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# *New formulas around Laplace transforms of quadratic forms for general Gaussian sequences*

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# New formulas around Laplace transforms of quadratic forms for general Gaussian sequences

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**Abstract:** Various methods to derive new formulas for the Laplace transforms of some quadratic forms of Gaussian sequences are discussed. In the general setting, an approach based on the resolution of an appropriate auxiliary filtering problem is developed; it leads to a formula in terms of the solutions of Voterra type recursions describing characteristics of the corresponding optimal filter. In the case of Gauss-Markov sequences, where the previous equations reduce to ordinary forward recursive equations, an alternative approach provides another formula; it involves the solution of a backward recursive equation. Comparing the different formulas for the Laplace transforms, various relationships between the corresponding entries are identified. In particular relationships between the solutions of matched forward and backward Riccati equations are thus proved probabilistically; they are proved again directly. In various specific cases, a further analysis of the concerned equations leads to completely explicit formulas for the Laplace transform.

**Key-words:** Gaussian sequences. Quadratic forms. Laplace transform. Martingale. Optimal filtering. Filtering error.

*(Résumé : tsvp)*

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# Nouvelles formules autour des transformées de Laplace de formes quadratiques de suites gaussiennes générales

**Résumé :** Diverses méthodes pour obtenir de nouvelles formules pour les transformées de Laplace de certaines formes quadratiques de suites gaussiennes sont discutées. Dans le cas général, une approche fondée sur la résolution d'un problème de filtrage auxiliaire approprié est développée; elle conduit à une formule en termes des solutions de récurrences de type Volterra qui décrivent les caractéristiques du filtre optimal correspondant. Dans le cas des suites gaussiennes markoviennes, où les récurrences précédentes se réduisent à des récurrences progressives, une approche alternative fournit une autre formule; elle met en jeu la solution d'une récurrence rétrograde. La comparaison des différentes formules pour la transformée de Laplace fait apparaître des relations entre les solutions des récurrences progressive et rétrograde correspondantes. En particulier des relations entre les solutions d'équations de Riccati progressive et rétrograde appariées sont ainsi démontrées de manière probabiliste; elles sont redémontrées de manière directe. Dans différents cas particuliers, une analyse détaillée des équations concernées conduit à des formules complètement explicites pour la transformée de Laplace.

**Mots-clé :** Suites gaussiennes. Formes quadratiques. Transformée de Laplace. Martingale. Filtrage optimal. Erreur de filtrage.

# 1 Introduction

Quadratic functionals of Gaussian processes have been given a great deal of interest over the last decades. Numerous results have been already reported both in the general setting of abstract Gaussian spaces and in various specific models. Concerning continous-time processes, specially around the Brownian motion, Laplace transforms of such functionals have been extensively investigated further to the pioneer paper [1] of Cameron-Martin (see, *e.g.*, [2] - [6], [8] - [9] and references therein). Here we concentrate on Laplace transforms of quadratic forms (Ltqf for short) of Gaussian sequences.

In what follows all random variables, vectors and sequences are defined on a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. Let us start with the well-known fundamental formula which tells that when $X$ is a $n$-dimensional Gaussian vector with mean $\mu$ and covariance matrix $\Lambda$, then for any $n \times n$ non negative symmetric matrix $R$:

$$\mathbb{E}\exp\{-\frac{1}{2}X'RX\} = \{\det[I_n + R\Lambda]\}^{-\frac{1}{2}} \exp\{-\frac{1}{2}\mu'[I_n + R\Lambda]^{-1}R\mu\}\,, \tag{1}$$

where $I_n$ stands for the $n \times n$ identity matrix. Now let $(X_t,\ t = 0, 1, \ldots)$ be an arbitrary one-dimensional Gaussian sequence with mean function $(m_t,\ t = 0, 1, \ldots)$ and covariance function $(K(t,s),\ t, s = 0, 1, \ldots)$, *i.e.*,

$$\mathbb{E}X_t = m_t\,; \quad \mathbb{E}(X_t - m_t)(X_s - m_s) = K(t,s)\,,$$

and let $(Q(t),\ t = 0, 1, \ldots)$ be any fixed (deterministic) sequence of nonnegative real numbers. Then of course, from the formula (1), we get immediately that for all $t \geq 0$

$$\mathbb{E}\exp\{-\frac{1}{2}\sum_{s=0}^{t} Q(s)X_s^2\} = \{\det[I_{t+1} + \mathcal{Q}_t\mathcal{K}_t]\}^{-\frac{1}{2}} \exp\{-\frac{1}{2}\underline{\mathrm{m}}_t'[I_{t+1} + \mathcal{Q}_t\mathcal{K}_t]^{-1}\mathcal{Q}_t\underline{\mathrm{m}}_t\}\,, \tag{2}$$

where $\mathcal{Q}_t$ stands for the $(t+1)$-dimensional diagonal matrix with $Q(s),\ s = 0, 1, \ldots, t$ as diagonal entries, $\mathcal{K}_t$ denotes the $(t+1) \times (t+1)$ matrix $((K(r,s), r, s = 0, 1, \ldots, t))$ and $\underline{\mathrm{m}}_t$ is the $(t+1)$-dimensional vector with components $m_s,\ s = 0, 1, \ldots, t$.

Here we investigate alternative forms of the expression (2) for the Laplace transform and various methods to derive new formulas are discussed. The paper is organized as follows. At first, in Section 2, an approach which applies to arbitrary Gaussian sequences is developed; it is based on the matching of an appropriate auxiliary filtering problem and it leads to a formula in terms of the solutions of Voterra type recursions describing characteristics of the corresponding optimal filter. Then, in Section 3, the case of Gauss-Markov sequences, where the previous equations reduce to ordinary forward recursive equations, is considered; an alternative approach provides another formula which involves the solution of a backward recursive equation. Comparing the different formulas for the Laplace transforms, various relationships between the corresponding entries are identified. In particular relationships between the solutions of matched forward and backward Riccati equations are thus proved probabilistically; they are viewed within the scope of the usual mathematical duality between optimal control and optimal filtering. Section 4 is devoted to various specific cases where a further analysis of the concerned equations leads to completely explicit formulas for the Laplace transform. Finally, the auxiliary results, which are themselves of independent interest, are investigated in Appendices A and B: the filtering problem introduced in Section 2 is solved and identites connected with the Riccati equations are proved again directly.

