Elasticity on a Thin Shell: Formal Series Solution

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Abstract: The three-dimensional equations of elasticity are posed on a domain of $\mathbb{R}^3$ defining a thin shell of thickness 2ε. The traction free conditions are imposed on the upper and lower faces together with the clamped boundary conditions on the lateral boundary. After a scaling in the transverse variable, the elasticity operator admits a power series expansion in ε with intrinsic coefficients with respect to the mean surface of the shell. This leads to define a formal series problem in ε associated with the three-dimensional equations. The main result is the reduction of this problem to a formal series boundary value problem posed on the mean surface of the shell.

Key-words: Mechanics of solids, Linear elasticity, Boundary layers, Differential geometry

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Elasticité sur une coque mince:
Résolution en séries formelles

Résumé : Les équations de l'élasticité tridimensionnelle sont posées sur un domaine de $\mathbb{R}^3$ définissant une coque mince d'épaisseur 2ε. On impose des conditions aux limites de traction libre sur les faces supérieures et inférieures, ainsi que des conditions aux limites d'encastrement le long du bord latéral. Après un changement d'échelle dans la variable transverse, l'opérateur d'élasticité se développe en séries entières en puissances de ε. Les coefficients de ce développement sont des opérateurs intrinsèques par rapport à la surface moyenne de la coque. Ceci amène à définir un problème en séries formelles en puissances de ε associé au problème tridimensionnel. Le principal résultat est la réduction de ce problème à un problème en séries formelles avec conditions aux limites, posé sur la surface moyenne de la coque

Mots-clés : Mécanique des solides, Elasticité linéaire, Couches limites, Géométrie différentielle

1 Introduction

1.a Origin of the problem and main aims

This paper deals with shell theory, whose main aim is the approximation of the three-dimensional linear elastic shell problem by a two-dimensional problem posed on the mean surface. This is an old and difficult question.

Let us recall that a shell is a three-dimensional object represented as a surface S thickened in its normal direction. We suppose that S is a compact orientable smooth surface with boundary, embedded in $\mathbb{R}^3$. For $\varepsilon \leq \varepsilon_0$ sufficiently small, we define the shell as the image Ω^ε of the manifold $S \times (-\varepsilon, \varepsilon)$ via the application

$$\Phi^\varepsilon : S \times (-\varepsilon, \varepsilon) \ni (P, x_3) \mapsto P + x_3 \boldsymbol{n}(P) \in \mathbb{R}^3, \tag{1.1}$$

where $\boldsymbol{n}(P)$ is a unit normal vector field on S. If S is a planar domain then the shell is a plate.

We suppose that the material constituting the shell is homogeneous and isotropic, and we consider the linear equations of three-dimensional elasticity, together with traction free conditions on the upper and lower faces and clamped boundary conditions on the lateral boundary. The solution $\boldsymbol{u}$ is a three-dimensional displacement and is considered in the following as the "exact" solution to be approximated by a two-dimensional object defined on the mean surface S.

In the sixties, different models have been proposed: see in particular Koiter [20, 21, 22], Naghdi [23], John [18], Novozhilov [25]. Concerning plates the derivation of the first two-dimensional model is earlier, see Kirchhoff [19].

Most of the shell models rely on a 3×3 system of intrinsic equations on S depending on ε, and write

$$\mathsf{K}(\varepsilon) := \mathsf{M} + \varepsilon^2 \mathsf{B} \tag{1.2}$$

where M is the *membrane* operator on S and B is a *bending* operator. If all above authors agree with the definition of the membrane operator M, different expressions of B can be found in the literature. For general shell geometry, the most popular and natural model is the one proposed by Koiter. This model describes the displacement of the shell by two tensors representing the change of metric and change of curvature of the surface submitted to a displacement. Moreover this model is elliptic for $\varepsilon > 0$ (see [1]). However, for $\varepsilon = 0$, the nature of the membrane operator depends on the geometry of the surface. In particular, M is elliptic only at the points where S is elliptic. The Koiter model relies partly upon computations made by John in [18]. But the question of determining the *best* model was very controversial (see in particular the introduction in [2] and the discussion in [21, 23]).

Different ways were explored in order to estimate the precision of a two-dimensional model. One of the first attempts was the estimate given by Koiter. Starting from the solution $\boldsymbol{z}$ of a 2D-problem associated with an operator $\mathsf{K}(\varepsilon)$ of the type (1.2) he constructed a 3D-displacement polynomial in the transverse variable x_3, and gave an estimate in energy norm between this reconstructed displacement and the 3D-displacement $\boldsymbol{u}$. However, this estimate fails for plates and the reason for this is the presence of boundary layer in the vicinity of the lateral boundary. This problem was already pointed out by GOL'DENVEIZER [16] (see the works of NAZAROV & ZORIN [24] and DAUGE & GRUAIS [9] for explicit proof).

More recently, the works by SANCHEZ-PALENCIA [26] and CIARLET, LODS, MIARA [4, 6, 5] showed that the 3D-displacement $\boldsymbol{u}$ and the 2D-displacement $\boldsymbol{z}$ solution of a system associated with the Koiter model converge toward the same limit as ε tends to 0, but in a weaker norm than the energy norm. See [3] for a review on these results. The limit is identified with the solution of a bending equation associated with the operator B of the Koiter model.

When it is available, the use of complete asymptotic expansions allows to have an exact representation of the behavior of the 3D displacement with respect to the thickness. Up to now, this is only done for plates and clamped elliptic shells: see [9, 10, 7] for plates. The result concerning clamped elliptic shells is a consequence of the present work and will be developed in a next paper, see also [14, 15]. In these cases, we can derive sharp estimates in every norm and analyze the performance of a 2D model. For clamped elliptic shells, all the classical models of the type (1.2) have the same accuracy with respect to ε.

The present work has several goals and consequences.

1. It gives formal computations that can be compared to those made by JOHN in [18]. In particular, we give the most general "shell model", e.g. the most general bending operator appearing after the dimension reduction process. The main point is that the mathematical setting of the result has been made precise and powerful by the use of *formal series*. We also show how this general bending operator "contains" (see below) Koiter's bending operator. Note however that the computations of JOHN were made for a more general 3D nonlinear elasticity model.

2. The present analysis was developed in order to find a complete asymptotic expansion of the displacement in the case of a clamped elliptic shell. In particular, it incorporates boundary layer phenomena near the lateral boundary in the case of clamped boundary conditions. This is thus a first step in the direction of finding a complete asymptotic expansion of the displacement (see [7] for an application to plates eigenvalues). This formal series representation has also been used in various situations: see [11, 12].

3. As we will see, the main result is the reduction of the 3D problem in formal series to a 2D boundary value problem in formal series, posed on the mean surface. It appears that the 2D formal series problem has strong similarities with Koiter's 2D problem. This fact can be used to state and prove a valid energy estimate in the spirit of Koiter. This work is presented in [8]. Thus the formal series approach can lead to *real* estimates and results.

1.B GENERAL CLASSICAL SETTING

We now sketch the main ideas. First of all, we set the equations in Cartesian coordinates on the domain Ω^ε. Our first goal is to write the equations in *normal coordinates*, where we agree that a normal coordinates system on Ω^ε is a system induced by the diffeomorphism (1.1) and of the form (x_α, x_3) where (x_α) is a coordinate system on S and x_3 is the transverse coordinate.

We first note that all tensor fields on Ω^ε can be decomposed into several tensors fields on the surfaces $S_{x_3} := \Phi^\varepsilon(S, x_3)$ for fixed x_3. For example the displacement field $\boldsymbol{u}$ decomposes into the surfacic 1-form (u_α) and the function u_3. After that, we show that any tensor field on S_{x_3} can be seen as a tensor field on S depending on x_3. Finally, the natural spaces involved in the equations are of the type $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T^q_pS_0)\big)$ where S_0 is identified with the surface S, and where $\Gamma(T^q_pS_0)$ is the space of tensor fields of type (p,q) on S_0.

In a normal coordinate system, we write $\mathsf{L}(x_\alpha, x_3; \mathrm{D}_\alpha, \partial_3)$ the three-dimensional operator, where D_α is the covariant derivative on S and ∂_3 the partial derivative with respect to x_3. Similarly, the traction operator on the upper and lower faces writes $\mathsf{T}(x_\alpha, x_3; \mathrm{D}_\alpha, \partial_3)$.

