second result is investigating the problem of independent sampling at subexponential times that was recently studied in [2]. We provide an alternative set of conditions, that appear easier to verify, under which Theorem 3.6 of [2] holds. In addition, our Proposition 3 fully generalizes Proposition 3.1 of [2]. The proofs below use the regenerative structure of B only to apply the CLT and Proposition 1. Hence, any process B satisfying the CLT and Proposition 1 is admissible, e.g. certain Gaussian processes as considered in [5].

Define the maximum $M_t = \sup_{0\leq s\leq t} B_s$. Note that M_t inherits the regenerative structure of B_t, but has the additional property that its sample paths are non-decreasing. Since B_t has positive drift, heuristically, B_t is not expected to be much smaller than M_t. Our theorem below shows that M_T and B_T have similar tail behavior. For convenience, we assume that mean rate $b = 1$.

Theorem 3 *Let $\mathbb{E}e^{Q(\gamma_i^*)} < \infty$, $i = 0, 1$ for some $Q \in \mathcal{SC}$ and $\mathbb{E}\nu_1^2 < \infty$. If $T \in \mathcal{S}$ is square-root insensitive and $\mathbb{P}[T > x] = e^{-o(Q(x))}$, then as $x \to \infty$*

$$\mathbb{P}[B_T > x] \sim \mathbb{P}[M_T > x] \sim \mathbb{P}[T > x].$$

Proof: Since $B_T \leq M_T$, it suffices to provide an upper bound for $\mathbb{P}[M_T > x]$ and a lower bound for $\mathbb{P}[B_T > x]$.

Upper bound. Write for $\delta < 1$

$$\mathbb{P}[M_T > x] \leq \mathbb{P}\left[T > x - k\sqrt{x}\right] + \mathbb{P}\left[M_T > x,\ \delta x < T \leq x - k\sqrt{x}\right] + \mathbb{P}[M_{\delta x} > x]. \tag{17}$$

One needs to show that the last two terms are $o(\mathbb{P}[T > x])$ as $x \to \infty$. Note that, by Proposition 1 and Lemma 3,

$$\begin{aligned} \mathbb{P}[M_{\delta x} > x] &\leq C\left(e^{-cx} + xe^{-cQ(x)}\right) \\ &\leq Cxe^{-cQ(x)} = o(\mathbb{P}[T > x]). \end{aligned}$$

To deal with the second term in (17), note that, in view of Proposition 1,

$$\begin{aligned}
\mathbb{P}\left[M_T > x,\ \delta x < T \le x - k\sqrt{x}\right] &= \int_{\delta x}^{x-k\sqrt{x}} \mathbb{P}[M_u > x]\, d\mathbb{P}[T \le u] \\
&\le \int_{\delta x}^{x-k\sqrt{x}} \mathbb{P}\left[\sup_{0\le s\le u}\{B_s - s\} > x-u\right] d\mathbb{P}[T \le u] \\
&\le C\int_{\delta x}^{x-k\sqrt{x}} \left(e^{-c\frac{(x-u)^2}{u}} + e^{-cu} + ue^{-cQ(x-u)}\right) d\mathbb{P}[T \le u] \\
&\le C\int_{\delta x}^{x-k\sqrt{x}} \left(e^{-c\frac{(x-u)^2}{x}} + e^{-c\delta x} + xe^{-cQ(x-u)}\right) d\mathbb{P}[T \le u].
\end{aligned}$$

Now, proceed exactly as in bounding f_2 in the proof of the upper bound of Theorem 2.

Lower bound. Following the steps of [2] we write

$$\begin{aligned}
\mathbb{P}[B_T > x] &\ge \int_{x+k\sqrt{x}}^{\infty} \mathbb{P}[B_u > x] d\mathbb{P}[T \le u] \\
&\ge \inf_{u > x+k\sqrt{x}} \mathbb{P}[B_u > x]\ \ \mathbb{P}[T > x + k\sqrt{x}].
\end{aligned}$$

Note that due to the monotonicity of $(x-u)/\sqrt{u}$ in u one obtains for $x > k^2$

$$\begin{aligned}
\inf_{u > x+k\sqrt{x}} \mathbb{P}[B_u > x] &\ge \inf_{u > x+k\sqrt{x}} \mathbb{P}\left[\frac{B_u - u}{\sqrt{u}} > \frac{-k}{\sqrt{1 + k/\sqrt{x}}}\right] \\
&\ge \inf_{u > x+k\sqrt{x}} \mathbb{P}\left[\frac{B_u - u}{\sqrt{u}} > -\frac{k}{2}\right].
\end{aligned}$$