## 2 Ltqf of arbitrary Gaussian sequences – A filtering approach

Here we continue with the general setting introduced in Section 1. From now on we use the following notation for the Ltqf corresponding to the Gaussian sequence $(X_t,\ t=0,1,\ldots)$ and the given deterministic sequence $(Q(t),\ t=0,1,\ldots)$ :

$$\mathcal{L}(t) = \mathbb{E}\exp\{-\frac{1}{2}\sum_{s=0}^{t} Q(s)X_s^2\}\,.$$

We state our main result:

**Theorem 1** *For any $t \geq 0$ the following equality holds:*

$$\mathcal{L}(t) = \prod_{s=0}^{t}[1+Q(s)\gamma(s,s)]^{-1/2}\exp\left\{-\frac{1}{2}\sum_{s=0}^{t}\frac{Q(s)z_s^2}{1+Q(s)\gamma(s,s)}\right\}, \tag{3}$$

*where $(\gamma(t,s),\ 0\leq s\leq t)$ is the unique solution of the equation*

$$\gamma(t,s) = K(t,s) - \sum_{r=0}^{s-1}\frac{Q(r)\gamma(s,r)}{1+Q(r)\gamma(r,r)}\gamma(t,r)\,,\ 1\leq s\leq t\,;\quad \gamma(t,0)=K(t,0)\,, \tag{4}$$

*and $(z_s,\ 0\leq s\leq t)$ is the unique solution of the equation*

$$z_s = m_s - \sum_{r=0}^{s-1}\frac{Q(r)\gamma(s,r)}{1+Q(r)\gamma(r,r)}z_r\,,\ 1\leq s\leq t\,;\quad z_0 = m_0\,. \tag{5}$$

**Remark 1** *Observe that if $m_s=0,\ s=0,1,\ldots$ then $z_s=0,\ s=0,1,\ldots$ and hence the formula (3) reduces to:*

$$\mathcal{L}(t) = \prod_{s=0}^{t}[1+Q(s)\gamma(s,s)]^{-1/2}.$$

*Since here the formula (2) tells nothing else but*

$$\mathcal{L}(t) = \{\det[I_{t+1}+\mathcal{Q}_t\mathcal{K}_t]\}^{-1/2}\,,$$

*of course we get the identity*

$$\det[I_{t+1} + \mathcal{Q}_t \mathcal{K}_t] = \prod_{s=0}^{t} [1 + Q(s)\gamma(s,s)] \,. \tag{6}$$

*Consequently, continuing the comparison of formulas (2) and (3) for a non centered sequence, it gives also the identity*

$$\underline{m}_t^{'}[I_{t+1} + \mathcal{Q}_t \mathcal{K}_t]^{-1} \mathcal{Q}_t \underline{m}_t = \sum_{s=0}^{t} \frac{Q(s) z_s^2}{1 + Q(s)\gamma(s,s)} \,. \tag{7}$$

*It is worth to emphasize that identities (6) and (7) tell that one may compute the determinant and quadratic form appearing in the left-hand sides (and hence also the Laplace transform) by means of running the procedures (4) and (5). These comments will be complemented in the subsection 3.3 for the particular case of Markov sequences and also in Remark 5 at the end of the Appendix A.*

The key point in the proof of Theorem 1 is the link between the computation of the Laplace transform and the resolution of an appropriate filtering type problem. Recall that if $(U,Y) = ((U_t, Y_t),\ t = 0,1,\ldots)$ is a pair of processes, supposing that only $Y$ is observed but one wishes to know $U_t$, the classical problem of filtering (resp. one-step prediction of) the signal $U$ at time $t$ from the observation of $Y$ up to time $t$ (resp. $t-1$) occurs. The solution to this problem is the conditional distribution of $U_t$ given the $\sigma$-field $\mathcal{Y}_t = \sigma(\{Y_s\,,\ 0 \le s \le t\})$ (resp. $\mathcal{Y}_{t-1}$) which is called the optimal filter (resp. the optimal one-step predictor). Of course, if the joint distribution of $(U,Y)$ is Gaussian, then the optimal filter and predictor are Gaussian distributions. Hence the resolution of the filtering and prediction problems can be reduced to the derivation of equations for first and second order conditional moments. In the sequel, for any random sequence $U = (U_t\,;\, t \ge 0)$ such that $\mathbb{E}|U_t| < +\infty$, for all $t \ge 0$ and $0 \le s \le t$ the notation $\pi_s(U_t)$ is used for the conditional expectation of $U_t$ given $\mathcal{Y}_s$ :

$$\pi_s(U_t) = \mathbb{E}(U_t / \mathcal{Y}_s) \,.$$

Moreover we make the convention that $\pi_{-1}(U_t) = \mathbb{E}U_t$.

Now we introduce the problems appropriate for computing the Ltqf. Let $(\varepsilon_t,\ t = 0,1,\ldots)$ be a sequence of i.i.d. standard Gaussian random variables which is independent of the given process $(X_t,\ t = 0,1,\ldots)$. Let us define the auxiliary sequences $(Y_t,\ t = 0,1,\ldots)$ and $(\xi_t,\ t = 0,1,\ldots)$ by

$$\begin{aligned} Y_t &= Q(t) X_t + \sqrt{Q(t)}\,\varepsilon_t \,, \\ \xi_t &= \sum_{s=0}^{t} X_s Y_s \,. \end{aligned} \tag{8}$$

We shall be concerned with one-step prediction for $X$ from $Y$ and with filtering $\xi$ from $Y$. Here, clearly the pair $(X,Y)$ is jointly Gaussian, and hence the optimal one-step predictor is the Gaussian distribution defined by the conditional mean $\pi_{t-1}(X_t)$ and the conditional variance $\gamma_{XX}(t) = \mathbb{E}[(X_t - \pi_{t-1}(X_t))^2/\mathcal{Y}_{t-1}]$ which actually is deterministic *i.e.*,

$$\gamma_{XX}(t) = \mathbb{E}[X_t - \pi_{t-1}(X_t)]^2\,,\ t \geq 1\,;\quad \gamma_{XX}(0) = K(0,0)\,. \tag{9}$$

Of course, the joint distribution of $(X, \xi, Y)$ is not Gaussian, but we observe that the conditional distribution of $(X_t, \xi_{t-1})$ given $\mathcal{Y}_{t-1}$ is Gaussian. Hence, in particular, the optimal filter for $\xi$ is the Gaussian distribution defined by the conditional mean $\pi_t(\xi_t)$ and the corresponding conditional covariance (which is random). Actually the other main characteristics which is involved in the sequel is the following conditional covariance :

$$\gamma_{X\xi}(t) = \mathbb{E}[(X_t - \pi_{t-1}(X_t))(\xi_{t-1} - \pi_{t-1}(\xi_{t-1}))/\mathcal{Y}_{t-1}]\,,\ t \geq 1\,;\quad \gamma_{X\xi}(0) = 0\,. \tag{10}$$

Equations governing the first and second order conditional moments involved in definitions (9)-(10) will be derived in Appendix A.