It is important to note that even if these operators are written in a coordinate system, they are in fact *intrinsic*, and express with respect to tensor operators on S_0. For ease of use, we consider the ***shifted displacement*** $\boldsymbol{w}$ obtained by multiplying the surfacic components of $\boldsymbol{u}$ by the Jacobin of the application $\Phi(\,\cdot\,, x_3)$ on S. This is a standard change (see [23]). Thus the operators L and T act on the shifted displacement $\boldsymbol{w}$, and are intrinsic in normal coordinates. Moreover, we made a change of sign, and the shifted displacement $\boldsymbol{w}$ satisfies the inner equation $\mathsf{L}\boldsymbol{w} = -\boldsymbol{f}$ in Ω^ε if $\boldsymbol{f}$ is a 1-form field representing the loading forces.

In the equation (1.1) we note that the definition of the shell is analytic in x_3. It is easy to show that all the natural tensors in Ω^ε and the covariant derivative expand in convergent power series of x_3. Hence we can show that the operators L and T expand in power series of x_3 with intrinsic coefficients with respect to S_0. Now in order to work on a manifold independent on ε we make the scaling $X_3 = \varepsilon^{-1}x_3$ to state the problem on the manifold $\Omega := S \times (-1, 1)$. The 3D elasticity operator are written $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$. These operators clearly expand in power series of ε with coefficients intrinsic operators on the manifold Ω.

Theorem 3.3 provides the expressions of the operators L^k and T^k appearing in the expansions

$$\mathsf{L}(\varepsilon) = \varepsilon^{-2}\sum_{k=0}^{\infty}\varepsilon^k\mathsf{L}^k \quad \text{and} \quad \mathsf{T}(\varepsilon) = \varepsilon^{-1}\sum_{k=0}^{\infty}\varepsilon^k\mathsf{T}^k.$$

This result, even if stated for the first time in this way, can be obtained using standard expansions of the covariant derivative and the metric. Most of these expressions can be found in [23].

1.c Formal series

Now with the operators $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$ we associate two *formal series* in powers of ε, written $\mathsf{L}[\varepsilon] = \varepsilon^{-2}\sum_{k\geq 0}\varepsilon^k\mathsf{L}^k$ and $\mathsf{T}[\varepsilon] = \varepsilon^{-1}\sum_{k\geq 0}\varepsilon^k\mathsf{T}^k$. Considering a formal series $\boldsymbol{f}[\varepsilon] = \sum_{k\geq 0}\varepsilon^k\boldsymbol{f}^k$ with 1-form field coefficients in Ω, we state the following formal series problem: Find a formal series $\boldsymbol{w}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{w}^k$ with 1-form field coefficients, such that

$$\begin{aligned} \mathsf{L}[\varepsilon]\boldsymbol{w}[\varepsilon] &= -\boldsymbol{f}[\varepsilon] && \text{in } \Omega, \\ \mathsf{T}[\varepsilon]\boldsymbol{w}[\varepsilon] &= 0 && \text{on } \Gamma_\pm, \\ \boldsymbol{w}[\varepsilon] &= 0 && \text{on } \Gamma_0, \end{aligned} \tag{1.3}$$

where $\Gamma_\pm$ are the upper and lower faces of Ω and Γ_0 the lateral boundary. Here, the product between two formal series is the Cauchy product.

The equations (1.3) are in fact a collection of equations. Up to multiplication by a constant, the first terms of the formal series $\mathsf{L}[\varepsilon]$ and $\mathsf{T}[\varepsilon]$ are $\partial^2_{X_3}$ and ∂_{X_3} respectively. But the operator $(\partial^2_{X_3}, \partial_{X_3})$ on $(-1,1)$ has non-trivial kernel and co-kernel. In the manifold Ω, the kernel is the space of displacements independent of X_3, and is denoted by $\Sigma(S_0)$. Hence, if $\boldsymbol{w}[\varepsilon]$ is a formal series solution of the first two equations in (1.3), the displacements $\boldsymbol{w}^k$ are determined up to elements $\boldsymbol{z}^k$ of the kernel. Moreover, solving successively for the displacements $\boldsymbol{w}^k$ requires compatibility conditions on the right-hand sides, e.g. on the $\boldsymbol{w}^\ell$ for $\ell < k$. This conditions are in fact equations on the $\boldsymbol{z}^k$ and form a formal series equation on S_0.

Theorems 4.1 and 4.3 reduce the two first equations in (1.3) to a two-dimensional problem. We show the existence of formal series operators $\mathsf{V}[\varepsilon]$, $\mathsf{Q}[\varepsilon]$, $\mathsf{A}[\varepsilon]$ and $\mathsf{G}[\varepsilon]$ such that if $\boldsymbol{z}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{z}^k$ is a formal series with coefficients in $\Sigma(S_0)$ satisfying the equation

$$\mathsf{A}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathsf{G}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{1.4}$$

then we can construct a formal series $\boldsymbol{w}[\varepsilon]$ by the equation

$$\boldsymbol{w}[\varepsilon] = \mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{1.5}$$

solution of the problem

$$\begin{aligned} \mathsf{L}[\varepsilon]\boldsymbol{w}[\varepsilon] &= -\boldsymbol{f}[\varepsilon] && \text{in } \Omega, \\ \mathsf{T}[\varepsilon]\boldsymbol{w}[\varepsilon] &= 0 && \text{on } \Gamma_\pm. \end{aligned} \tag{1.6}$$

Here, the coefficients of the formal series $\mathsf{V}[\varepsilon]$ are polynomial in X_3, and V^0 coincides with the identity. Similarly, the coefficients of $\mathsf{Q}[\varepsilon]$ are operators acting on the 1-form fields space on Ω. The coefficients of the formal series $\mathsf{A}[\varepsilon]$ are 2D operators acting on $\Sigma(S_0)$ and the coefficients of $\mathsf{G}[\varepsilon]$ take values into this space. Note that the first coefficient G^0 is the mean value across $(-1,1)$.

The equation (1.4) is a two dimensional formal series problem set on the mean surface. We show that the formal series $\mathsf{A}[\varepsilon]$ writes

$$\mathsf{A}[\varepsilon] = \mathsf{M} + \varepsilon^2 \mathsf{A}^2 + \cdots,$$

where M is the membrane operator. The operator A^2 is a sort of bending operator. The exact expression of A^2 is given in Theorem 4.4. Proposition 4.5 gives an estimate of the difference between A^2 and the bending operator B of the Koiter model. We obtain in particular that these operators coincide on the space of inextensional displacements. This has to be related with the convergence result (see [3]). Hence in the formal series $\mathsf{A}[\varepsilon]$ the first term M is not elliptic for every geometry of S, but the Koiter operator $\mathsf{K}(\varepsilon) = \mathsf{M} + \varepsilon^2\mathsf{B}$ is always elliptic and we can estimate the difference between $\mathsf{K}(\varepsilon)$ and the operator $\mathsf{M} + \varepsilon^2\mathsf{A}^2$.

Note that in the case where the boundary ∂S_0 is empty, no boundary conditions are present, but orthogonality conditions to the rigid displacements are imposed to the loading forces and the displacement. These conditions can also be expressed as formal series conditions.

The second step of this work (Theorem 5.3) deals with boundary layer formal series. In general, if $\boldsymbol{z}[\varepsilon]$ is a solution of (1.4), the reconstructed displacement (1.5) cannot satisfy the condition $\boldsymbol{w}[\varepsilon] = 0$ on the lateral boundary. Indeed the operators in the formal series $\mathsf{V}[\varepsilon]$ have increasing orders of surfacic derivatives, and the condition $\boldsymbol{w}[\varepsilon]\big|_{\Gamma_0} = 0$ implies an infinity of boundary conditions on the coefficients of $\boldsymbol{z}[\varepsilon]$. But even if we consider that the formal series $\mathsf{A}[\varepsilon]$ has $\mathsf{K}(\varepsilon)$ as first term, this operator of order 2 in z_α and 4 in z_3 cannot solve for an infinity of boundary condition for the coefficients $\boldsymbol{z}^k$.

