Study of the Quasi-Static Norton-Hoff Heat Problem

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Abstract: We are interested in the study of a quasistatic visco-plastic flow with thermal effects. The fluid motion is governed by the incompressible Norton-Hoff model coupled with the dynamic heat equation whose the dissipation mecanichal power is the source term. The viscosity of the fluid is modeled by the Arrhenuis law.
One of the principal aims of this paper is to prove an existence result to the considered problem and the compactness of the solutions set.

Key-words: Visco-plastic fluid, Arrhenius law, Schauder operator, Galerkin method, Sobolev interpolation

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L'Étude du Problème de Norton-Hoff Thermique Dans le Cas Quasi-Statique

Résumé : On s'interesse dans ce papier à l'étude d'un fluide visco-plastique dans le cas quasi-static en tenant compte des effets thermiques. L'écoulement est gouverné par le modèle de Norton-Hoff incompressible couplé avec l'équation de chaleur dynamique dont la puissance mécanique dissipée est le second terme. La viscosité est modélisée par la loi d'Arrhenius.
Le but pricipal de ce papier est de montrer un résultat d'existence du problème consideré et la compacité de l'ensemble des solutions.

Mots-clés : Fluide visco-plastic, loi d'Arrhenius, Schauder opérateur, méthode de Galerkin, interpolation des Sobolev

Contents

1 Introduction

In this paper, we are interested in the study of a quasistatic Norton-Hoff flow with thermal effects. The fluid motion is governed by the incompressible Norton-Hoff model (see [7],[8] [19]) coupled with the dynamic heat equation whose the dissipation mecanichal power is the source term. The viscosity of the fluid is modeled by the Arrhenuis law (see [19], [21]).
One of the principal aims of this chapter is to prove an existence result to the considered problem.
The main idea, in this context, is to adapt the schauder fixed point result (see [22]) which requires, withal, an extention of a compact operator defined in a particular convex subset of a Banach-space.
There are many several steps in this study. Notably the optimum space-regularity of the heat equation source term is rather $L^1(D)$, as far as we know this case was never studied. That's why, as a first principal step, we investigate the dynamic heat equation, with the conduction term, with non-smooth datum (the considered space-regularity of the source terms is only H^{-1}). In order to prove the corresponding existence and uniqueness result we use the Galerkin method (see [16]) via a mere change of functions. Then we establish an abstract result providing the optimal regularity of the heat equation, without the conduction term, with L^1 space-regularity of the source term. For this, we treat the problem with more regularity data (L^2 space-regularity) and there after we use the Sobolev interpolation (see [1]) in order to recover the hopened optimal regularity.
As for the conduction term, since its space-regularity is at least L^1 it builds with the source term a L^1 space-regularity data. Hence the boostraping method provides the same optimal regularity result recovered in the case without conduction term. Indeed, this optimal regularity of the heat solution has to involve the uniform regularity which is needed when dealing with the continuity and compacity of the Schauder operator where the scale factor has to have an uniform behavor.
The second important difficulty is generated by the fact that the scale factor associated to the Norton-Hoff law has to be uniformly bounded in space and in time. In a fisrt time, we prove that the heat solution is a non-negative function. For this we supply an implicit schema in time and we consider the associated problem to which the maximum principle provides a positivity result to each solution via a inductive proof. Therefore, we introduce a parameter family of interpolated functions (see [16]) linked to each solution of the discretized problem. Then, due to an a priori estimate we prove a convergence result to the family of the interpolated functions with respect to the parameter and via the Banach-Zucks theorem (see [6]) we check that the limit function is non negative and fulfills the heat equation. Hence, uniqueness result provides that the solution of the heat equation is non-negative.
Thus, we come to the last several step which deals with the main result of this chapter: the existence at least of one solution to the Norton-Hoff heat problem. The main idea consists to adapt the Schauder fixed-point.

We begin by establishing an iteratif process undertaking the existing coupling between the temperature and the velocity of the fluid. While, the hopened fixed-point of a such process will depends on the continuity and the compacity of the extended mapping. The continuity result is recovered in two substeps, the first one concerns the velocity of the fluid in which we give an a priori estimate and with the fact that the set of admissible velocities is a Banach-space uniformly convex (see [3]) we get the continuity with respect to the velocity. The second one is devoted to the continuity with respect to the temperatue, indeed we proceed as above with the fact that the set of admissible temperatures is an Hilbert-space (see [3]).
As for the compacity result, its proof is basicly supplied by a priori estimates through the uniform convexity and the reflexivity properties of the corresponding spaces (see [3]).
Thus, we recover the expected existence result.
Finally, we end by proving that the set of solutions to Norton-Hoff heat problem is strongly compact. The proof is mainly based on the technical reults established in the study of the Schauder operator.

2 Notations and Hypothesis

2.1 Notations

D: regular domain,

I: a time interval of $I\!R$

$\partial D = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$

$\Sigma_i = I \times \Gamma_i$

σ: Cauchy stress tensor,

$\varepsilon(u) = 1/2(D(u) + D(u)^*)$: linearized strain velocity tensor,

Du: the differential of the velocity u.

K: viscosity of the material,

K_c: consistence of the material,

γ: thermodependence coefficient,

θ_0: strict non-negative function,

p: exponent of the material; $1 < p < 2$: it is the sensibility coefficient of the material to the strain velocity tensor,

P: hydrostatic pression,

Id: identity tensor,

f: the density of the gravitation acting on the fluid,

$\sigma \cdot \cdot \varepsilon$: dissipation mecanichal power issu from the Joule effect,

λ: diffusion coefficient,

k: conductivity coefficient,

q_i: heat flux.

2.2 Hypothesis

i)The domain D is locally in one side of its boundary ∂D which is twicely continuously differentiable.
ii)Assume that λ belongs to $L^\infty(I,\ W^{1,\infty}(D))$ and $min\lambda(t,x) = \mu > 0$ for all $(t,x) \in I \times D$.

2.3 The Arrhenius Law

The viscosity of the fluid is modeled by the Arrhenius law (see [19], [21]). A such law, issu of the metallurgy, has a phenomenologic and describing fundamenttally the plastic-behavors of the materials.

$$(1) \qquad K = K(t,x) \quad = \quad K_c exp(\tfrac{\gamma}{\theta+\theta_0})$$

The coefficient $K(\theta(t,x))$ is linked to the viscosity of the fluid at the instant t on the position x with temperature $\theta(t,x)$.