Now we can state the announced key property :

**Lemma 1** *For any $t = 0, 1, \ldots$ the following equality holds:*

$$\mathcal{L}(t) = \prod_{s=0}^{t}[1 + Q(s)\gamma_{XX}(s)]^{-1/2}\exp\{-\frac{1}{2}\sum_{s=0}^{t}\frac{Q(s)[\pi_{s-1}(X_s) - \gamma_{X\xi}(s)]^2}{1 + Q(s)\gamma_{XX}(s)}\}\,. \tag{11}$$

Before turning to the proof of this lemma it is worth to mention that the equality (11) tells in particular that the quantity $[\pi_{s-1}(X_s) - \gamma_{X\xi}(s)]^2$ is deterministic. Actually, it will appear that the difference $\pi_{s-1}(X_s) - \gamma_{X\xi}(s)$ is itself deterministic. Comparing equations (3) and (11), it is clear that, starting from Lemma 1, to prove Theorem 1 it will be sufficient to check that the quantities $\gamma_{XX}(s)$ and $\pi_{s-1}(X_s) - \gamma_{X\xi}(s)$ are nothing but $\gamma(s, s)$ and $z_s$ where $\gamma(t, s)$ is the unique solution of equation (4) and $z_s$ is the unique solution of equation (5). This will be done in Appendix A and now we prove Lemma 1.

**Proof of Lemma 1** Setting

$$I_{t-1} = \frac{1}{2}\sum_{s=0}^{t-1} Q(s)X_s^2\,,$$

we can write

$$\frac{\mathcal{L}(t)}{\mathcal{L}(t-1)} = \frac{\mathbb{E}(\exp\{-I_{t-1} - \frac{1}{2}Q(t)X_t^2\})}{\mathbb{E}(\exp\{-I_{t-1}\})}\,. \tag{12}$$

Let us define a new probability measure $\tilde{\mathbb{P}}$ by

$$d\tilde{\mathbb{P}} = \exp\{-\zeta_{t-1}\}.d\mathbb{P}\,;\quad \zeta_{t-1} = \sum_{s=0}^{t-1}\sqrt{Q(s)}X_s\varepsilon_s + \frac{1}{2}\sum_{s=0}^{t-1}Q(s)X_s^2\,. \tag{13}$$

Under $\tilde{\mathbb{P}}$ the distribution of $X$ is the same as under $\mathbb{P}$ and $X$ is independent of $(Y_s, 0 \leq s \leq t-1)$. Hence we can rewrite the equality (12) as

$$\frac{\mathcal{L}(t)}{\mathcal{L}(t-1)} = \frac{\tilde{\mathbb{E}}(\exp\left\{-I_{t-1} - \frac{1}{2}Q(t)X_t^2\right\}/\mathcal{Y}_{t-1})}{\tilde{\mathbb{E}}(\exp\{-I_{t-1}\}/\mathcal{Y}_{t-1})},$$

where $\tilde{\mathbb{E}}(./\mathcal{Y}_{t-1})$ denotes a conditional expectation computed with respect to $\tilde{\mathbb{P}}$. Then, using the classical Bayes formula, again we can rewrite (12) as

$$\frac{\mathcal{L}(t)}{\mathcal{L}(t-1)} = \frac{\mathbb{E}(\exp\left\{-I_{t-1} - \frac{1}{2}Q(t)X_t^2\right\}\exp\{-\zeta_{t-1}\}/\mathcal{Y}_{t-1})}{\mathbb{E}(\exp\{-I_{t-1}\}\exp\{-\zeta_{t-1}\}/\mathcal{Y}_{t-1})}.$$

Since from the definitions (8) and (13) we have $\xi_{t-1} = I_{t-1} + \zeta_{t-1}$, this means that

$$\frac{\mathcal{L}(t)}{\mathcal{L}(t-1)} = \frac{\mathbb{E}(\exp\left\{-\xi_{t-1} - \frac{1}{2}Q(t)X_t^2\right\}/\mathcal{Y}_{t-1})}{\mathbb{E}(\exp\{-\xi_{t-1}\}/\mathcal{Y}_{t-1})}.$$

Now, we observe that under $\mathbb{P}$ the conditional distribution of the pair $(X_t, \xi_{t-1})$ given $\mathcal{Y}_{t-1}$ is Gaussian. But for a Gaussian pair $(U, V)$ of random variables we have

$$\frac{\mathbb{E}e^{-V-\frac{1}{2}U^2}}{\mathbb{E}e^{-V}} = [1 + \mathbb{E}(U - \mathbb{E}(U))^2]^{-1/2}\exp\{-\frac{1}{2}\frac{[\mathbb{E}U - \mathbb{E}(U - \mathbb{E}(U))(V - \mathbb{E}(V))]^2}{1 + \mathbb{E}(U - \mathbb{E}(U))^2}\}.$$

Therefore, we get

$$\frac{\mathcal{L}(t)}{\mathcal{L}(t-1)} = [1 + Q(t)\gamma_{XX}(t)]^{-1/2}\exp\{-\frac{1}{2}\frac{Q(t)[\pi_{t-1}(X_t) - \gamma_{X\xi}(t)]^2}{1 + Q(t)\gamma_{XX}(t)}\}.$$

Finally, this gives immediately equation (11) which achieves the proof of the lemma. ■

Before turning to more particular examples (see Section 4), now we analyze the case of a general Gauss-Markov process.

## 3 Ltqf of Gauss-Markov sequences – Two approaches

In this part we concentrate on the case of a Gaussian AR(1) process $X$, *i.e.*, a Gauss-Markov process driven by

$$X_t = A_t X_{t-1} + D_t^{1/2}\tilde{\varepsilon}_t\,,\ t \geq 1\,; \quad X_0 = \eta\,, \tag{14}$$

where $(\tilde{\varepsilon}_t,\ t = 0, 1, \ldots)$ is a sequence of i.i.d. standard Gaussian random variables which is independent of the initial condition $\eta$. Moreover $\eta$ is assumed to be a Gaussian variable with mean $m_0$ and variance $k(0)$ and $(A_t,\ t \geq 1)$, $(D_t,\ t \geq 1)$ are (deterministic) sequences of real numbers such that $D_t \geq 0$ for $t \geq 0$. In this setting, it is easy to check that the mean and covariance functions of $X$ are given by

$$m_t = [\prod_{u=1}^{t} A_u] m_0 \, ; \quad K(t,s) = [\prod_{u=s+1}^{t} A_u] k(s) \, , \; 0 \leq s \leq t \, ,$$

where

$$k(t) = [\prod_{u=1}^{t} A_u^2] k(0) + \sum_{s=1}^{t} [\prod_{u=s+1}^{t} A_u^2] D_s \, , \; t \geq 0 \, ,$$

and the conventions $\sum_{u=1}^{0} = 0$ and $\prod_{u=t+1}^{t} = 1$ are made. Of course, inserting this into the formula (2), one obtains a first expression of the Ltqf. Now we investigate alternative forms.

