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***Pancyclic arcs and connectivity in tournaments***

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## Pancyclic arcs and connectivity in tournaments

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**Abstract:** A *tournament* is an orientation of the edges of a complete graph. An arc is *pancyclic* in a digraph  $D$  if it is contained in a cycle of length  $l$ , for every  $3 \leq l \leq |D|$ . In [4], Moon showed that every strong tournament contains at least three pancyclic arcs and characterized the tournaments with exactly three pancyclic arcs. All these tournaments are not 2-strong. In this paper, we are interested in the minimum number  $p_k(n)$  of pancyclic arcs in a  $k$ -strong tournament of order  $n$ . We conjecture that (for  $k \geq 2$ ) there exists a constant  $\alpha_k > 0$  such that  $p_k(n) \geq \alpha_k n$ . After proving that every 2-strong tournament has a hamiltonian cycle containing at least five pancyclic arcs, we deduce that for  $k \geq 2$ ,  $p_k(n) \geq 2k + 3$ . We then characterize the tournaments having exactly four pancyclic arcs and those having exactly five pancyclic arcs.

**Key-words:** pancyclic, tournament, strong connectivity, strongness

## Arcs pancycliques et connexité dans les tournois

**Résumé :** Un *tournoi* est une orientation des arêtes du graphe complet. Un arc est *pancyclique* dans un graphe orienté  $D$  s'il est contenu dans un cycle de longueur  $l$ , pour toute longueur  $3 \leq l \leq |D|$ . Dans [4], Moon a montré que tout tournoi fortement connexe contenait au moins trois arcs pancycliques et il a caractérisé les tournois ayant exactement trois arcs pancycliques. Tous ces tournois ne sont pas 2-fortement connexes. Dans ce rapport, nous étudions le nombre minimum  $p_k(n)$  d'arcs pancycliques dans un tournoi  $k$ -fortement connexe. Nous conjecturons que (pour  $k \geq 2$ ), il existe une constante  $\alpha_k$  telle que  $f_k(n) \geq \alpha_k n$ . Nous prouvons d'abord que tout tournoi 2-fortement connexe possède un circuit hamiltonien contenant au moins cinq arcs pancycliques. Nous en déduisons ensuite que pour  $k \geq 2$ ,  $p_k(n) \geq 2k + 3$ . Enfin, nous caractérisons les tournois ayant exactement quatre arcs pancycliques et ceux ayant exactement cinq arcs pancycliques.

**Mots-clés :** pancyclique, tournoi, forte connexité

## 1 Introduction

A *tournament* is an orientation of the arcs of a complete graph. Paths and cycles are always directed. A  $l$ -cycle is a cycle of length  $l$ .

An arc or a vertex is *pancyclic* in a digraph  $D$  if, for every  $3 \leq l \leq |D|$ , it is contained in an  $l$ -cycle.

A tournament is *strong* (or *strongly connected*) if for any two vertices  $x$  and  $y$  there exists a path beginning in  $x$  and terminating at  $y$ . A nonstrong tournament is said to be *reducible*. A tournament is  $k$ -strong, if  $T - Y$  is strong for any set  $Y$  of  $k - 1$  vertices. A tournament is  $(= k)$ -strong or *exactly  $k$ -strong*, if it is  $k$ -strong and not  $(k + 1)$ -strong.

To contain a pancyclic arc or vertex, a tournament must contain a hamiltonian cycle. Therefore, it must be strong according to the well known theorem of Camion [2]: *A tournament has a hamiltonian cycle if and only if it is strong*. Moon [3] gave an alternative proof of Camion's theorem by proving that every vertex of a strong tournament is pancyclic.

Analogously, one may wonder whether there are pancyclic arcs in tournament and how many. Moon [4] showed that every strong tournament contains at least three pancyclic arcs. Actually, he proved a somewhat stronger result : indeed, instead of considering the number  $p(T)$  of pancyclic arcs in the tournament  $T$ , he proved that  $h(T)$  the maximum number of pancyclic arcs contained in some hamiltonian cycle of  $T$  is at least 3.

**Theorem 1 (Moon, [4])** *Let  $T$  be a strong tournament with  $n \geq 3$  vertices.*

$$h(T) \geq 3$$

*with equality holding only if  $T \in \mathcal{P}_3$ .*

A tournament is in  $\mathcal{P}_3$  if there is a vertex  $v$  such that  $T - v$  is the transitive tournament  $TT[t_1, t_2, \dots, t_m]$  ( $(t_i, t_j)$  is an arc if and only if  $i < j$ ), and an integers  $1 < i_1 \leq m$  such that  $v \rightarrow t_j$  if and only if  $1 \leq j < i_1$ .

Let  $p_k(n)$  be minimum number  $p_k(n)$  of pancyclic arcs in a  $k$ -strong tournament of order  $n$  and let  $h_k(n) := \min\{h(T); T \text{ } k\text{-strong of order } n\}$ .

Because the tournaments of  $\mathcal{P}_3$  are  $(= 1)$ -strong, we have  $p_1(n) = h_1(n) = 3$  and if  $k \geq 2$ ,  $p_k(n) \geq h_k(n) \geq 4$ . However we consider that this lower bound 4 is not tight.

In this paper, we show Section 3 sufficient conditions for an arc to be pancyclic in a tournament. Using these conditions, we give an easy alternative proof of Theorem 1. Moreover our method allow us to go further. In Section 4, we prove that for  $k \geq 2$ ,  $h_k(n) \geq 5$  and then deduce that  $p_k(n) \geq 2k + 3$ . Finally, we characterize the tournaments with exactly four pancyclic arcs and those with exactly five pancyclic arcs.

However, our lower bound for  $h_k$  and  $p_k$  seems to be still far from the exact value. We conjecture that for  $k \geq 2$ ,  $p_k(n)$  tends linearly to infinity :

**Conjecture 1** For  $k \geq 2$ , there exists a constant  $\alpha_k > 0$  such that  $p_k(n) \geq \alpha_k n$ .

We cannot expect to have more pancyclic arcs since there are  $k$ -strong tournaments having less than  $2kn$  pancyclic arcs.

**Proposition 1**  $p_k(n) \leq 2kn - 2k^2 - k$

**Proof.** Let  $T_n$  be the  $k$ -strong tournament obtained from the rotative tournament on  $2k + 1$  vertices by blowing up a vertex with a transitive tournament  $TT$  of order  $n - 2k$ . Every arc in  $TT$  is not pancyclic in  $T$  since it is contained in no 3-cycle. Thus  $(n - 2k)(n - 2k - 1)/2$  arcs are not pancyclic. ■

**Proposition 2**

$$h_k(n) \leq 3k$$

**Proof.** If  $n \leq 3k$ , we have trivially the answer. Suppose now that  $n > 3k$ . Consider the  $k$ -strong tournament  $T_n$  obtained from two  $TT_k$ ,  $A$  and  $B$ , and one  $TT_{n-2k}$   $C$  such that  $A \rightarrow B \rightarrow C \rightarrow A$ . It is easy to see that every arc contained in one of the three subtournaments  $A$ ,  $B$  and  $C$  is not pancyclic because it is contained in no 3-cycle. It follows that  $h(T_n) \leq 3k$ . ■

The bound  $3k$  is not tight because of the 2-strong tournaments of order  $2k + 1 \leq n < 3k$ . However, we think that if  $n$  is large enough, the above example are extremal.

**Conjecture 2** For  $n$  sufficiently large,  $h_k(n) = 3k$ .

Alspach [1] showed that every arc of a regular tournament is pancyclic. This implies that  $h_k(2k + 1) = 2k + 1$ . In order to avoid the exception of small order in Conjecture 2, it would make sense to first try to prove that  $h_k(n) \geq 2k + 1$ .

## 2 Definition and preliminaries

Let  $T$  be a tournament. Let  $x$  and  $y$  be two vertices of  $T$ . We write  $x \rightarrow y$  if  $(x, y)$  is an arc of  $T$ . Likewise, let  $X$  and  $Y$  be two subdigraphs of  $T$ . We write  $X \rightarrow Y$  if  $x \rightarrow y$  for all pairs  $(x, y) \in V(X) \times V(Y)$ .

Let  $A_1, A_2, \dots, A_k$  be a family of subdigraphs of  $T$ . We denote by  $T[A_1, A_2, \dots, A_k]$  the subtournament induced by  $T$  on the set of vertices  $\bigcup_{1 \leq i \leq k} V(A_i)$  and by  $T - [A_1, A_2, \dots, A_k]$

the subtournament induced by  $T$  on the set of vertices  $V(T) \setminus \bigcup_{1 \leq i \leq k} V(A_i)$ .

$A(X, Y)$  denotes the set of arc  $(x, y)$  with  $x \in X$  and  $y \in Y$ .  $A^+(X)$  is the set of arcs outgoing from  $X$ , that is  $A(X, T - X)$  and  $A^-(X)$  is the set of arcs ingoing into  $X$ , that is  $A(T - X, X)$ .

The *dual* of a digraph  $D$  is the digraph  $-D$  on the same set of vertices such that  $x \rightarrow y$  is an arc of  $-D$  if and only if  $y \rightarrow x$  is an arc of  $D$ .

An  $(x, y)$ -path is a path that begins in  $x$  and terminates at  $y$ .

One can easily show the following result (whose proof is left to the reader) that allows to extend an  $(x, y)$  path to a longer one under condition.