Therefore, the square-root insensitivity results in, for an appropriate $\sigma > 0$,

$$\underline{\lim}_{x\to\infty} \frac{\mathbb{P}[B_T > x]}{\mathbb{P}[T > x]} \ge 1 - \Phi\left(\frac{-k}{2\sigma}\right),$$

where $\Phi(\cdot)$ is the distribution function of the standard normal r.v. Letting $k \to \infty$ concludes the proof. □

Proposition 3 *If* $\mathbb{E}e^{\theta\sqrt{\gamma_i^*}} < \infty$, $i = 0, 1$ *for some* $\theta > 0$, $\mathbb{E}\nu_1^2 < \infty$ *and* T *is square-root insensitive, then as* $x \to \infty$

$$\mathbb{P}[B_T > x] \sim \mathbb{P}[M_T > x] \sim \mathbb{P}[T > x].$$

Remark 4 This result shows that Theorems 3.8, 3.10 and 3.11 of [2] hold under less restrictive conditions.

Proof: We follow the same steps as in the proof of Theorem 3. The only difference is that a bound on $C\int_{\delta x}^{x-k\sqrt{x}} xe^{-c\sqrt{x-u}} d\mathbb{P}[T \le u]$ is obtained using the same arguments as in bounding f_{23} in the proof of Proposition 2. □

Appendix

Proof of Lemma 5: Let $Y_i = X_i \mathbf{1}\{X_i \le u\}$ and

$$1/u \le s \le Q(u)/u. \tag{18}$$

Then

$$\mathbb{P}\left[\max_{1\le n\le x}\left\{\sum_{i=1}^{n} X_i - n\mathbb{E}X\right\} > u\right] \le \mathbb{P}\left[\max_{1\le n\le x}\left\{\sum_{i=1}^{n} Y_i - n\mathbb{E}Y_1\right\} > u\right] + x\mathbb{P}[X>u]$$
$$\le \mathbb{P}\left[\max_{1\le n\le x} \exp\left\{\sum_{i=1}^{n} s(Y_i - \mathbb{E}Y_1)\right\} > e^{su}\right] + x\mathbb{P}[X>u].$$

Next, note that $\exp\{\sum_{i=1}^{n} s(Y_i - \mathbb{E}Y_1)\}$ is a submartingale. Therefore, applying a submartingale inequality (e.g. see Theorem 9.4.1 in [8] or Theorem 35.3 in [3]) in the preceding equation leads to

$$\mathbb{P}\left[\max_{1\le n\le x}\left\{\sum_{i=1}^{n} X_i - n\mathbb{E}X\right\} > u\right] \le e^{-su}\mathbb{E}\left[e^{s(Y_1-\mathbb{E}Y_1)}\right]^x + x\mathbb{P}[X>u]$$
$$\le e^{-su-sx\mathbb{E}Y_1}(\mathbb{E}e^{sY_1})^x + Cxe^{-Q(u)}; \tag{19}$$

the last bound is due to Markov's inequality. By repeating exactly the same steps of the proof of Theorem 3.2 in [11] one can show that there exists a constant C^* such that for all s in the given range (18)

$$\mathbb{E}e^{sY_1} \le 1 + s\mathbb{E}Y_1 + C^* s^2.$$

Substituting the preceding bound in (19) yields

$$\mathbb{P}\left[\max_{1\le n\le x}\left\{\sum_{i=1}^{n} X_i - n\mathbb{E}X\right\} > u\right] \le e^{-su+s^2xC^*} + Cxe^{-Q(u)}$$

The rest of the proof is exactly the same as in Theorem 3.2 of [11]: choose $s = Q(u)/u$ if $x \le u^2/(2C^*Q(u))$ and $s = u/(2xC^*)$ otherwise. See the proof in [11] for the details. □

Proof of Lemma 6: The statement follows from

$$\mathbb{P}\left[N_x - x/\mathbb{E}X > u\right] = \mathbb{P}\left[\sum_{i=1}^{\lfloor u+x/\mathbb{E}X\rfloor} X_i < x\right] \le \mathbb{P}\left[\sum_{i=1}^{\lfloor u+x/\mathbb{E}X\rfloor} (\mathbb{E}X - X_i) > (u-1)\mathbb{E}X\right]$$

and the following lemma. □

Lemma 8 *If* $\mathbb{E}e^{\theta X} < \infty$ *for some* $\theta > 0$*, then there exists* $\delta > 0$ *such that for all* x *and* $0 \le u \le \delta x$

$$\mathbb{P}\left[\sum_{i=1}^{x} X_i - x\mathbb{E}X > u\right] \le Ce^{-cu^2/x}.$$