3 Norton-Hoff Heat Monophasic Problem

We consider a regular domain D occupied by a viscoplastic fluid. The fluid motion is governed by the quasistatic incompressible Norton-Hoff model coupled with the dynamic heat equation taking in account the conduction term within the time dependant dissipation mecanichal power as a volume source term. The viscosity of the fluid is modeled by the Arrhenius law. We impose a boundary homogeneous Dirichlet condition to the velocity for all t in I. Asfor the heat conditions, we impose a heat flux on a part of the boundary and on the rest the rate of heat is adiabatic. The corresponding initial conditions given are enough-regular.
The flow problem consists to look for a velocity field and a temperature function defined on $\overline{D}$ and fullfil the hereafter equations.

$$(2) \qquad \mathcal{P} \left\{ \begin{array}{rcll} K(\theta(t,x))|\varepsilon(u)|^{p-2}\varepsilon(u) \ + \ PId & = & \sigma & \text{in} \quad I \times D \\ -div(\sigma) & = & f & \text{in} \quad I \times D \\ div(u) & = & 0 & \text{in} \quad I \times D \\ \theta_t \ + \ ku.\nabla\theta \ - \ div(\lambda\nabla\theta) & = & \sigma(u) \cdot \cdot \varepsilon(u) & \text{in} \quad I \times D \end{array} \right.$$

with the boundary conditions:

$$(3) \qquad \mathcal{BC} \left\{ \begin{array}{rcll} u & = & 0 & \text{on} \quad I \times \partial D \\ \dfrac{\partial\theta}{\partial n} & = & 0 & \text{on} \quad \Sigma_1 \\ \dfrac{\partial\theta}{\partial n} & = & -q_i & \text{on} \quad \Sigma_2 \end{array} \right.$$

and let consider the following initial conditions:

$$(4) \qquad \mathcal{IC} \left\{ \begin{array}{rcll} u(0,.) & = & u_0(.) & \text{in} \quad D \\ \theta(0,.) & = & 0 & \text{in} \quad D \end{array} \right.$$

Functional Setting We denote

$$\mathcal{W} = W_0^{1,p}(D) = \{v \in W^{1,p}(D) \ \text{ s.t } \ v = 0 \ \text{ on } \ \partial D\}$$

and

$$\mathcal{W}_{div} = \{v \in \mathcal{W} \ \text{ s.t } \ div\, v = 0 \ \ in \ \ D\}$$

which are the natural spaces involved in the study of static Norton-Hoff problem (see also [9]). The following proposition provides a Banach space structure to the above spaces.

Proposition 3.1 *The mapping $\|.\|$ defined from $\mathcal{W}$ to $I\!R^+$ by*

$$\|v\| = (\int_D |\varepsilon(v)|^p)^{1/p}$$

is a norm on $\mathcal{W}$, equivalent to the one induced by the canonical one of $W^{1,p}(D)$.

For this norm $\mathcal{W}_{div}$ is closed in $\mathcal{W}$ so also it is a Banach space for the induced norm, still denoted $\|.\|$ for shortness (see also [12]).

Remark 3.1 *We shall notice that the optimum regularity of the heat equation source term is rather $L^2(I, L^1(D))$, as far as we know this case was never studied. That's why we will investigate the heat equation with non-smooth datum.*

In a first principal step, we investigate the dynamic heat equation, with the conduction term, with non-smooth datum (with as H^{-1} space-regularity of the source terms). In order to prove the corresponding existence and uniqueness result we use the Galerkin method via a mere change of functions.

4 Heat Equation With L^1 Source Term

We consider a bounded smooth domain D with boundary $\partial D = (\bar{\Gamma})_1 \cup (\bar{\Gamma})_2$, a time interval $I =]0, \tau[$ and the evolution problem

$$(5) \qquad \left\{ \begin{array}{rcll} \dfrac{\partial \theta}{\partial t} + ku.\nabla\theta - div(\lambda \nabla \theta) & = & g & \text{in} \quad I \times D \\ \theta & = & 0 & \text{on} \quad \Sigma_1 = I \times \Gamma_1 \\ \dfrac{\partial \theta}{\partial n} & = & -q_i & \text{on} \quad \Sigma_2 = I \times \Gamma_2 \\ \theta(0) & = & \varphi_0 & \text{in} \quad D \end{array} \right.$$

4.1 Non-Smooth Datum

Let us consider the hereafter database

(6) $$g \in L^2(0,\tau,L^1(D)) \text{ and } q_i \in L^2(0,\tau,L^1(\Sigma_2)).$$

In the sequel we shall assume that the open set Σ_1 in ∂D may be empty. So that we are never able to use the term $(\int_D |\nabla\theta|^2 dx)^{1/2}$ as a norm. We will proceed as follows. Let us consider the following mere change of functions, for each constant $a > 0$.

$$\psi(t,x) \;=\; e^{-at}\,\theta(t,x)$$

then the problem (5) can be rewritten as follows:

(7) $$\begin{cases} \frac{\partial}{\partial t}\psi \;+\; a\,\psi \;+\; ku.\nabla\psi \;-\; div(\lambda\nabla\psi) &= g^a \quad \text{in} \quad]0,\tau[\times D \\ \psi &= 0 \quad \text{on} \quad \Sigma_1 \\ \frac{\partial}{\partial n}\psi &= -q_i^a \quad \text{on} \quad \Sigma_2 \\ \psi(0) &= \psi_0 \quad \text{in} \quad D \end{cases}$$

Where

(8) $$g^a \;=\; e^{-at}g \in L^2(0,\tau,L^1(D)) \text{ and } q_i^a = e^{-at}q_i \in L^2(0,\tau,L^1(\Sigma_2)).$$

We will use the Galerkin method in order to get an existence and uniqueness result to the last system.

4.2 Galerkin Method

We start with assuming that both g^a and q_i^a are in $L^2(0,\tau,E)$ where E is a Banach space of distributions, respectively $E = H^{-1}(D)$ and $E = H^{-1}(\Sigma_2)$. We consider the problem (7) with such datum. We use now the usual Galerkin approximation method. Let $e_1,..,e_m,....$ be a dense family in $H^1(D)$ with $e_i = 0$ on Σ_1 and each e_i being a smooth function. We look for approximation in form of expansions in the following classical form

(9) $$\Psi^m(t,x) = \Sigma_{i=1}^m \;\; \psi_i^m(t)\, e_i(x)$$

the function $\Psi^m(t) \;=\; (\psi_1^m(t),...,\psi_m^m(t))^*$ in $C^2([0,\tau], I\!R^m)$ being the solution of the following ordinary linear differential system , $1 \le j \le m,\; \forall\;\; t \in (0,\tau)$,