## 3.1 Forward approach

Here, as an immediate consequence of the filtering approach developed in Section 2, we get a second formula for the Ltqf :

**Corollary 1** *For all $t \geq 0$ the following equality holds :*

$$\mathcal{L}(t) = \prod_{s=0}^{t} [1 + Q(s)\gamma_s]^{-1/2} \exp\{-\frac{1}{2} \sum_{s=0}^{t} \frac{Q(s) Z_s^2 m_0^2}{1 + Q(s)\gamma_s}\} \, , \tag{15}$$

*where $(\gamma_s \, , \; 0 \leq s \leq t)$ is the unique solution of the equation*

$$\gamma_s = \frac{A_s^2 \gamma_{s-1}}{1 + Q(s-1)\gamma_{s-1}} + D_s \, , \; 1 \leq s \leq t \, ; \quad \gamma_0 = k(0) \, , \tag{16}$$

*and $(Z_s \, , \; 0 \leq s \leq t)$ is defined by*

$$Z_s = \prod_{r=1}^{s} \frac{A_r}{1 + Q(r-1)\gamma_{r-1}} \, , \; 0 \leq s \leq t \, . \tag{17}$$

**Proof** At first, we notice that the corresponding one-step prediction problem for $X$ in view of $Y$ is quite standard (see, *e.g.*, [8]) and it is well-known that the variance $\gamma_{XX}(s)$ is nothing else but the solution $\gamma_s$ of the Riccati type equation (16). Then one can check that the solution of equation (4) is given by

$$\gamma(t,s) = [\prod_{r=s+1}^{t} \frac{A_r}{1 + Q(r-1)\gamma_{r-1}}] \gamma_s \, .$$

Moreover, inserting this into (5), it is readily found that $z_s$ is also given by $z_s = Z_s m_0$ with $Z_s$ given by (17). Then, from (3), we get (15) immediately. ∎

**Remark 2** *(a) Observe that* $(Z_s,\ 0 \leq s \leq t)$ *defined by (17) is nothing else but the solution of the recursive equation*

$$Z_s = \frac{A_s}{1+Q(s-1)\gamma_{s-1}} Z_{s-1}\,,\ 1 \leq s \leq t\,;\quad Z_0 = 1\,. \tag{18}$$

*Then Remark 1 can be revisited in terms of the procedures (16) and (18) to compute the left-hand sides of (6) and (7).*

*(b) Clearly the above filtering approach to derive the Ltqf, which here leads to the expression (15) in terms of the solutions of the ordinary forward recursions (16) and (18), is really a forward approach in the sense that it is based on a recursion giving* $\mathcal{L}(t)$ *in terms of* $\mathcal{L}(t-1)$ *(see the proof of Lemma 1).*

## 3.2 Backward approach

Now we turn to a backward approach which leads to an expression of the Ltqf in terms of the solution of a backward recursion. Precisely, we have the following alternative expression for the Ltqf :

**Theorem 2** *For all* $t \geq 0$ *the following equality holds :*

$$\mathcal{L}(t) = \prod_{r=0}^{t-1} [1 + D_{r+1}\Gamma(t,r+1)]^{-1/2}[1+k(0)\Gamma(t,0)]^{-1/2}\exp\{-\frac{1}{2}\frac{\Gamma(t,0)m_0^2}{[1+k(0)\Gamma(t,0)]}\}\,, \tag{19}$$

*where* $(\Gamma(t,s)\,,\ 0 \leq s \leq t+1)$ *is the solution of the equation*

$$\Gamma(t,s) = \frac{A_{s+1}^2\Gamma(t,s+1)}{1+D_{s+1}\Gamma(t,s+1)} + Q(s)\,,\ 0 \leq s \leq t\,;\quad \Gamma(t,t+1) = 0\,. \tag{20}$$

**Proof** We introduce the quantity $\mathcal{L}(t;s,x)$ as the analogue of the Ltqf $\mathcal{L}(t)$ for an AR(1) process $X^{s,x}$ which is driven by the same equation as $X$ but starts at time $s \leq t$ from a fixed point $x$ , *i.e.*,

$$\mathcal{L}(t;s,x) = \mathbb{E}\exp\{-\frac{1}{2}\sum_{r=s}^{t} Q(r)(X_r^{s,x})^2\}.$$

Clearly, due to the Markov property, we have

$$\mathcal{L}(t;s,x) = \exp\{-\frac{1}{2}Q(s)x^2\}\mathbb{E}\mathcal{L}(t;s+1,X_{s+1}^{s,x})\,,$$

where the distribution of $X_{s+1}^{s,x}$ is Gaussian with mean $A_{s+1}x$ and variance $D_{s+1}$. Recall that the fundamental formula (1) says in particular that if $U$ is a real Gaussian random variable with mean $\mu$ and variance $\sigma^2$ then

$$\mathbb{E}\exp\{-\frac{\lambda}{2}U^2\} = [1+\lambda\sigma^2]^{-\frac{1}{2}}\exp\{-\frac{1}{2}[1+\lambda\sigma^2]^{-1}\lambda\mu^2\}\,;\quad \lambda \geq 0\,.$$

Then, looking for $\mathcal{L}(t;s,x)$ in the form

$$\mathcal{L}(t;s,x) = \exp\{-\frac{1}{2}\Gamma(t,s)x^2\}r_s\,,$$

it is readily seen that $\Gamma(t,s)$ and $r_s$ must satisfy (20) and

$$r_s = [1 + D_{s+1}\Gamma(t,s+1)]^{-1/2}r_{s+1}\,,\ 0 \leq s \leq t\,;\quad r_{t+1} = 1\,.$$

So, we obtain

$$\mathcal{L}(t;0,x) = \prod_{r=0}^{t}[1 + D_{s+1}\Gamma(t,r+1)]^{-1/2}\exp\{-\frac{1}{2}\Gamma(t,0)x^2\}\,.$$

But, since of course we have also

$$\mathcal{L}(t) = \mathbb{E}\mathcal{L}(t;0,\eta)\,,$$

again using the one-dimensional version of (1), we can easily conclude that (19) holds. ■

**Remark 3** (*a*) *Again Remark 1 can be revisited in terms of the procedure (20) to compute the left-hand sides of (6) and (7).*

(*b*) *Actually the equation (20), involved in the expression of the Ltqf which has just been derived through the backward approach, belongs to the world of optimal control. Namely, let us consider the stochastic optimal control problem for a signal $S$ governed by*

$$S_s = A_s S_{s-1} + B_s U_s ds + V_s\,,\ \ 1 \leq s \leq t\,,\ \ S_0 = x\,,$$

*where $V$ is a Gaussian noise with $\mathbb{E}V_r V_s' = \delta_{rs}D_s$ and $U$ is the (adapted) control policy, with the payoff*

$$\frac{1}{2}\mathbb{E}\{\sum_0^t Q(s)S_s^2 + \sum_1^t R(s)U_s^2\}\,.$$

*Then (see,* e.g. *[8]), if $B^2/R \equiv D$, the quantity $-\log\mathcal{L}(t;0,x)$ is nothing else but the minimal cost and moreover the optimal policy is given by a feedback which can also be expressed in terms of $\Gamma(t,s)$.*