**Proposition 3** *Let  $P = (v_1, v_2, \dots, v_m)$  be a path in a tournament  $T$  and  $x$  a vertex of  $T - P$ .*

*If there exist  $1 \leq i < j \leq m$  such that  $v_i \rightarrow x \rightarrow v_j$ , then in  $T$  there is a path with  $m + 1$  vertices starting in  $v_1$  and ending in  $v_m$ .*

A non-strong tournament  $T$  is said to be *reducible*. It admits a *reduction* into two subtournaments  $T_1$  and  $T_2$  such that  $V(T_1) \cup V(T_2) = V(T)$  and  $T_1 \rightarrow T_2$ ; in this case, we write  $T = T_1 \rightarrow T_2$ .

A (strong) *component* of  $T$  is a strong subtournament of  $T$  which is maximal by inclusion. Let  $T_1, T_2, \dots, T_m$  be the components of  $T$ . Then  $(V(T_1), V(T_2), \dots, V(T_m))$  is a partition of  $V(T)$  and without loss of generality, we may suppose that  $T_i \rightarrow T_j$  whenever  $i < j$ . In this case we say that  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  is the *decomposition* of  $T$ . The component  $T_1$  (resp.  $T_m$ ) is called the *outsection* (resp. *insection*) of  $T$ , denoted by  $out(T)$  (resp.  $In(T)$ ) and its vertices are called the *outgenerators* of  $T$  (resp. *ingenerators* of  $T$ ).

**Proposition 4** *In a tournament, a vertex is the beginning of a hamiltonian path if and only if it is an outgenerator.*

**Proof.** Let  $T$  be a tournament. Then  $T = Out(T) \rightarrow T - Out(T)$  (with  $T - Out(T)$  empty when  $T$  is strong).  $T[Out(T)]$  is strong, and thus by Camion's theorem admits a hamiltonian cycle  $C$ . Then every outgenerator  $v$  is the origin of a hamiltonian path  $Q$  of  $T[Out(T)]$ . And  $T - Out(T)$  has a hamiltonian path  $P$ . So  $(P, Q)$  is hamiltonian path of  $T$  with origin  $v$ . ■

**Proposition 5** *Let  $T$  be a reducible tournament of order  $n$ . Let  $u$  be an ingenerator and  $t$  an outgenerator of  $T$ . For any  $1 \leq l \leq n - 1$ , there is a  $(t, u)$ -path of length  $l$ .*

**Proof.** Let  $T_1 \rightarrow T_2$  be a reduction of  $T$  and for  $i = 1, 2$ , let  $n_i$  be the order of  $T_i$ . Clearly,  $t$  is an ingenerator of  $T_2$  and  $u$  an outgenerator of  $T_1$ . Therefore, by Proposition 4,  $t$  is the end of a hamiltonian path  $(t_{n_2-1}, t_{n_2-2}, \dots, t_1, t)$  of  $T_2$  and  $u$  is the origin of a hamiltonian path  $(u, u_1, \dots, u_{n_1-2}, u_{n_1-1})$  of  $T_1$ . Now, for any  $1 \leq l \leq n - 1$ , pick  $0 \leq l_1 \leq n_1 - 1$  and  $0 \leq l_2 \leq n_2 - 1$  such that  $l_1 + l_2 = l - 1$ . Then  $(u, u_1, \dots, u_{l_1}, t_{l_2}, t_{l_2-1}, \dots, t)$  is a path of length  $l$ . ■

A *reductor* of a tournament is a smallest subtournament  $X$  such that  $T - X$  is reducible. If  $T$  is  $(= k)$ -strong then a reductor has  $k$  vertices.

**Proposition 6** *Let  $X$  be a reductor of a tournament  $T$ . Every element of  $X$  dominates an outgenerator of  $T - X$  and is dominated by an ingenerator of  $T - X$ .*



**Proof.** Let  $x$  be an element of  $X$ . Let  $Y$  be the set of vertices that are not outgenerator of  $T - X$ . If  $Out(T - X) \rightarrow x$ , then  $T - [X \setminus x]$  is reducible with reduction  $Out(T - X) \rightarrow T[Y, x]$ . This contradicts that  $X$  is a reductor.

Analogously, we prove that  $x$  is dominated by an ingenerator of  $T - X$ . ■

**Proposition 7** *Let  $x$  and  $y$  be two vertices of a reductor  $X$  of a 2-tournament  $T$ . If  $in(T - X) \geq 3$ , there are two distinct vertices  $z_x$  and  $z_y$  of  $In(T - X)$  such that  $z_x \rightarrow x$  and  $z_y \rightarrow y$ .*

**Proof.** Suppose that two such vertices do not exist, then by Proposition 6, there is a vertex  $u \in In(T - X)$  such that  $u \rightarrow \{x, y\}$  and  $In(T - X) \setminus u \leftarrow \{x, y\}$ . Then  $X \setminus \{x, y\} \cup \{u\}$  is a reductor of  $T$ , which is a contradiction. ■

### 3 Lower bounds for $p(T)$ and $h(T)$

#### 3.1 Sufficient conditions for an arc to be pancyclic

**Lemma 1** *Let  $X$  be a subtournament such that  $T - X$  is reducible and every vertex of  $X$  dominates an outgenerator of  $T - X$ . Let  $v$  be an outgenerator of  $X$  and  $u$  an ingenerator of  $T - X$ . If  $u \rightarrow v$ , then the arc  $(u, v)$  is pancyclic.*

**Proof.** By Proposition 4,  $v = v_0$  is the origin of a hamiltonian path  $(v_0, v_1, \dots, v_{k-2}, v_{k-1})$  of  $T[X]$ . For  $0 \leq i \leq k - 1$ , let  $t_i$  be an outgenerator of  $T - X$  dominated by  $v_i$ . For  $3 \leq l \leq n$ , take  $0 \leq k' \leq k - 1$  and  $1 \leq l \leq n - k - 1$  such that  $k' + l + 2 = l$ . Then by Proposition 5, in  $T - X$ , there is a  $(t_{k'}, u)$  path  $Q$  of length  $l$ . Thus  $(v_0, v_1, \dots, v_{k'}, Q, v_0)$  is a cycle of length  $l$  going through  $(u, v) = (u, v_0)$ . ■

**Corollary 1** *Let  $X$  be the reductor of a tournament  $T$ . Let  $v$  be an outgenerator of  $X$  and  $u$  an ingenerator of  $T - X$ . If  $u \rightarrow v$ , then the arc  $(u, v)$  is pancyclic.*

**Corollary 2** *Let  $T$  be a strong tournament.*

$$p(T) \geq h(T) \geq 2$$

**Proof.** Let  $X$  be a reductor of  $T$ . Let  $P$  be a hamiltonian path of  $X$  with beginning  $v$  and end  $w$ . Clearly,  $v$  is an outgenerator of  $X$  and  $w$  is an ingenerator of  $X$ . Then by Corollary 1 (and its dual),  $(u, v)$  and  $(w, t)$  are pancyclic.

And by Proposition 5, there is a  $(t, u)$ -path  $Q$  that is hamiltonian in  $T - X$ , thus  $(P, Q, v)$  is a hamiltonian cycle containing  $(u, v)$  and  $(w, t)$ . ■

Let  $T$  be a strong tournament  $T$  with reductor  $X$ . Let  $X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_l$  be a decomposition of  $X$  and  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  be a decomposition of  $T - X$ .

In the remaining of this section, we examine the number of pancyclic arcs in the different parts of  $T$ .

**Lemma 2** For  $1 \leq i \leq l$ , if an arc is pancyclic in  $X_i$ , then it is also pancyclic in  $T$ .

**Proof.** Let  $e$  be a pancyclic arc in one of the  $X_i$ . Then  $e$  is contained in a 3-cycle. It is also contained in a hamiltonian cycle and then a hamiltonian path of  $X_i$ . This hamiltonian path may be extended to a hamiltonian path  $(v_0, v_1, \dots, v_{k-1})$  of  $X$  using hamiltonian paths of the  $X_{i'}$ , for  $i' \neq i$ . Let  $j$  be the index such that  $e = (v_j, v_{j+1})$ . Let  $4 \leq l \leq n$ . Choose  $0 \leq l_1 \leq j$ ,  $0 \leq l_2 \leq k - j - 2$  and  $1 \leq l_3 \leq n - k - 1$  such that  $l_1 + l_2 + l_3 + 3 = l$ . By Proposition 6, there is an ingenerator  $u$  of  $T - X$  dominating  $v_{j-l_1}$  and an outgenerator  $t$  of  $T - X$  dominated by  $v_{j+1+l_2}$ . Then by Proposition 5, in  $T - X$  there is a  $(t, u)$ -path of length  $l_3$ . Hence  $(v_{j-l_1}, v_{j-l_1+1}, \dots, v_{j+1+l_2}, P, v_{j-l_1})$  is an  $l$ -cycle containing  $e$ . ■

**Lemma 3** Suppose that  $T - X$  is the transitive tournament  $TT(t_1, t_2, \dots, t_m)$ .

For every  $1 \leq i \leq m - 1$ , the arc  $(t_i, t_{i+1})$  is pancyclic if and only if it is contained in a 3-cycle.

**Proof.** By Proposition 6,  $X \rightarrow t_1$  and  $t_m \rightarrow X$ . Let  $(v_0, v_1, \dots, v_{k-2}, v_{k-1})$  be a hamiltonian path of  $X$ .