(10) $$\int_D \left(\frac{\partial}{\partial t}\psi^m e_j + a\psi^m e_j + k(u.\nabla\psi^m)e_j + \lambda\,\nabla\psi^m.\nabla e_j\;\right)dx \;=\; \int_D g^a e_j dx + \int_{(\partial D)_2} q_i^a\, e_j\, ds$$

from which we derive these a priori estimates:

$$(11)\qquad \begin{aligned}&\int_0^\tau \int_D (\frac{\partial}{\partial t}\psi^m \psi^m \; + \; a(\psi^m)^2 \; + \; k(u.\nabla\psi^m)\psi^m \; + \; \lambda|\nabla\psi^m|^2)dxdt \\ &\leq \int_0^\tau (||g^a||_{H_{\bar{D}}^{-1}(R^N)} \; + \; C_{\Sigma_2}\; ||q_i^a||_{H_{oo}^{-1/2}(\Sigma_2)}\;)\;||\psi^m||_{H^1(D)}\; dt\end{aligned}$$

where the constant C_{Σ_2} is related to the continuity of the trace operator on Σ_2.
Then setting

$$a = \mu$$

we get

$$(12)\qquad ||\psi^m||_{L^2(0,\tau,H^1(D))} \leq \frac{1}{\mu}(\int_0^\tau (||g^\mu||_{H_{\bar{D}}^{-1}(R^N)} + C_{\Sigma_2}\; ||q_i^\mu||_{H_{oo}^{-1/2}(\Sigma_2)})^2\; dt\;)^{1/2}$$

And then, assuming for shortness that $\varphi_0 = \varphi_i^m(0) = 0$,

$$(13)\qquad \frac{1}{2}||\psi^m||^2_{L^\infty(0,\tau,L^2(D))} \; \leq \; \frac{1}{\mu}\int_0^\tau (||g^\mu||_{H_{\bar{D}}^{-1}(R^N)} + C_{\Sigma_2}\; ||q_i^\mu||_{H_{oo}^{-1/2}(\Sigma_2)})^2\; dt$$

4.3 Existence and Uniqueness Result

We deduce the following result

Proposition 4.1 *For any g^μ, q_i^μ verifying the previous H^{-1}-assumptions we get a unique solution ψ in $E = L^\infty(0,\tau,L^2(D)) \cap L^2(0,\tau,H^1(D))$ to problem (7), (we refer to Lions [16]).*

From the equation itself we get

$$\frac{\partial}{\partial t}\psi \;\; = \;\; -\mu\psi + div(\lambda\nabla\psi) + g^\mu - ku.\nabla\psi^m$$

so that if $\lambda \in L^\infty(0,\tau,L^\infty(D))$ as assuming, it yields

$$\psi \in H^1(0,\tau,H^{-1}(D))$$

As a consequence we get an isomorphism

$$\zeta : \;\; (g^\mu,\; q_i^\mu) \longrightarrow \; \psi$$

From $F = H_{\bar{D}}^{-1}(R^N) \times H_{oo}^{-1/2}(\Sigma_2)$ onto $\mathcal{Q} = H^1(0,\tau,H^{-1}(D)) \cap L^2(0,\tau,H^1(D))$.
Then the transpose ζ^* is also an isomorphism between the dual spaces.
Thus, we recover the existence and uniqueness result to the dynamic heat equation with less-regular data.

In the sequel we establish an abstruct result providing the optimal regularity of the heat equation, without the conduction term, with L^1 space-regularity of the source term. For this, we treat the problem with more regularity data (L^2 space-regularity) and there after we use the Sobolev interpolation (see [1]) in order to recover the hopened optimal regularity.
As for the conduction term, since its space-regularity is at least L^1 it builds with the source term a L^1 space-regularity data. Hence the boostraping method provides the same optimal regularity result recovered in the case without conduction term.

4.4 Optimal Regularity of the Heat Equation with L^1 Source Term

4.4.1 An Abstract Result

First, We establish the expected optimal regularity result of the considered heat equation in an abstract framework.

Theorem 4.1 *Let θ be the solution of the heat equation whose the second term f belongs to $L^1(D)$ for all t in I with initial condition $\theta_0(x)$ belongs to $H^1(D)$:*

$$(14) \qquad \left\{ \begin{array}{rcll} \partial_t\theta - div(\lambda\nabla\theta) & = & f & in\ I\times D \\ \dfrac{\partial\theta}{\partial n} & = & -q_i & on\ \Sigma_2 \\ \dfrac{\partial\theta}{\partial n} & = & 0 & on\ \Sigma_1 \\ \theta(0,x) & = & \theta_0(x) & in\ D \end{array} \right.$$

Then, θ lays in $L^2(I,\ L^\infty(D))$.

Basically the proof is made up of two steps. The first one consists to treat the problem with more regular source data, as for the second one is devoted to the Sobolev interpolation.
We start by dealing with the regularity of θ with second term f belongs to $L^2(D)$ for all t in I.
Being given R any standing of θ fulfilling the following equation

$$(15) \qquad \left\{ \begin{array}{rcll} -div(\lambda\nabla R) & = & f & in\ D \\ \dfrac{\partial R}{\partial n} & = & -q_i & on\ \Sigma_2 \\ \dfrac{\partial R}{\partial n} & = & 0 & on\ \Sigma_1 \\ R(0,x) & = & \theta(0,x) & in\ D \end{array} \right.$$

lemma 4.1 *It is easy to see that at least $\partial_t R$ lays in $L^2(D)$ for all t in I.*

Let $\Phi = \theta - R$, then Φ fulfills the hereafter problem:

$$(16)\qquad \left\{\begin{array}{rcll} \partial_t\Phi - div(\lambda\nabla\Phi) &=& -\partial_t R & in\ D\\ \dfrac{\partial\Phi}{\partial n} &=& 0 & on\ \Sigma = \bar{\Sigma}_1 \cup \bar{\Sigma}_2\\ \Phi(0,x) &=& 0 & in\ D \end{array}\right.$$

lemma 4.2 *The solution Φ belongs to $L^2(I,\ H^2(D))$.*

Proof of lemma 4.2

Under the domain's regularity, since Φ belongs to $L^2(D)$ and by referring to [17], it will be enough to prove that $\Delta\Phi$ lays in $L^2(D)$.
Let us multiply the two terms of the problem (16) by $(-\Delta\Phi)$, it follows for all t in I.

$$(17)\quad \frac{1}{2}\int_0^\tau\int_D \partial_t|\nabla\Phi|^2 dxdt + \int_0^\tau\int_D \lambda|\Delta\Phi|^2 dxdt = \int_0^\tau\int_D \partial_t R\Delta\Phi dxdt - \int_0^\tau\int_D (\nabla\lambda\nabla\Phi)\Delta\Phi dxdt$$

hence, the regularity of λ and the Cauchy-Schwartz inequality (see [3]) recover the hopened estimate. It exists a constant c such that

$$\|\Delta\Phi\|_{L^2(D)} \leq c$$

which involve the result.