Proof: Markov's inequality yields for $0 < s \le \theta/2$

$$\mathbb{P}\left[\sum_{i=1}^{x} X_i - x\mathbb{E}X > u\right] \le e^{-s(u+x\mathbb{E}X)} \left(\mathbb{E}e^{sX}\right)^x. \tag{20}$$

Next,

$$\begin{aligned}\mathbb{E}e^{sX} &= \mathbb{E}\left[e^{sX}\mathbf{1}\{sX \le 1\}\right] + \mathbb{E}\left[e^{sX}\mathbf{1}\{sX > 1\}\right] \\ &\le 1 + s\mathbb{E}X + s^2\mathbb{E}X^2 + \mathbb{E}\left[e^{sX}\mathbf{1}\{sX > 1\}\right],\end{aligned} \tag{21}$$

since $e^x \le 1 + x + x^2$ for $x \le 1$. A bound on the last term in (21) follows from integration by parts and Markov's inequality

$$\begin{aligned}\mathbb{E}\left[e^{sX}\mathbf{1}\{sX > 1\}\right] &= e\mathbb{P}[sX > 1] + s\int_{1/s}^{\infty} e^{su}\mathbb{P}[X > u]\,du \\ &\le s^2 e\mathbb{E}X^2 + \frac{s\mathbb{E}e^{\theta X}}{\theta - s}e^{1-\theta/s} \\ &\le Cs^2,\end{aligned} \tag{22}$$

where in the last inequality we used $e^{-x} \le 1/x$ for $x > 0$ and the range of s. Substituting (22) in (21) and then (21) in (20) results in

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^{x} X_i - x\mathbb{E}X > u\right] &\le e^{-s(u+x\mathbb{E}X)} e^{x\log(1+s\mathbb{E}X+Cs^2)} \\ &\le e^{-su+xs^2C},\end{aligned}$$

which after setting $s = u/(2Cx) \le \delta/(2C) \le \theta/2$ yields the statement of the lemma. □

Proof of Lemma 7: Let $0 < 3\delta < 1 - a - b$. Then

$$\begin{aligned}\mathbb{P}\left[\sup_{t>lx}\{A_t + B_t - t\} > x\right] &\le \mathbb{P}\left[A_{lx} + B_{lx} - lx + \sup_{t\ge lx}\{(A_t - A_{lx}) + (B_t - B_{lx}) - (t - lx)\} > x\right] \\ &\le \mathbb{P}\left[A_{lx} + B_{lx} > (1-\delta)lx\right] + \mathbb{P}\left[W_{A+B} > (1 + l\delta)x\right] \\ &\le \mathbb{P}\left[W_{A+B}^{1-2\delta} > \delta lx\right] + \mathbb{P}\left[W_{A+B} > \delta lx\right] \\ &\le 2\mathbb{P}\left[W_{A+B}^{1-2\delta} > \delta lx\right] \\ &\le 2\mathbb{P}\left[W_A^{a+\delta} > \frac{\delta lx}{2}\right] + 2\mathbb{P}\left[W_B^{1-3\delta-a} > \frac{\delta lx}{2}\right],\end{aligned}$$

where we repeatedly used the fact that for any two functions $f(t), g(t)$, $\sup_t\{f(t) + g(t)\} \le \sup_t f(t) + \sup_t g(t)$. Now, since $1 - 3\delta - a > b$, the second term in the preceding equation is $o(\mathbb{P}[W_A^{1-b} > x])$ as $x \to \infty$ by (10) and Lemmas 4, 3. Hence,

$$\overline{\lim_{x\to\infty}} \frac{\mathbb{P}\left[\sup_{t>lx}\{A_t + B_t - t\} > x\right]}{\mathbb{P}[W_A^{1-b} > x]} \le 2\,\overline{\lim_{x\to\infty}} \frac{\mathbb{P}\left[W_A^{a+\delta} > \frac{\delta lx}{2}\right]}{\mathbb{P}[W_A^{1-b} > x]};$$

passing $l \to \infty$ and using the assumption yield the statement of the lemma. □

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