Suppose first that  $i = 1$ . Let  $4 \leq l \leq n$ . Take  $0 \leq k' < k$ , and  $1 \leq l' < m - 2$  such that  $k' + l' + 3 = l$ . By Proposition 5, there is a  $(t_2, t_m)$ -path  $P$  of length  $l'$  in  $T[t_2, \dots, t_m]$ . Then  $(t_1, P, v_0, v_1, \dots, v_{k'}, t_1)$  is a cycle of length  $l$ . Thus if  $(t_1, t_2)$  is contained in a 3-cycle then it is pancyclic.

By duality, we have the result if  $i = m - 1$ .

Suppose now that  $1 < i < m - 1$ . Let  $v$  be the vertex such that  $(t_i, t_{i+1}, v, t_i)$  is a 3-cycle. Necessarily,  $v$  is in  $X$ . Then  $(t_i, t_{i+1}, v, t_1, t_i)$  is a 4-cycle. Let  $5 \leq l \leq n$ . The arc  $(t_i, t_{i+1})$  is contained in a cycle of length  $l$ . Indeed, take  $0 \leq k' < k$ ,  $0 < l_1 < i$  and  $0 \leq l_2 < m - i$  such that  $k' + l_1 + l_2 + 3 = l$ . By Proposition 5, there is a  $(t_1, t_i)$ -path  $P_1$  of length  $l_1$  in  $T[t_1, \dots, t_i]$ , and a  $(t_{i+1}, t_m)$ -path  $P_2$  of length  $l_2$  in  $T[t_{i+1}, \dots, t_m]$ . Then  $(t_i, P_2, v_0, v_1, \dots, v_{k'}, P_1)$  is an  $l$ -cycle containing  $(t_i, t_{i+1})$ . Thus  $(t_i, t_{i+1})$  is pancyclic. ■

**Lemma 4** For  $1 < i < m$ , if  $|T_i| \geq 4$ , then every  $T_i$ -pancyclic arc is pancyclic in  $T$ .

**Proof.** Let  $(v_0, v_1, \dots, v_{k-2}, v_{k-1})$  be a hamiltonian path of  $X$ .

Let  $(a_1, a_2)$  be a pancyclic arc in  $T_i$ . It is contained in a hamiltonian cycle of  $T_i$ ,  $C_i = (a_1, a_2, \dots, a_{n_i}, a_1)$ .

Let us prove that  $(a_1, a_2)$  is contained in a cycle of length  $l$  for all  $5 \leq l \leq n$ . Let  $S_1 = T[T_1, T_2, \dots, T_{i-1}, a_1]$  and  $S_2 = T - [X, S_1]$ . Clearly,  $a_1$  is an ingenerator of  $S_1$  and  $a_2$  an outgenerator of  $S_2$ . There exists three integers  $1 \leq l_1 \leq |S_1| - 1$ ,  $1 \leq l_2 \leq |S_2| - 1$  and  $0 \leq k' \leq k - 1$  such that  $l_1 + l_2 + k' = l - 3$ . By Proposition 6, there is an ingenerator  $u_0$  of  $T - X$  and then also of  $S_2$  which dominates  $v_0$  and an outgenerator  $t_{k'}$  of  $S_1$  which is dominated by  $v_{k'}$ . By Proposition 5, there is a  $(t_{k'}, a_1)$ -path  $P_1$  of length  $l_1$  in  $S_1$  and a  $(a_2, u_0)$ -path  $P_2$  of length  $l_2$  in  $S_2$ . Then  $(P_1, P_2, v_0, v_1, \dots, v_{k'}, t_{k'})$  is the desired  $l$ -cycle.

And because it is pancyclic in  $T_i$ ,  $(a_1, a_2)$  is contained in an  $l$ -cycle, for  $3 \leq l \leq n_i$ . ■

**Lemma 5** For any  $1 < i < m$ , if  $T_i$  is a 3-cycle, then two arcs of  $T_i$  are pancyclic in  $T$ .

**Proof.** Let  $(v_0, v_1, \dots, v_{k-2}, v_{k-1})$  be a hamiltonian path of  $X$ . The component  $T_i$  is the 3-cycle  $(a_1, a_2, a_3, a_1)$ .

In the same way as in the proof of Lemma 4 the arc  $(a_1, a_2)$ , is contained in cycle of length  $l$  for all  $5 \leq l \leq n$ . Analogously,  $(a_2, a_3)$  and  $(a_3, a_1)$  are contained in cycle of length  $l$  for all  $5 \leq l \leq n$ .

Hence, it suffice to prove that two arcs of  $T_i$  is contained in a 4-cycle. Without loss of generality, we may assume that  $\{a_1, a_2\} \leftarrow v_0$  or  $\{a_1, a_2\} \rightarrow v_0$ . If  $\{a_1, a_2\} \leftarrow v_0$ , then  $(a_1, a_2, u_0, v_0, a_1)$  and  $(a_2, a_3, u_0, v_0, a_2)$  are 4-cycles. If  $\{a_1, a_2\} \rightarrow v_0$ , then  $(a_3, a_1, v_0, t_0, a_3)$  and  $(a_1, a_2, v_0, t_0, a_1)$  are 4-cycles. ■

**Lemma 6** *Let  $C = (a_1, a_2, \dots, a_{n_1}, a_1)$  be a hamiltonian cycle of  $T_1$  such that  $a_1$  is dominated by an ingenerator of  $X$ .*

*If  $(a_j, a_{j+1})$  is pancyclic in  $T_1$  and  $1 \leq j \leq n_1 - 2$  then  $(a_j, a_{j+1})$  is pancyclic in  $T$ .*

**Proof.** By  $T_1$ -pancyclicity,  $(a_j, a_{j+1})$  is contained in a cycle of length  $l$  for  $3 \leq l \leq n_1$ .

Let now  $l$  be an integer of  $[n_1 + 1, n]$ .

Let  $v_{k-1}$  be an ingenerator of  $X$  dominating  $a_1$ . By Proposition 4, there exists a hamiltonian path  $(v_0, v_1, \dots, v_{k-2}, v_{k-1})$  of  $T[X]$ . By Proposition 6, for every  $0 \leq i \leq k-1$ , there is an ingenerator  $u_i$  of  $T - X$  dominating  $v_i$ .

Let  $1 \leq l_1 \leq n - k - j - 1$  and  $0 \leq k' \leq k - 1$  such that  $k' + l_1 = l - j - 2$ . Obviously,  $a_{j+1}$  is an outgenerator of  $T' = T - [X, a_1, a_2, \dots, a_j]$ . And  $u_{k-1-k'}$  is an ingenerator of  $T'$ . Thus, by Proposition 5 there is an  $(a_{j+1}, u_{k-1-k'})$  path  $P$  of length  $l_1$ . Hence  $(P, v_{k-1-k'}, v_{k-1-k'}, \dots, v_{k-1}, a_1, a_2, \dots, a_{j+1})$  is a cycle of length  $l$ . ■

**Lemma 7** *There are at least  $h(T_1) - 1$  pancyclic arcs in  $T_1$ .*

**Proof.** Let  $C = (a_1, a_2, \dots, a_{n_1}, a_1)$  be a hamiltonian cycle of  $T_1$  containing  $h(T_1)$   $T_1$ -pancyclic arcs. Without loss of generality, we may suppose that  $a_1$  is dominated by an ingenerator  $v$  of  $X$ .

Since  $j_1 \leq n_1 - 2$ , by Lemma 6, every pancyclic arc in  $P = (a_1, a_2, \dots, a_{n_1-1})$  is also pancyclic in  $T$ . If  $P$  contains  $h(T_1) - 1$   $T_1$ -pancyclic arcs, we have the result.

Hence we may assume that  $(a_{n_1-1}, a_{n_1})$  and  $(a_{n_1}, a_1)$  are pancyclic in  $T_1$ .

If  $v \rightarrow a_{n_1}$ , again by Lemma 6, the arc  $(a_{n_1}, a_1)$  is pancyclic in  $T$ .

If  $v \leftarrow a_{n_1}$  then let us prove that  $e_2 = (a_{n_1-1}, a_{n_1})$  is pancyclic in  $T$ . By pancyclicity in  $T_1$ ,  $e_2$  is contained in a cycle of length  $l$  for  $3 \leq l \leq n_1$ . And  $(a_1, a_2, \dots, a_{n_1}, v, a_1)$  is an  $(n_1 + 1)$ -cycle containing  $e_2$ .

Let now  $l$  be an integer of  $[n_1 + 2, n]$ .

By Proposition 4, there is a hamiltonian path such that  $(v_0, v_1, \dots, v_{k-2}, v_{k-1})$  be a hamiltonian path of  $X$ . By Proposition 6, for every  $0 \leq i \leq k-1$ , there is an ingenerator  $u_i$  of  $T - X$  dominating  $v_i$ .

Let  $1 \leq l_1 \leq n - k - n_1 - 1$  and  $0 \leq k' \leq k - 1$  such that  $k' + l_1 = l - n_1 - 1$ . Obviously,  $a_{n_1}$  is an outgenerator of  $T' = T - [X, Out(X) \setminus a_{n_1}]$ . And  $u_{k-1-k'}$  is an ingenerator of  $T'$ . Thus, by Proposition 5 there is a path  $P$  of length  $l_1$  with start  $a_{n_1}$  and end  $u_{k-1-k'}$ . Hence  $(P, v_{k-1-k'}, v_{k-1-k'}, \dots, v_{k-1}, a_1, a_2, \dots, a_{n_1})$  is a cycle of length  $l$ . ■

### 3.2 The lower bounds

Using the above lemmas, we derive lower bounds for  $p(T)$  and  $h(T)$ .

