Free Boundary Problemin Norton-Hoff Steady Flow with Thermal Effects

Jamel Ferchichi, Jean-Paul Zolésio

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Free Boundary Problem in Norton-Hoff Steady Flow with Thermal Effects

Jamel Ferchichi — Jean-Paul Zolésio

N° 4417

Mars 2002

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4417--FR+ENG

Free Boundary Problem
in Norton-Hoff Steady Flow with Thermal Effects

Jamel Ferchichi , Jean-Paul Zolésio

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Opale

Rapport de recherche n° 4417 — Mars 2002 — 25 pages

Abstract: We consider the free boundary identification problem in a steady viscoplastic flow. The fluid motion is governed by the incompressible Norton-Hoff model coupled with the heat equation. The viscosity of the fluid is modeled by the Arrhenius law.
Our point of view is to treat the problem as a shape sensitivity of a cost functional formulated on the free boundary and governed by the Norton-Hoff state. One of the main results of this paper is to provide a shape sensitivity result to the considered cost functional.

Key-words: Free boundary, identification, visco-plastic fluid, Arrhenius law, inverse problem, shape optimization, functional cost.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Frontière Libre dans less Écoulements Stationnaires de Type Norton-Hoff Thermique

Résumé : On s'intéresse dans ce papier à l'étude du problème inverse de l'identification des frontières libres dans les écoulements visco-plastiques. L'écoulement du fluide est gouverné par le modele de Norton-Hoff incompressible couplé à l'équation de chaleur avec condition de Robin. La viscosité du fluide est modélisée par la loi d'Arrhenius.
Le but pricipal de ce papier est de fournir un résultat de sensibilité permettant d'identifier la frontière libre.

Mots-clés : Frontière libre, identification, fluide visco-plastic, loi d'Arrhenius, problème inverse, optimisation de forme, fonctionelle coùt.

Contents

1 Introduction and Motivation

We consider a regular domain occupied by a viscoplastic fluid whose the boundary is split in three parts: the Dirichlet boundary over which the velocity and a heat flux rate related to the Robin condition are specified, the outflow boundary is adiabatic and over which the velocity is also specified Finally the free boundary subject the classical conditions: zero normal velocity component, stress and heat flux rate.
On this domain, we formulate the Norton-Hoff heat steady problem subject to the above boundary conditions.
We notice that here the coupling occurs between a non homogeneous Norton-Hoff equation and the heat equation with Robin condition. This setting a novel existence result to the corresponding coupling problem when the geometry of the domain is specified. In fact, it is suuficient to prove existence and uniqueness result to each equation separately. Due to this we prove existence to the heat equation via the Lax-Milgram theorem (see [5]). We provide a positivity result to the heat equation and we establish the optimal regularity of the solution taking in account the regularity of the associated right hand side, the proofs are similar than the ones given in [11]. Besides, we supply an existence result to the non-homogeneous Norton-Hoff problem through a variational method via a regularity result linked to the scale factor. Whereas, due to the existence result provided, in a general setting, we recover the desiried result.
The free boundary identification problem can be considered as an inverse problem consisting in searching for a domain Ω such that the complementarary of its free boundary is known. Our point of view is to treat the problem as a shape sensitivity of a cost functional formulated on the free boundary and governed by the Norton-Hoff state. From a heuristic point of view, looking for the shape sensitivity consists in observing the perturbation effect on the solution defined in a moving domain (see [8], [20]). However, Norton-Hoff model is non-linear and not regular enough which implies a number of technical difficulties: mainly, we are not able to differentiate the considered cost functional. Obviously, numerous many differentiation results exist concerning non-linear problems, in particular for the steady Navier-Stokes equations, in which the linearized problem is well posed and regular (see [4]). However our corresponding linearized problem is neither well posed nor regular.The main idea is to avoid differentiating the cost functional via a differentiation of the state. Instead, we introduce a combination between the min-max derivation (see [2]) and a parameter penalization. In fact, the Norton-Hoff equation is non-linear but it is variational, we penalize it with the so-called compliance functional which is the minimum of the associated energy. As the heat equation is linear this enables us to formulate a min-max where the heat equation plays the role of a constraint. Therefore, we establish a parameter penalized functional formulated as an infemum. First, we prove that the parameter infemum is reached, second that the penalized functional converges towards the cost one when the parameter goes to zero. Then, we give an abstract differentiation result concerning a class of shape functionals

formulated as minimum. Thus we get the shape gradient of the penalized functional. Finally, we provide a weak existence of the shape gradient of the cost functional.

2 Free Boundary Problem in the Static Case

2.1 The Norton-Hoff Heat Static Problem

We consider a C^k-domain Ω occupied by a viscoplastic fluid. Its boundary is splitting in three parts:
the Dirichlet boundary γ_d on which, on the one hand, the velocity is given and equal to u_d, on the other hand we impose a heat flux related to the Robin condition. The out flow boundary γ_s on which the rate of heat is adiabatic, the velocity is too equal to u_d; the last condition is provided by making a stopper. Finally the free boundary endowed by the classical free boundary conditions; on which the normal component of the velocity field and of the stress one are equal to zero with an adiabatic rate. The free boundary problem consists to search for a velocity field, a temperature function and the geometry of a boundary γ_L fullfiling the hereafter equations.

$$(1) \qquad \mathcal{SP}\left\{\begin{array}{rcll} K(\theta(t,x))|\varepsilon(u)|^{p-2}\varepsilon(u) \;+\; PI & = & \sigma & \text{in } \Omega \\ -div(\sigma) & = & f & \text{in } \Omega \\ div(u) & = & 0 & \text{in } \Omega \\ -\; div(\lambda\nabla\theta) & = & \sigma(u)\cdot\cdot\varepsilon(u) & \text{in } \Omega \end{array}\right.$$

with the boundary conditions:

$$(2) \qquad \mathcal{BC}\left\{\begin{array}{rcll} u & = & u_d & \text{on } \gamma_d\cup\gamma_s \\ \dfrac{\partial\theta}{\partial n} & = & 0 & \text{on } \gamma_L \\ \rho\theta \;+\; \dfrac{\partial\theta}{\partial n} & = & -q_i & \text{on } \gamma_d \\ (\sigma\cdot n,\tau) & = & 0 & \text{on } \gamma_L \\ (\sigma\cdot n,n) & = & c_\sigma & \text{on } \gamma_L \end{array}\right.$$

where c_σ is an unknown constant. With the free boundary classical conditions:

$$(3) \qquad \left\{\begin{array}{rcll} u\cdot n & = & 0 & \text{on } \gamma_L \\ (\sigma\cdot n,n) & = & 0 & \text{on } \gamma_L \end{array}\right.$$

with $\partial\Omega \;=\; \gamma_d\cup\gamma_s\cup\gamma_L$, where
γ_d is the Dirichlet boundary, γ_s is the leaving boundary on which we make a stopper which oblige the fluid to flow with the same velocity u_d, γ_L designates the free boundary and ρ is the Robin coefficient.