■

4.4.2 Sobolev Interpolation

We have seen, in the previous subsection, that if the second term of the heat equation belongs to $H^{-1}(D)$ (respect in $L^2(D)$) the corresponding solution is in $H^1(D)$ (respect in $H^2(D)$). This allows us to use the linear Sobolev-spaces interpolation (see [1])in order to get the proof of theorem 4.1.

Proof of theorem 4.1
In a general setting, the second term belongs to $L^1(D)$ so also in any Sobolev space $H^{-\varepsilon}(D)$ for any non-negative parameter ε such that $H^{2-\varepsilon}(D)$ is a linear interpolated Sobolev-space of $(H^1,\ H^2)$. Whereas $H^{2-\varepsilon}(D) = W^{2-\varepsilon,2}(D)$ is embedding in $W^{s,\infty}(D)$ for all $s \leq 1/2 - \varepsilon$, so also in $W^{0,\infty}(D)$
Thus, the announced result is supplied.

■

4.4.3 Boostraping Method

Theorem 4.2 *Let θ be solution of the heat equation (with conduction term) coupled with the Norton-Hoff one. Then θ belongs to $L^2(I,\ L^\infty(D))$.*

Proof of theorem 4.2

Mainly, the proof will be given by boostraping the abstract result in the considered case. Since the conduction term $ku.\nabla\theta$ belongs to $L^1(D)$, hence the data $g - ku.\nabla\theta$ belongs also to $L^1(D)$. Hence the abstract theorem (4.1) provides the result and thus we comeover the first main difficulty.

■

The second important difficulty in this study is generated by the fact that the scale factor associated to the Norton-Hoff law has to be uniformly bounded in space and in time. In a fisrt time, we prove that the heat solution is a non-negative function. For this we supply an implicit schema in time and we consider the associated problem to which the maximum principle (see [3]) provides a positivity result to each solution via a recurent reasoning. Therefore, we introduce a parameter family of interpolated functions linked to each solution of the discretized problem. Then, due to an a priori estimate we prove a convergence result to the family of the interpolated functions with respect to the parameter and via the Banach-Zucks theorem (see [19]) we check that the limit function is non negative and fulfills the heat equation. Hence, uniqueness result provides that the solution of the heat equation is non-negative.

5 Regularity of The Arrhenius Law

Theorem 5.1 *Let $K = K_c \; exp(\frac{\gamma}{(\theta_0+\theta)})$ being the scale factor associated to the Norton-Hoff law, where K_c is the consistency of the material, θ is the solution of the heat equation and θ_0 is a strict non-negative function. Then*

$$K \text{ belongs to } L^\infty(0,\tau;L^\infty(D))$$

The following technical result will be needed for the proof.

5.1 The Implicit Schema

We consider the following implicit-schema:

Let $\rho_m = \frac{\tau}{m}$ being a subdivision of the time interval $[0,\tau]$, so

$$[0,\tau] \;=\; \cup_{n=0}^{m}[n\rho_m,(n+1)\rho_m]$$

let

$\forall m \in IN, \theta^m = (\theta_0^m, \theta_1^m, ..., \theta_m^m)^* \in H^1(D)^{m+1}$ so also

$$(18)\quad \left\{ \begin{array}{rcll} \dfrac{\theta_{n+1}^m - \theta_n^m}{\rho_m} + ku.\nabla\theta_{n+1}^m - div(\lambda\nabla\theta_{n+1}^m) & = & g & in\ D \\ \dfrac{\partial\theta_{n+1}^m}{\partial n} & = & 0 & on\ (\partial D)_1 \\ \dfrac{\partial\theta_{n+1}^m}{\partial n} & = & -q_i & on\ (\partial D)_2 \\ n = 1,...,m & & & \end{array} \right.$$

where g is equal to $\sigma(u)\cdot\cdot\varepsilon(u)$.

5.2 The Maximum Principle

lemma 5.1 *Via the maximum principle and a recurrent reasoning, we get*

$$\text{if } \theta_n^m \geq 0 \text{ then } \theta_{n+1}^m \geq 0$$

Proof of lemma 5.1
In fact, θ_n^m satisfies the hereafter optimization problem:

$$\Phi(\theta_{n+1}^m) = min_{\varphi \in H^1} \Phi(\varphi)$$

where Φ is a convex lci mapping and Gateaux differentiable given by

$$\Phi(\varphi) = \int_D \frac{1}{2}(\varphi^2 + \frac{1}{2}\rho_m \lambda |\nabla \varphi|^2 - \theta_n^m \varphi - \rho_m g \varphi - \rho_m q_i \varphi)$$

since $q_i > 0$ and $g > 0$, then if $\theta_n^m \geq 0$ we come to

$$\Phi(\theta_{n+1}^m) \geq \Phi(|\theta_{n+1}^m|)$$

accordingly, by the uniqueness of the minimum;

$$\theta_{n+1}^m = |\theta_{n+1}^m|$$

■

5.3 The Interpolation Method

Let $\overline{\theta}^m$ being a piecewise interpolated of θ^m in $L^2([0,\tau], H^1(D))$ fulfilling the below assumption (see [16]):

$$\text{for all instant } t \in [n\rho_m, (n+1)\rho_m[; \quad \bar{\theta}^m(t) = \theta_n^m$$

so, it yields that

lemma 5.2

$$\overline{\theta}^m \geq 0; \quad \forall m$$

and

$$\begin{cases} \dfrac{\overline{\theta}^m(t+\rho_m) - \overline{\theta}^m(t)}{\rho_m} + ku.\nabla\overline{\theta}^m(t+\rho_m) - div(\lambda\nabla(\overline{\theta}^m(t+\rho_m)) = g(t) & in \ D \\ \dfrac{\partial \overline{\theta}^m}{\partial n} = 0 & on \ (\partial D)_1 \\ \dfrac{\partial \overline{\theta}^m}{\partial n} = -q_i & on \ (\partial D)_2 \\ \forall t \in [0,\tau] \end{cases} \tag{19}$$

