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On the Approximation of the Area of a Surface

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On the Approximation of the Area of a Surface

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Abstract: We compare the area of a smooth surface S with the area of a surface M which is *close to it*, in terms of the local structure of S and the distance between S and M . We obtain convergence results, in particular when the surface approaching S is the restricted Delaunay complex associated to an ϵ -sample \mathcal{S} of S .

Key-words: Surface, area, curvature, Hausdorff distance, Delaunay triangulation, sample.

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Sur l'approximation de l'aire d'une surface

Résumé : Nous comparons l'aire d'une surface lisse S et l'aire d'une surface M qui lui est proche, au moyen de la structure locale de S et de la distance entre S et M . Nous obtenons des résultats de convergence, en particulier lorsque la surface qui approche S est le complexe de Delaunay restreint associé à un ϵ -échantillonnage \mathcal{S} de S .

Mots-clés : Surface, aire, courbure, distance de Hausdorff, triangulation de Delaunay, échantillonnage.

1 Introduction

Consider a smooth (compact orientable) surface S of the Euclidean space \mathbb{E}^3 , and a subset M , "close to S " smooth almost everywhere. When does the area $\mathcal{A}(M)$ of M approximate the area $\mathcal{A}(S)$ of S ?

Before dealing with this problem, we must remark the following:

- Instead of surfaces, consider a curve C in \mathbb{E}^3 and a sequence of polygons P_n whose vertices lie in C , and whose initial and final points are the initial and final points of C . If the length of the edges of the polygons tends to zero, then *by definition of the length*, the length of C is the limit of the length of P_n .
- The situation is completely different for surfaces: the well known "*Lampion de Schwarz*" shows that the area of a sequence of triangulations inscribed in a fixed cylinder can tend to infinity. In order to describe succinctly this phenomena, let us build a 2-parameter family of *generalized Lampions de Schwarz*. Let C be a cylinder of finite height H and radius R in \mathbb{E}^3 . Let $P(n, N)$ denote the triangulation inscribed in C defined as follows: consider N circles on the cylinder C obtained by intersecting C by 2-planes orthogonal to the axis of C . Inscribe on each circle a regular n -gon such that the n -gon on the slice k is deduced from the n -gon of the slice $k - 1$ by a rotation of angle $\frac{2\pi}{n}$. Then join each vertex v of the slice $k - 1$ to the two vertices of the slice k which are on each side of v . We obtain a triangulation whose vertices $v_{i,j}$ are defined as follows:

$$\begin{aligned} \forall i \in \{0, \dots, n-1\}, \quad v_{i,j} &= (R \cos(i\alpha), R \sin(i\alpha), jh) \text{ if } j \text{ is even,} \\ \forall j \in \{0, \dots, N\}, \quad v_{i,j} &= (R \cos(i\alpha + \frac{\alpha}{2}), R \sin(i\alpha + \frac{\alpha}{2}), jh) \text{ if } j \text{ is odd,} \end{aligned}$$

and whose faces are:

$$\begin{aligned} v_{i,j} \ v_{i+1,j} \ v_{i,j+1}, \\ v_{i,j} \ v_{i-1,j+1} \ v_{i,j+1}, \end{aligned}$$

where $\alpha = \frac{\pi}{n}$ and $h = \frac{H}{N}$.

Then when n tends to infinity, the area $\mathcal{A}(P(n, n^3))$ of $P(n, n^3)$ tends to infinity.

J. Fu [6] obtained a result of convergence of the area of a sequence of triangulations which converges to a smooth surface. In this short note, we obtain a general result: we give an upper bound and a lower bound of the area of S in terms of the area of M and geometric quantities related to S , as soon as M is "close enough" to S , in a sense which will be precised in a moment. (In the case where M is a triangulation, we do not need every angle of M to be quite large -compare with [9]-) . We give an application of this result in the theory of surface reconstruction from scattered sample points: if S is a ϵ -sample of a (closed) surface, (in the sense of [1], [3]), then the limit of the area $\mathcal{A}(T_\epsilon)$ of the Delaunay triangulation T_ϵ satisfies:

$$\lim_{\epsilon \rightarrow 0} \mathcal{A}(T_\epsilon) = \mathcal{A}(S).$$

2 The General Framework

2.1 Smooth surfaces

In the following, by a *smooth surface* S we mean a compact (oriented) regular \mathcal{C}^2 surface with or without boundary ∂S , embedded in \mathbb{E}^3 . S is endowed with the Riemannian structure induced by the standard scalar product of \mathbb{E}^3 . We denote by da_S the area form on S and by ds the canonical orientation of ∂S . Let ν be the unitary normal vector field on S (compatible with the orientation of S). Since S is \mathcal{C}^2 , we can define the second fundamental form h of S in the direction ν , and the Weingarten endomorphism A_p (which is the adjoint of h_p). At each point p of S , we denote by λ_p^1, λ_p^2 the principal curvatures of S at p , that is, the eigenvalues of h_p . The determinant of h_p is the *Gauss curvature* G_p of S at p , its trace is the *mean curvature* H_p of S at p . The *maximal curvature* of S at p is $\rho_p = \max(|\lambda_p^1|, |\lambda_p^2|)$. The function ρ is continuous function on S . For more details on smooth surfaces, one may refer to [4] or [2].

To compare the area of two surfaces close from one another (in a sense that we shall precise later), we need to introduce the projection function ξ : let m be any point of \mathbb{E}^3 , and consider the subset of points of S whose distance to m is minimum. If S is compact, this subset is not empty. If it is reduced to a single point, we call it $\xi(m)$, and then define the projection function ξ .

The existence of ξ and its regularity have been extensively studied. The following result (see [5]) ensures that, when S is smooth, ξ is well defined in a small neighborhood of S .

Proposition 1 *Let S be a smooth compact surface of \mathbb{E}^3 . Then there exists an open set U_S of \mathbb{E}^3 containing S where ξ is well defined. Moreover ξ is continuous in U_S .*

The open set U_S depends both locally and globally on the smooth surface S . Globally, U_S depends on points which can be far from one another on the surface, but which are close in \mathbb{E}^3 . Locally, the normals of S do not intersect in U_S .

For the following, we need the

Definition 1 *Let S be a smooth surface of \mathbb{E}^3 .*

- *The function $\omega_S : U_S \rightarrow \mathbb{E}$ defined by*

$$\omega_S(m) = \|\xi(m) - m\| \rho_{\xi(m)}$$

is called the relative curvature function of S on U_S .

- *The real number*

$$\omega_S(M) = \sup_{m \in M} \omega_S(m),$$

is called the relative curvature of M with respect to S .

We shall also need the notion of *reach of a surface*, introduced in [5].

Definition 2 The reach r of a surface S is the largest $r > 0$ for which ξ is defined on the (open) tubular neighborhood $U_r(S)$ of radius r of S .

Remark in particular that if M lies in $U_r(S)$, then $\omega_S(M) \leq 1$, (see [8] for instance). (The notion of reach can be related to the notion of *local feature size -LFS-*, (see [1] and [3]), but we don't need this relation in our context. Finally, we give the following

Definition 3 A subset M of \mathbb{E}^3 is closely near S if it lies in $U_r(S)$, where r is the reach of S and if the restriction of ξ to M is one-to-one.

If M is closely near S and differentiable almost everywhere, one can define at almost every point m of M the angle $\alpha_m \in [0, \frac{\pi}{2}]$ between the tangent planes $T_m M$ and $T_{\xi(m)} S$. We put

$$\alpha_{\max} = \sup_{m \in M} \alpha_m,$$

$$\alpha_{\min} = \inf_{m \in M} \alpha_m.$$

3 The Approximation of the Area

The following result shows that if M is closely near S and has enough regularity to have a tangent plane almost everywhere, then the area of S is bounded from above and from below by quantities depending on **the area of M , the Hausdorff distance between M and S , the curvature of S , and the angle between the corresponding tangent planes.**

Theorem 1 Let S be a (compact orientable) C^2 surface in \mathbb{E}^3 . Let M be a surface which is differentiable almost everywhere, and closely near S . Then the area satisfies:

$$\mathcal{A}(S) = \int_M \frac{\cos \alpha_m}{1 + \delta_m \epsilon_m H_{\xi(m)} + \delta_m^2 G_{\xi(m)}} da_M(m),$$

where $\epsilon_m = \langle \nu_{\xi(m)}, \frac{\xi(m) - m}{\|\xi(m) - m\|} \rangle \in \{-1, +1\}$ and $\delta_m = \|\xi(m) - m\|$.