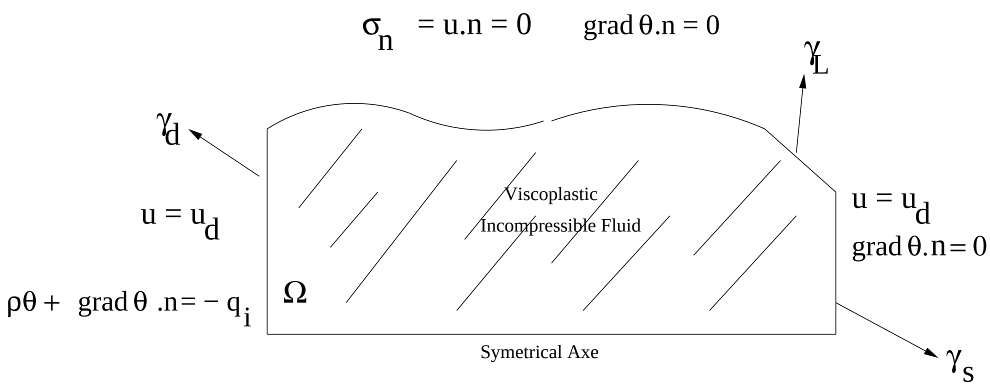

Figure 1: Visco-plastic Fluid

We begin by investigate the associated heat equation with Robin condition. We prove an existence result due to the Lax-Milgram theorem, we establish an optimal regularity and a positivity result which will be needed to recover a regularity result linked to the scale factor issue from the Arrhenius law.

2.1.1 Heat Equation

$$
(4) \qquad \begin{cases} -\, div(\lambda \nabla \theta) & = \quad \sigma(u)..\varepsilon(u) \quad \text{in} \quad \Omega \\ \dfrac{\partial \theta}{\partial n} & = \quad 0 \quad \text{on} \quad \gamma_L \\ \rho\theta \; + \; \dfrac{\partial \theta}{\partial n} & = \quad -q_i \quad \text{on} \quad \gamma_d \end{cases}
$$

Proposition 2.1 *The heat equation has a unique solution in $H^1(\Omega)$. And there exists $\varepsilon > 0$ such that the solution belongs to $W^{2-\varepsilon,2}(\Omega)$.*

For the proof it is enough to refer to [11].

Proposition 2.2 *The solution of the heat equation is non-negative*

$$\theta(x) \geq 0; \; a.e.$$

Proof of proposition

Indeed, the heat problem is variational: the solution minimize the associated energy which is coercive in $H^1(\Omega)$ as $\rho > 0$,

$$\theta = argmin(\frac{1}{2}\int_\omega \lambda|\nabla\varphi|^2) + \frac{1}{2}\rho\int_{\gamma_d}\varphi^2 - \int_\Omega g\varphi + \int_{\gamma_d} q_i\varphi)$$

since $-q_i$ and g are non-negative then we get for all φ in $H^1(\Omega)$;

$$\frac{1}{2}\int_\Omega \lambda|\nabla\varphi|^2 + \rho\int_{\gamma_d}\varphi^2 - \int_\Omega g\varphi + \int_{\gamma_d} q_i\varphi \geq \frac{1}{2}\int_\Omega \lambda|\nabla|\varphi||^2 + \rho\int_{\gamma_d}|\varphi|^2 - \int_\Omega g|\varphi| + \int_{\gamma_d} q_i|\varphi|$$

hence we check

$$\theta \geq |\theta| \ ; \quad a, e$$

which achieves the proof

■

In the sequel, we deal with the non homogeneous Norton-Hoff problem. We prove an existence result via a variational study through an established regularity result related to the scale factor.

2.1.2 Non Homogeneous Norton-Hoff Problem

$$(5) \qquad \left\{ \begin{array}{rcll} K_c|\varepsilon(u)|^{p-2}\varepsilon(u) \ + \ PI & = & \sigma & \text{in} \ \ \Omega \\ -div(\sigma) & = & f & \text{in} \ \ \Omega \\ div(u) & = & 0 & \text{in} \ \ \Omega \\ u & = & u_d & \text{on} \ \ \gamma_d \cup \gamma_s \\ (\sigma.n, \tau) & = & 0 & \text{on} \ \ \partial\Omega \end{array} \right.$$

Functional Setting Let

$$\mathcal{W}_{div} = \{v \in W^{1,p}(\Omega); \ \ div(v) = 0\}$$

$$\mathcal{C}(\Omega) = \{v \in \mathcal{W}_{div}; \ \ v = u_d \ \ on \ \ \gamma_d \cup \gamma_s\}$$

$\mathcal{C}$ is a closed convex subset of the Banach-space $\mathcal{W}_{div}$.

Regularity of the Arrhenius Law

Proposition 2.3 *Let* $K = K_c \ exp(\frac{\gamma}{(\theta_0+\theta)})$ *being the scale factor associated to the Norton-Hoff law, where* K_c *is the consistency of the material,* θ *is the solution of the heat equation and* θ_0 *is a strict non-negative function. Then*

$$K \ \ belongs \ to \ \ L^\infty(\Omega)$$

Proof of proposition 2.3
In fact, it is enough to mention that the consistency K_c and the heat solution θ are non-negatives and via the fact that the function θ_0 is strict non-negative, then we get the proof.

■

Existence Result

Proposition 2.4 *The non-homogeneous Norton-Hoff problem has a unique solution in the convex subset* $\mathcal{C}$.

Proof of proposition 2.4

Let us introduce the following mapping; for any given f in $\mathcal{W}'$

$$\begin{aligned} \Phi : \mathcal{C} \ &\rightarrow \quad IR \\ v \rightarrow &\int_\Omega \frac{K}{p}|\varepsilon(v)|^p - fv \end{aligned} \tag{6}$$

The functional Φ is convex, lower semi-continuous, coercive and Gateaux differentiable. Its Gateaux derivative at u in direction v is

$$\Phi'(u;v) = \int_\Omega K|\varepsilon(u)|^{p-2}\varepsilon(u)..\varepsilon(v) - fv$$

where the expression $|\varepsilon(u)|^{p-2}\varepsilon(u)..\varepsilon(v)$ has to be understood as continuously extended with 0 at any point x with $|\varepsilon(u)|(x) = 0$.
It remains to prove that Φ is coercive. Indeed, let u_n be a sequence in $\mathcal{C}$ such that $\|u_n\|_{\mathcal{W}_{div}}$ tends to infinity, therefrom let w be a raising of u_n in $\mathcal{C}$ and $v_n = u_n - w$ hence v_n belongs to $W_0^{1,p}$, then

$$\Phi(u_n) = (|\frac{K}{p}|^{\frac{1}{p}}|\varepsilon(v_n + w)|_{L^p})^p - \int_\Omega f(v_n + w)$$

so

$$\Phi(u_n) \geq c||v_n| - |w||^p - \|g\|_{L^{p^*}}\|v_n\|_{L^p}$$

then

$$|v_n|_{L^p} \leq (\Phi(u_n) + C_P\|g\|_{L^{p^*}}|v_n|_{L^p})^{\frac{1}{p}}$$

while, if u_n tends to infinity then $\Phi(u_s)$ has to do too, otherwise it is impossible due to the last inequality.
Thus, there exists a unique u in $\mathcal{C}$ such that

$$\Phi(u) = \min_{v \in \mathcal{C}} \Phi(v)$$

Which ends the proof.