5.4 A Priori Estimate

lemma 5.3 *There exists a constant c such that*

$$\|\overline{\theta}^m\|_{L^2(0,\tau;H^1(D))} \leq c$$

Proof of lemma 5.3
Let us assume that $(\theta_n^m, \ \ \theta_{n+1}^m)_{L^2,L^2,D} \ = \ 0$
by Green's formula we come to

$$\int_D |\theta_{n+1}^m|^2 + \rho_m \int_D \lambda |\nabla \theta_{n+1}^m|^2 = \rho_m \int_D g \theta_{n+1}^m + \rho_m \int_{(\partial D)} q_i \theta_{n+1}^m$$

one can check easily

$$\|\theta^m\|_{H^1(D)} \ \ \leq max(\frac{1}{\mu}, \ \tau)\{\|f\|_{H^{-1}(D)} + c\|q_i\|_{H_{00}^{\frac{-1}{2}}}\}$$

otherwise

$$\|\theta^m\|_{L^2(0,\tau;H^1(D))} \ = \ \rho_m \|\theta^m\|_{H^1(D)}$$

by compacity argument we recover a weakly convergent subsequence towards π in $L^2(0,\tau;H^1(D))$ and it converges strongly in $L^2(0,\tau;L^2(D))$. Let us denote the subsequence by $\overline{\theta}^m$ itself.

■

In order to have an idea about the sign of the limit function π, we may use the Banach-Zucks theorem (see [6]).

Theorem 5.2 *(convex-subsequence, see [6]) Let H be a Hilbert space and e_m a sequence which converges weakly towards e in H.*
Then it exists a family of constant λ_i^m such that

$$0 \leq \lambda_i^m, \ \ 1 \leq i \leq n_m, \ \ \Sigma_i \lambda_i^m = 1$$

and also

$$h^m = \Sigma_i \lambda_i^m e_m \longrightarrow e, \ \ \textit{strongly in} \ \ H$$

lemma 5.4 *As a consequence,*
since $\overline{\theta}^m \geq 0$ then $h_m \geq 0$ hence $\pi \geq 0$.

Hereafter, we will try to show up that $\pi = \theta$ solution of the heat equation.

Proposition 5.1 *The limit function π is solution of the heat equation, furthermore*

$$\pi \ = \ \theta$$

Proof of proposition 5.1
We have, for all φ in $C_c^\infty(D)$

$$\int_0^\tau \int_D \frac{\overline{\theta}^m(t+\rho_m) - \overline{\theta}^m(t)}{\rho_m}\varphi(t) \;=\; \int_0^\tau \int_D \overline{\theta}^m(t)\frac{\varphi(t-\rho_m)-\varphi(t)}{\rho_m}$$

it follows

$$\int_0^\tau \int_D \overline{\theta}^m(t)\frac{\varphi(t-\rho_m)-\varphi(t)}{\rho_m} \longrightarrow \int_0^\tau \int_D \pi(-\varphi)_t \quad ; m \uparrow \infty$$

while

$$\int_0^\tau \int_D \lambda \nabla \overline{\theta}^m(t) \nabla \varphi(t-\rho_m) \longrightarrow \int_0^\tau \int_D -div(\lambda \nabla \varphi)\pi \quad ; m \uparrow \infty$$

and

$$\int_0^\tau \int_D [ku.\nabla \overline{\theta}^m(t)]\varphi(t-\rho_m) \longrightarrow \int_0^\tau \int_D [ku.\nabla \pi]\varphi \quad ; m \uparrow \infty$$

Hence one can easily check that π is solution of the heat equation with the same limit and initial conditions recovered bu Green formula, so also by uniqueness $\pi = \theta$. Thus, the solution of the heat equation is non-negative, i.e.

$$\theta(t,\ x) \geq 0; \text{ for all } (t,\ x) \in (0,\ \tau) \times D$$

■

5.5 Proof of theorem 5.1

Proof of theorem 5.1
Indeed, since K_c lays in $L^\infty(D)$ and $\theta \geq 0$ with $\theta_0 > 0$, the proof is obvious.

■

6 Existence Result

In order to get the existence result of the monophasic Norton-Hoff heat problem, the main idea consists to adapt the Schauder fixed-point theorem which requires, withal, an extention of a compact operator defined in a particular convex subset of a Banach-space. A such result allows us to prove that an established iteratif process undertaking the stand coupling has at least a fixed point.

6.1 Process

Let φ^n be a given temperature then the Arrhenius law provides the viscosity $K^{\varphi^n} = K^{n+1}$. While, by resolving the Norton-Hoff equation we recover the fluid's velocity $u^{\varphi^n} = u^{n+1}$. Hence by the Joule effect we get the dissapation mecanichal power which allows us to resolve the heat equation and to get the temperature $\theta^{\varphi^n}_{u^n}$, and the process continue.

$$\begin{array}{ccc} \varphi^n \text{ given} \overset{Viscosity}{\longrightarrow} K^{\varphi^n} = K^{n+1} \overset{Norton-Hoff}{\longrightarrow} u^{\varphi^n} = u^{n+1} & & \\ & & \overset{Joule effects}{\downarrow} \\ \theta^{n+1} = \theta^{\varphi^n}_{u^{n+1}} \overset{heat\, equation}{\longleftarrow} \sigma(u^{n+1}) \cdot \cdot \varepsilon(u^{n+1}) & & \end{array}$$

The couple (u^{n+1}, θ^{n+1}) fulfills the hereafter problem with the same boundary and initial conditions (3), (4).

$$\mathcal{P}(u^n, \theta^n) \left\{ \begin{array}{rcll} -div(K^{n+1}|\varepsilon(u^{n+1})|^{p-2}\varepsilon(u^{n+1}) \ + \ PI) & = & f & \text{in } I \times D \\ div(u^{n+1}) & = & 0 & \text{in } I \times D \\ (\theta^{n+1})_t \ + \ ku^n.\nabla\theta^{n+1} \ - \ div(\lambda\nabla\theta^{n+1}) & = & \sigma(u^{n+1}) \cdot \cdot \varepsilon(u^{n+1}) & \text{in } I \times D \end{array} \right.$$

(20)

Remark 6.1 *We note here that the heat equation is linear with respect to* θ^{n+1}.