In the case of plates, the operator $\mathsf{K}(\varepsilon)$ decouples into the membrane and the bending operator respectively. These operators are elliptic. In this case, there exists an asymptotic expansion of the displacement in powers of ε with two scales (see [24, 9]). The first scale consists of terms independent of ε, and the second of boundary layer terms near the lateral boundary. If we set r the distance to the lateral boundary and s the arc length along ∂S_0, these terms are of the form $\varphi(\varepsilon^{-1}r, s, \varepsilon^{-1}x_3)$ on Ω^ε, and are exponentially decreasing with respect to $R = \varepsilon^{-1}r$.

In our case, we introduce a new formal series problem including a new scale of boundary layer: Find a formal series $\boldsymbol{\varphi}[\varepsilon]$ whose terms are functions $\boldsymbol{\varphi}^k(R, s, X_3)$ *exponentially decreasing with respect to* R, such that

$$\big(\mathcal{L}[\varepsilon], \mathcal{T}[\varepsilon]\big)\boldsymbol{\varphi}[\varepsilon] = 0 \quad \text{and} \quad \boldsymbol{w}[\varepsilon]\big|_{\Gamma_0} + \boldsymbol{\varphi}[\varepsilon]\big|_{R=0} = 0, \tag{1.7}$$

where the formal series $\mathcal{L}[\varepsilon]$ and $\mathcal{T}[\varepsilon]$ are induced by Taylor expansions in $R = 0$ of the operators L and T in coordinates (R, s, X_3), and where the formal series $\boldsymbol{w}[\varepsilon]$ is given by (1.5). Note that $R = 0$ coincides with the lateral boundary Γ_0.

Theorem 5.3 shows that the existence of a formal series $\boldsymbol{\varphi}[\varepsilon]$ solution of (1.7) relies upon compatibility conditions on $\boldsymbol{z}[\varepsilon]$ on the boundary ∂S_0: There exist formal series operators $\mathsf{d}[\varepsilon]$ and $\mathsf{h}[\varepsilon]$ whose coefficients define four trace operators on the boundary ∂S_0, such that if $\boldsymbol{z}[\varepsilon]$ satisfies the equation

$$\mathsf{d}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathsf{h}[\varepsilon]\boldsymbol{f}[\varepsilon] \tag{1.8}$$

on the boundary ∂S_0, then we can construct a formal series $\boldsymbol{\varphi}[\varepsilon]$ solution of the problem (1.7). Moreover, the first term of the formal series $\mathsf{d}[\varepsilon]$ writes

$$\mathsf{d}^0\boldsymbol{z} = (z_r, z_s, z_3, \partial_r z_3)\big|_{\partial S_0}$$

where r is the geodesic distance to ∂S_0 in S_0. This operator is the natural Dirichlet operator associated with the Koiter model $\mathsf{K}(\varepsilon)$ for $\varepsilon > 0$.

The equations (1.4) and (1.8) form the two-dimensional *reduced problem.* If $\boldsymbol{z}[\varepsilon]$ is a solution of the reduced problem, then we can construct two formal series $\boldsymbol{w}[\varepsilon]$ and $\boldsymbol{\varphi}[\varepsilon]$ satisfying the equations (1.6) and (1.7).

Various difficulties arise when trying to solve the reduced equation. In particular, the first terms $(\mathsf{M}, \mathsf{d}^0)$ do not define an invertible operator, even if the surface S_0 is elliptic. Using the estimate for $\mathsf{A}^2 - \mathsf{B}$ where B is the bending Koiter operator, we can however see the formal series $(\mathsf{A}[\varepsilon], \mathsf{d}[\varepsilon])$ as a formal series with first term $(\mathsf{K}(\varepsilon), \mathsf{d}^0)$ that defines an invertible operator for every geometry of S_0. This fact is used in [8] to obtain an estimate in the spirit of Koiter.

In the case of clamped elliptic shells, the membrane operator with boundary conditions $z_s = z_s = 0$ on ∂S_0 is elliptic, and we can consider the problem (1.4), (1.8) as a singularly perturbed formal series problem. We have to introduce a new boundary layer scale of formal series to obtain a solution. This work will be presented in a next paper (see also [15, 14]).

In the case of plates, the operator $\mathsf{M} + \varepsilon^2 \mathsf{A}^2$ is triangular with respect to (z_α, z_3) and we can show that the reduced 2D problem has a solution. This yields the construction of a complete asymptotic expansion of the displacement as in [9, 7].

2 Normal coordinates and tensors

2.a Three-dimensional problem

Recall that the domain Ω^ε defining the shell is given by (1.1). This domain has a lateral boundary Γ_0^ε image of $\partial S \times (-\varepsilon, \varepsilon)$ by the application Φ^ε. The upper and lower faces $S_{\pm\varepsilon}$ are the images of $S \times \{\pm\varepsilon\}$. We suppose that the material constituting the shell is homogeneous and isotropic, characterized by its two Lamé coefficients λ and μ. The loading forces applied to the shell are represented by a smooth vector field $\boldsymbol{f}$ defined on Ω^ε. We suppose that the shell is clamped along Γ_0^ε and we imposed the traction free condition on $S_{+\varepsilon}$ and $S_{-\varepsilon}$. The displacement of the shell is represented by the 1-form field $\boldsymbol{u}$. In Cartesian coordinates $\{t^i\}$ the problem then writes

$$\begin{aligned} -\partial_j A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) &= f^i \quad \text{in} \quad \Omega^\varepsilon, \\ \mathsf{T}^i(\boldsymbol{u}) &= 0 \quad \text{on} \quad S_{\pm\varepsilon}, \\ \boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_0^\varepsilon, \end{aligned} \tag{2.1}$$

where $A^{ijk\ell} = \lambda\delta^{ij}\delta^{k\ell} + \mu(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk})$, where ∂_j is the partial derivative with respect to t^i and $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ with $\boldsymbol{u} = u_i \mathrm{d}t^i$ in Cartesian coordinates. On the same way f^i denote the components of the vector field in the basis $\frac{\partial}{\partial t^i}$. The operator $\mathsf{T}^i(\boldsymbol{u})$ is the natural traction operator on the faces $S_{\pm\varepsilon}$ appearing after integration by parts in the associated bilinear form:

$$(\boldsymbol{u}, \boldsymbol{v}) \mapsto \int_{\Omega^\varepsilon} A^{ijk\ell} e_{ij}(\boldsymbol{u}) e_{k\ell}(\boldsymbol{u}) \, \mathrm{d}t^1 \, \mathrm{d}t^2 \, \mathrm{d}t^3.$$

This is the classical problem of linear elasticity set in Cartesian coordinates on a shell-shaped domain of $\mathbb{R}^3$. Korn inequality [13] implies that this problem has a unique solution in $\mathrm{H}^1(\Omega^\varepsilon)^3$.

2.B Normal coordinates

The diffeomorphism Φ^ε of the equation (1.1) is called the *normal parametrization* of Ω^ε . A *normal coordinate system* is a coordinate system on $\Omega^\varepsilon = \Phi^\varepsilon\big(S \times (-\varepsilon, \varepsilon)\big)$ induced by a coordinate system on S . If $(U_i, \varphi_i)_{i\in I}$ is an atlas of local charts on S , a natural atlas on Ω^ε is given by the charts $\Phi^\varepsilon\big(U_i \times (-\varepsilon, \varepsilon)\big)$ together with the applications

$$\mathbb{R}^3 \supset \Phi^\varepsilon\big(U_i \times (-\varepsilon, \varepsilon)\big) \stackrel{(\Phi^\varepsilon)^{-1}}{\longrightarrow} U_i \times (-\varepsilon, \varepsilon) \stackrel{\varphi_i \times \mathrm{Id}}{\longrightarrow} \varphi_i(U_i) \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3. \tag{2.2}$$

Moreover for fixed x_3 these charts induce local charts on S_{x_3} with domains $\Phi^\varepsilon(U_i, x_3)$ and applications

$$S_{x_3} \supset \Phi^\varepsilon(U_i, x_3) \stackrel{F_{x_3}^{-1}}{\longrightarrow} U_i \stackrel{\varphi_i}{\longrightarrow} \varphi_i(U_i) \subset \mathbb{R}^2, \tag{2.3}$$

where $F_{x_3} : S \to S_{x_3}$ is defined as $\Phi^\varepsilon(\,\cdot\,, x_3)$.