■

Remark 2.1 *In vertu of the existence result to the monophasic Norton-Hoff heat problem supplied in [?], w e shall notice here that problem (1), (2) has at least a solution when the geometry of domain is known.*

2.2 Free Boundary Problem

We look for a domain Ω with boundary $\partial\Omega = \gamma_d \cup \gamma_s \cup \gamma_L$ such that:

$$u \cdot n = (\sigma \cdot n,\ n) = 0 \ \ on\ \gamma_L$$

The hereafter lemma will be so useful; it allows us to lighten the equations of the free boundary problem.

lemma 2.1 *Assume that $u \cdot n = 0$ on γ_L, then since*

$$u = u_d \ \ on \ \ \gamma_d \cup \gamma_s$$

hence

$$\sigma_n \ = \ (\sigma \cdot n, n) \ = \ 0 \ \ on \ \ \gamma_L.$$

Proof of lemma

In fact, we have for all v in the convex subset $C(\Omega)$;

$$\int_\Omega |\varepsilon(u)|^{p-2}\varepsilon(u) \cdot \cdot \varepsilon(v-u) - f(v-u) \geq 0$$

Green's formula yields, for all v in $C(\Omega)$

$$-\int_\Omega div(\sigma)(v-u) - f(v-u) + \int_{\partial\Omega} \sigma \cdot n(v-u) \geq 0$$

Since the friction is zero we get, for all v in $C(\Omega)$

$$\int_{\partial\Omega} (\sigma \cdot n, n)(v-u) \cdot n = 0$$

but $v - u = 0$ on $\gamma_d \cup \gamma_s$, so

$$\forall \ \ v \ \in \ C \ ; \int_{\gamma_L} (\sigma \cdot n, n)(v-u) \cdot n = 0$$

as $u.n = 0$ on γ_L and $div(v-u) = 0$, hence for all v in C ; $\int_{\gamma_L} (v-u) \cdot n = \int_{\gamma_L} v \cdot n = 0$, also $\int_{\gamma_L} (\sigma \cdot n, n) v \cdot n = 0$. This means that $(\sigma \cdot n, n)$ belongs to the dual of the set of functions whom the mean on γ_L is vanished, then there exists a constant c such that:

$$(\sigma \cdot n, n) = ([K|\varepsilon(u)|^{p-2}\varepsilon(u) + p] \cdot n, n) \ = \ c_\sigma$$

But the pression P is defined in $L^{p'}$ to within a constant. One may choose $P = P' + c_\sigma$, whence

$$(\sigma \cdot n, n) \ = \ 0$$

which achieves the proof.

■

In a general setting, an identification problem can be considered as an inverse problem.

2.2.1 Inverse Problem

Accordingly, the inverse problem consists to look for a domain Ω within the boundary $\partial\Omega = \bar{\gamma}_d \cup \bar{\gamma}_b \cup \bar{\gamma}_L$ such that:

$$u \cdot n \;=\; 0 \;\; on\; \gamma_L$$

The following proposition proves that the free-boundary problem is well posed (in the sense that the free boundary is building by an unique velocity).
Our point of view is to treat the inverse problem as a shape control of a cost functional formulated on the free boundary and governed by the Norton-Hoff state.

2.2.2 Cost Functional

The cost functional is given via a mapping F_Ω from $\mathcal{W}_{div}(\Omega)$ to IR.

$$J(\Omega) \;=\; F(\Omega; u_\Omega) \;=\; \int_{\gamma_l} |u.n|^p$$

then the free boundary γ_L will be the set of points where $u \cdot n$ vanishes.

In fact, Norton-Hoff is not enough regular and non linear which involves many technical difficulties mainly we are not able to differentiate the considered cost functional. Obviously there exists many differentiation results concerning non-linear problems, notably for the Navier-stokes problem in the static case, in which they use the linearized problem which is well posed and regular (see [4]). However our corresponding linearized problem is neither well posed nor regular. The main idea here is to avoid differentiating the cost functional via a differentiation of the state. That is why we introduce a combination between the min-max derivation and a parameter penalization. In fact, Norton-Hoff equation is non linear but it is varitonal, we penalize it with the so-called compliance functional which is the minimum of the associated energy. As for the heat equation, it is linear, which enables us to formulate it in a min-max where we use the heat equation as a contrainst. Therefrom, we establish a parameter penalized functional formulated in an infimum.

2.2.3 Penalized Functional

Let q being a given reel such that the embedding of $W^{q,2}$ in $W^{0,\infty}$ is compact where $W^{q,2}$ is a subset of the Sobolev-space $W^{2-\varepsilon,2}$ for all $\varepsilon > 0$.
Let us consider a set of admissible viscosities:

$$\mathcal{A} = \{k \in W^{q,2}(\Omega);\;\; k \geq \frac{1}{2}\}$$

For α in $]0,1[$, we introduce the following penalization of the functional J:

$$J^\alpha(\Omega) = \inf_{(v,k)\in\mathcal{W}_{div}\times\mathcal{A}} \mathcal{F}_{\Omega,\alpha}(v,k)$$

with

$$\mathcal{F}_{\Omega,\alpha}(v,k) \;=\; \inf_{\varphi\in\mathcal{Q}}\sup_{\psi\in\mathcal{Q}} L(\varphi,\psi)$$

where

$$L(\varphi,\psi) \;=\; j(\varphi) \;+\; H(\varphi,\psi)$$

$$j(\varphi) \;=\; \int_{\gamma_l} |v.n|^p \;+\; \frac{1}{\alpha}[\Phi_\Omega(v,k) - e_K(\Omega)] \;+\; \frac{1}{\alpha}\|k - A(\varphi)\|^2_{W^{q,2}}$$

$$H(\varphi,\psi) \;=\; \int_\Omega \lambda\nabla\varphi\nabla\psi - k|\varepsilon(v)|^p\psi + \int_{\gamma_d}\rho\varphi\psi - \int_{\gamma_d} q_i\psi$$

and

$$A(\varphi) \;=\; K_c exp(\frac{\gamma}{\varphi+\theta_0})$$

$$\Phi_\Omega(v,k) = \int_\Omega \frac{k}{p}|\varepsilon(v)|^p - fv$$

$$e_{K_\Omega}(\Omega) = min\{\Phi_\Omega(v,k),\ (v,k)\in\mathcal{C}(\Omega)\times\mathcal{A}(\Omega)\}$$

Remark 2.2 *We shall notice here that, on the one hand, the penalization will oblige the mapping Φ to reach its minimum and the reel k to recover the Arrhenius low associated to the temperature φ. On the other hand, via the inf-sup the map H has to vanish which requires the couple (φ,ψ) to reach the corresponding saddle-point associated to the heat equation. These reasons explain the choice of the penalized functional.*

2.2.4 Existence of Minima

Proposition 2.5 *The functional L has saddle-points. Then*

$$\mathcal{F}_{\Omega,\alpha}(v,k) \;=\; \min_{\varphi\in\mathcal{Q}}\max_{\psi\in\mathcal{Q}} L(\varphi,\psi)$$

Proposition 2.6 *The functional $\mathcal{F}_{\Omega,\alpha}$ has a minimum in $\mathcal{W}_{div}(\Omega)\times\mathcal{A}$.*

Proof of Proposition

Let (v_n,k_n) be a minimizing sequence of the functional $\mathcal{F}$. This means that

$$\forall(v,k)\in\mathcal{W}_{div}(\Omega)\times\mathcal{A};\;\; \mathcal{F}(v_n,k_n)\le\mathcal{F}(v,k)$$

then

$$\mathcal{F}(v_n,k_n)\le\mathcal{F}(u,A(\theta))$$

$$\mathcal{F}(v_n,k_n)\le\int_{\gamma_l}|u.n|^p$$

therefore

$$(7)\quad \int_{\gamma_l} |v_n \cdot n|^p \ + \ \frac{1}{\alpha}[\Phi_\Omega(v_n, k_n) - e_{K_\Omega}(\Omega)] \ + \ \frac{1}{\alpha}\|k_n - A(\theta)\|^2_{W^{q,2}} \leq \int_{\gamma_l} |u \cdot n|^p$$

accordingly, on the one hand there exists $c > 0$ such that

$$\|v_n\|_{\mathcal{W}_{div}} \leq c$$

so one can extract a subsequence denoted also v_n which converges weakly towards v^* in $\mathcal{W}_{div}$. Then $|\varepsilon(v_n)|^p$ converges strongly towards $|\varepsilon(v^*)|^p$ in $L^1(\Omega)$.
On the other hand, there exists $c \geq 0$ such that

$$\|k_n\|_{W^{q,2}} \leq c$$

then we can extract a subsequence denoted also k_n which converges weakly towards k^* in $\mathcal{A}$. Since the embedding of $W^{q,2}$ in L^∞ is compact, we get; k_n converges strongly towards k^* in L^∞. Whence

$$k_n|\varepsilon(v_n)|^p \to k^*|\varepsilon(v^*)|^p, n \uparrow \infty \ \textit{strongly in} \ L^1(\Omega)$$

with the truth that the functionals $j(\varphi)$ and H are weakly l.s.c. on $\mathcal{W}_{div}(\Omega) \times \mathcal{A}(\Omega)$ we get

$$\mathcal{F}(v^*, k^*) \leq \liminf_{n\uparrow\infty} \mathcal{F}(v_n, k_n)$$

Thus the minimum's existence. Let $(u^\alpha_\Omega, K^\alpha_\Omega)$ denote a minimizer:

$$J^\alpha \ = \ \mathcal{F}(u^\alpha_\Omega, K^\alpha_\Omega)$$

Which achieves the proof.