## 3.3 Matched Riccati recursive equations

It has been mentioned in Remarks 2 and 3 that the Riccati equations (16) and (20) belong to the world of optimal filtering and to the world of optimal control respectively. Usually, links between matched forward and backward Riccati equations come naturally within the scope of the mathematical duality between these two worlds in the linear-quadratic Gaussian theory of dynamical systems. It is the case here since it is readily seen, from the formulas (15) and (19) for the Ltqf, that the following statement holds :

**Corollary 2** *The following relations hold:*

$$\prod_{s=0}^{t}[1+Q(s)\gamma_s]=\prod_{s=0}^{t-1}[1+D_{s+1}\Gamma(t,s+1)]\,[1+k(0)\Gamma(t,0)]\,, \tag{21}$$

$$\sum_{s=0}^{t}\frac{Q(s)Z_s^2}{1+Q(s)\gamma_s}=\frac{\Gamma(t,0)}{1+k(0)\Gamma(t,0)}\,. \tag{22}$$

Actually we can give direct proofs of the identities (21)-(22) which have just been derived probabistically. This is done in Appendix B.

# 4 Particular cases

In this part we investigate some examples of processes $X$ for which we can provide completely explicit formulas for the Laplace transform

$$\mathcal{L}(t;\mu)=\mathbb{E}\exp\{-\frac{\mu}{2}\sum_{s=0}^{t}X_s^2\}\,,\ \mu>0\,.$$

In the further analysis of these examples, a common key point is the resolution of a Riccati equation of the form (16) using the so-called linearization method. We shall be concerned only with the case when coefficients $A_s$ are all nonzero and of course here $Q(s)=\mu$ for all $s$. Then, if the pair $((\Psi_s^1,\Psi_s^2),\ s=0,1,\dots)$ is governed by the linear recursions

$$\begin{cases}\Psi_{s+1}^1=A_{s+1}^{-1}\Psi_s^1+\mu A_{s+1}^{-1}\Psi_s^2\,, & \Psi_0^1=1\,,\\ \Psi_{s+1}^2=D_{s+1}\Psi_{s+1}^1+A_{s+1}\Psi_s^2\,, & \Psi_0^2=k(0)\,,\end{cases} \tag{23}$$

the corresponding solution $(\gamma_t,\quad t=0,1,\dots)$ of (16) is given by $\gamma_t=(\Psi_t^1)^{-1}\Psi_t^2$. Moreover, the following equality holds:

$$\prod_{s=0}^{t}(1+\mu\gamma_s)=[\prod_{s=1}^{t+1}A_s]\Psi_{t+1}^1\,. \tag{24}$$

Now we turn to the examples, beginning with Markovian cases.

## 4.1 Homogeneous first order autoregressive processes

Here, for some fixed real number $\theta\neq 0$, in the AR(1) model (14), we take $A_t\equiv\theta$ and $D_t\equiv 1$, *i.e.*, $X_t=\theta X_{t-1}+\tilde{\varepsilon}_t$. Of course, if the initial condition $\eta$ has mean $m_0$ and variance $k(0)$, then the mean and covariance functions of $X$ are given by

$$m_t=\theta^t m_0\,;\quad K(t,s)=\theta^{|t-s|}k(s)\,;\quad k(s)=\theta^{2s}k(0)+\sum_{l=1}^{s}\theta^{2(s-l)}\,.$$

Solving (23) for $k(0)=0$ we obtain

$$\Psi_t^1 = \theta^{-t}\frac{(1-\lambda_-)\lambda_+^t + (\lambda_+ - 1)\lambda_-^t}{\lambda_+ - \lambda_-},$$

where

$$\lambda_\pm = \frac{\mu + 1 + \theta^2 \pm \sqrt{(\mu + (\theta+1)^2)(\mu + (\theta-1)^2)}}{2}.$$

*Homogeneous AR(1) process starting from zero* – If we take $\eta = 0$, *i.e.*, $m_0 = 0$ and $k(0)=0$, then from Corollary 1 and (24) we get immediately the corresponding Laplace transform, $\mathcal{L}_0(t;\mu)$ say, as

$$\mathcal{L}_0(t;\mu) = \{\frac{(1-\lambda_-)\lambda_+^{t+1} + (\lambda_+ - 1)\lambda_-^{t+1}}{\lambda_+ - \lambda_-}\}^{-1/2}.$$

This is nothing else but the result obtained in [7] through another approach. It is interesting to note that for $\theta = 1$, *i.e.*, when $X$ is simply a random walk, we have the limiting behavior

$$\lim_{N\to\infty} \mathcal{L}_0([Nt];\frac{\mu}{N^2}) = (\cosh(\mu t))^{-1/2}.$$

Actually, since the sequence $\{N^{-1/2}X_{[Nt]}\,, t \geq 0\}$ converges in distribution to a standard Brownian motion $B$, not surprisingly this limit gives the well-known Cameron-Martin formula for the Laplace transform of $\int_0^t B_s^2\, ds$ (see, *e.g.*, [1] and [6] for other approaches to this result).

*Homogeneous AR(1) process starting from $x$* – Now, for some real number $x \neq 0$, we take $\eta = x$, *i.e.*, $m_0 = x$ and $k(0) = 0$. In order to apply Corollary 1 we need to calculate the quadratic form involving $Z_s$ which satisfies (17). From (17) and (24), we get that $Z_s = 1/\Psi_s^1$. Then it can be checked that

$$\sum_{s=0}^{t} \frac{Z_s^2}{1+\mu\gamma_s} = 1 + \frac{\theta^2}{\mu} - \frac{\theta}{\mu}\frac{\Psi_t^1}{\Psi_{t+1}^1}.$$

Hence, applying (15), we obtain the Laplace transform, $\mathcal{L}_x(t;\mu)$ say, as

$$\mathcal{L}_x(t,\mu) = \mathcal{L}_0(t,\mu)\exp\left\{-\frac{x^2}{2}\left\{\mu + \theta^2[1 - \frac{(1-\lambda_-)\lambda_+^t + (\lambda_+ - 1)\lambda_-^t}{(1-\lambda_-)\lambda_+^{t+1} + (\lambda_+ - 1)\lambda_-^{t+1}}]\right\}\right\}. \qquad (25)$$

*Stationary AR(1) process* – Finally, we deal with the case where $-1 < \theta < 1$ and the process $X$ is stationary. It means that for $\eta$ we choose the mean $m_0 = 0$ and the variance $k(0) = 1/(1-\theta^2)$. Of course here the Laplace transform can be computed as

$$\mathcal{L}(t;\mu) = \mathbb{E}\mathcal{L}_\eta(t;\mu),$$

where $\mathcal{L}_x(t;\mu)$ is given by (25). Then, integrating the right hand side of (25) with respect to the distribution of $\eta$ it is readily seen that

$$\mathcal{L}(t;\mu) = \{d_+\lambda_+^{t+1} + d_-\lambda_-^{t+1}\}^{-1/2},$$

where

$$d_+ = (\frac{\mu+1-\theta^2}{1-\theta^2} - \lambda_-)/(\lambda_+ - \lambda_-)\,; \quad d_- = (\lambda_+ - \frac{\mu+1-\theta^2}{1-\theta^2})/(\lambda_+ - \lambda_-)\,.$$

Note that this formula can also be derived directly from (15) and (24) by the resolution of (23) with the initial condition $1/(1-\theta^2)$ for $\Psi^2$.