Theorem 6.1 *The Norton-Hoff heat problem has at least a solution in* $\mathcal{W} \times \mathcal{Q}$*: there exists* (V, φ) *such that*

$$(u^{\varphi}, \theta^{\varphi}_V) \ = \ (V, \varphi)$$

6.2 The Schauder Fixed-Point Theorem

Theorem 6.2 *Let* M *be a nonempty, closed, bounded, convex subset of a Banach-space* X*, and suppose* $\mathcal{T} : M \to M$ *is a compact operator. Then* $\mathcal{T}$ *has a fixed point, (see also [22]).*

In the sequel, we denote $\mathcal{O}_{lip}$ the set of Lipschitz domains included in D.
With the previous results, we can extend the mapping

$$\mathcal{G} : \quad (V, \varphi) \longrightarrow (u^{\varphi}, \theta^{\varphi}_V)$$

so it takes value into the closed, bounded, convex ball $\mathcal{B}$ included in $\mathcal{W} \times \mathcal{Q}$. Let $\mathcal{B} = \mathcal{B}_u \times \mathcal{B}_\theta$, where $\mathcal{B}_u$ and $\mathcal{B}_\theta$ are the closed balls of $\mathcal{V}$ and $\mathcal{Q}$ whose radius are given subsequently.
The fixed-point will depend on the continuity and the compacity of the extended mapping.

6.3 Strong Continuity Result

Proposition 6.1 *Let* $(\Omega_n, \varphi_n)_{n \in IN}$ *be a sequence in* $\mathcal{W}_{div} \times \mathcal{Q}$ *such that* (V_n, φ_n) *converges strongly in* $\mathcal{W}_{div} \times \mathcal{Q}$ *towards* (V_*, φ_*)*. Then*
$(u^{\varphi_n}, \theta^{\varphi_n}_{V_n})$ *solution of* $\mathcal{P}(V_n, \varphi_n)$ *converges strongly in* $\mathcal{W}_{div} \times \mathcal{Q}$ *towards* $(u^{\varphi_*}, \theta^{\varphi_*}_{V_*})$ *solution of* $\mathcal{P}(V_*, \varphi_*)$.

The proof of proposition (6.1) can be given in three steps. We begin by giving a priori estimates linked to the velocity u solution of the Norton-Hoff equation and to the temperature solution of the heat one. We also give the weak continuity of the extended mapping in the second step. In the third one we derive the strong continuity throught the continuity of the dissipation mecanichal power.

First Step: A Priori Estimate We have the following estimates.

lemma 6.1 *There exists a constant c such that if u is the solution of the Norton-Hoff problem then:*

$$\|u\|_{L^2(0,\tau;W)} \leq \frac{c}{K_c}\|f\|^{\frac{1}{p-1}}_{L^\infty(0,\tau;L^{p'})}$$

Let $\frac{c}{K_c}\|f\|^{\frac{1}{p-1}}_{L^\infty(0,\tau;L^{p'})}$ be the radius of the above mentioned ball $\mathcal{B}_u$.

Proof of lemma 6.1

It is easy to derive via the Cauchy Schwartz inequality, (see [3])

$$\int_D K|\varepsilon(u)|^p \leq \|f\|_{L^{p'}(D)}\|u\|_{L^p(D)}$$

else Poincare's inequality (see [1]) provides

$$K_c\|u\|_{\mathcal{W}} \leq c\|f\|^{\frac{1}{p-1}}_{L^{p'}} \quad \forall t \in I$$

then

$$\|u\|_{\mathcal{W}} \leq \frac{c}{K_c}\|f\|^{\frac{1}{p-1}}_{L^\infty(0,\tau;L^{p'})} \ \forall t \in I$$

moreover

$$\|u\|_{L^\infty(0,\tau;\mathcal{W})} \leq \frac{c}{K_c}\|f\|^{\frac{1}{p-1}}_{L^\infty(0,\tau;L^{p'})}$$

since L^∞ is continuously embedding in L^2, it follows

$$\|u\|_{L^2(0,\tau;\mathcal{W})} \leq \frac{c}{K_c}\|f\|^{\frac{1}{p-1}}_{L^\infty(0,\tau;L^{p'})}$$

and the proof is achived

■

As a consequence, one can easily deduce that

lemma 6.2

$$\int_D \sigma(u)\cdot\cdot\varepsilon(u) \leq c\|f\|^{\frac{p}{p-1}}_{L^\infty(0,\tau;L^{p'})}$$

and so also

$$\|g\|_{H^{-1}_{\bar{D}}(R^N)} \leq c\|f\|^{\frac{p}{p-1}}_{L^\infty(0,\tau;L^{p'})}$$

lemma 6.3 *There exists a constant c such that if θ is the solution of the heat equation, then we get*

$$(21)\quad \|\theta\|_{L^2(0,\tau,H^1(D))} \leq \frac{1}{\mu}\Big(\int_0^\tau (c\|f\|^{\frac{p}{p-1}}_{L^\infty(0,\tau;L^{p'})} + C_{\Sigma_2}\ \|q_i\|_{H^{-1/2}_{oo}(\Sigma_2)})^2\, dt\ \Big)^{1/2} = R_1$$

and then

$$(22)\quad \|\theta\|_{L^\infty(0,\tau,L^2(D))} \leq \Big(\frac{2}{\mu}\int_0^\tau (c\|f\|^{\frac{p}{p-1}}_{L^\infty(0,\tau;L^{p'})} + C_{\Sigma_2}\ \|q_i\|_{H^{-1/2}_{oo}(\Sigma_2)})^2\, dt\ \Big)^{1/2} = R_2$$

Let the $min(R_1,\ R_2)$ be the radius of the above mentioned ball $\mathcal{B}_\theta$.

Second Step: Weak Continuity We denote

$$(u_n, \theta_n) = (u^{\varphi_n}, \theta^{\varphi_n}_{V_n}) \ \textit{and}\ K_n = K^{\varphi_n}$$

Under previous lemma (6.1) the sequence $(u_n)_{n\in IN}$ is bounded in the reflexif Banach space $\mathcal{W}_{div}$ and thus one can extract a subsequence $(u_{n_k})_{k\in IN}$ which converges weakly in $\mathcal{W}_{div}$. Let u_* this weak limit, and assume $(u_n)_{n\in IN}$ itself converges weakly towards u_*.

lemma 6.4 *The vector u_* is the solution of $\mathcal{NH}(K^{\varphi_*})$, moreover $u_* = u^{\varphi_*}$.*