Let us fix a local chart (U, φ) of S and the associated coordinate system (x_α) . In the following Latin indices will always refer to three-dimensional indices (here 1,2,3) while Greek indices will refer to two-dimensional indices (1 and 2). Using the application (2.2), the system (x_α, x_3) is hence a local 3D coordinate system on the shell. We set

$$\frac{\partial}{\partial x_i} = \mathbf{x}_i, \quad i \in \{1, 2, 3\}$$

the associated coordinate vector fields. It is clear that $\mathbf{x}_3(x_\alpha, x_3) = \boldsymbol{n}(x_\alpha)$ and that the vectors $\mathbf{x}_\alpha(x_\alpha, x_3)$ are tangent to S_{x_3} : they are the coordinates vector fields associated with the local map (2.3). In this coordinate system on Ω^ε the metric is defined by

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle_{\mathbb{R}^3}. \tag{2.4}$$

For fixed x_3 , the surface S_{x_3} is embedded in the domain Ω^ε of $\mathbb{R}^3$ and the metric of S_{x_3} is thus the restriction of the metric g_{ij} to S_{x_3} . In the following, we usually identify the abstract manifold S with the embedded manifold S_0 . We set ∇ the standard connexion on Ω^ε associated with the Euclidean scalar product on $\mathbb{R}^3$. The Christoffel symbols associated with ∇ vanish in Cartesian coordinates. We set D^{x_3} the connexion on S_{x_3} induced by ∇ . A first step is to show how the 3D tensor fields on Ω^ε yield naturally tensor fields on the surfaces S_{x_3} . After that, we show that in fact all the tensor fields on S_{x_3} can be seen as tensor fields on $S = S_0$ depending on x_3 . These ideas and *normal reduction* of tensor fields are already explained in [23].

2.c Normal reduction of tensor fields

Let (U,φ) and (V,ψ) two local charts of S around $P\in S$. Writing F_{x_3} the application $\Phi^\varepsilon(\,\cdot\,,x_3)$, these two charts induce local charts $(F_{x_3}(U),\varphi\circ F_{x_3}^{-1})$ and $(F_{x_3}(V),\psi\circ F_{x_3}^{-1})$ on the surface S_{x_3} around the point $F_{x_3}(P)$. The changing chart application is then $\varphi\circ F_{x_3}^{-1}\circ(\psi\circ F_{x_3}^{-1})^{-1}=\varphi\circ\psi^{-1}$ from $\psi(V\cap U)$ into $\varphi(V\cap U)$.

On the other hand, these two local charts induce local charts of the type (2.2) on Ω^ε. We verify that the changing chart application on Ω^ε is simply

$$\mathbb{R}^3\supset\psi(V\cap U)\times(-\varepsilon,\varepsilon)\stackrel{\psi^{-1}\times\mathrm{Id}}{\longrightarrow}V\cap U\times(-\varepsilon,\varepsilon)\stackrel{\varphi\times\mathrm{Id}}{\longrightarrow}\varphi(V\cap U)\times(-\varepsilon,\varepsilon)\subset\mathbb{R}^3.$$

The Jacobian matrix of this application writes

$$A^i_j=\begin{pmatrix} j^\alpha_\beta & 0\\ 0 & 1\end{pmatrix},$$

where j^α_β is the Jacobian matrix of the application $\varphi\circ\psi^{-1}$ from $\psi(V\cap U)$ into $\varphi(V\cap U)$.

Let $\boldsymbol{u}$ be a 1-form field on Ω^ε. Let u_i and $\overline{u}_i$ be the components of $\boldsymbol{u}$ in both coordinate system of type (2.2) induced by (U,φ) and (V,ψ) respectively. We consider the two families of functions u_α and $\overline{u}_\alpha$ where α is a surfacic varying index. These two families define local 1-form fields on S_{x_3} in two different local basis. However, as $\boldsymbol{u}$ is a tensor field on Ω^ε the expression of the components of $\boldsymbol{u}$ in the two basis are related by the matrix A^j_i. Using the special form of this matrix, we compute that

$$j^\alpha_\gamma\,u_\alpha=A^i_\gamma u_i=\overline{u}_\gamma.$$

As j^α_β is the Jacobian matrix of changing chart on S_{x_3}, we conclude that the components u_α and $\overline{u}_\alpha$ are the components of a 1-form field defined globally on S_{x_3}. Similarly, we have $u_3=A^3_3u_3=A^i_3u_i=\overline{u}_3$, and thus the component u_3 in any coordinate system is a global function defined on S_{x_3}.

Similarly, consider the deformation tensor of type $(2,0)$ on Ω^ε written e_{ij} and $\overline{e}_{ij}$ in both coordinate systems induced by (U,φ) and (V,ψ) respectively. We consider the two families of functions $e_{\alpha3}$ and $\overline{e}_{\alpha3}$ where α is a surfacic varying index while 3 is fixed. Using the fact that the deformation tensor is a tensor field on Ω^ε, the components e_{ij} and $\overline{e}_{ij}$ are linked and we have $j^\alpha_\gamma\,e_{\alpha3}=A^i_\gamma\,A^j_3\,e_{ij}=\overline{e}_{\gamma3}$, and we conclude that the components $e_{\alpha3}$ define a 1-form field globally on S_{x_3}. Similarly, we see that the components $e_{\alpha\beta}$, where α and β are surfacic varying indices, are the components of a tensor field of type $(2,0)$ on S_{x_3}, and the component e_{33} defines globally a function on S_{x_3}.

We can obviously generalize this fact to other type of tensor fields. The result is that each tensor field on Ω^ε can be decomposed into several tensors fields on S_{x_3} by fixing some indices to the value 3 and letting the other vary into surfacic indices.

2.d Shifter and projections on the mean surface

The mean surface S_0 is characterized by its metric $a_{\alpha\beta}(x_\sigma)=g_{\alpha\beta}(x_\sigma,0)$ and its curvature tensor $b_{\alpha\beta}(x_\sigma)$ which is symmetric. The *Codazzi-Mainardi* equation, expressing the fact that the curvature tensor of Ω^ε vanishes, yields that

$$\mathrm{D}_\alpha b_{\beta\sigma} = \mathrm{D}_\beta b_{\alpha\sigma},$$

where $\mathrm{D} = \mathrm{D}^0$ is the connexion on S_0 (see [23]). Recall that on a Riemannian manifold, we use the metric to obtain isomorphisms between covariant and contravariant vector fields spaces. In coordinate system, this means that we can lower or upper the indices, which correspond to contraction with the metric tensor. For example the curvature tensor $b_{\alpha\beta}$ can be viewed as a tensor of type $(1,1)$ and in this case the components are written $b_\alpha^\beta = a^{\beta\sigma} b_{\sigma\alpha}$ where $a^{\alpha\beta}$ is the inverse of the metric tensor. Note that even if an expression is written in coordinate system, the equation is intrinsic provided that the indexed object are tensor fields on S_0 .

Consider a vector field $\boldsymbol{Y}$ on Ω^ε . In a local basis $\mathbf{x}_i = (\mathbf{x}_\alpha, \mathbf{x}_3)(x_\alpha, x_3)$ this vector field writes

$$\boldsymbol{Y} = Y^\sigma(x_\alpha, x_3)\mathbf{x}_\sigma(x_\alpha, x_3) + Y^3(x_\alpha, x_3)\mathbf{x}_3(x_\alpha, x_3).$$

For $x_3 = 0$, the basis $(\mathbf{x}_i(x_\alpha, 0))$ consists simply of a local basis $\mathbf{x}_\sigma(x_\alpha, 0)$ on a domain U of S_0 and of the normal vector field $\mathbf{x}_3(x_\alpha, 0) = \boldsymbol{n}(x_\alpha, 0)$. However, as $U \times (-\varepsilon, \varepsilon)$ is embedded in $\mathbb{R}^3$, this basis extends by translation over the domain corresponding to $U \times (-\varepsilon, \varepsilon)$ in Ω^ε . Hence we can decompose $\boldsymbol{Y}$ as

$$\boldsymbol{Y} = \tilde{Y}^\sigma(x_\alpha, x_3)\mathbf{x}_\sigma(x_\alpha, 0) + \tilde{Y}^3(x_\alpha, x_3)\mathbf{x}_3(x_\alpha, 0).$$