**Definition 1** Let  $k$  be an integer and  $T$  a tournament.  $\epsilon(k, T) = 1$  if  $|T| \geq k$  and 0 otherwise.

**Theorem 2** If  $T - X$  is transitive then

$$p(T) \geq |A(\text{In}(T - X); \text{Out}(X))| + |A(\text{In}(X); \text{Out}(T - X))| + \sum_{j=1}^l p(X_j) + 1 \quad (1)$$

Otherwise

$$\begin{aligned} p(T) \geq & |A(\text{In}(T - X); \text{Out}(X))| + |A(\text{In}(X); \text{Out}(T - X))| + \sum_{j=1}^l p(X_j) \\ & + \sum_{i=2}^{m-1} \{2\epsilon(3, T_i) + (p(T_i) - 2)\epsilon(4, T_i)\} \\ & + \epsilon(3, T_1)(h(T_1) - 1) + \epsilon(3, T_m)(h(T_m) - 1) \end{aligned} \quad (2)$$

**Proof.** Since every outgenerator of  $T - X$  is dominated by an ingenerator of  $T - X$ , according to Corollary 1, every arc of  $A(\text{In}(T - X); \text{Out}(X))$  is pancyclic. By duality, every arc of  $A(\text{In}(X); \text{Out}(T - X))$  is pancyclic. According to Lemma 2, there are at least  $\sum_{j=1}^l p(X_j)$   $T$ -pancyclic arcs in  $X$ .

Suppose now that  $T - X$  is a transitive tournament. Then since  $v_0 \rightarrow t_1$  and  $v_0 \rightarrow t_m$  then there is an index  $i$  such that  $(t_i, t_{i+1}, v_0, t_i)$  is a 3-cycle. So by Lemma 3, this arc is pancyclic. So we obtain Equation 1.

To obtain Equation 2, let us now count the number of pancyclic arcs contained in each  $T_i$  such that  $|T_i| \geq 3$ . If  $i = 1$  or  $i = m$ , by Lemma 7 (or its dual),  $h(T_i) - 1$  arc are  $T$ -pancyclic.

If  $1 < i < m$ , then, if  $T_i$  is a 3-cycle then by Lemma 5, 2 arcs of  $T_i$  are pancyclic in  $T$ ; and if  $|T_i| \geq 4$ , by Lemma 4, each  $T_i$ -pancyclic arc is pancyclic in  $T$ . ■

**Theorem 3** If  $T - X$  is transitive then

$$h(T) \geq 3 + \sum_{j=1}^l \epsilon(3, X_j) \cdot \min\{h(X_j); |X_j| - 1\} \quad (3)$$

otherwise

$$\begin{aligned} h(T) \geq & 2 + \sum_{j=1}^l \epsilon(3, X_j) \cdot \min\{h(X_j); |X_j| - 1\} + \sum_{i=2}^{m-1} \epsilon(3, T_i) \cdot \min\{h(T_i); |T_i| - 1\} \\ & + \epsilon(3, T_1)(h(T_1) - 1) + \epsilon(3, T_m)(h(T_m) - 1) \end{aligned} \quad (4)$$

**Proof.** For  $1 \leq j \leq l$ , let  $P_j$  be a hamiltonian path of  $X_j$  defined as follows :

- If  $X_j$  is reduced to a single vertex  $x_j$  then  $P_j = (x_j)$ .
- if  $|X_j| \leq 3$ , let  $P_j$  is obtained from a hamiltonian cycle of  $X_j$  containing  $h(X_j)$   $X_j$ -pancyclic arcs by removing a non  $X_j$ -pancyclic arc if  $h(X_j) < |X_j|$  or any arc if  $h(X_j) = |X_j|$ .

By Lemma 2, each  $P_j$  contains  $\epsilon(3, X_j) \cdot \min\{h(X_j); |X_j| - 1\}$   $T$ -pancyclic arcs. Let  $v$  be the beginning of  $P_1$  and  $w$  the end of  $P_l$ .

Suppose first that  $T - X$  is the transitive tournament  $TT(t_1, t_2, \dots, t_m)$ . By Lemma 3, there exists  $i$  such that  $(t_i, t_{i+1})$  is pancyclic and by Corollary 1,  $(w, t_1)$  and  $(t_m, v)$  are pancyclic. Thus, the hamiltonian cycle  $((t_1, t_2, \dots, t_m, P_1, P_2, \dots, P_l, t_1))$  contains  $3 + \sum_{j=1}^l \epsilon(3, X_j) \cdot \min\{h(X_j); |X_j| - 1\}$ .

Suppose now that  $T - X$  is not transitive.

For  $1 < i < m$ , let  $Q_i$  be a hamiltonian path of  $T_i$  defined as follows :

- If  $T_i$  is reduced to a single vertex  $t_i$  then  $Q_i = (t_i)$ .
- If  $|T_i| = 3$ , then  $Q_i$  is a path formed by two arcs of  $T_i$  that are pancyclic in  $T$ . (Such arcs exists according to Lemma 5.)
- If  $|T_i| \leq 4$ , let  $Q_i$  is obtained from a hamiltonian cycle of  $T_i$  containing  $h(T_i)$   $T_i$ -pancyclic arcs by removing a non  $T_i$ -pancyclic arc if  $h(T_i) < |T_i|$  or any arc if  $h(T_i) = |T_i|$ .

By Lemma 4, each  $Q_i$  contains  $\epsilon(3, T_i) \cdot \min\{h(T_i); |T_i| - 1\}$   $T$ -pancyclic arcs.

Let  $C_1 = (a_1, a_2, \dots, a_{n_1}, a_1)$  be a hamiltonian path of  $T_1$  containing  $h(T_1)$   $T_1$ -pancyclic arcs. Then set  $Q_1 := ((a_1, a_2, \dots, a_{n_1}))$ . Analogously define  $Q_m$ . By (the proof of) Lemma 6  $Q_1$  (resp.  $Q_m$ ) contains at least  $h(T_1) - 1$  (resp.  $h(T_m) - 1$ ) pancyclic arcs in  $T$ .

Then the hamiltonian cycle  $(w, Q_1, Q_2, \dots, t_m, P_1, P_2, \dots, P_l)$  gives Equation 4. ■

### 3.3 Tournaments with $h(T) = 3$

From the lower bounds, one can easily derive Theorem 1 :

**Proof of Theorem 1** Let us prove the result by induction on the order  $n$  of  $T$ . If  $n = 3$ , it is obviously true.

Suppose now that it is true for strong tournaments of order less than  $n$ . Let  $X$  be a reductor of  $T$  and  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  be a decomposition of  $T - X$ .

If  $T - X$  is transitive, Equation 3 yields the result and if it is not then Equation 4 gives  $h(T) \geq 4$ .

Suppose now  $T$  is a tournament such that  $h(T) = 3$ . Let  $X$  be a reductor of  $T$ . By Equation 4 then  $X$  and  $T - X$  are a transitive tournaments. Set  $X = TT[v_0, v_1, \dots, v_{k-1}]$  and  $T - X = TT[t_1, t_2, \dots, t_{n-k}]$ . By Corollary 1,  $(t_{n-k}, v_0)$  and  $(v_{k-1}, t_1)$  are pancyclic. Therefore there is at most one pancyclic arc on the hamiltonian path of  $T - X$ . Thus by

Lemma 3, there is an index  $i$  such that  $X \rightarrow t_j$  if and only if  $j \leq i$ . And  $(t_i, t_{i+1})$  is pancyclic.

Suppose that  $T$  is 2-strong. Then  $T - t_1$  and  $T - t_{n-k}$  are strong, so  $2 \leq i \leq n - k - 2$ . By Lemma 1,  $(t_{n-k-1}, v_0)$  is pancyclic in  $T - t_{n-k}$  and  $(t_{n-k}, v_1)$  is pancyclic in  $T - v_0$ . Thus two arcs are  $T$ -pancyclic because they are contained in the  $n$ -cycle  $C_3 = (t_{n-k-1}, v_0, v_{k-1}, t_2, t_3, \dots, t_{n-k-2}, t_{n-k}, v_1, v_2, \dots, v_{k-2}, t_1, t_{n-k-1})$  if  $k \geq 3$  or  $C_2 = (t_{n-3}, v_0, t_1, t_{n-2}, v_1, t_2, t_3, \dots, t_{n-3})$  if  $k = 2$ . Analogously,  $(v_{k-1}, t_2)$  and  $(v_{k-2}, t_1)$  are also  $T$ -pancyclic. Hence the cycle  $C_2$  or  $C_3$  contains four pancyclic arcs. This is a contradiction.