## 4.2 Gaussian bridge between 0 and $N$

Here, given a sequence $(W_t,\ t=1,\dots,N)$ of i.i.d. standard Gaussian random variables, we consider the process $X$ defined by:

$$X_t = \sum_{s=1}^{t} W_s - \frac{t}{N}\sum_{s=1}^{N} W_s\,; \quad 0 \le t \le N\,.$$

Clearly, by the definition, we have $X_0 = 0$ and $X_N = 0$ and the process $X$ can be seen as a discrete time analogue of the standard Brownian bridge, which we may call the Gaussian bridge between 0 and $N$. Actually $X$ is a centered Markovian process with the covariance function

$$K(t,s) = s(1-\frac{t}{N})\,; \quad 0 \le s \le t \le N\,.$$

It is easy to check that it is a nonhomogeneous AR(1) process driven by (14) with $A_t = D_t = (N-t)/(N+1-t)$. The resolution of the corresponding equation (23) leads to

$$\Psi_t^1 = \frac{1}{\sqrt{\mu(\mu+4)}}\{(\delta_+^{t+1} - \delta_-^{t+1}) - (\mu + A_{t+1})(\delta_+^t - \delta_-^t)\}\,,\ 0 \le t \le N-1\,,$$

where

$$\delta_\pm = \frac{\mu + 2 \pm \sqrt{\mu(\mu+4)}}{2}\,.$$

Then, applying (15) and (24), we can obtain the Laplace transform as

$$\mathcal{L}(t;\mu) = \{(1-\frac{t}{N})\frac{\delta_+^{t+1} - \delta_-^{t+1}}{\sqrt{\mu(\mu+4)}} - (1-\frac{t+1}{N})\frac{\delta_+^t - \delta_-^t}{\sqrt{\mu(\mu+4)}}\}^{-1/2}\,.$$

Again we have the limiting behavior

$$\lim_{N\to\infty} \mathcal{L}([Nt];\frac{\mu}{N^2}) = \left\{(1-t)\cosh(\sqrt{\mu}\,t) + \frac{\sinh(\sqrt{\mu}\,t)}{\sqrt{\mu}}\right\}^{-1/2}\,.$$

Actually, since here the sequence $\{N^{-1/2}X_{[Nt]}\,, t \ge 0\}$ converges in distribution to a standard Brownian bridge $B^*$, this limit gives the Laplace transform for $\int_0^t (B_s^*)^2\,ds$ (see, *e.g.*, [6] for an other approach to this result).

### 4.3 Moving average of order 1

Here we consider the case of a MA(1) process, *i.e*, a non Markovian process $X$ defined by

$$X_t = W_t + W_{t-1}\,; t \geq 0\,,$$

where $(W_{-1}, W_0, W_1, \ldots)$ is a sequence of i.i.d. standard Gaussian variables. Of course $X$ is centered and has the covariance function $K(t,s) = 2$ if $s = t$, 1 if $s = t-1$ and 0 if $s < t-1$. In order to solve equation (4) we can take

$$\gamma(t,s) = 0\,,\ s < t-1\,;\quad \gamma(t,t-1) = 1\,,\ t \geq 1\,,$$

and $\gamma(t,t) = \gamma_t$ where $\gamma_t$ is the solution of the equation:

$$\gamma_t = 2 - \frac{\mu}{1+\mu\gamma_{t-1}}\,,\ t \geq 1\,;\quad \gamma_0 = 2\,.$$

Actually this equation can be rewritten as (16) with $A_t = \mu$ and $D_t = 2-\mu$. The resolution of the corresponding equation (23) leads to

$$\Psi_t^1 = \mu^{-t}\frac{\rho_+^{t+1} - \rho_-^{t+1}}{\sqrt{4\mu+1}}\,,$$

where

$$\rho_\pm = \frac{2\mu + 1 \pm \sqrt{4\mu+1}}{2}\,.$$

Then, applying (3) and (24), we can obtain the Laplace transform as

$$\mathcal{L}(t;\mu) = \{\frac{\rho_+^{t+1} - \rho_-^{t+1}}{\sqrt{4\mu+1}}\}^{-1/2}\,.$$

## Appendix A – Solution of the auxiliary filtering type problems

Here, for an arbitrary Gaussian sequence $X$, we deal with the one-step prediction and filtering problems of the signals $X$ and $\xi$ given by (8) respectively from the observation of $Y$ defined in (8). Recall that the solutions can be reduced to equations for the conditional moments. The following statement provides the equations for the characteristics which give the solution of the prediction problem and the equation for the other quantity $\pi_{t-1}(X_t) - \gamma_{X_\xi}(t)$ appearing in the expression (11) of the Ltqf :

**Theorem 3** *The conditional mean $\pi_{t-1}(X_t)$ and the variance of the one-step prediction error $\gamma_{XX}(t)$ are given by the equations*

$$\pi_{t-1}(X_t) = m_t + \sum_{s=0}^{t-1}\frac{\gamma(t,s)}{1+Q(s)\gamma_{XX}(s)}[Y_s - Q(s)\pi_{s-1}(X_s)]\,, t \geq 0\,, \tag{26}$$

$$\gamma_{XX}(t) = \gamma(t,t)\,,\quad t \geq 0\,, \tag{27}$$

*where $\gamma$ is the unique solution of equation (4). Moreover, with $\gamma_{X_\xi}(t)$ defined by (10), the difference $\pi_{t-1}(X_t) - \gamma_{X_\xi}(t)$ is the solution $z_t$ of equation (5).*

**Proof** Since for the general setting the analysis is quite parallel, for simplicity of notation we deal only with the case $Q \equiv 1$, *i.e.*, $Y_t = X_t + \varepsilon_t$. Since the joint distribution of $(X_r, Y_s)$ for any $r\,, s$ is Gaussian we can apply the Note following Theorem 13.1 in [8]. For any $l$ we can write

$$\left\{ \begin{array}{l} \pi_l(X_t) = \pi_{l-1}(X_t) + \dfrac{\gamma(t,l)}{\langle\nu\rangle_l}\nu_l, \\ \pi_{-1}(X_t) = m_t, \end{array} \right. \tag{28}$$

where $\nu_l = Y_l - \mathbb{E}(Y_l/\mathcal{Y}_{l-1}) = Y_s - \pi_{l-1}(X_l)$ is the innovation and $\langle\nu\rangle_l$ is its variance

$$\langle\nu\rangle_l = 1 + \gamma(l,l)\,,$$

with

$$\gamma(t,l) = \mathbb{E}(X_t - \pi_{l-1}(X_t))(X_l - \pi_{l-1}(X_l))\,. \tag{29}$$