∎

lemma 2.2 *Let $\theta_n = \theta(v_n, k_n)$ be the solution of the heat equation with right hand side $k_n|\varepsilon(v_n)|^p$. The above result provides; θ_n converges weakly towards $\theta(v^*, k^*) = \theta^*$ in $W^{q,2}$, where q is a reel chosen as above.*

Let θ^α_Ω be the solution of the heat equation with right hand-side $K^\alpha_\Omega|\varepsilon(u^\alpha_\Omega))|^p$.

2.2.5 Convergence of the Penalized Functional

Proposition 2.7 *When α tends to zero*

$$J^\alpha(\Omega) \longrightarrow J(\Omega)$$

and

$$\sup \|(u^\alpha_\Omega, K^\alpha_\Omega) - (u_\Omega, K_\Omega)\|_{\mathcal{W}_{div}(\Omega)\times\mathcal{A}(\Omega)} \to 0.$$

Proof of proposition 2.7

Let α_n be a sequence in $]0,1[$ which tends to zero. Any sequence $(u_\Omega^{\alpha_n}, K_\Omega^{\alpha_n})_{n\in IN}$, where $(u_\Omega^{\alpha_n}, K_\Omega^{\alpha_n})$ is a minimizer of $\mathcal{F}_{\Omega,\alpha}$, converges weakly (up to passing to a subsequence as before) to (u^*, k^*) in $\mathcal{W}_{div}(\Omega)\times\mathcal{A}(\Omega)$.
Since the embedding of $W^{q,2}$ in L^∞ is compact then $K_\Omega^{\alpha_n}$ converges strongly towards k^* in L^∞. This yields that $\theta(u_\Omega^{\alpha_n}, K_\Omega^{\alpha_n})$ converges towards $\theta(u^*, k^*)$ in $W^{q,2}$ then $A(\theta(u_\Omega^{\alpha_n}, K_\Omega^{\alpha_n}))$ converges to $A(\theta(u^*, k^*))$ a.e.
Since $\mathcal{F}_{\Omega,\alpha_n}(u_\Omega^{\alpha_n}, K_\Omega^{\alpha_n}) \le \mathcal{F}_{\Omega,\alpha_n}(u_\Omega, K_\Omega)$, we have

$$J^{\alpha_n}(\Omega) \le J(\Omega)\ ;\ \forall\, n$$

and so, on the one hand

$$\|K_\Omega^{\alpha_n} - A(\varphi)\|_{W^{q,2}} \le \alpha_n J(\Omega)$$

then by passing to the limit with respect to n via the lower semi-continuity property of the sobolev-space $W^{q,2}$, we get

$$k^* = A(\theta(u^*, k^*))$$

on the other hand

$$\Phi_\Omega(u_\Omega^{\alpha_n}, K_\Omega^{\alpha_n}) \le \alpha_n J(\Omega) \quad + \quad e_{K_\Omega}(\Omega)$$

by passing to the limit when n tends to infinity, the lower semi-continuity of Φ on $\mathcal{W}_{div}(\Omega)\times\mathcal{A}(\Omega)$ yields

$$\Phi_\Omega(u^*, A(\theta(u^*, k^*))) \le e_{K_\Omega}(\Omega)$$

but $e_{K_\Omega}(\Omega)$ being the unique minimum of Φ, then $(u^*, k^*) = (u_\Omega, K_\Omega)$.
Hence, by uniqueness of the heat solution, $\theta(u^*, k^*) = \theta$
From the weak lower-semi continuity of $\mathcal{F}$, we then have

$$J(\Omega) \le lim \inf_{n\to\infty} J^{\alpha_n}(\Omega)$$

hence, we derive

$$J^{\alpha_n}(\Omega) \to J(\Omega)$$

Equation (7) also prove that $\Phi_\Omega(u_\Omega^{\alpha_n}, K_\Omega^{\alpha_n})$ tends to $e_{K_\Omega}(\Omega)$. As we know already, this yields $\|u_\Omega^{\alpha_n}\|_{\mathcal{W}_{div}}$ converges to $\|u_\Omega\|_{\mathcal{W}_{div}}$, and thus $u_\Omega^{\alpha_n}$ converges towards u_Ω strongly in $\mathcal{W}_{div}$.
Thus, the proof is given.

■

We provide some abstract results which are the tools in order to solve the shape sensitivity problem with respect to the free boundary.

3 Abstract Results

In order to differentiate the penalized cost functional we adopt the so-called Velocity method introduced by Cea and Zolésio. It consists to perturb domains via a vector field to which we can associate a flow mapping.

3.1 The Velocity Method

Let D be a subset of IR^N. We assume it is bounded and has a smooth boundary. We denote by $\mathcal{O}_{lip}$ the set of all Lipschitz domains in D. Let I be a closed interval of IR^+ which contains 0. Let k be a non-negative integer. The purpose of the speed method is to provide one-parameter deformations of a domain in order to formulate continuity and differentiability properties in a "computable way". This method allows arbitrary large deformation of the domain. We will briefly remind some results about this method which will be used throughout this work. We refer to [20] for the proves and further details.
We will choose speed-field in the space

$$\mathcal{V}_k = \{V \in C^0(I, C^k(\overline{D}, IR^N));\ \ V \cdot n_{\partial D} = 0 \text{ on } \partial D\}$$

where $n_{\partial D}$ denotes the out unit normal of ∂D. This space is endowed with the uniform convergence topology, and thus is a Banach space when I is bounded.
The following proposition yields from O.D.E. theory. It allows us to derive from a field V a one-parameter deformation of the domains.

Proposition 3.1 *Each V in $\mathcal{V}_k$ has a unique flow mapping $T(V)$ in $C^1(I, C^k(\overline{D}, IR^N))$. The image T_s of any s in I:*
i) maps $\overline{D}$ onto $\overline{D}$.
ii) has an inverse T_s^{-1} and $T^{-1} : s \to T_s^{-1}$ belongs to $\mathcal{V}_k(I)$
iii) the application T is solution of the equation

$$\partial_s T(s) = V_s \circ T_s \ \text{ and } \ T_0 = Id_{\overline{D}}$$

Moreover, flows have the semi-group property:

$$\forall V \in \mathcal{V}_k(I), \forall s, t \in I,\ s + t \in I;\ \ T_0(V_{s+t}) = T_t(V_s)$$

For any domain ω in D, we can consider the family

$$\Omega_s(V) = T_s(V)(\Omega_0),\ \ \Omega_0(V) = \Omega$$

When $\mathcal{O}$ is a subset of $\mathcal{O}_{lip}$, we define a shape functional on $\mathcal{O}$ as a mapping $\mathcal{O} \to IR^N$. The family $\mathcal{O}$ has to be stable under the flow transformations i.e.

$$\forall V \in \mathcal{V}_k(I), \forall s \in I,\ T_s(V)(\mathcal{O}) \subset \mathcal{O}$$

In order to give sense to the shape analysis, we briefly revisit here.