But we compute easily using the form of the diffeomorphism (1.1) and the properties of the curvature that we have for all x_3

$$\mathbf{x}_3(x_\alpha, x_3) = \mathbf{x}_3(x_\alpha, 0) \quad \text{and} \quad \mathbf{x}_\sigma(x_\alpha, x_3) = \mathbf{x}_\sigma(x_\alpha, 0) - x_3 b_\sigma^\beta(x_\alpha)\mathbf{x}_\beta(x_\alpha, 0), \tag{2.5}$$

and this implies the relation

$$\tilde{Y}^3(x_\alpha, x_3) = Y^3(x_\alpha, x_3) \quad \text{and} \quad \tilde{Y}^\sigma(x_\alpha, x_3) = \mu_\beta^\sigma(x_\alpha, x_3) Y^\beta(x_\alpha, x_3),$$

where μ_σ^β is the *shifter* (see [23]) defined by

$$\mu_\sigma^\beta(x_\alpha, x_3) = \delta_\sigma^\beta - x_3 b_\sigma^\beta(x_\alpha).$$

Hence a vector field $\boldsymbol{Y}$ can be represented by its components (Y^i) or $(\tilde{Y}^i)$ and the shifter appears as the Jacobian of a change of coordinates. Similarly, a displacement (which is a covariant tensor of order 1) can be represented by its coordinates (v_σ, v_3) along the basis induced by the diffeomorphism (1.1) or by its coordinates $(\tilde{v}_\sigma, \tilde{v}_3)$ along the coordinates associated with the fact that S is included in $\mathbb{R}^3$, and we have the relations

$$\tilde{v}_3 = v_3 \quad \text{and} \quad \tilde{v}_\sigma = (\mu^{-1})_\sigma^\beta v_\beta,$$

where $(\mu^{-1})_\sigma^\beta$ is the inverse of the shifter.

Now the fact that the change of coordinates application are the same for S_0 or for S_{x_3} implies that the coordinates $\tilde{v}_\alpha$ and v_α are in fact both the coordinates of 1-form fields on S_0 depending on x_3. More generally, if we consider a tensor field on Ω^ε, then by fixing some indices to the transverse index 3 it gives a tensor field on S_{x_3} of type (p,q). But these components yield also a tensor field on S_0 of type (p,q) depending on x_3 and thus, an element of $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T_p^q S_0)\big)$ where $\Gamma(T_p^q S_0)$ is the fiber bundle of tensor fields of type (p,q) on S_0. The shifter appears as an endomorphism of these spaces.

As it will be of constant use, we define the space $\Sigma(S_0) := \Gamma(T_1 S_0)\times\mathcal{C}^\infty(S_0)$. Thus the natural space for a 3D 1-form field $\boldsymbol{u}$ after the normal reduction on the surface will be the space $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big)$.

Recall that $\boldsymbol{u} = (u_\alpha, u_3)$ denotes the 3D displacement solution of (2.1). The 3D equations are simpler when expressed with respect to the shifted displacement $\tilde{\boldsymbol{u}}$. That is why we will always denote by $\boldsymbol{w}$ the shifted displacement associated with $\boldsymbol{u}$.

Hence, in normal coordinates, the problem (2.1) can be written

$$\begin{aligned} \mathsf{L}(x_\alpha, x_3; \mathrm{D}_\alpha, \partial_3)\boldsymbol{w} &= -\boldsymbol{f} \quad \text{in} \quad \Omega^\varepsilon, \\ \mathsf{T}(x_\alpha, x_3; \mathrm{D}_\alpha, \partial_3)\boldsymbol{w} &= 0 \quad \text{on} \quad S_{\pm\varepsilon}, \\ \boldsymbol{w} &= 0 \quad \text{on} \quad \Gamma_0^\varepsilon, \end{aligned} \tag{2.6}$$

where $\boldsymbol{w}$ is the shifted displacement, and where the operators

$$\mathsf{L} : \mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big) \to \mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big), \quad \mathsf{T} : \mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big) \to \Sigma(S_{\pm\varepsilon})$$

are *intrinsic* operators. Note that the fact that we decided to define the operators L and T as taking values in the space $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big)$ and $\Sigma(S_{\pm\varepsilon})$ implies that the $\boldsymbol{f}$ appearing in the equation (2.6) is considered as a element of $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big)$. Hence, in terms of components, the covariants components of $\boldsymbol{f}$: $f_i = g_{ij} f^j$ are involved in the equations.

3 Expansion of the operators

In this section, we give the expansions of the operators L and T. All the framework for these computations can be found in [23] or in [14] for a similar presentation.

3.a Expansion of the connexion

In the following, we only write the dependence on x_3 of the tensor fields. Using the equations (2.4) and (2.5) we have that

$$g_{3i}(x_3) = \delta_{i3} \quad \text{and} \quad g_{\alpha\beta}(x_3) = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta}, \tag{3.1}$$

where δ_{ij} is the Kronecker tensor, and where we used the classical notation: $c_{\alpha\beta} = b_\alpha^\sigma b_{\sigma\beta}$. The equation (3.1) is valid in the space $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T_2 S_0)\big)$ and hence is *intrinsic* and does not depend on the coordinate system on S_0. Note that the relation (3.1) can be written $g_{\alpha\beta}(x_3) = \mu^\sigma_\alpha(x_3)\mu^\gamma_\beta(x_3)a_{\sigma\gamma}$. It is clear that for ε_0 sufficiently small, the shifter is invertible, and that we have the expansion:

$$(\mu^{-1})^\beta_\alpha(x_3) = \sum_{k=0}^{\infty} x_3^k (b^k)^\beta_\alpha, \tag{3.2}$$

where we write $(b^k)^\beta_\alpha$ for $b^{\nu_1}_\alpha b^{\nu_2}_{\nu_1}\cdots b^\beta_{\nu_{k-1}}$ with the convention $(b^0)^\beta_\alpha = \delta^\beta_\alpha$. Hence it is clear that the metric tensor is invertible, and we compute that:

$$g^{3i}(x_3) = \delta^{i3} \quad \text{and} \quad g^{\alpha\beta}(x_3) = \sum_{k=0}^{\infty}(k+1)x_3^k(b^k)^{\alpha\beta}. \tag{3.3}$$

Recall that the Christoffel symbols in any coordinate system are defined by the formula

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}),$$

where g^{ij} is the inverse of the metric tensor. From the previous equations we have that

$$\begin{aligned}
\Gamma^i_{33}(x_3) &= 0, \\
\Gamma^3_{\alpha 3}(x_3) &= 0, \\
\Gamma^3_{\alpha\beta}(x_3) &= -\tfrac{1}{2}g^{33}(x_3)\partial_3 g_{\alpha\beta}(x_3) = b_{\alpha\sigma}\mu^\sigma_\beta(x_3) = b_{\alpha\beta} - x_3 c_{\alpha\beta}, \\
\Gamma^\beta_{\alpha 3}(x_3) &= \tfrac{1}{2}g^{\beta\sigma}(x_3)\partial_3 g_{\alpha\sigma}(x_3) = -b^\sigma_\alpha(\mu^{-1})^\beta_\sigma(x_3) = -\textstyle\sum_{k=0}^{\infty} x_3^k(b^{k+1})^\beta_\alpha.
\end{aligned} \tag{3.4}$$

Moreover, it is clear that the Christoffel symbols $\Gamma^\sigma_{\alpha\beta}(x_3)$ are the Christoffel symbols of the connexion D^{x_3} on S_{x_3} . In the following we identify the connexion D and D^0 . Hence in a fixed normal coordinate system, the Christoffel symbols $\Gamma^\gamma_{\alpha\beta} := \Gamma^\gamma_{\alpha\beta}(0)$ are the Christoffel symbols associated with the connexion D . The following proposition gives the expansion of the Christoffel symbols (see [23, 14]):

Proposition 3.1 *In a fixed normal coordinate system, we have the expansion*

$$\Gamma^\gamma_{\alpha\beta}(x_3) = \Gamma^\gamma_{\alpha\beta} - \sum_{n=1}^{\infty} x_3^n (b^{n-1})^\gamma_\delta \mathrm{D}_\alpha b^\delta_\beta. \tag{3.5}$$

Note that this equation can also be written

$$\Gamma^\gamma_{\alpha\beta}(x_3) = \Gamma^\gamma_{\alpha\beta} - x_3(\mu^{-1})^\gamma_\delta(x_3)\mathrm{D}_\alpha b^\delta_\beta. \tag{3.6}$$

The most important fact is that the difference $\Gamma^\gamma_{\alpha\beta}(x_3) - \Gamma^\gamma_{\alpha\beta}$ is an element of the space $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T^1_2 S_0)\big)$ and thus is intrinsic. In particular, this implies that the covariant derivative of a tensor field on S_{x_3} expands with respect to x_3 in an intrinsic way. Let us take for example a tensor field $T_\alpha^\beta(x_3)$ viewed as a tensor field on S_{x_3} or as an element of $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T_1^1 S_0)\big)$. The covariant derivative $\mathrm{D}_\sigma^{x_3} T_\alpha^\beta(x_3)$ defines an element of $\Gamma(T_2^1 S_{x_3})$. But the previous expansion implies that in the space $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T_2^1 S_0)\big)$ we have the expansion

$$\mathrm{D}_\sigma^{x_3} T_\alpha^\beta(x_3) = \mathrm{D}_\sigma T_\alpha^\beta(x_3) + x_3(\mu^{-1})_\delta^\gamma(x_3) T_\gamma^\beta(x_3)\mathrm{D}_\sigma b_\alpha^\delta - x_3(\mu^{-1})_\delta^\beta(x_3) T_\alpha^\gamma(x_3)\mathrm{D}_\sigma b_\gamma^\delta,$$

and thus the covariant derivative D^{x_3} expands with respect to D . Similar formula can be found for other type of tensors.