By the definition (29), we see for $l = t$ that the variance $\gamma_{XX}(t)$ is given by (27). Now, equality (28) implies

$$\pi_l(X_t) = m_t + \sum_{r=0}^{l} \frac{\gamma(t,r)}{1+\gamma_{XX}(r)}[Y_r - \pi_{r-1}(X_r)]\,,$$

and putting $l = t-1$ we get nothing else but equation (26). Concerning the solution of the one-step prediction problem, it just remains to show that the covariance $\gamma(t,s)$ satisfies equation (4). Let us define

$$\delta_X(t,l) = X_t - \pi_l(X_t)\,.$$

According to (28) we can write

$$\delta_X(t,l) = \delta_X(t,l-1) - \frac{\gamma(t,l)}{\langle\nu\rangle_l}\nu_l,$$

and so

$$\mathbb{E}\delta_X(t^1,l)\delta_X(t^2,l) = \mathbb{E}\delta_X(t^1,l-1)\delta_X(t^2,l-1) - \frac{\gamma(t^1,l)\gamma(t^2,l)}{\langle\nu\rangle_l}\,,$$

or

$$\mathbb{E}\delta_X(t^1,l)\delta_X(t^2,l) = \mathbb{E}\delta_X(t^1,-1)\delta_X(t^2,-1) - \sum_{r=0}^{l} \frac{\gamma(t^1,r)\gamma(t^2,r)}{\langle\nu\rangle_r}. \tag{30}$$

Taking $t^1 = t\,, t^2 = s\,, l = s-1$ in (30), it is readily seen that equation (4) holds for $\gamma(t,s)$. Now we analyze the difference $\pi_{t-1}(X_t) - \gamma_{X\xi}(t)$. Using the representation $\xi_t = \sum_{r=0}^{t} Y_r X_r$ we can rewrite $\gamma_{X\xi}(t)$ in the following form

$$\begin{aligned} \gamma_{X\xi}(t) &= \pi_{t-1}(\xi_{t-1} - \pi_{t-1}(\xi_{t-1}))(X_t - \pi_{t-1}(X_t)) \\ &= \sum_{r=0}^{t-1} \pi_{t-1}((X_r - \pi_{t-1}(X_r))(X_t - \pi_{t-1}(X_t)))Y_r \\ &= \sum_{r=0}^{t-1} \mathbb{E}((X_r - \pi_{t-1}(X_r))(X_t - \pi_{t-1}(X_t)))Y_r\,. \end{aligned}$$

So we have

$$\gamma_{X_\xi}(t) = \sum_{r=0}^{t-1} \tilde{\gamma}(t,r) Y_r \,, \tag{31}$$

where

$$\tilde{\gamma}(t,r) = \mathbb{E}((X_r - \pi_{t-1}(X_r))(X_t - \pi_{t-1}(X_t))) = \gamma(r,t)\,. \tag{32}$$

Using the definitions (29) and (32) we can write

$$\tilde{\gamma}(t,r) - \gamma(t,r) = -\mathbb{E} X_t (\pi_{t-1}(X_r) - \pi_{r-1}(X_r)).$$

Again, applying the Note following Theorem 13.1 in [8], we can write also

$$\pi_l(X_r) = \pi_{l-1}(X_r) + \frac{\gamma(r,l)}{\langle \nu \rangle_l} \nu_l,$$

This means that

$$\pi_{t-1}(X_r) - \pi_{r-1}(X_r) = \sum_{l=r}^{t-1} \frac{\gamma(r,l)}{\langle \nu \rangle_l} \nu_l \,,$$

or equivalently

$$\pi_{t-1}(X_r) - \pi_{r-1}(X_r) = \sum_{l=r}^{t-1} \frac{\tilde{\gamma}(l,r)}{\langle \nu \rangle_l} \nu_l \,.$$

Then, mutltiplying by $X_t$ and taking expectations in both sides, we get

$$\mathbb{E} X_t (\pi_{t-1}(X_r) - \pi_{r-1}(X_r)) = \sum_{l=r}^{t-1} \frac{\tilde{\gamma}(l,r)}{\langle \nu \rangle_l} \gamma(t,l).$$

Hence we have proved the following relation

$$\tilde{\gamma}(t,r) - \gamma(t,r) = -\sum_{l=r}^{t-1} \frac{\tilde{\gamma}(l,r)}{\langle \nu \rangle_l} \gamma(t,l)\,. \tag{33}$$

Now we can show that the difference $z_t = \pi_{t-1}(X_t) - \gamma_{X_\xi}(t)$ satisfies the equation (5). Using (26), (31) and (33), we obtain

$$\begin{aligned}
z_t &= m_t + \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle \nu \rangle_r} (Y_r - \pi_{r-1}(X_r)) - \sum_{r=0}^{t-1} \tilde{\gamma}(t,r) Y_r \\
&= m_t - \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle \nu \rangle_r} \pi_{r-1}(X_r) + \sum_{r=0}^{t-1} ( \frac{\gamma(t,r)}{\langle \nu \rangle_r} - \tilde{\gamma}(t,r)) Y_r \\
&= m_t - \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle \nu \rangle_r} \pi_{r-1}(X_r) + \sum_{r=0}^{t-1} ( \frac{\gamma(t,r)}{\langle \nu \rangle_r} - (\gamma(t,r) - \sum_{l=r}^{t-1} \frac{\tilde{\gamma}(l,r)}{\langle \nu \rangle_l} \gamma(t,l))) Y_r \\
&= m_t - \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle \nu \rangle_r} \pi_{r-1}(X_r) - \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle \nu \rangle_r} \gamma(r,r)) Y_r + \sum_{l=0}^{t-1} \frac{\gamma(t,l)}{\langle \nu \rangle_l} \sum_{r=0}^{l} \tilde{\gamma}(l,r) Y_r \,,
\end{aligned}$$

with, in the last step, the use of the equality $\langle\nu\rangle_r = 1+\gamma(r,r)$. Now, using (31) again and the property $\gamma(r,r)=\widetilde{\gamma}(r,r)$, we can write

$$\begin{aligned} z_t &= m_t - \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle\nu\rangle_r}\pi_{r-1}(X_r) + \sum_{l=0}^{t-1}\frac{\gamma(t,l)}{\langle\nu\rangle_l}\gamma_{X\xi}(l) \\ &= m_t - \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle\nu\rangle_r}(\pi_{r-1}(X_r) - \gamma_{X\xi}(r)) \\ &= m_t - \sum_{r=0}^{t-1} \frac{\gamma(t,r)}{\langle\nu\rangle_r} z_r\,, \end{aligned}$$