Definition 3.1 *A shape functional J is directionally-shape continuous at Ω_0 in $\mathcal{O}$ (with respect to $\mathcal{V}_k$) iff for any V in $\mathcal{V}_k$, the mapping $s \to J(\Omega_s(V))$ is continuous at 0.*
A shape functional J is shape differentiable at Ω_s iff
i) For any V in $\mathcal{V}_k$ the Eulerian derivative of J at Ω_0 in direction V in $\mathcal{V}_k$

$$dJ(\Omega_0; V) = \lim_{s \to 0} \frac{J(\Omega_s(V)) - J(\Omega_0)}{s}$$

exists.
ii)The mapping $V \to dJ(\Omega_0; V)$ is linear and continuous from $\mathcal{V}_k$ to $I\!R^N$.

In the following, we exhibit an abstract result concerning a class of shape functionals; mainly the functionals formulated in a minimum. This result will allows us to establish the hopened shape gradient of the penalized functional.

3.2 Eulerian Derivative of a Class of Shape Functionals

We are interested in shape functional given via a variational principle. Its shape differentiability is obtained by the following result.

Let $(F_\Omega : \mathcal{W}_{div}(\Omega) \longrightarrow I\!R)_{\Omega \in \mathcal{O}}$ be a family of functionals on the set $\mathcal{W}_{div} \times \mathcal{A}$.

Assumption 3.1 *There exists Ω_0 in $\mathcal{O}$ and V in $\mathcal{V}_k$ such that*
i) There exists $s_0 > 0$ such that for any $0 \leq s \leq s_0$, the mapping $(v, k) \to (DT_s^{-1}.v, k) \circ T_s(V)$ is an isomorphism between $\mathcal{W}_{div}(\Omega_s(V)) \times \mathcal{A}(\Omega_s(V))$ and $\mathcal{W}_{div}(\Omega_0) \times \mathcal{A}(\Omega_0)$.
ii) For any $0 \leq s \leq s_0$, $J_{\Omega_s(V)}$ is a weakly l.s.c. functional on $\mathcal{W}_{div}(\Omega_s(V)) \times \mathcal{A}(\Omega_s(V))$.
iii) For any (v, k) in $\mathcal{W}_{div}(\Omega_0) \times \mathcal{A}(\Omega_0)$; $s \to J_{\Omega_s(V)}((DT_S.v, k) \circ T_s^{-1}(V))$ lays in $C^1([0, s_0], I\!R)$.
iv) The mapping $(s, v, k) \to \partial_s J_{\Omega_s(V)}((DT_s.v, k) \circ T_s^{-1}(V))$ is weakly l.s.c. on $[0, s_0] \times \mathcal{W}_{div}(\Omega_0) \times \mathcal{A}(\Omega_0)$ at any point $(0, v, k)$.

Theorem 3.1 *With previous assumption, the shape functional $J(\Omega) = \min\limits_{(v,k)} J_\Omega(v, k)$ has an Eulerian derivative at Ω_0 in direction V.*
This Eulerian derivative is linear with respect $V(0)$ iff there exists a unique minimizer for J_{Ω_0}.

Proof of Theorem

The field V being fixed, we omit its reference in the notations. By previous results the functional has a minimum for any $s \leq s_0$. It is reached on a subset W of $\mathcal{W}_{div}(\Omega_s) \times \mathcal{A}(\Omega_s)$. This set may be transported:

$$W^s = \{(DT_s.v, k) \circ T_s^{-1}; (v, k) \in W_s\} \subset \mathcal{W}_{div}(\Omega_0) \times \mathcal{A}(\Omega_0).$$

For any (v^s, k^s) in W^s , (v^0, k^0) in W^0 , (v, k) in $\mathcal{W}_{div}(\Omega_0) \times \mathcal{A}(\Omega_0)$ and $0 < s \le s_0$, we have

$$\begin{aligned}\frac{J(\Omega_s) - J(\Omega_0)}{s} &= \frac{J_{\Omega_s}((DT_s.v^s, k^s) \circ T_s^{-1}) - J_{\Omega_0}(v^0, k^0)}{s} \\ &\le \frac{J_{\Omega_s}((DT_s.v^0, k^0) \circ T_s^{-1}) - J_{\Omega_0}(v^0, k^0)}{s}\end{aligned}$$

Passing to the limit when s tends to zero, we come to

$$\overline{\partial} J(\Omega_0; V) = \limsup_{s \to 0} \frac{J(\Omega_s) - J(\Omega_0)}{s} \le \partial_s J_{\Omega_s}((DT_s.v^0, k^0) \circ T_s^{-1})|_{s=0}$$

since $W^0 = W_0$, hence

$$\overline{\partial} J(\Omega_0; V) \ \le \ \inf_{(v_0,k_0)} \partial_s J_{\Omega_s}((DT_s.v^0, k^0) \circ T_s^{-1})|_{s=0} \tag{8}$$

But we also have, for any (v^s, k^s) in W^s , (v^0, k^0) in W^0 , (v, k) in $\mathcal{W}_{div}(\Omega_0) \times \mathcal{A}(\Omega_0)$ and $0 < s \le s_0$,

$$\begin{aligned}\frac{J_{\Omega_s}((DT_s.v^s, k^s) \circ T_s^{-1}) - J_{\Omega_s}(v^s, k^s)}{s} &\le \frac{J(\Omega_s) - J(\Omega_0)}{s} \\ &= \frac{J_{\Omega_s}((DT_s.v^s, k^s) \circ T_s^{-1}) - J_{\Omega_0}(v^0, k^0)}{s}\end{aligned}$$

Since $[s \to J_{\Omega_s}((DT_s.v^s, k^s) \circ T_s^{-1})] \in C^1([0, s_0], IR)$, for any $s \in [0, s_0]$, there exits σ (depending on s) with $|\sigma| < s$ such that

$$J_{\Omega_s}((DT_s.v^s, k^s) \circ T_s^{-1}) - J_{\Omega_0}(v^0, k^0) = s \partial_s J_{T_s(V)(\Omega)}((DT_s.v^s, k^s) \circ T_s^{-1}|_{s=\sigma})$$

passing to the limit when s tends to zero we come to

$$\partial_s J_{\Omega_s}((DT_s.v^0, k^0) \circ T_s^{-1})|_{s=0}) \le \underline{\partial} J(\Omega_0; V) = \liminf_{s \to 0} \frac{J(T_s(V)(\Omega)) - J(\Omega)}{s}$$

and thus, since $W^0 = W_0$,

$$\sup_{(v_0,k_0)} \partial_s J_{\Omega_s}((DT_s.v^0, k^0) \circ T_s^{-1})|_{s=0}) \ \le \ \underline{\partial} J(\Omega_0; V) \tag{9}$$

Finally, equations (8) and (9) yield the existence of the Eulerian derivative of J at Ω_0 in direction V.

$$dJ(\Omega_0; V) \ = \ \inf_{(v_0,k_0)} \partial_s J_{\Omega_s}((DT_s.v_0, k_0) \circ T_s^{-1}(V))|_{s=0}$$

■

4 Weak-Differentiability of State Dependent Functional

We shall notice here that it is easy to see that the considered cost functional F_Ω fulfills the hereafter assumption.