As an obvious consequence, one has the following

Corollary 1 Let M be a surface closely near S . Then,

$$\frac{\cos \alpha_{\max}}{(1 + \omega_S(M))^2} \mathcal{A}(M) \leq \mathcal{A}(S) \leq \frac{\cos \alpha_{\min}}{(1 - \omega_S(M))^2} \mathcal{A}(M).$$

The Hausdorff distance δ_{Haus} between two subsets A and B of \mathbb{E}^3 is defined by

$$\delta_{Haus}(A, B) = \max(\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)).$$

We deduce from the previous corollary the following

Corollary 2 *Let S be a (compact orientable) C^2 surface in \mathbb{E}^3 . Let M_n be a sequence of surfaces closely near S which are differentiable almost everywhere. If*

1. *the Hausdorff limit of M_n is S , when n goes to infinity,*
2. *the angle between tangent planes $T_m M_n$ and $T_{\xi(m)} S$ tends to 0 almost everywhere, when n goes to infinity,*

then,

$$\lim_{n \rightarrow \infty} \mathcal{A}(M_n) = \mathcal{A}(S).$$

We present now two examples showing the relevance of the introduced geometric quantities.

1. First of all, consider two spheres S and M with same centers and different radius 1 and R . The angle between the normals of S and M is 0, but the relative curvature of M with respect to S is equal to $1 - R$. If R tends to 0 then $\omega_S(M)$ tends to 1 and the area of M to 0.

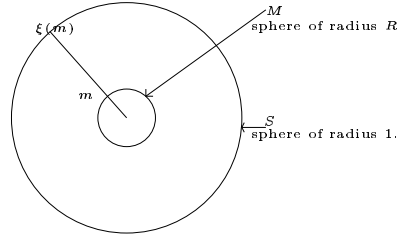


Figure 1: $\cos \alpha_{max} = 1$ and $\omega_S(M)$ is large

2. Consider now a flat surface S , and a sequence $(M_n)_{n \geq 1}$ of triangulations such that $\omega_S(M_n)$ is equal to 0 and α_{max} is equal to $\frac{\pi}{4}$. The area of M_n is equal to $\sqrt{2}$ times the area of S .

4 An application in Surface Reconstruction

Let S be a finite set of points inscribed in a (compact) surface S . We remind the following

Definition 4 *The set T of Delaunay simplices of S whose dual intersects S is called the restricted Delaunay complex of S with respect to S .*

If the sample has suitable properties, then T has interesting topological and geometrical properties. We will use the concept of ϵ -sample, introduced in [1]:

Definition 5 A set of points S on S is an ϵ -sample of S if and only if for every point m of S , the ball $B(m, \epsilon LFS(m))$ encloses at least one point of S .

Let us summarize the main properties of the restricted Delaunay complex associated to an ϵ -sample, [1].

Proposition 2 Let S be a compact surface of \mathbb{E}^3 . Let T_{S_ϵ} be the restricted Delaunay complex of an ϵ -sample S_ϵ with respect to S .

1. If $\epsilon < 0.1$, then T_{S_ϵ} is homeomorphic to S .
2. The maximum angle α_{\max} between the corresponding tangent planes of T_{S_ϵ} and S (via ξ) satisfies:

$$\alpha_{\max} = O(\epsilon).$$

If ϵ is small enough, it is easy to check that the map ξ is well defined and one-to-one. As an immediate consequence of Proposition (2) and Theorem (1), we get the following

Corollary 3 Let S be a compact surface of \mathbb{E}^3 . Let T_{S_ϵ} be the restricted Delaunay complex of an ϵ -sample S_ϵ with respect to S . Then

$$\lim_{\epsilon \rightarrow 0} \mathcal{A}(T_{S_\epsilon}) = \mathcal{A}(S).$$

5 Proof of Theorem 1

The following proposition gives the main differential properties of the function ξ when it is defined. It has been studied in [7] in the case where S is the boundary of a convex set of \mathbb{E}^3 and in [9].

Proposition 3 Let S be a smooth surface of \mathbb{E}^3 without boundary, U_S an open subset of \mathbb{E}^3 where the map $\xi : U_S \rightarrow S$ is well defined. Then,

1. the map ξ is C^1 in U_S and satisfies for every $m \in U_S$:

$$D\xi(m)(Z_m) = 0, \forall Z_m \text{ orthogonal to } T_{\xi(m)}S,$$

$$D\xi(m)(X_m) = (Id + \delta_m \epsilon_m A_{\xi(m)})^{-1}(X_m), \forall X_m \text{ parallel to } T_{\xi(m)}S,$$

where $\epsilon_m = \langle \nu_{\xi(m)}, \frac{\xi(m)-m}{\|\xi(m)-m\|} \rangle \in \{-1, +1\}$ and $A_{\xi(m)} = -D\nu(\xi(m))$ is the Weingarten endomorphism of S at the point $\xi(m)$.