Proof of lemma 6.4

Since Norton-Hoff is variational (i.e. the solution realize the minimum of the associated Energy), it is easy to check the following inequality, for any $n \in IN$:

$$\int_D \frac{K_n}{p}|\varepsilon(u_n)|^p - \int_D f u_n \leq \int_D \frac{K_n}{p}|\varepsilon(v)|^p - \int_D fv = \Phi_n(v) \quad \forall v \in \mathcal{W}_{div}$$

where Φ_n is a convex and Gateaux differentiable mapping.
Let $K_* = K^{\varphi_*}$, accordingly the sequence K_n converges strongly towards K_* in $L^2(D)$ and so also in $L^{p'}(D)$.
it yields

$$\liminf_{n\uparrow\infty}\{\int_D \frac{K_n}{p}|\varepsilon(u_n)|^p - \int_D f u_n\} \leq \int_D \frac{K_*}{p}|\varepsilon(v)|^p - \int_D fv \quad \forall v \in \mathcal{W}_{div}$$

then

$$\liminf_{n\uparrow\infty}\int_D \frac{K_n}{p}|\varepsilon(u_n)|^p + \liminf_{n\uparrow\infty}(-\int_D f u_n) \leq \int_D \frac{K_*}{p}|\varepsilon(v)|^p - \int_D fv \quad \forall v \in \mathcal{W}_{div}$$

which involves

$$\liminf_{n\uparrow\infty}(\int_D \frac{K_n}{p}|\varepsilon(u_n)|^p) - \int_D f u_* \leq \int_D \frac{K_*}{p}|\varepsilon(v)|^p - \int_D fv \quad \forall v \in \mathcal{W}_{div}$$

it arises that

$$\int_D \frac{K_n}{p}|\varepsilon(u_n)|^p \;=\; \|(\frac{K_n}{p})^{1/p}\varepsilon(u_n)\|^p_{L^p(D)}$$

whereas the mapping $v \longrightarrow \|(\frac{K_n}{p})^{1/p}\varepsilon(v)\|_{L^p(D)}$ is weakly lsc, convex onto $\mathcal{W}_{div}$. Indeed, the function $\frac{K_n}{p}|\varepsilon(u_n)|^p$ is positive and $sup_n \int_D \frac{K_n}{p}|\varepsilon(u_n)|^p < \infty$ since $K_n \leq K_c exp(\frac{\gamma}{\theta_0})$ while; $(\frac{K_n}{p})^{1/p}\varepsilon(u_n) \rightharpoonup (\frac{K_*}{p})^{1/p}\varepsilon(u_*)$, $\;n \uparrow \infty$ in $L^p(D)$, whence using Fatou's lemma (see also [3]),

$$\int_D \frac{K_*}{p}|\varepsilon(u_*)|^p \;\leq\; \liminf_{n\uparrow\infty} \int_D \frac{K_n}{p}|\varepsilon(u_n)|^p$$

it follows

$$\int_D \frac{K_*}{p}|\varepsilon(u_*)|^p - \int_D f u_* \;\leq\; \int_D \frac{K_*}{p}|\varepsilon(v)|^p - \int_D f v = \Phi_*(v) \quad \forall v \in \mathcal{W}_{div}$$

thus, u_* realize the minimum of Φ_*, by uniqueness $u_* = u^{\varphi_*}$ (see also [12]) and so the lemma (6.4) is proved.

■

Third Step: Strong Continuity We begin by giving this fundamental lemma.

lemma 6.5 *The sequence* $(\sigma(u_n)\cdot\cdot\varepsilon(u_n))_{n\in IN}$ *converges towards* $\sigma(u_*)\cdot\cdot\varepsilon(u_*)$ *, strongly in* $L^1(D)$ *and so also in* $H^{-1}(D)$*; when* $n \uparrow \infty$

Proof of lemma 6.5

In veru of Green's formula, we have briefly.

$$\int_D K_n|\varepsilon(u_n)|^p \;=\; \int_D f u_n$$

whereas,

$$\int_D f u_n \longrightarrow \int_D f u_*, \quad when \;\; n \uparrow \infty$$

while $\int_D f u_*$ is equal to $\int_D K_*|\varepsilon(u_*)|^p$, whence

$$\int_D K_n|\varepsilon(u_n)|^p \longrightarrow \int_D K_*|\varepsilon(u_*)|^p; \;\; n \uparrow \infty$$

accordingly

$$\|\sigma(u_n)\cdot\cdot\varepsilon(u_n) - \sigma(u_*)\cdot\cdot\varepsilon(u_*)\|_{L^1(D)} \longrightarrow 0; \;\; n \uparrow \infty$$

thus the lemma is proved.

■

Corollary 6.1 *Indeed, since the Banach space $\mathcal{W}$ is uniformly convex and moreover*

$$limSup_{n\uparrow\infty}\|u_n\|_{\mathcal{W}} \leq \|u_*\|_{\mathcal{W}},$$

we deduce the strong convergence of u_n to u_* in $\mathcal{W}_{div}$ (see also [3]).

lemma 6.6 *The limit function θ_* is solution of the heat equation, further more $\theta_* = \theta_{V_*}^{\varphi_*}$.*

Proof of lemma 6.6

Let us denote by g_n the associated seuqnece of dissipation mecanichal power $\sigma(u_n)\cdot\cdot\varepsilon(u_n)$.
since

$$\|\theta_n\|_{L^2(0,\tau,H^1(D))} \leq \frac{1}{\mu}(\int_0^\tau (\|g_n\|_{H_{\bar{D}}^{-1}(R^N)} + C_{\Sigma_2}\ \|q_i\|_{H_{oo}^{-1/2}(\Sigma_2)})^2\, dt\ )^{1/2}$$

and then

$$\|\theta_n\|_{L^\infty(0,\tau,L^2(D))} \leq \frac{2}{\mu}(\int_0^\tau (\|g_n\|_{H_{\bar{D}}^{-1}(R^N)} + C_{\Sigma_2}\ \|q_i\|_{H_{oo}^{-1/2}(\Sigma_2)})^2\, dt\ )$$

else

$$\int_D \sigma(u_n)\cdot\cdot\varepsilon(u_n) \leq c\|f\|_{L^\infty(0,\tau;L^{p'})}^{\frac{p}{p-1}}$$

then

$$\|g_n\|_{H_{\bar{D}}^{-1}(R^N)} \leq c\|f\|_{L^\infty(0,\tau;L^{p'})}^{\frac{p}{p-1}}$$

hence by compacity argument, one can extract a subsequence $(\theta_{n_k}, g_{n_k})_{k\in IN}$ which converges weakly towards (θ_*, g_*) in $\mathcal{Q}\times L^\infty(0,\tau;H^{-1}(D))$. One can assume that $(\theta_n, g_n)_{n\in IN}$ itself converges weakly towards (θ_*, g_*).