Assumption 4.1 *We assume that for any* Ω
i) F_Ω *is weakly continuous on* $\mathcal{W}_{div}(\Omega)$.
ii) There exist constants $\beta \geq 0$ *and* $q \leq p$ *such that*

$$\forall v \in \mathcal{W}_{div}(\Omega), F_\Omega(v) \geq -\beta \|v\|^q_{\mathcal{W}_{div}(\Omega)}$$

iii) For any Ω_0*, any* V*, any* $v \in \mathcal{W}_{div}(\Omega)$*, the mapping* $s \to F_{\Omega_s(V)}(DT_s.v \circ T_s^{-1}(V))$ *is differentiable near zero.*

The first hypothesis stands here for shortness and may be weakened.

4.1 Differentiability of the Penalized Functional

In the sequel, we will be interested to the differentiability with respect to the domain of the penalized functional. The velocity method will provide the shape differentiability and an expression of the shape gradient. We point out that our study needs only to move the free boundary γ_l.
For the following, we fix Ω_0 in $\mathcal{O}$ and a speed-field V in $\mathcal{V}_k$. Let T belongs to $\mathcal{T}_k$ such that $T_t(V)$ is the associated flow mapping to V; for any $s \in I$:

$$\Omega_s = T_t(V)(\Omega), \;\; T_t(V)|_{\gamma_d \cup \gamma_s} = Id, \;\; \gamma_l^s = T_t(V)(\gamma_l)$$

The transported problem will naturally be defined in transport sets:

$$\mathcal{C}_s = \mathcal{C}(\Omega_s), \;\; \mathcal{Q}_s = \mathcal{Q}(\Omega_s), \;\; \mathcal{A}_s = \mathcal{A}(\Omega_s)$$

Each of these sets inherits of all the properties of $\mathcal{C}, \mathcal{Q}$ and $\mathcal{A}$.

Transported Static Problem The transport of the problem can not be done without the transport of the data. In our situation, the right hand side is the only data to be transported. A rather general way do deal with this problem is to consider that the data is a family $\{f_\Omega\}$ indexed by all the admissible domains Ω such that, at least, f_Ω belongs to $\mathcal{W}'_{div}(\Omega)$ for any ω. In this section, for the sake of simplicity, we make the following assumption, which is a very simple particular case.

Assumption 4.2 *Let* f *being in* $L^{p'}(D, IR^N)$.

Thus we are able to formulate the transported static problem. It has a solution (u_s, θ_s) in $\mathcal{C}_s \times \mathcal{Q}_s$.

$$(10)\quad (\mathcal{S}P)_s \begin{cases} K(\theta(x)_s)|\varepsilon(u_s)|^{p-2}\varepsilon(u_s) + P_s I = \sigma_s & \text{in } \Omega_s \\ -div(\sigma_s) = f & \text{in } \Omega_s \\ div(u_s) = 0 & \text{in } \Omega_s \\ -\ div(\lambda \nabla \theta_s) = \sigma(u_s)..\varepsilon(u_s) & \text{in } \Omega_s \end{cases}$$

with the boundary conditions:

$$(11)\quad (\mathcal{C}L)_s \begin{cases} u_s = u_d & \text{on } \gamma_d \cup \gamma_s \\ \dfrac{\partial \theta_s}{\partial n_s} = 0 & \text{on } \gamma_l^s \\ \rho\theta_s + \dfrac{\partial \theta_s}{\partial n_s} = -q_i & \text{on } \gamma_d \\ (\sigma_s \cdot n_s, \tau_s) = 0 & \text{on } \gamma_l^s \end{cases}$$

Remark 4.1 *We recall here that our aim is to identify the free boundary γ_l that is why we only move it.*

We have the following transport lemma:

lemma 4.1 *Let v in the convex $C(\Omega)$, then*

$$div(v) = 0 \ \text{ iff } \ div[(DT_s.v) \circ T_s^{-1}] = 0$$

Shape Gradient of the Compliance

Proposition 4.1 *The mapping $s \longrightarrow \Phi(s, v, k)$ is differentiable in 0 for any (v, k) in $\mathcal{W}_{div} \times \mathcal{A}$. We will be interested in the expression of this derivative for u_{Ω_0}, the solution of Norton-Hoff in Ω_0.*

$$(12)\quad \begin{aligned} \partial_s \Phi_{\Omega_s}((DT_s.v, k) \circ T_s^{-1})|_{s=0} = & -\int_{\Omega_0} k|\varepsilon(v)|^{p-2}\varepsilon(v)..s(DvDV_0 - D(DV_0.v)) \\ & \int_{\Omega_0} (\frac{k}{p}|\varepsilon(v)|^p - fv)divV_0 - Df.V_0.v - fDV_0.v \end{aligned}$$

Proof of Proposition 12

The transport lemma and a mere change of variable provide;

$$(13)\quad \begin{aligned} \Phi_{\Omega_s}((DT_s.v, k) \circ T_s^{-1}) &= \int_{\Omega_s} \frac{k \circ T_s^{-1}}{p}|\varepsilon(DT_s.v \circ T_s^{-1})|^p - f(DT_S.v \circ T_s^{-1}) \\ &= \int_{\Omega} [\frac{k}{p}|s(DT_s^{-1}.D(DT_s.v))|^p - (f \circ T_s)DT_S.v]j(s) \end{aligned}$$

where, for any operator π; $s(\pi) = \frac{1}{2}(\pi +^* \pi)$.
Then the result is supplied, since the mapping $s \longrightarrow (DT_s, DT_S^{-1}j(s))$ is differentiable in 0.

■

Proposition 4.2 *The compliance functional*

$$e(\Omega_s) = \min\{\Phi_{\Omega_s}(v,K); (v,k) \in C(\Omega_s) \times \mathcal{A}\}$$

is shape differentiable

$$\partial_s e(\Omega_0) = \partial_s \Phi((DT_s.u_{\Omega_0}, K_{\Omega_0}) \circ T_s^{-1})|_{s=0}$$

This gradient is given by expression (12).

lemma 4.2 *For all (v,k) in $\mathcal{W}_{div} \times \mathcal{A}$, the mapping*

$$s \to \Phi_{\Omega_s}((DT_s.v,k) \circ T_s^{-1})$$

is weakly-l.s.c. at $(0,v,k)$.

Min-Max Derivation

Proposition 4.3 *Let y be the adjoint state of the heat equation. The mapping*

$$s \longrightarrow \inf_{\varphi_s \in \mathcal{Q}_s} \sup_{\psi_s \in \mathcal{Q}_s} L(s, \varphi_s, \psi_s)$$

is differentiable in 0. Its derivative is given by

$$\partial_s L((\theta, y) \circ T_s^{-1})|_{s=0} = \partial_s j(\theta \circ T_s^{-1})|_{s=0} + \partial_s H((\theta, y) \circ T_s^{-1})|_{s=0}$$

where (θ, y) is an associated saddle-point solution of the corresponding coupled state-adjoint problem.

Proof of proposition 4.3

Since the functional $L(s, \varphi_s, \psi_s)$ is given as follows;

$$L(s, \varphi_s, \psi_s) = j(s, \varphi_s) + H(s, \varphi_s, \psi_s)$$

then, on the one hand, with the fact that the function $j(s, \varphi_s)$ is quasi-convex with respect to φ_s then on the other hand $H(s, \varphi, \psi)$ is convex-concave with respect to (φ_s, ψ_s). It is enough to refer to [20].

■

lemma 4.3 *For any (φ, ψ) in $\mathcal{Q}^2$, the mapping*

$$s \to \inf_{\varphi \in \mathcal{Q}} \sup_{\psi \in \mathcal{Q}} L(s, \varphi \circ T_s^{-1}, \psi \circ T_s^{-1})$$

is weakly-l.s.c. at $(0, \varphi, \psi)$.