Thus  $T$  is  $(= 1)$ -strong. Then by Lemma 3 it is in  $\mathcal{P}_3$ . ■

Note that in the proof of Theorem 1, we also show the following :

**Proposition 8** *Let  $X$  be a reductor of a strong tournament. There is at least one  $T$ -pancyclic arc in  $T - X$ .*

## 4 Number of pancyclic arcs in $k$ -strong tournaments

**Theorem 4 (Yao, Guo and Zhang, [5])** *Every tournament contains a vertex  $x$  such that every outgoing arc is pancyclic.*

From this result, we derive lower bounds on the number of pancyclic arcs in a strong tournament :

**Lemma 8**

$$p(T) \geq h(T) + \delta^+ - 1 \tag{5}$$

$$p(T) \geq h(T) + \delta^+ + \delta^- - 3 \tag{6}$$

**Proof.** By Theorem 4 and its dual, there is a vertex  $x$  such that every outgoing arc is pancyclic and a vertex  $y$  such that every ingoing arc is pancyclic. Obviously,  $|A^+(x) \cap A^-(y)| \leq 1$ . There are  $h(T)$  pancyclic arcs on a hamiltonian cycle. At most one of them is in  $A^+(X)$  and at most two of them are in  $A^+(x) \cup A^-(y)$ . Hence,  $p(T) \geq h(T) + d^+(x) - 1 \geq h(T) + \delta^+ - 1$ , and

$$p(T) \geq d^+(x) + d^-(y) - |A^+(x) \cap A^-(y)| + h(T) - 2 \tag{7}$$

$$\geq d^+(x) + d^-(y) + h(T) - 3 \tag{8}$$

■

**Lemma 9** *Let  $X$  be the reductor of a 2-strong tournament. Suppose that  $X$  is the transitive tournament  $TT(v_0, v_1, \dots, v_{k-1})$ .*

*For every  $0 \leq j \leq k - 2$ , the arc  $(v_j, v_{j+1})$  is pancyclic if and only if it is contained in a 3-cycle.*

**Proof.** Let  $0 \leq j \leq k-2$  and  $4 \leq l \leq n$ . Let us prove that  $(v_j, v_{j+1})$  is contained in an  $l$ -cycle. There exist  $0 \leq l_1 \leq j$ ,  $0 \leq l_2 \leq k-j-2$  and  $1 \leq l_3 \leq n-k$  such that  $l_1 + l_2 + l_3 + 3 = l$ . By Proposition 6, there is an ingenerator  $u$  of  $T-X$  dominating  $v_{j-l_1}$  and an outgenerator  $t$  of  $T-X$  dominated by  $v_{j+1+l_2}$ . And by Proposition 5, in  $T-X$ , there is an  $(t, u)$ -path  $P$  of length  $l_3$ . Therefore,  $(P, v_{j-l_1}, v_{j-l_1+1}, \dots, v_{j+1+l_2}, t)$  is the desired  $l$ -cycle.  $\blacksquare$

**Theorem 5** For every a 2-strong tournament  $T$ ,  $h(T) \geq 5$ .

**Proof.** Let  $T$  be a 2-strong tournament such that  $h(T) = 4$ . Let  $X$  be a reductor of  $T$  and  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  be a decomposition of  $T-X$ . By Equations 3 and 4, we may assume that  $X$  is a transitive tournament, say  $TT(v_0, v_1, \dots, v_{k-1})$ , and that at most one of the  $T_i$  is not reduced to a single vertex  $t_i$ .

I) Suppose that for some  $2 \leq i \leq m-1$ ,  $T_i$  is not reduced to a single vertex. Then by Equation 4, we may assume that  $T_i$  is a 3-cycle  $(a, b, c, a)$ . By Lemma 10, for  $1 \leq j \leq i-1$ ,  $X \rightarrow t_j$  and for  $i+1 \leq j \leq m$ ,  $X \leftarrow t_j$ . Without loss of generality, we may assume that both  $a$  and  $b$  dominate a vertex in  $X$ . Then by Lemma 10,  $(t_{i-1}, a)$  and  $(t_{i-1}, b)$  are pancyclic and by (the proof of ) Lemma 5, the arcs  $(a, b)$  and  $(c, a)$  are pancyclic. If  $c$  dominates a vertex in  $X$  then  $(b, c)$  is pancyclic, and if  $c$  is dominated a vertex in  $X$  then  $(c, t_{i+1})$  is pancyclic. In any case, the hamiltonian cycle  $(t_1, t_2, \dots, t_{i-1}, a, b, c, t_{i+1}, \dots, t_m, v_0, v_1, \dots, v_{k-1}, t_1)$  contains five pancyclic arcs.

II) Suppose that  $|T_1| \geq 3$ .

By Equation 4, we may assume that  $h(T_1) = 3$ . So  $T_1 \in \mathcal{P}_3$  according to Theorem 1. Let  $w$  be the reductor of  $T_1$  such that  $T_1 - w$  is the transitive tournament.

If for some  $2 \leq j < m$ ,  $t_j$  is dominated by a vertex of  $X$ , then by Lemma 10 (dual), there is a pancyclic arc in the path  $(t_j, t_{j+1}, \dots, t_m)$ . Let  $P$  be a hamiltonian path of  $T_1$  beginning at an outneighbour of  $V_{k-1}$ . The hamiltonian cycle  $C = (v_{k-1}, P, t_2, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_m, v_0, v_1, \dots, v_{k-1})$  contains five pancyclic arcs. Hence we may assume that  $X \leftarrow t_j$ , for  $2 \leq j \leq m$ .

If there is a pancyclic arc in the path  $(v_0, v_1, \dots, v_{k-1})$ , the cycle  $C$  contains five pancyclic arcs. So, by Lemma 9, we may assume that  $N_{T_1}^+(v_{k-1}) \subseteq N_{T_1}^+(v_{k-2}) \subseteq \dots \subseteq N_{T_1}^+(v_0)$ . There are two distinct vertices  $r_1$  and  $r_2$  of  $T_1$  that are dominated by  $v_{k-1}$ . Let  $C_1$  be a hamiltonian cycle of  $T_1$  containing three  $T_1$ -pancyclics arcs. Let  $r_1^-$  (resp.  $r_2^+$ ) be the vertex dominating  $r_1$  (resp.  $r_2$ ) in  $C$ . Let  $P_1$  (resp.  $P_2$ ) be the subpaths of  $C_1$  with beginning  $r_1$  (resp.  $r_2$ ) and end  $r_2^-$  (resp.  $r_1^-$ ).

1) Suppose first that  $m \geq 3$ .

Let  $C'$  be the cycle defined as follows:

- If  $k \geq 3$ , then  $C' := (v_0, v_{k-1}, P_1, t_m, v_1, v_2, \dots, v_{k-2}, P_2, t_2, t_3, \dots, t_{m-1}, v_0)$ ;
- if  $k = 2$ , then  $C' := (v_0, P_2, t_m, v_1, P_1, t_2, t_2, t_3, \dots, t_{m-1}, v_0)$ .

Since  $C_1$  contains at least three  $T_1$ -pancyclic arcs, one of them, say  $e$  is contained in  $P_1$  or  $P_2$ . So, by Lemma 6,  $e$  is pancyclic in  $T$ . Moreover  $C'$  contains the four arcs  $(t_m, v_1)$ ,  $(t_{m-1}, v_0)$ ,  $(v_{k-2}, r_2)$  and  $(v_{k-1}, r_1)$ . By Lemma 1, these arcs are pancyclic in respectively in  $T - [v_0]$ ,  $T - [t_m]$ ,  $T - [v_{k-1}]$  and  $T$ . Hence because they are contained in  $C'$ , these four arcs are pancyclic. Thus  $C'$  contains five pancyclic arcs.

- 2) Suppose now that  $m = 2$ . Then  $v_0$  (and then every  $v_j$ ,  $0 \leq j \leq k-1$ ) has an inneighbour in  $T_1$  which dominates  $X$ . Without loss of generality we may suppose that this inneighbour  $s_2^-$  is between  $r_1$  and  $r_2$  in  $C_1$ , that  $s_2$  the successor of  $s_2^-$  along  $C_1$  dominates  $v_0$ . Let  $Q_1$  (resp.  $Q_2$ ) be the path beginning in  $r_1$  (resp.  $s_2$ ) and ending in  $s_2^-$  (resp.  $r_1^-$ ).

Set  $C_0 := (v_0, Q_2, t_2, v_1, v_2, \dots, v_{k-1}, Q_1, v_0)$ . It is easy to check that  $(v_0, s_2)$  and  $(t_2, v_1)$  and  $(v_{k-1}, r_1)$  are pancyclic in  $T$ . If  $P_1 \cup P_2$  contains two  $T_1$ -pancyclic arcs then these two are also pancyclic in  $T$  by Lemma 6. So  $C_0$  contains five pancyclic arcs.

Hence we may assume that both  $(r_1^-, r_1)$  and  $(s_2^-, s_2)$  are pancyclic. Let  $e$  be the third  $T_1$ -pancyclic arc. Let  $x$  be a vertex of  $Q_1$  such that  $v_0 \rightarrow x$  and its successor  $x^+$  in  $Q_1$  dominates  $v_0$ . Such a vertex exists since  $v_0 \rightarrow r_1$  and  $s_2^- \rightarrow v_0$ . The arc  $(x, x^+)$  of  $Q_1$  is pancyclic. Indeed ...

Thus if  $e = (x, x^+)$ , so  $e \in Q_1$ .