3.B 2D-operators and deformation tensor

We call 2D-operator an operator independent of x_3 acting on $\Sigma(S_0)$ and taking values in a tensor field space on S_0 . We will see that the expansions of the operators L and T involve naturally 2D operators on S_0 . Let us define the following classical 2D operators: we first recall that the change of metric tensor is the 2D-operator $\boldsymbol{\gamma} : \Sigma(S_0) \to \Gamma(T_2 S_0)$ defined by

$$\gamma_{\alpha\beta}(\boldsymbol{z}) = \tfrac{1}{2}(\mathrm{D}_\alpha z_\beta + \mathrm{D}_\beta z_\alpha) - b_{\alpha\beta} z_3, \tag{3.7}$$

for $\boldsymbol{z} \in \Sigma(S_0)$. The change of curvature tensor is the 2D operator $\boldsymbol{\rho} : \Sigma(S_0) \to \Gamma(T_2 S_0)$ defined by:

$$\rho_{\alpha\beta}(\boldsymbol{z}) = \mathrm{D}_\alpha \mathrm{D}_\beta z_3 - c_{\alpha\beta} z_3 + b_\alpha^\sigma \mathrm{D}_\beta z_\sigma + \mathrm{D}_\alpha b_\beta^\sigma z_\sigma. \tag{3.8}$$

Moreover we define the operator $\theta_\alpha(\boldsymbol{z}) = \mathrm{D}_\alpha z_3 + b_\alpha^\sigma z_\sigma$ and the operator

$$\Lambda_{\alpha\beta}(\boldsymbol{z}) = \tfrac{1}{2}(b_\alpha^\sigma \mathrm{D}_\sigma z_\beta - b_\beta^\sigma \mathrm{D}_\alpha z_\sigma). \tag{3.9}$$

All these operators are intrinsic with respect to the mean surface S_0 . These operators act naturally by extension on the space $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big)$. Remark that the operator $\Lambda_{\alpha\beta}$ is not symmetric. Thus we write $\Lambda^{\alpha}_{\cdot\beta} = a^{\alpha\sigma}\Lambda_{\sigma\beta}$ for the corresponding $(1,1)$ tensor field.

The deformation tensor on Ω^ε writes $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$ in any coordinate system. We define the shifted deformation tensor as: $\tilde{e}_{ij}(\boldsymbol{w}) = e_{ij}(\boldsymbol{u})$ where $\boldsymbol{w}$ is the shifted displacement associated with $\boldsymbol{u}$. The following result gives the expansion of the operators $\tilde{e}_{ij}$:

Proposition 3.2 *For* $\boldsymbol{w} \in \mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big)$ *we have the expansions*

$$\begin{aligned} \tilde{e}_{33}(\boldsymbol{w}) &= \partial_3 w_3, \\ \tilde{e}_{\alpha 3}(\boldsymbol{w}) &= \tfrac{1}{2}\big(\partial_3 w_\alpha - x_3 b_\alpha^\beta \partial_3 w_\beta + \theta_\alpha(\boldsymbol{w})\big), \\ \tilde{e}_{\alpha\beta}(\boldsymbol{w}) &= \gamma_{\alpha\beta}(\boldsymbol{z}) + x_3\big(c_{\alpha\beta} w_3 - \tfrac{1}{2} b_\alpha^\sigma \mathrm{D}_\beta w_\sigma - \tfrac{1}{2} b_\beta^\sigma \mathrm{D}_\alpha w_\sigma\big). \end{aligned} \tag{3.10}$$

Remark that using the normal reduction of tensor fields, the first equation of (3.10) is valid in $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Sigma(S_0)\big)$, the second in $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T_1 S_0)\big)$ and the third in the space $\mathcal{C}^\infty\big((-\varepsilon,\varepsilon),\Gamma(T_2 S_0)\big)$.

Proof. (i) In a normal coordinate system we have using the equation (3.4),

$$e_{33}(\boldsymbol{u}) = \partial_3 u_3 - \Gamma^i_{33}(x_3) u_i = \partial_3 u_3.$$

As $u_3 = w_3$ we get the result.
(ii) On the same way we compute that

$$2e_{\alpha 3}(\boldsymbol{u}) = \partial_3 u_\alpha + \partial_\alpha u_3 - 2\Gamma^\beta_{\alpha 3}(x_3) u_\beta$$

using the fact that $\Gamma^3_{\alpha 3}(x_3) = 0$. Using the relations $u_\alpha = \mu^\beta_\alpha(x_3) w_\beta$ and (3.4), we have

$$\begin{aligned} 2\tilde{e}_{\alpha 3}(\boldsymbol{w}) &= \partial_3 \mu^\beta_\alpha(x_3) w_\beta + \partial_\alpha w_3 + 2b^\beta_\alpha w_\beta \\ &= \partial_3 w_\alpha - \partial_3(x_3 b^\beta_\alpha w_\beta) + \partial_\alpha w_3 + 2b^\beta_\alpha w_\beta, \end{aligned}$$

and we get the result.
(iii) We have that

$$e_{\alpha\beta}(\boldsymbol{u}) = \tfrac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^\gamma_{\alpha\beta}(x_3) u_\gamma - \Gamma^3_{\alpha\beta}(x_3) u_3.$$

Using (3.6) and (3.4) we get

$$e_{\alpha\beta}(\boldsymbol{u}) = \tfrac{1}{2}(\mathrm{D}_\alpha u_\beta + \mathrm{D}_\beta u_\alpha) + x_3 w_\delta \mathrm{D}_\alpha b^\delta_\beta - b_{\alpha\beta} w_3 + x_3 c_{\alpha\beta} w_3.$$

We find the result using $u_\alpha = w_\alpha - x_3 b^\beta_\alpha w_\beta$ and the Codazzi-Mainardi equation. ■

Recall that $\tilde{e}^\beta_\alpha(\boldsymbol{w}) = g^{\beta\delta}(x_3)\tilde{e}_{\delta\alpha}(\boldsymbol{w})$. Using Proposition 3.2 and the expansion (3.3) of $g^{\alpha\beta}(x_3)$ we can show that we have expansion (see [14]):

$$\tilde{e}^\alpha_\beta(\boldsymbol{w}) = \gamma^\alpha_\beta(\boldsymbol{w}) + \sum_{n=1}^{\infty} x_3^n (b^n)^\alpha_\delta \gamma^\delta_\beta(\boldsymbol{w}) + \sum_{n=1}^{\infty} n x_3^n (b^{n-1})^\alpha_\delta \Lambda^\delta{}_{\cdot\beta}(\boldsymbol{w}). \tag{3.11}$$