which is nothing else but equation (5). ■

**Remark 4** *It is worth mentioning that, paralleling the proof of Theorem 3, one can extend the result to the case when the observation equation for the signal $X$ is*

$$Y_t = C_t X_t + B_t^{1/2}\varepsilon_t\,,$$

*instead of the first equation in (8). Then the conditional mean $\pi_{t-1}(X_t)$ and the variance of the one-step prediction error $\gamma_{XX}(t)$ are given by the equations*

$$\pi_{t-1}(X_t) = m_t + \sum_{s=0}^{t-1}\frac{C_s\gamma(t,s)}{B_s + C_s^2\gamma_{XX}(s)}[Y_s - C_s\pi_{s-1}(X_s)]\,, t\geq 0\,,$$

$$\gamma_{XX}(t) = \gamma(t,t)\,, \quad t \geq 0\,,$$

*where $\gamma$ is the unique solution of equation*

$$\gamma(t,s) = K(t,s) - \sum_{r=0}^{s-1}\frac{C_r^2\gamma(s,r)}{B_r + C_r^2\gamma(r,r)}\gamma(t,r)\,,\ 1\leq s\leq t\,; \quad \gamma(t,0) = K(t,0)\,.$$

■

**Remark 5** *Here we visit again the identities (6) and (7) in terms of the characteristics of the process $Y$. Without loss of generality, we concentrate on the case where $Q(s)=1$ for all $s$. At first let us observe that the matrix $I_{t+1}+\mathcal{K}_t$ which is involved in identities (6) and (7) is nothing else but the covariance matrix of the vector $\underline{Y}_t = (Y_0,\ldots,Y_t)'$. Moreover, we see that the sequence of innovations $\nu_t = Y_t - \pi_{t-1}(Y_t)$ is generated by the recursion:*

$$\nu_t = (Y_t - m_t) - \sum_{s=0}^{t-1}\frac{\gamma(t,s)}{\langle\nu\rangle_s}\nu_s; \quad 1\leq s\leq t; \quad \nu_0 = Y_0 - m_0, \tag{34}$$

*where $\gamma(t,s)$ is the solution of equation (4) (with $Q(s)=1$) and the variance $\langle\nu\rangle_s = \mathbb{E}(\nu_s)^2$ is given by $\langle\nu\rangle_s = 1+\gamma(s,s)$. It appears that the recursion (5) generating the sequence*

*$(z_s\,, s=0,1,\ldots)$ from $(m_s\,, s=0,1,\ldots)$ is exactly the same as recursion (34) generating $(\nu_s\,, s=0,1,\ldots)$ from $(Y_s - m_s\,, s=0,1,\ldots)$. Actually (34) and (5) can be rewritten for $\underline{\nu}_t = (\nu_0,\ldots,\nu_t)'$ and $\underline{z}_t = (z_0,\ldots,z_t)'$ as :*

$$\underline{\nu}_t = T_t(\underline{Y}_t - \underline{m}_t)\,;\quad \underline{z}_t = T_t\underline{m}_t\,,$$

*where $T_t$ is a $(t+1)\times(t+1)$ lower triangular matrix with ones as diagonal entries. Of course $T_t$ satisfies*

$$\begin{pmatrix} \langle\nu\rangle_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \langle\nu\rangle_t \end{pmatrix} = T_t(I_{t+1}+\mathcal{K}_t)T_t'\,,$$

*and also*

$$(I_{t+1}+\mathcal{K}_t)^{-1} = T_t' \begin{pmatrix} \langle\nu\rangle_0^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \langle\nu\rangle_t^{-1} \end{pmatrix} T_t\,,$$

*which is the Choleski decomposition of the matrix $(I_{t+1}+\mathcal{K}_t)^{-1}$. Therefore we get that*

$$\det[I_{t+1}+\mathcal{K}_t] = \prod_{s=0}^{t}\langle\nu\rangle_s\,;\quad \underline{m}_t'[I_{t+1}+\mathcal{K}_t]^{-1}\underline{m}_t = \sum_{s=0}^{t}\frac{z_s^2}{\langle\nu\rangle_s}\,,$$

*which can be rewritten as (6) and (7).*

## Appendix B – Direct proof of Corollary 2

We start with the following Hamiltonian system for the pair $((x_s, p_s),\ s=0,\ldots,t+1)$ :

$$\left\{\begin{array}{ll} x_{s+1} = A_{s+1}x_s + D_{s+1}p_{s+1}\,, & x_0 = x \neq 0\,, \\ p_s = A_{s+1}p_{s+1} - Q(s)x_s\,, & p_{t+1} = 0. \end{array}\right. \tag{35}$$

Using direct calculations one can prove the following representations :

$$\left\{\begin{array}{l} x_s = x\,(1+k(0)\Gamma(t,0))\,Z_s + \gamma_s p_s\,, \\ p_s = -\Gamma(t,s)\,x_s\,, \end{array}\right. \tag{36}$$

where $\gamma_s$ , $Z_s$ and $\Gamma(t,s)$ are defined by (16), (17) and (20) respectively.
It follows from (17), (35) and (36) that

$$\frac{A_{s+1}}{1+Q(s)\gamma_s} = \frac{Z_{s+1}}{Z_s}\,;\qquad \frac{A_{s+1}}{1+D_{s+1}\Gamma(t,s+1)} = \frac{x_{s+1}}{x_s}\,.$$

Hence we can write

$$\prod_{s=0}^{t}(1+Q(s)\gamma_s) = \frac{\prod_{s=0}^{t}A_{s+1}}{Z_{t+1}};\qquad \prod_{s=0}^{t-1}(1+\Gamma(t,s+1)D_{s+1}) = \frac{\prod_{s=0}^{t}A_{s+1}}{x_{t+1}}\,x\,,$$

which, due to the final condition $x_{t+1} = x\,(1 + k(0)\Gamma(t,0))Z_{t+1}$ , gives (21).
To prove (22) we notice that $p_0 = -\Gamma(t,0)x$. But it follows from (35), (36) and (17) that

$$p_s = \frac{Z_{s+1}p_{s+1}}{Z_s} - \frac{Q(s)Z_s}{1+Q(s)\gamma_s} x(1 + k(0)\Gamma(t,0)) \,,$$

$$p_s Z_s - Z_{s+1}p_{s+1} = -\frac{Q(s)Z_s^2}{1+Q(s)\gamma_s} x(1 + k(0)\Gamma(t,0)) \,,$$

$$-\frac{p_0 x}{1 + k(0)\Gamma(t,0)} = \frac{\Gamma(t,0)}{1 + k(0)\Gamma(t,0)}\, x = \sum_{s=0}^{t} \frac{Q(s)Z_s^2}{1+Q(s)\gamma_s}\, x \,.$$

Therefore identity (22) holds.

■

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