Three cases may arise:

- a) Suppose that  $w = s_2 = r_1^-$ . Then  $T_1 - [s_2^-, s_2] \rightarrow s_2^-$ , thus  $(s_2^-, v_0, r_1, s_2^-)$  is a 3-cycle. For  $4 \leq l \leq n_1 + 1$ ,  $(s_2^-, v_0)$  is contained in the  $l$ -cycle obtained by replacing the arc  $(s_2^-, s_2)$  by  $(s_2^-, v_0, s_2)$  in the  $l-1$ -cycle. Now since  $v_0 \rightarrow X - v_0$  and  $X \rightarrow r_1$ , by Proposition 3 (applied for every vertex of  $X - v_0$  one after another),  $(s_2^-, v_0)$  is contained in an  $l$ -cycle for  $n_1 + 2 \leq l \leq n - 1$ . And it is contained in  $C_0$ . Hence,  $(s_2^-, v_0)$  is pancyclic and  $C_0$  contains five pancyclic arcs.
- b) Suppose that  $w = r_1$ . Then  $r_1 \rightarrow T_1 - Q_2$  and  $Q_2 \rightarrow r_1$ . Then  $(s_2^-, v_0, r_1, s_2^-)$  is a 3-cycle. For  $4 \leq l \leq n_1 + 1$ , pick  $Q'_1$  a subpath of  $Q_1$  ending in  $s_2^-$  of length  $l_1 < |Q_1| - 2$  and  $Q'_2$  a subpath of  $Q_2$  beginning in  $s_2$  of length  $l_2 < |Q_2| - 2$  such that  $l_1 + l_2 + 4 = l$ , then  $(s_2^-, v_0, Q'_2, r_1, Q'_1)$  is an  $l$ -cycle. Analogously to the end of Case a), we obtain that  $(s_2^-, v_0)$  is pancyclic and  $C_0$  contains five pancyclic arcs.
- Suppose that  $w = s_2^- = x^+$ . Then  $r_1 \rightarrow w$ , so  $(s_2^-, v_0, r_1, s_2^-)$  is a 3-cycle. And  $T_1 - s_2^-$  is a transitive tournament with outgenerator  $s_2$  and ingenerator  $x$ . Thus for  $4 \leq l \leq n_1 + 1$ , in  $T_1 - s_2^-$ , there is an  $(s_2, x)$ -path  $P$  of length  $l - 2$ . Hence,  $(Q, s_2^-, v_0, s_2)$  is an  $l$ -cycle. Again, analogously to the end of Case a), we obtain that  $(s_2^-, v_0)$  is pancyclic and  $C_0$  contains five pancyclic arcs.

- III) Suppose now that  $T - X$  is a transitive tournament.



If  $v_0 \rightarrow t_{n-k-1}$ , then  $X' = T[t_{n-k}, v_1, v_2, \dots, v_{k-1}]$  is a reductor of  $T$  and  $T - X'$  is not a transitive. Indeed  $v_0$  is dominated by some vertex  $t_i$  and then  $(t_i, v_0, t_1, t_i)$  is a 3-cycle. So we have the result by one of the previous case.

Analogously, we obtain the result if  $v_{k-2} \leftarrow t_2$ . Thus we may assume that  $v_0 \leftarrow t_{n-k-1}$  and  $v_{k-2} \rightarrow t_2$ .

By Lemma 1,  $(t_{n-k-1}, v_0)$  is pancyclic in  $T - t_{n-k}$  and  $(t_{n-k}, v_1)$  is pancyclic in  $T - v_0$ . Thus two arcs are  $T$ -pancyclic because they are contained in the  $n$ -cycle  $C_3 = (t_{n-k-1}, v_0, v_{k-1}, t_2, t_3, \dots, t_{n-k-2}, t_{n-k}, v_1, t_{n-k-1})$  if  $k \geq 3$  or  $C_2 = (t_{n-3}, v_0, t_1, t_{n-2}, v_1, t_2, t_3, \dots, t_{n-3})$  if  $k = 2$ . Analogously,  $(v_{k-1}, t_2)$  and  $(v_{k-2}, t_1)$  are also  $T$ -pancyclic.

Suppose that  $k \geq 3$ , then the 4 pancyclic arcs  $(t_{n-k-1}, v_0)$ ,  $(t_{n-k}, v_1)$ ,  $(v_{k-1}, t_2)$  and  $(v_{k-2}, t_1)$  are contained in the two  $n$ -cycles  $C_3$  and  $C'_3 = (t_{n-k-1}, v_0, v_{k-1}, t_2, t_{n-k}, v_1, v_2, \dots, v_{k-2}, t_1, t_3, t_4, \dots, t_{n-k-1})$ . Therefore no arcs in  $\{(t_i, t_{i+1}), 2 \leq i \leq n-k-2\}$  is  $T$  pancyclic. By Lemma 3,  $t_i \rightarrow v_0$  for  $2 \leq i \leq n-k$  and  $(t_1, t_2)$  is pancyclic, and  $v_{k-1} \rightarrow t_i$  for  $1 \leq i \leq n-k-1$  and  $(t_{n-k-1}, t_{n-k})$  is pancyclic. Thus at least one of the arcs  $(v_i, v_{i+1})$  is in a 3-cycle is pancyclic by Lemma 9. Hence the cycle  $C = (v_0, v_1, \dots, v_{k-1}, t_1, t_2, \dots, t_{n-k}, v_0)$  contains five pancyclic arcs.

Thus we may assume that  $k = 2$ . We may assume that there is no pancyclic arcs in  $\{(t_i, t_{i+1}), 2 \leq i \leq n-4\}$ , otherwise  $C_2$  contains five pancyclic arcs. Then by Lemma 3,  $t_i \rightarrow v_0$  for  $2 \leq i \leq n-2$  and  $(t_1, t_2)$  is pancyclic, and  $v_1 \rightarrow t_i$  for  $1 \leq i \leq n-3$  and  $(t_{n-3}, t_2)$  is pancyclic. Hence  $(v_0, v_1)$  is in the 3-cycle  $(v_0, v_1, t_2, v_0)$ , so by Lemma 9, it is pancyclic. Hence the cycle  $C = (v_0, v_1, t_1, t_2, \dots, t_{n-2}, v_0)$  contains five pancyclic arcs. ■

This result is best possible since the regular tournament  $R_5$  on five vertices is 2-strong and obviously satisfies  $h(R_5) = 5$ .

It follows directly from Theorems 5 and Equation 6, that for  $k \geq 2$ , every  $k$ -strong tournament has at least  $2k + 2$  pancyclic arcs.

**Corollary 3** *Every  $k$ -strong tournament has at least  $2k + 2$  pancyclic arcs.*

We now prove a slightly better result.

**Theorem 6** *Every  $k$ -strong ( $k \geq 2$ ) tournament has at least  $2k + 3$  pancyclic arcs.*

**Proof.** Let  $x$  and  $y$  be vertices such that the arcs of  $A^+(x) \cup A^-(y)$  are pancyclic. By Equation 7, we have the result, if  $d^+(x) \geq k + 1$  or  $d^-(y) \geq k + 1$  or  $A^+(x) \cap A^-(y) = \emptyset$ . Thus we may assume that  $x \rightarrow y$ , and  $d^+(x) = d^-(y) = k$ .

Then  $X = N^+(x)$  is a reductor of  $T$  containing  $y$ . We have  $2k - 1$  pancyclic arcs in  $A^+(x) \cup A^-(y)$  and by Corollary 1, there is at least one pancyclic arc  $e_1$  in  $A(X, Out(T - X))$ .

If  $X$  is not transitive, then according to Lemma 2, there are at least three pancyclic arcs in  $X$ , with at most one of them in  $A^-(y)$ . And by Proposition 8, there is at most one pancyclic arc in  $T - X$ . It follows that  $p(t) \geq 2k + 3$ . Hence we may assume that  $X$  is the transitive tournament  $TT(v_0, v_1, \dots, v_{k-1})$ .

Let  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  be a decomposition of  $T - X$ . By Lemmas 4, 5 and 6, at most one of the  $T_i$  is not reduced to the vertex  $t_i$ .

Suppose that for some  $2 \leq i \leq m-1$ ,  $T_i$  is not reduced to a single vertex. If  $|T_i| \geq 4$  then by Lemma 4, there are at least three pancyclic arcs in  $T_i$ . Thus  $p(T) \geq 2k+3$ . Assume now that  $T_i$  is a 3-cycle. Two arcs of  $T_i$  are pancyclic by Lemma 5. Let  $t_i$  be a vertex of  $T_i$ . By Lemma 10, if  $t_i \rightarrow v_0$ , then  $(t_{i-1}, t_i)$  is pancyclic and if  $t_i \leftarrow v_0$ , then  $(t_i, t_{i+1})$  is pancyclic. Hence,  $p(T) \geq 2k+3$ .

Suppose now that  $T_1$  is not reduced to a single vertex. By Lemma 7, we have the result if  $h(T_1) \geq 4$ . So we may assume that  $T_1 \in \mathcal{P}_3$ . Also by Lemma 7, there are two pancyclic arcs in  $T_1$ . According to Proposition 7 (dual), there are two distinct vertices  $w_1$  and  $w_2$  of  $T_1$  such that  $v_{k-1} \rightarrow w_1$  and  $v_{k-2} \rightarrow w_2$ . Let  $C_1$  be a hamiltonian cycle of  $T_1$ .