Let us apply the Green's formula in order to get the weak formulation related to the heat equation whose θ_n is solution.

$$\int_0^\tau\int_D(\frac{\partial}{\partial t}\theta_n)\psi + \int_0^\tau\int_D k(u.\nabla\theta_n)\psi + \int_0^\tau\int_D \lambda\nabla\theta_n\nabla\psi = \int_0^\tau\int_D g_n\psi - \int_{\Sigma_2} q_i\psi \quad \forall\ \psi\in\mathcal{Q}$$

under the previous compacity argument, we deduce

$$\int_0^\tau\int_D(\frac{\partial}{\partial t}\theta_*)\psi + \int_0^\tau\int_D k(u.\nabla\theta_*)\psi + \int_0^\tau\int_D \lambda\nabla\theta_*\nabla\psi = \int_0^\tau\int_D g_*\psi - \int_{\Sigma_2} q_i\psi \quad \forall\ \psi\in\mathcal{Q}$$

moreover, it is relatively easy to check that:

$$\frac{\partial}{\partial n}\theta_* = 0\ \ on\ \ \Sigma_1, \quad \frac{\partial}{\partial n}\theta_* = -q_i\ \ on\ \ \Sigma_2,\ \ and\ \ \theta_*(0,.) = 0\ \ in\ \ D$$

whence θ_* is solution of the heat equation, thanks to uniqueness argument $\theta_* = \theta_{\Omega_*}^{\varphi_*}$. On the other hand, we are able to prove that θ_n converges strongly towards θ_*, in fact, Green's formula yields to

$$\int_0^\tau \int_D (\frac{\partial}{\partial t}\theta_n)\theta_n \;+\; \int_0^\tau \int_D \lambda|\nabla\theta_n|^2 \;=\; \int_0^\tau \int_D g_n\theta_n \;-\; \int_{\Sigma_2} q_i\theta_n$$

while θ_n converges weakly towards θ_* in $L^2(I,\ H^1(D))$ and due to the theorem supplied by [17] θ_n converges strongly towards θ_* in $L^2(I,\ L^2(D))$, it follows that

$$\|\theta_n\|_{L^2(I,\ L^2(D))} \longrightarrow \|\theta_*\|_{L^2(I,\ L^2(D))} \text{ when } n\uparrow\infty$$

$$\int_{\Sigma_2} q_i\theta_n \longrightarrow \int_{\Sigma_2} q_i\theta_*, \text{ when } n\uparrow\infty$$

and since g_n converges strongly to g_* in $L^2(I, H^{-1}(D))$, then

$$\int_0^\tau \int_D g_n\theta_n \longrightarrow \int_0^\tau \int_D g_*\theta_*, \text{ when } n\uparrow\infty$$

while θ_* is solution of the heat equation, one can check

$$\int_0^\tau \int_D \lambda|\nabla\theta_n|^2 \longrightarrow \int_0^\tau \int_D \lambda|\nabla\theta_*|^2 \text{ when } n\uparrow\infty$$

hence we get the norm's convergence

$$\|\theta_n\|_{L^2(I,\ H^1(D))} \longrightarrow \|\theta_n\|_{L^2(I,\ H^1(D))} \text{ when } n\uparrow\infty$$

whence

$$\theta_n \text{ converges strongly towords } \theta_* \text{ in the Hilbert space } L^2(I,\ H^1(D))$$

■

Thus, the strong continuity result is provided.

6.4 Strong Compactness Result

We will prove that the set $\mathcal{G}(\mathcal{B})$ is strongly relatively-compact in $\mathcal{W}_{div} \times \mathcal{Q}$.

Proposition 6.2 *For any integer $n \in IN$ we denote $(V_{n+1}, \theta_{n+1})_{n\in IN} = \mathcal{G}(V_n, \theta_n)$ being a sequence of $\mathcal{G}(\mathcal{B}) \subset \mathcal{W}_{div} \times \mathcal{Q}$, then one can extract a subsequence $(V_{n_k}, \theta_{n_k})_{k\in IN}$ which converges strongly in $\mathcal{G}(\mathcal{B})$.*

Proof of Proposition 6.2

Indeed, $(V_{n+1}, \theta_{n+1})_{n\in IN} : t \longrightarrow (u^{\theta_n}, \theta_{V_n}^{\theta_n})$. According lemmas (6.1) and (6.3), it transpires that:

$$\|(V_{n+1}, \theta_{n+1})\|_{\mathcal{W}\times\mathcal{Q}} \leq c\,; \;\; \textit{where } c \textit{ is independent on } n.$$

Compacity argument, supplied by the considered Sobolev spaces, yields to an extracted subsequence $(V_{n_k}, \theta_{n_k})_{k\in IN}$ which converges weakly to (V_*, θ_*) in $\mathcal{W}_{div} \times \mathcal{Q}$. We result by proposition (6.1) that $(u^{\theta_{n_k}}_{\Omega_t(V_{n_k})}, \theta^{\theta_{n_k}}_{\Omega_t(V_{n_k})})$ converges strongly towards $(u^{\theta_*}, \theta^{\theta_*}_{V_*})$ in $\mathcal{B} \subset \mathcal{V} \times \mathcal{Q}$ and accordingly $(V_{n_k}, \theta_{n_k})_{k\in IN}$ converges strongly to (V_*, θ_*). Whereas

$$(V_*, \theta_*) : t \longrightarrow (u^{\theta_*}, \theta^{\theta_*}_{V_*})$$

furthermore, strong countinuity of the operator $\mathcal{G}$ yields that (V_*, θ_*) belongs to $\mathcal{G}(\mathcal{B})$.
Whence the strong compactness result of the operator $\mathcal{G}$ is supplied.

■

Thus, the Schauder fixed-point theorem provides an existence result for the Monophasic Norton-Hoff heat problem.

7 Compacity of the Solutions Set

Theorem 7.1 *Let S be the set of solutions to the Norton-Hoff heat problem.*

$$S = \{(u, \theta) \in \mathcal{V} \times \mathcal{Q}; \;\; s,t \;\; (u, \theta) \;\; \textit{solution of } \mathcal{P}\}$$

Then S is compact for the strong topology induced by $\mathcal{V} \times \mathcal{Q}$.

Proof of Theorem 7.1

We use the same setting as before, let (u_n, θ_n) any subsequence belongs to S.
Then under the above lemmas 6.1 and 6.3, the couple (u_n, θ_n) belongs to the closed bounded ball $\mathcal{B} \subset \mathcal{V} \times \mathcal{Q}$. Therefore, by compacity argument one can extract a subsequence, also denoted (u_n, θ_n), which converges weakly towards (u_*, θ_*) in $\mathcal{V} \times \mathcal{Q}$. Moreover, lemmas 6.4, 6.5, 6.6 and corollary 6.1 provide that, on one hand, (u_n, θ_n) converges strongly to (u_*, θ_*) and, on the other hand, (u_*, θ_*) is a solution of the problem $\mathcal{P}$. Accordingly, (u_*, θ_*) belongs to S.
Thus, S is a compact subset of $\mathcal{V} \times \mathcal{Q}$.

■

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