3.c Expansion of the operator

In normal coordinates, the system (2.1) writes

$$\begin{aligned} -\nabla_j A^{ijk\ell} e_{k\ell}(\boldsymbol{u}) &= f^i \quad \text{in} \quad \Omega^\varepsilon \\ A^{3jk\ell} e_{k\ell}(\boldsymbol{u}) &= 0 \quad \text{on} \quad S_{\pm\varepsilon} \\ \boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma^\varepsilon_0, \end{aligned}$$

where $A^{ijk\ell} = \lambda g^{ij}(x_3) g^{k\ell}(x_3) + 2\mu(g^{ik}(x_3) g^{j\ell}(x_3) + g^{i\ell}(x_3) g^{jk}(x_3))$. Hence, using the definitions of L and T and the fact that the covariant derivative ∇ commutes with the metric, we have:

$$\mathsf{L}_i(\boldsymbol{w}) = A_{i\cdots}^{\cdot jk\ell} \nabla_j \tilde{e}_{k\ell}(\boldsymbol{w}) \quad \text{and} \quad \mathsf{T}_i(\boldsymbol{w}) = A_{i\cdots}^{\cdot 3k\ell} \tilde{e}_{k\ell}(\boldsymbol{w}).$$

Hence we have

$$\mathsf{L}_i(\boldsymbol{w}) = \lambda \nabla_i \tilde{e}^\ell_\ell(\boldsymbol{w}) + 2\mu \nabla_\ell \tilde{e}^\ell_i(\boldsymbol{w}) \quad \text{and} \quad \mathsf{T}_i(\boldsymbol{w}) = \lambda \delta^3_i \tilde{e}^\ell_\ell(\boldsymbol{w}) + 2\mu \tilde{e}^3_i(\boldsymbol{w}). \tag{3.12}$$

It is clear that the operators L and T admit power series expansions with respect to x_3. In order to set the problem in a manifold independent on ε, we make the following scaling on the transverse variable: We define the manifold $\Omega = S\times(-1,1)$ and we set $I=(-1,1)$. A normal coordinate system on Ω is a coordinate system of the form (x_α, X_3) where $\{x_\alpha\}$ is a coordinate system on S and where $x_3 = \varepsilon X_3$ is the corresponding point in Ω^ε. We note $\Gamma_\pm$ the upper and lower faces of Ω and Γ_0 the lateral boundary corresponding to $\partial S\times I$.

We then define the 3D elasticity operator on Ω as the operators $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$:

$$\mathsf{L}(\varepsilon): \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \quad \text{and} \quad \mathsf{T}(\varepsilon): \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \Sigma(\Gamma_\pm),$$

obtained from L and T after the scaling $x_3 = \varepsilon X_3$. Hence in any normal coordinate system, we have

$$\big(\mathsf{L}(\varepsilon), \mathsf{T}(\varepsilon)\big)(x_\alpha, X_3; \mathrm{D}_\alpha, \partial_{X_3}) = (\mathsf{L},\mathsf{T})(x_\alpha, \varepsilon X_3; \mathrm{D}_\alpha, \varepsilon^{-1}\partial_{X_3}).$$

In the following, if $\boldsymbol{w}\in\mathcal{C}^\infty\big(I,\Sigma(S_0)\big)$ is independent on ε, we set $\tilde{e}_\alpha^\beta(\varepsilon)\boldsymbol{w}$ the deformation tensor after the scaling. Hence we can write the equation (3.11) as

$$\tilde{e}_\beta^\alpha(\varepsilon)\boldsymbol{w} = \sum_{k=0}^\infty \varepsilon^k \big(\tilde{e}_\beta^\alpha(\varepsilon)\boldsymbol{w}\big)^k \tag{3.13}$$

where for all $k\geq 0$,

$$\big(\tilde{e}_\beta^\alpha(\varepsilon)\boldsymbol{w}\big)^k = X_3^k (b^k)_\delta^\alpha \gamma_\beta^\delta(\boldsymbol{w}) + k\, X_3^k (b^{k-1})_\delta^\alpha \Lambda^\delta_{\cdot\,\beta}(\boldsymbol{w}). \tag{3.14}$$

Using the previous expansions, it is easy to show that the operators $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$ expand in power series of ε. The following theorem gives the expressions of these expansions. As it is a simple computation, the proof is given in Appendix A. Note that all the framework required to obtain these expansions is present in [23].

Theorem 3.3 *The operators $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$ expand in power series of ε for $\varepsilon\leq\varepsilon_0$ and write*

$$\mathsf{L}(\varepsilon) = \varepsilon^{-2}\sum_{k=0}^\infty \varepsilon^k \mathsf{L}^k \quad \textit{and} \quad \mathsf{T}(\varepsilon) = \varepsilon^{-2}\sum_{k=0}^\infty \varepsilon^k \mathsf{T}^k,$$

where we have the following expressions: for $k=0$, we have $\mathsf{L}^0 = \ell\circ\partial_{X_3}^2$ et $\mathsf{T}^0 = \ell\circ\partial_{X_3}$, where $\ell(\boldsymbol{w}) = \big(\mu w_\alpha, (\lambda+2\mu)w_3\big)$, moreover

$$\begin{aligned} \mathsf{L}_\sigma^1(\boldsymbol{w}) &= -\mu b_\alpha^\alpha \partial_{X_3} w_\sigma + (\lambda+\mu)\mathrm{D}_\sigma \partial_{X_3} w_3 - X_3 \mu b_\sigma^\alpha \partial_{X_3}^2 w_\alpha, \\ \mathsf{L}_3^1(\boldsymbol{w}) &= -\mu b_\alpha^\alpha \partial_{X_3} w_3 + (\lambda+\mu)\gamma_\alpha^\alpha(\partial_{X_3}\boldsymbol{w}), \end{aligned} \tag{3.15}$$

and

$$\mathsf{L}^2_\sigma(\boldsymbol{w}) = -\mu X_3 c^\alpha_\alpha \partial_{X_3} w_\sigma + \mu X_3 b^\alpha_\alpha b^\beta_\sigma \partial_{X_3} w_\beta - \mu b^\alpha_\alpha \mathrm{D}_\sigma w_3 - \mu b^\beta_\beta b^\alpha_\sigma w_\alpha + \lambda \mathrm{D}_\sigma \gamma^\alpha_\alpha(\boldsymbol{w}) + 2\mu \mathrm{D}_\alpha \gamma^\alpha_\sigma(\boldsymbol{w}), \quad (3.16)$$

$$\mathsf{L}^2_3(\boldsymbol{w}) = -\mu X_3 c^\alpha_\alpha \partial_{X_3} w_3 + (\lambda+\mu) b^\beta_\alpha \gamma^\alpha_\beta\big(\partial_{X_3}(X_3 \boldsymbol{w})\big) + \mu b^\beta_\alpha \gamma^\alpha_\beta(\boldsymbol{w}) + \mu \mathrm{D}^\alpha \theta_\alpha(\boldsymbol{w}).$$

Moreover, for $n \geq 3$ *, we have*

$$\begin{aligned} \mathsf{L}^n_\sigma(\boldsymbol{w}) = & -\mu X_3^{n-1} (b^n)^\alpha_\alpha \partial_{X_3} w_\sigma + \mu X_3^{n-1} (b^{n-1})^\alpha_\alpha b^\delta_\sigma \partial_{X_3} w_\delta - \mu X_3^{n-2} (b^{n-1})^\alpha_\alpha \theta_\sigma(\boldsymbol{w}) \\ & + \lambda \mathrm{D}_\sigma (b^{n-2})^\alpha_\beta \gamma^\beta_\alpha (X_3^{n-2} \boldsymbol{w}) + 2\mu \mathrm{D}_\alpha \big(\tilde{e}^\alpha_\sigma(\varepsilon)\boldsymbol{w}\big)^{n-2} \\ & + 2\mu \textstyle\sum_{k=1}^{n-2} X_3^k (b^{k-1})^\delta_\nu \big(\tilde{e}^\alpha_\delta(\varepsilon)\boldsymbol{w}\big)^{n-2-k} \mathrm{D}_\alpha b^\nu_\sigma \\ & - 2\mu \textstyle\sum_{k=1}^{n-2} X_3^k (b^{k-1})^\alpha_\nu \big(\tilde{e}^\delta_\sigma(\varepsilon)\boldsymbol{w}\big)^{n-2-k} \mathrm{D}_\alpha b^\nu_\delta, \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \mathsf{L}^n_3(\boldsymbol{w}) = & -\mu X_3^{n-1} (b^n)^\alpha_\alpha \partial_{X_3} w_3 + (\lambda+\mu)(b^{n-1})^\alpha_\beta \gamma^\beta_\alpha \big(\partial_{X_3}(X_3^{n-1}\boldsymbol{w})\big) \\ & + \mu(n-1)(b^{n-1})^\alpha_\beta \gamma^\beta_\alpha (X_3^{n-2}\boldsymbol{w}) + \mu \textstyle\sum_{\ell=0}^{n-2} (b^\ell)^\delta_\alpha \mathrm{D}^\alpha (b^{n-2-\ell})^\beta_\delta \theta_\beta (X_3^{n-2}\boldsymbol{w}), \end{aligned} \quad (3.18)$$

where for all $k \geq 0$ *,* $\big(\tilde{e}^\alpha_\beta(\varepsilon)\boldsymbol{w}\big)^k$ *is given by the equation* (3.14)*. For the traction operator, we have*

$$\mathsf{T}^1_\sigma(\boldsymbol{w}) = \mu\theta_\sigma(\boldsymbol{w}) - \mu X_3 b^\alpha_\sigma \partial_{X_3} w_\alpha, \quad \textit{and} \quad \mathsf{T}^n_\sigma(\boldsymbol{w}) = 0 \quad \textit{for} \quad n \geq 2, \quad (3.19)$$

and

$$\mathsf{T}^n_3(\boldsymbol{w}) = \lambda (b^{n-1})^\alpha_\beta \gamma^\beta_\alpha (X_3^{n-1}\boldsymbol{w}) \quad \textit{for} \quad n \geq 1. \quad (3.20)$$