- Suppose first that  $m \geq 3$ . If a vertex  $t_i$  with  $2 \leq i \leq m-1$  dominates an element of  $X$  then by Lemma 10 one of the arcs  $(t_j, t_{j+1})$  with  $i \leq j \leq m-1$  is pancyclic. Hence  $p(T) \geq 2k+3$ . So we may assume that  $T - [T_1, X] \rightarrow X$ . Let us prove that  $(v_{k-2}, w_2)$  is pancyclic : it is contained in the hamiltonian cycle  $(P_2, t_m, v_{k-1}, P_1, t_2, t_3, \dots, t_{m-1}, v_0, v_1, \dots, v_{k-2}, w_2)$ , where  $P_1$  (resp.  $P_2$ ) is the subpath along  $C_1$  of  $T_1$  starting in  $w_1$  (resp.  $w_2$ ) and ending in the predecessor of  $w_2$  (resp.  $w_1$ ). For  $3 \leq l \leq n-1$ , let  $0 \leq k' \leq k-2$ ,  $0 \leq m' \leq m-2$  and  $0 \leq l_1 \leq n_1-1$  such that  $l_1+k'+m'+3 = l$ . There is a path  $Q_1$  of length  $l_1$  starting in  $w_2$ . Thus  $(v_{k-2}, w_2)$  is contained in the  $l$ -cycle  $(v_{k-2}, Q_2, t_{m-m'}, t_{m-m'+1}, \dots, t_m, v_{k-2-k'}, v_{k-1-k'}, \dots, v_{k-2})$ . Hence  $T$  has at least  $2k+3$  pancyclic arcs.

- Suppose now that  $m = 2$ . Since  $d_X^-(v_0) \leq k-2$ ,  $v_0$  is dominated by at least one vertex in  $T_1$ . Hence there exists a vertex  $w_0$  of  $T_1$  dominated by  $v_0$  such that its predecessor  $w_0^-$  along  $C_1$  dominates  $v_0$ .

Because  $d_X^+(v_{k-1}) \leq k-2$ ,  $v_{k-1}$  dominates at least two vertices of  $T_1$ , so at least one, say  $w_1$ , distinct of  $w_0$ .

Let us now prove that  $(v_0, w_0)$  is pancyclic. For  $3 \leq l \leq n_1+1$ , let  $Q_0$  be a path in  $T_1$  starting at  $w_0$  of length  $l-3$ . Then  $(v_0, Q_0, t_2, v_0)$  is an  $l$ -path. Let  $Q_1$  (resp.  $Q_2$ ) be the subpath of  $C_1$  starting at  $w_1$  (resp.  $w_0$ ) and ending in  $w_0^-$  (resp. the predecessor of  $w_1$ ) along  $C_1$ . Then for  $n_1+2 \leq l \leq n$ ,  $(Q_1, v_0, Q_2, t_2, v_{n+1-l}, v_{n-l}, \dots, v_{k-1}, w_1)$  is an  $l$ -path containing  $(v_0, w_0)$ .

Hence  $T$  has at least  $2k+3$  pancyclic arcs.

Suppose now that  $T - X$  is the transitive tournament  $TT(t_1, t_2, \dots, t_{n-k})$ .

Let  $i_0$  be the smallest integer  $i > 1$  such that  $v_{k-1} \rightarrow t_i$ .

- Suppose that  $i_0 = 2$ . Since  $T$  is 2-strong,  $v_0$  is dominated by vertex  $t_{i_1}$  distinct from  $t_{n-k}$ . By Lemma 1,  $(v_{k-1}, t_2)$  and  $(v_{k-2}, t_1)$  are respectively  $(T - t_1)$ - and  $(T - v_{k-1})$ -pancyclic, thus they are contained in  $l$ -cycle for any  $3 \leq l \leq n-1$ . And they both are in the following hamiltonian cycle :  $(v_{k-1}, t_2, t_3, \dots, t_{i_1}, v_0, v_1, \dots, v_{k-2}, t_1, t_{i_1+1}, t_{i_1+2}, \dots, t_{n-k}, v_{k-1})$ . Thus they both are pancyclic in  $T$ . Moreover, by Proposition 8, there is a  $T$ -pancyclic arc in  $T - X$ . Hence,  $p(T) \geq 2k+3$ .

- Suppose now that  $i_0 > 2$ . Then  $X' = T[X - v_{-1}, t_1]$  is a reductor of  $X$ . And by Lemma 3,  $(t_1, t_2)$  is  $T$ -pancyclic.

The subtournament  $T' = T[t_{i_0}, t_{i_0+1}, \dots, t_{n-k}, v_{k-1}]$  is a strong component of  $T - X'$ . By Lemma 7, there are  $h(T') - 1$   $T'$ -pancyclic arcs in  $T'$ . And at most one of them is in  $A^+(x)$ . Thus, if  $h(T') \geq 4$ , we obtain  $p(T) \geq 2k + 3$ . So we may assume that  $h(T') = 3$ , so  $T' \in \mathcal{P}_3$ . Since  $T' - v_{k-1}$  is transitive,  $(t_{n-k}, v_{k-1})$  and  $(v_{k-1}, t_{i_0})$  are  $T'$ -pancyclic. Let  $e_2$  be the third  $T'$ -pancyclic arc. By Lemma 6,  $e_2$  is also  $T$ -pancyclic because  $t_{n-k} = x$  dominates  $v_0$ .

If  $v_0$  is dominated by a vertex of  $T' - [x, v_{k-1}]$  then by Lemma 6,  $(v_{k-1}, t_{i_0})$  is pancyclic in  $T$  and so  $p(T) \geq 2k + 3$ .

If  $v_0$  dominates  $T' - [x, v_{k-1}]$ , it must have an in neighbour  $t_{i_2}$  with  $2 \leq i_2 \leq i_0 - 1$ . Hence, by Proposition 10, there is a pancyclic arc  $(t_i, t_{i+1})$  with  $1 \leq i \leq i_0 - 2$ . So,  $p(T) \geq 2k + 3$ .

■

## 5 Tournaments with exactly four pancyclic arcs

We now prove a generalization of Lemma 3.

**Lemma 10** *Let  $T$  be a strong tournament,  $X$  a reductor of  $T$  and  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  be a decomposition of  $T - X$ .*

- i) For  $1 < i < m - 1$ , if there exist  $t_i \in T_i$  and  $t_{i+1} \in T_{i+1}$  such that  $(t_i, t_{i+1})$  is in a 3-cycle then  $(t_i, t_{i+1})$  is pancyclic.*
- ii) If  $m \geq 3$  and  $T_1 = \{t_1\}$  and there is a vertex  $t_2 \in T_2$  such that  $(t_1, t_2)$  is in a 3-cycle then  $(t_1, t_2)$  is pancyclic.*

**Proof.** Let  $(v_0, v_1, \dots, v_{k-1})$  be a hamiltonian path of  $X$  and for  $1 \leq i \leq m$ , set  $n_i = |T_i|$ .

i) Let  $v$  be the vertex of  $X$  such that  $(t_i, t_{i+1}, v, t_i)$  is a 3-cycle. Then  $v$  belongs to  $X$  and then by Proposition 6, dominates a vertex  $t \in T_1$ . Then  $(t_i, t_{i+1}, v, t, t_i)$  is a 4-cycle. Let  $t_1$  be an element of  $T_1$  that is dominated by  $v_{k-1}$  and  $t_m$  be an element of  $T_m$  dominating  $v_0$ . Let  $5 \leq l \leq n$ . The arc  $(t_i, t_{i+1})$  is contained in a cycle of length  $l$ . Indeed, take  $0 \leq k' < k$ ,  $0 < l_1 < \sum_{j=1}^i n_j$  and  $0 \leq l_2 < \sum_{j=i+1}^m n_j$  such that  $k' + l_1 + l_2 + 3 = l$ . By Proposition 5, there is a  $(t_1, t_i)$ -path  $P_1$  of length  $l_1$  in  $T[T_1, \dots, T_i]$ , and a  $(t_{i+1}, t_m)$ -path  $P_2$  of length  $l_2$  in  $T[T_{i+1}, \dots, T_m]$ . Then  $(t_i, P_2, v_0, v_1, \dots, v_{k'}, P_1)$  is an  $l$ -cycle containing  $(t_i, t_{i+1})$ . Hence  $(t_i, t_{i+1})$  is pancyclic.

ii) Let  $4 \leq l \leq n$ . Let us prove that  $(t_1, t_2)$  is contained in an  $l$ -cycle. By Proposition 6,  $v_{k-1} \rightarrow t_1$ . Let  $t_m$  be an element of  $T_m$  dominating  $v_0$ . Since  $m \geq 3$ ,  $T - [X, t_1]$  is reducible and  $t_2$  is one of its outgenerators and  $t_m$  one of its ingenerators. Therefore, by Proposition 5, there is a  $(t_1, t_m)$ -path  $P$  of length  $l - 3$  in  $T - [X, t_1]$ . Then  $(P, v_0, v_1, \dots, v_{k-1}, t_1, t_2)$  is the desired  $l$ -cycle. ■

**Definition 2** A tournament is in  $\mathcal{P}_4$  if there is a vertex  $v$  such that  $T - v$  is the transitive tournament  $TT[t_1, t_2, \dots, t_m]$  and three integers  $1 < i_1 < i_2 < i_3 \leq m$  such that  $v \rightarrow t_j$  if and only if  $1 \leq j < i_1$  or  $i_2 \leq j < i_3$ .

**Theorem 7** A tournament has exactly 4 pancyclic arcs if and only if it is in  $\mathcal{P}_4$ .

**Proof.** It is easy to check that every tournament of  $\mathcal{P}_4$  has exactly four pancyclic arcs.

Let  $T$  be a tournament with exactly 4 pancyclic arcs. By Corollary 3,  $T$  is (= 1)-strong. Let  $\{v\}$  be a reductor of  $X$  and  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  be a decomposition of  $T - v$ . By Equation 2, at most one of the  $T - i$ ,  $1 \leq i \leq m$  is not reduced to a single vertex.