Shape Gradient of the Arrhenius Law

Definition 4.1 *Let* $(q,r)=(\frac{8}{5},\frac{31}{5})$, *then*

$$\|k-A(\varphi)\|^2_{W^{q,2}(\Omega)}=\|k-A(\varphi)\|^2_{W^{1,2}(\Omega)}+\int_\Omega\int_\Omega\frac{|(k-A(\varphi))(x-y)|^2}{|x-y|^r}dxdy$$

Proposition 4.4 *The mapping* $s\longrightarrow\|k\circ T_s^{-1}-A(\varphi\circ T_s^{-1})\|^2_{W^{q,2}(\Omega_s)}$ *is differentiable for all* (k,φ) *in* $\mathcal{A}\times\mathcal{Q}$ *in* 0. *Its derivative is given as follows:*

$$\begin{aligned}\partial_s\|k\circ T_s^{-1}-A(\varphi\circ T_s^{-1})\|^2_{W^{q,2}(\Omega_s)}|_{s=0}= & -r\int_\Omega\int_\Omega\frac{|(k-A(\varphi))(x-y)|^2}{|x-y|^{r+2}}<V_0(x)-V_0(y),x-y>dxdy\\ & +\int_\Omega\int_\Omega\frac{|(k-A(\varphi))(x-y)|^2}{|x-y|^r}divV_0dxdy\\ & +\int_\Omega[|k-A(\varphi)|^2+|\nabla(k-A(\varphi))|^2]divV_0\\ & -2\int_\Omega<\varepsilon(V_0)\nabla(k-A(\varphi)),\nabla(k-A(\varphi))>\end{aligned}\tag{14}$$

lemma 4.4 *For any* (k,φ) *in* $\mathcal{A}(\Omega)\times\mathcal{Q}(\Omega)$, *the mapping*

$$I\times\mathcal{A}(\Omega)\times\mathcal{Q}(\Omega)\to I\!R;\ \ (s,k,\varphi)\to\|k\circ T_s^{-1}-A(\varphi\circ T_s^{-1})\|^2_{W^{q,2}(\Omega_s)}$$

is weakly-l.s.c. at $(0,k,\varphi)$ *(see [1]).*

Differentiability of the Penalized Functional

Theorem 4.1 *Under assumption 3.1, the penalized functional* J^α *is shape differentiable in* $\mathcal{O}_k$ *with respect to* $\mathcal{V}_k$.

Proof of theorem 4.1

Indeed, the functional J^α satisfies the assumptions of theorem 3.1: for any initial domain Ω_0 in $\mathcal{O}_k$ and any V in $\mathcal{V}_k$,
i) For any s in I the functional $\mathcal{F}_{\Omega_s(V),\alpha}$ is weakly l.s.c. on $\mathcal{W}_{div}(\Omega)$.
ii) From propositions (4.1),(4.3), (4.4) and assumption3.1, there exists a neighborhood I' in I such that

$$\forall(v,k)\in\mathcal{W}_{div}\times\mathcal{A}(\Omega_0):\ [s\to\mathcal{F}_{\Omega_s(V),\alpha}((DT_s.v,k)\circ T_s^{-1}(V))]\in C^1(I')$$

iii) From lemmas (4.2), (4.3), (4.4) and assumption3.1,
the mapping $(s,v,k)\to\partial_s\mathcal{F}_{\Omega_s(V),\alpha}((DT_s.v,k)\circ T_s^{-1}(V))$ is weakly-l.s.c. on $I'\times\mathcal{W}_{div}(\Omega_0)\times\mathcal{A}$.
Accordingly, the shape functional J^α has an Eulerian derivative $dJ^\alpha(\Omega_0;V)$ at Ω_0 in direction V in $\mathcal{V}_k$: if we denote $S^\alpha=\{(v^\alpha,k^\alpha);\mathcal{F}_{\Omega_0,\alpha}(v^\alpha,k^\alpha)=J^\alpha(\Omega_0)\}$, then

$$dJ^\alpha(\Omega_0;V)=\min\{\partial_s\mathcal{F}_{\Omega_s(V),\alpha}((DT_s.v^\alpha,k^\alpha)\circ T_s^{-1})_{|s=0};\ \ (v^\alpha,k^\alpha)\in S^\alpha\}$$

with

$$\begin{aligned}
\partial_s \mathcal{F}_{\Omega_s(V),\alpha}((DT_s.u^\alpha, K^\alpha)\circ T_s^{-1})_{|s=0} = & \\
& \int_{\gamma_l} |u_{\Omega_0}.n|^p div V_0 \\
& -\tfrac{1}{\alpha}\int_{\Omega_0} K^\alpha |\varepsilon(u^\alpha)|^{p-2}\varepsilon(u^\alpha)..s(Du^\alpha DV_0 - D(DV_0.u^\alpha)) \\
& +\tfrac{1}{\alpha}\int_{\Omega_0} (\frac{K^\alpha}{p}|\varepsilon(u^\alpha)|^p - fu^\alpha) div V_0 - Df.V_0.u^\alpha - fDV_0.u^\alpha \\
& +\tfrac{1}{\alpha}\int_{\Omega_0} K_{\Omega_0}|\varepsilon(u_{\Omega_0})|^{p-2}\varepsilon(u_{\Omega_0})..s(Du_{\Omega_0}DV_0 - D(DV_0.u_{\Omega_0})) \\
& -\tfrac{1}{\alpha}\int_{\Omega_0} (\frac{K_{\Omega_0}}{p}|\varepsilon(u_{\Omega_0})|^p - fu_{\Omega_0}) div V_0 - Df.V_0.u_{\Omega_0} - fDV_0.u_{\Omega_0} \\
(15) \qquad & -2\int_{\Omega_0} \lambda(\varepsilon(V_0).\nabla\theta^\alpha, \nabla y^\alpha) - [\lambda\nabla\theta^\alpha\nabla y^\alpha - K^\alpha|\varepsilon(u^\alpha)|^p y^\alpha] div V_0 \\
& +\int_{\Omega_0} pK^\alpha|\varepsilon(u^\alpha)|^{p-2}\varepsilon(u^\alpha)..s(Du^\alpha DV_0 - D(DV_0.u^\alpha))y^\alpha \\
& -\tfrac{r}{\alpha}\int_{\Omega_0}\int_{\Omega_0} \frac{|(K^\alpha - A(\theta^\alpha))(x-y)|^2}{|x-y|^{r+2}} < V_0(x) - V_0(y), x-y > dxdy \\
& +\tfrac{1}{\alpha}\int_{\Omega_0}\int_{\Omega_0} \frac{|(K^\alpha - A(\theta^\alpha))(x-y)|^2}{|x-y|^r} div V_0 dxdy \\
& +\tfrac{1}{\alpha}\int_{\Omega_0} [|K^\alpha - A(\theta^\alpha)|^2 + |\nabla(K^\alpha - A(\theta^\alpha))|^2] div V_0 \\
& -\tfrac{2}{\alpha}\int_{\Omega_0} < \varepsilon(V_0)\nabla(K^\alpha - A(\theta^\alpha)), \nabla(K^\alpha - A(\theta^\alpha)) >
\end{aligned}$$

This Eulerian derivative is linear and continuous with respect to V_0, hence J^α is shape differentiable at Ω_0 with respect to $\mathcal{V}_k$: there exists a distribution ∇J^α such that for any domain Ω_0 and for any V in $\mathcal{V}_k$,

$$\forall t \in IR, \;\; dJ^\alpha(\Omega_t(V), V(t)) = < \nabla J^\alpha(\Omega_t(V)), \; V(t) >$$

■

Remark 4.2 *We are not able to get a boundary expression in the last shape gradient because of the less-regularity of Norton-Hoff state. But by the abstract result given by Hadamard, we already know that the hole of the shape gradient can be given as a boundary one.*

Remark 4.3 *We shall notice here that we are not able to compute explicitly the limit of the shape gradient when the parameter α goes to zero. In fact, it is in the same context of the non regularity of Norton-Hoff state, otherwise we are able to derive.*

4.2 Weak Eulerian Derivative of State Dependent Functional

We have the following weak differentiability result.