2. In particular, the matrix of $D\xi(m) : \mathbb{E}^3 \rightarrow T_{\xi(m)}S$ (in local orthonormal frames $(e_{\xi(m)}^1, e_{\xi(m)}^2, \nu_{\xi(m)}^S)$ and $(e_{\xi(m)}^1, e_{\xi(m)}^2)$) is given by:

$$\begin{pmatrix} \frac{1}{1 + \delta_m \epsilon_m \lambda_{\xi(m)}^1} & 0 & 0 \\ 0 & \frac{1}{1 + \delta_m \epsilon_m \lambda_{\xi(m)}^2} & 0 \end{pmatrix}.$$

where $e_{\xi(m)}^1$ and $e_{\xi(m)}^2$ are unitary principal vectors and $\nu_{\xi(m)}^S$ is the oriented normal of S at $\xi(m)$.

Sketch of proof of Proposition 3

- For every $m \in U_S$, the point $\xi(m)$ is defined by the following relation:

$$\forall m \in U_S, \forall X \in T_{\xi(m)}S, \langle \xi(m) - m, X_{\xi(m)} \rangle = 0.$$

Consequently, for every $m \in T_m S$, the function ξ is constant on the orthogonal of $T_m S$.

- Consider now a vector $X_m \in T_m U_S$ which is parallel to $T_{\xi(m)}S$. We have:

$$D\xi(m)(X) = X + \delta_m \epsilon_m D\nu(\xi(m)) \circ D\xi(m)(X).$$

The endomorphism $(I + \delta_m \epsilon_m A_{\xi(m)})$ is clearly invertible, and consequently,

$$D\xi(m)(X) = (I + \delta_m \epsilon_m A_{\xi(m)})^{-1}(X).$$

- The rest of the proof is obvious.

Now consider the surface M lying in U_S . By assumption, the restriction $\xi|_M$ of ξ to M is one-to-one, and then is a diffeomorphism. We shall apply the classical area formula:

$$\mathcal{A}(S) = \int_S da_S = \int_M \xi_{|M}^* da_S = \int_M |D\xi_{|M}(m)| da_S(\xi(m)),$$

which is pertinent since we are in the case where ξ is a diffeomorphism. The following lemma evaluates the Jacobian of $\xi|_M$.

Lemma 1 *Let S be a smooth surface of \mathbb{E}^3 without boundary, U_S an open subset of \mathbb{E}^3 where the map $\xi : U_S \rightarrow S$ is well defined. Let $M \subset U_S$ be a smooth surface. Then the Jacobian of the differential of $\xi|_M$ is given by:*

$$|D\xi_{|M}(m)| = \frac{\cos \alpha_m}{(1 + \delta_m \epsilon_m \lambda_{\xi(m)}^1)(1 + \delta_m \epsilon_m \lambda_{\xi(m)}^2)},$$

where $\epsilon_m = \langle \nu_{\xi(m)}, \frac{\xi(m) - m}{\|\xi(m) - m\|} \rangle \in \{-1, +1\}$, and $\delta_m = \|\xi(m) - m\|$.

This lemma is obvious, by considering the restriction $\xi|_M$ of ξ to M . Remark that the condition of the theorem on the angle between corresponding tangent planes implies that $\xi|_M$ is a diffeomorphism between M and S .

End of the proof of Theorem 1 Since M is closely inscribed in S , $\xi|_M$ is one-to-one between M and S . Therefore:

$$\mathcal{A}(S) = \int_M |D\xi_{|M}(m)| da_M(m) = \int_M \frac{\cos \alpha_m}{(1 + \delta_m \epsilon_m \lambda_{\xi(m)}^1)(1 + \delta_m \epsilon_m \lambda_{\xi(m)}^2)} da_M(m).$$

6 Conclusion

In this paper, we obtain an approximation of the area of a surface when this surface is "closed" to another surface whose area is known, using geometric quantities. This is a first step in the study of the approximation of geometric invariants on a surface (the curvatures for instance) with the corresponding invariants of a surface closed to it. In particular, if we deal with smooth surfaces approximated by discrete ones, it would be interesting to have a unified geometric theory including both discrete and smooth objects.

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