Let us prove that there exists a decomposition such that each  $T_i$  is reduced to a single vertex.

1. If  $T - v$  is a transitive tournament, then by Lemma 3,  $T$  is in  $\mathcal{P}_4$ .
2. Suppose that there exist  $1 < i < m$ , such that  $T_m$  is not reduced to a vertex. By Corollary 1,  $(t_m, v)$  and  $(v, t_1)$  are pancyclic. By Equation 2, then  $|T_i| = 3$ . If  $T_i \rightarrow v$  or  $v \rightarrow T_i$  then the three arcs of  $T_i$  are pancyclic, then  $T$  has 5 pancyclic arcs. This is a contradiction. Then there is a vertex  $t_i \in T_i$  such that  $t_i \rightarrow v$ . Thus there is an arc  $(t_j, t_{j+1})$  with  $1 \leq j < i$  such that  $(v, t_j, t_{j+1}, v)$  is a 3-cycle. So by Lemma 10,  $(t_j, t_{j+1})$  is pancyclic. Again  $T$  has 5 pancyclic arcs which is a contradiction.
3. Suppose that  $T_m$  is not reduced to a vertex. By Equation 2,  $v$  has a unique inneighbour  $u$  in  $T_m$ . Set  $T_i = \{t_i\}$  for  $1 \leq i \leq m - 1$ . By Corollary 1,  $(u, v)$  and  $(v, t_1)$  are pancyclic and by Lemma 6 dual, there are two pancyclic arcs in  $T_m$ . Therefore, there is no other pancyclic arcs in  $T$ . Thus according to Lemma 10,  $v \rightarrow \{t_i, 1 \leq i \leq m - 1\}$ . Then  $u$  is a reductor. And there are 3 pancyclic arcs which are not in  $T_m - u$ . Then by Lemmas 4, 5 and 7,  $T - u$  is a transitive tournament. So we have the result by Case 1.

■

## 6 Tournaments with exactly five pancyclic arcs

**Definition 3** A tournament is in  $\mathcal{P}_5$  if there is a vertex  $v$  such that  $T - v$  is the transitive tournament  $TT[t_1, t_2, \dots, t_m]$  and three integers  $1 < i_1 < i_2 < i_3 < i_4 < i_5 \leq m$  such that  $v \rightarrow t_j$  if and only if  $1 \leq j < i_1$  or  $i_2 \leq j < i_3$  or  $i_4 \leq j < i_5$ .

The tournament  $Q(n)$  ( $n \geq 5$ ) is the tournament constructed from the transitive tournament  $TT_{n-4} = TT(t_1, t_2, \dots, t_{n-4})$  and four vertices  $a, b, c$  and  $v$  such that  $(a, b, c, a)$  is a 3-cycle,  $v \rightarrow TT_{n-4}$ ,  $TT_{n-4} \rightarrow \{a, b, c\}$ ,  $\{a, b\} \rightarrow v$  and  $c \leftarrow v$ .

**Proposition 9**

$$p(Q(5)) = 5$$

For  $n \geq 6$ ,  $p(Q(n)) = 6$ .

**Proof.**  $v$  is a reductor of  $T$ . By Corollary 1,  $(a, v)$ ,  $(b, v)$  and  $(v, t_1)$  are pancyclic. And By Lemma 6,  $(a, b)$  and  $(c, a)$  are pancyclic. Thus  $p(Q(n)) \geq 5$ .

Now an arc in  $TT_{n-4}$  is not pancyclic in  $T$  because it is contained in no 3-cycle ;  $(b, c)$  is not pancyclic because it is contained in no 4-cycle ; and  $(v, c)$  is not pancyclic because it is contained in no  $l$ -cycle for  $l \geq 5$ . For  $1 \leq i < n - 4$  the arc  $(t_i, x)$  with  $x \in \{a, b, c\}$  is not pancyclic because it is contained in no hamiltonian cycle. The arc  $(t_{n-4}, a)$  is not pancyclic since it is contained in no hamiltonian cycle and  $(t_{n-4}, c)$  is not pancyclic since it is contained in no 3-cycle

For  $3 \leq l \leq n - 2$ , the arc  $(t_{n-4}, b)$  is contained in the  $l$ -cycle  $(t_{n-4}, b, v, t_{n-1-l}, \dots, t_{n-4})$ . For  $5 \leq l \leq n$ , the arc  $(t_{n-4}, b)$  is contained in the  $l$ -cycle  $(t_{n-4}, b, c, a, v, t_{n+1-l}, \dots, t_{n-4})$ . Thus, if  $n \geq 6$ ,  $p(Q(n)) = 6$ . It is easy to check that if  $n = 5$ , the arc  $(t_1, b)$  is contained in no 4-cycle. Thus  $p(Q(5)) = 5$ .  $\blacksquare$

**Theorem 8** *A tournament has exactly 5 pancyclic arcs if and only if it is in  $\mathcal{P}_5$  or is  $Q(5)$ .*

**Proof.** It is easy to check that every tournament of  $\mathcal{P}_5 \cup \{Q(5)\}$  has exactly five pancyclic arcs.

Let  $T$  be a tournament with exactly 5 pancyclic arcs. By Corollary 3,  $T$  is  $(= 1)$ -strong. By Equation 2, at most one of the  $T_i$ ,  $1 \leq i \leq m$  is not reduced to a single vertex.

- If  $T - v$  is a transitive tournament. Then by Lemma 3,  $T$  is in  $\mathcal{P}_5$ .
- Suppose that there exist  $1 < i < m$ , such that  $T_m$  is not reduced to a vertex. By Corollary 1,  $(t_m, v)$  and  $(v, t_1)$  are pancyclic. By directionnal duality, we may suppose that there is a vertex  $t_i \in T_i$  such that  $t_i \rightarrow v$ . Then by Lemma 10, there is a pancyclic arc in  $T[t_1, t_2, \dots, t_i]$ . Thus, there are at most two pancyclic arc in  $T_i$ . Then by Lemma 4 and the proof of Lemma 5,  $T_i$  is a 3-cycle containing 2 pancyclic arcs and there is a vertex  $s_i \in T_i$  dominated by  $v$ . Then by Lemma 10, there is a pancyclic arc in  $T[s_i, t_{i+1}, \dots, t_m]$ . So  $p(T) \geq 6$  which is a contradiction.
- Suppose that  $T_m$  is not reduced to a vertex.

Then  $\delta^+ \geq 2$ . So by Equation 5,  $h(T) \leq 4$  thus by Theorem 1,  $T_m \in \mathcal{P}_3$ . Let  $w$  be a reductor of  $T_m$  such that  $T_m - w$  is the transitive tournament  $TT[a_1, \dots, a_p]$ . Let  $i$  be the index such that  $w \rightarrow a_i$  and  $w \leftarrow a_{i+1}$ .

$v \rightarrow T_m \setminus \{a_i, w, a_p\}$ , otherwise by Lemma 6,  $h(T) \geq 5$  which is a contradiction. Also  $v \rightarrow T - [v, T_m]$  otherwise by Lemma 10, there is a pancyclic arc in  $\{(t_j, t_{j+1}), 1 \leq j \leq m - 2\}$ .

By Equation 2,  $h_{T_m}^-(v) \leq 2$ .

If  $h_{T_m}^-(v) = 1$ , then let  $u$  be the inneighbour of  $v$  in  $T_m$ . The vertex  $u$  is a reductor. And  $out(T - u) < out(T - v)$ . Iterating the process, we find a reductor  $v'$  such that  $T - v'$  is transitive or  $h_{T_m}^-(v') = 2$ . So we may assume that  $h_{T_m}^-(v) = 2$ .

- Suppose that  $N_{T_m}^-(v) = \{a_i, w\}$ . If  $i \neq 1$ , then by Lemma 6 dual, the three arcs  $(w, a_1)$ ,  $(a_p, w)$  and  $(a_i, a_{i+1})$  are pancyclic. So  $p(T) \geq 6$  which is a contradiction. Thus  $i = 1$ . It is easy to see that  $(a_1, a_2)$  is contained in every cycle of length  $l$ , for  $5 \leq l \leq n$ . It follows that  $|T_m| = 3$ . Then  $T = Q(n)$  and by Proposition 9,  $T = Q(5)$ .
- If  $N_{T_m}^-(v) = \{w, a_p\}$ , then by Lemma 6 dual,  $(a_p, w)$  and  $(a_i, a_{i+1})$  are pancyclic. It is easy to see that  $(w, a_1)$  is contained in every cycle of length  $l$ , for  $5 \leq l \leq n$ . And because  $(w, a_1)$  is pancyclic in  $T_m$ , it follows that  $|T_m| = 3$ . Thus  $T = Q(n)$  and by Proposition 9,  $T = Q(5)$ .
- Suppose that  $N_{T_m}^-(v) = \{a_i, a_p\}$ . If  $i \neq p - 1$ , then by Lemma 6 dual, the three arcs  $(w, a_1)$ ,  $(a_p, w)$  and  $(a_i, a_{i+1})$  are pancyclic. So  $p(T) \geq 6$  which is a contradiction. Thus  $i = p - 1$ . It is easy to see that  $(a_p, w)$  is contained in every cycle of length  $l$ , for  $5 \leq l \leq n$ . And because  $(a_p, w)$  is pancyclic in  $T_m$ , it follows that  $|T_m| = 3$ . So  $T = Q(n)$  and by Proposition 9,  $T = Q(5)$ .

■

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