Theorem 4.2 *For any Ω_0 in $\mathcal{O}_k$ and V in $\mathcal{V}_k$, the mapping $s \longrightarrow J(\Omega_s(V))$ has a weak Eulerian derivative $\delta J(\Omega_0, V)$ in the dual space $\mathcal{H}'$ with*

$$\mathcal{H} = \{\phi \in H^1([0,\tau]); \; \phi(\tau) = 0\}$$

Proof of theorem 4.2

For any $0 < \alpha < 1$ and any $0 \leq s \leq \tau$, we have

$$J^\alpha(\Omega_s) - J^\alpha(\Omega_0) = \int_0^s dJ^\alpha(\Omega_t; V_t)dt$$

Hence for any ξ in $L^2([0,\tau])$,

$$\begin{aligned}\int_0^\tau \xi(s)[J^\alpha(\Omega_s) - J^\alpha(\Omega_0)]ds &= \int_0^\tau \xi(s)[\int_0^s dJ^\alpha(\Omega_t; V_t)dt]ds \\ &= \int_0^\tau dJ^\alpha(\Omega_t; V_t)[\int_t^\tau \xi(s)ds]dt\end{aligned} \tag{16}$$

Since the element of $H^1([0,\tau])$ are absolutely continuous,

$$\mathcal{H} = \{\psi(t) = \int_t^\tau \xi(s)ds; \xi \in L^2(0,\tau)\}$$

Hence for any ψ in $\mathcal{H}$,

$$-\int_0^\tau [J^\alpha(\Omega_s) - J^\alpha(\Omega_0)]\psi'(s)ds = \int_0^\tau dJ^\alpha(\Omega_s; V_s)\psi(s)ds$$

With proposition 2.7 and the shape continuity of J. Lebesgue's theorem provides

$$lim_{\alpha\to 0}\int_0^\tau \xi(s)[J^\alpha(\Omega_s) - J^\alpha(\Omega_0)]ds = \int_0^\tau \xi(s)[J(\Omega_s) - J(\Omega_0)]ds$$

Hence uniform boundedness principle provides the existence of a linear operator δJ in $\mathcal{H}'$ such that

$$\forall\psi \in \mathcal{H}, \int_0^\tau \delta J(s)\psi(s)ds = lim_{\alpha\to 0}\int_0^\tau dJ^\alpha(\Omega_s; V_s)\psi(s)ds$$

and we have

$$-\int_0^\tau [J(\Omega_s) - J(\Omega_0)]\psi'(s)ds = \int_0^\tau \delta J(s)\psi(s)ds$$

Thus the proof is achieved.

■

Conclusion

We shall notice here, on the one hand, that we are not able to get a boundary expression in the penalized shape gradient because of the less-regularity of Norton-Hoff state. But, by the abstract result we know that the whole of the penalized shape gradient can be given as a boundary one and we are not able to compute explicitly the limit of the provided shape gradient when the penalization parameter goes to zero. In fact, by using the so-called extractor method we prove some hidden regularity on the flow under a questionable density property. In that context we would be able to get the limit with respect the penalization parameter in order to recover the boundary expression for the functional shape derivative.

References

[1] Adams, Robert A.
[B] Sobolev spaces.
Pure and Applied Mathematics, Vol. 65. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers],
New York-London, 1975. xviii+268 pp. (Reviewer: A. Kufner).

[2] Aubin J.P.
Applied Functional Analysis: Pure and Applied Mathematic. (English)
[B] A Wiley-Interscience Series of Texts, Monographs and Tracts.
Library of Congress Cataloging in Puplication Data, (1979).

[3] Argyris J. H. and Doltsinis J. St.
On the natural fomulation and analysis of large deformation coupled thermomechanical problems, Computer methods in applied mechanics and engineering, vol.25,pp. 195-253(1981).

[4] Boisgérault B. and Zolésio J. P.
Shape derivative of sharp functionals governed by Navier-Stokes flow.(English)
Partial differential equations (Praha, 1998), 49–63, Chapman Hall/CRC Res. Notes Math., 406, Chapman Hall/CRC, Boca Raton, FL, 2000. (Reviewer: Dorin Bucur) .

[5] Brezis H.
Analyse fonctionnelle. Theorie et applications. (French)
[B] Collection Mathematiques Appliquees pour la Maitrise. Paris etc.:
Masson. XIV, 233 p. FF 125.00 (1983).

[6] Courant R. and Hilbert D.
Methods of Mathematical Physics. Itersciences, New York, 1955.

[7] Cuer M. and Zolésio J.P.
Control of Singular problem via differentiation of a min-max,
[J] Syst. Control Lett. 11, No.2, 151-158 (1988).

[8] Delfour M. C.; Zolésio J.-P.
Shapes and Geometries. Analysis, Differential Calculus, and Optimization.
Advances in Design and Control. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA. 2001.

[9] Ekeland E. and Temam R.
Analyse convexe et problemes variationnels. (French)
[B] Etudes mathematiques. Paris - Bruxelles - Montreal: Dunod; Gauthier-Villars. IX, 340 p. F 220.00 (1974).

[10] Ferchichi J. and Zolésio J.P.
Free Boundary Problem in the Norton-Hoff Static Case
Congres AIP; Applied Inverse Problem; Theoretical and Computational Aspects. 18-22 Juin 2001, Montichattini Terme, Italie.

[11] Ferchichi J. and Zolésio J.P.
Study of the Quasi-static Norton-Hoff Heat Problem
Rapport de Recherche INRIA, à apparaitre.

[12] Daignieres M. ; Fremon M. and Friaa A.
Modèle de type Norton-Hoff généralisé pour l'étude des déformation lithosphériques.
C.R. Acad. Sc. Paris, t. 186, Série B. 1978, p. 371.

[13] Friaâ Ahmed
La loi de Norton-Hoff Généralisée en Plasticité et Viscoplasticité
Thèse de Doctorat dÉtat, Paris VI, 1979.

[14] Friaâ Ahmed
Le Matériau de Norton-Hoff Généralisé et ses Applications en Analyse Limite.
(French) C. R. Acad. Sci. Paris Sér. A-B 286 (1978), no. 20, A953–A956. 73E20 (00A89) .

[15] Gomez N.
Shape Control in Visco-Plastic Flows
Thèse ENSMP, 2000.

[16] Kobayashi S. and Rebelo N.
A coupled analysis of viscoplastic deformation and heat transfer:I Theoretical consideration, II Applications. Int.J. of Mech.Sci.22,699-705, 707-718(1980).

[17] Micheletti A. M.
Metrica per famiglie di domini limitati e proprieta generich degli autovalori.
Annali della Scuola Normale Superiore di Pisa, 26 (fasc.3) 1972.

[18] Murat F. and Simon J.
Sur le contrôle par un domaine geometrique. Publication de l'Université de Pierre et Marie Curie (Paris 6), 76015, 1977.

[19] Norton F.H.
The creep of steel at hight temperature.
Mc Graw Hill, New York, U.S.A., 1929.

[20] Sokolowski J. and Zolésio J.P.
Introduction to shape optimization: shape sensitivity analysis. (English)
[B] Springer Series in Computational Mathematics. 16. Berlin etc.:
Springer-Verlag, (ISBN 3-540-54177-2). 250 p. (1992).

[21] Zolésio J. P.
Optimal Control of partial differential equations (Chemnits, 1998), pages 167-176, Basel, 1999. Birkhausr.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399