4 Formal series solution

Using the theorem 3.3, we associate with the operators $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$ the *formal series* $\mathsf{L}[\varepsilon]$ and $\mathsf{T}[\varepsilon]$ defined by

$$\mathsf{L}[\varepsilon] = \varepsilon^{-2} \sum_{k\geq 0} \varepsilon^k \mathsf{L}^k \quad \text{and} \quad \mathsf{T}[\varepsilon] = \varepsilon^{-2} \sum_{k\geq 0} \varepsilon^k \mathsf{T}^k.$$

We recall that if E and F are two function spaces, if $a[\varepsilon] = \sum_{k\geq 0} \varepsilon^k a^k$ is a formal series with coefficients $a^k \in \mathcal{L}(E,F)$, and if $b[\varepsilon] = \sum_{k\geq 0} \varepsilon^k b^k$ is a formal series with coefficients $b^k \in E$, then the formal series $c[\varepsilon] = a[\varepsilon]b[\varepsilon]$ is defined by the equation $c[\varepsilon] = \sum_{k\geq 0} \varepsilon^k c^k$ where for all n , $c^n = \sum_{k=0}^n a^k b^{n-k}$.

Let $\boldsymbol{f}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{f}^k$ be a formal series with coefficients in $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$. The *three-dimensional formal series problem* is the problem of finding a formal series $\boldsymbol{w}[\varepsilon] = \sum_k \varepsilon^k \boldsymbol{w}^k$ with coefficients in $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$, solution of the equations

$$\begin{aligned} \mathsf{L}[\varepsilon]\boldsymbol{w}[\varepsilon] &= -\boldsymbol{f}[\varepsilon] \quad \text{in} \quad \Omega, \\ \mathsf{T}[\varepsilon]\boldsymbol{w}[\varepsilon] &= 0 \quad \text{on} \quad \Gamma_\pm, \\ \boldsymbol{w}[\varepsilon] &= 0 \quad \text{on} \quad \Gamma_0. \end{aligned} \tag{4.1}$$

In the following we denote by $\mathcal{I} : \Sigma(S_0) \to \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ the canonical injection.

4.a First terms

We will first consider the two first equations in (4.1):

$$\begin{aligned} \mathsf{L}[\varepsilon]\boldsymbol{w}[\varepsilon] &= -\boldsymbol{f}[\varepsilon] \quad \text{in} \quad \Omega, \\ \mathsf{T}[\varepsilon]\boldsymbol{w}[\varepsilon] &= 0 \quad \text{on} \quad \Gamma_\pm. \end{aligned} \tag{4.2}$$

These equations are in fact collections of equations written:

$$\forall n \in \mathbb{N}, \quad \left\{ \begin{aligned} \textstyle\sum_{\ell=0}^n \mathsf{L}^\ell \boldsymbol{w}^{n-\ell} &= -\boldsymbol{f}^{n-2} \quad \text{in} \quad \Omega, \\ \textstyle\sum_{\ell=0}^n \mathsf{T}^\ell \boldsymbol{w}^{n-\ell} &= 0 \quad \text{on} \quad \Gamma_\pm. \end{aligned} \right. \tag{4.3}$$

For $\ell < 0$, we pose $\boldsymbol{f}^\ell = 0$. According to [18], we pose $p = \lambda(\lambda + 2\mu)^{-1}$. For $n = 0$, the equations (4.3) write

$$\left\{ \begin{aligned} \mathsf{L}^0 \boldsymbol{w}^0 &= 0 \quad \text{in} \quad \Omega, \\ \mathsf{T}^0 \boldsymbol{w}^0 &= 0 \quad \text{on} \quad \Gamma_\pm. \end{aligned} \right. \tag{4.4}$$

The transverse components of this equation write

$$\left\{ \begin{aligned} (\lambda + 2\mu)\partial^2_{X_3} w^0_3 &= 0 \quad \text{in} \quad \Omega, \\ (\lambda + 2\mu)\partial_{X_3} w^0_3 &= 0 \quad \text{on} \quad \Gamma_\pm. \end{aligned} \right.$$

We thus get immediately

$$\int_{-1}^{X_3} \partial^2_{X_3} w^0_3 \,\mathrm{d}X_3 = 0 = \partial_{X_3} w^0_3(X_3),$$

whence $w^0_3(x_\alpha, X_3) = z^0_3(x_\alpha)$ where z^0_3 is a function independent on X_3 . Similarly, the surfacic components of (4.4) write

$$\left\{ \begin{aligned} \mu\partial^2_{X_3} w^0_\sigma &= 0 \quad \text{in} \quad \Omega, \\ \mu\partial_{X_3} w^0_\sigma &= 0 \quad \text{on} \quad \Gamma_\pm. \end{aligned} \right.$$

Hence in a normal coordinate system, we have $w^0_\sigma(x_\alpha, X_3) = z^0_\sigma(x_\alpha)$. Moreover, it is clear that the components z_α define a 1-form field on S_0, independent on X_3. Thus we have the intrinsic equation $\boldsymbol{w}^0 = \boldsymbol{z}^0$ in the space $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$.

For $n = 1$ the equations (4.3) write

$$\left\{\begin{array}{rcll} \mathsf{L}^0\boldsymbol{w}^1 &=& -\mathsf{L}^1\boldsymbol{w}^0 & \text{in} \quad \Omega, \\ \mathsf{T}^0\boldsymbol{w}^1 &=& -\mathsf{T}^1\boldsymbol{w}^0 & \text{on} \quad \Gamma_\pm. \end{array}\right. \tag{4.5}$$

But as $\boldsymbol{w}^0 = \boldsymbol{z}^0$ is independent on X_3, and using the expressions (3.15) of the operator L^1 and (3.19), (3.20) of T^1, we see that $\mathsf{L}^1\boldsymbol{w}^0 = 0$, $\mathsf{T}^1_\sigma(\boldsymbol{w}^0) = \mu\theta_\sigma(\boldsymbol{z}^0)$ and $\mathsf{T}^1_3(\boldsymbol{w}^0) = \lambda\gamma^\alpha_\alpha(\boldsymbol{z}^0)$. Thus the transverse components of the equation (4.5) write

$$\left\{\begin{array}{rcll} (\lambda + 2\mu)\partial^2_{X_3} w^1_3 &=& 0 & \text{in} \quad \Omega, \\ (\lambda + 2\mu)\partial_{X_3} w^1_3 &=& -\lambda\gamma^\alpha_\alpha(\boldsymbol{z}^0) & \text{on} \quad \Gamma_\pm. \end{array}\right.$$

These equations define a Neumann problem whose compatibility condition $\big[\gamma^\alpha_\alpha(\boldsymbol{z}^0)\big]^{+1}_{-1} = 0$ is always satisfied. Hence we have

$$\partial_{X_3} w^1_3(X_3) = \partial_{X_3} w^1_3(-1) = -p\gamma^\alpha_\alpha(\boldsymbol{z}^0),$$

and thus $w^1_3 = z^1_3 - X_3 p\gamma^\alpha_\alpha(\boldsymbol{z}^0)$ where, as before, the function z^1_3 is independent on X_3. Moreover, the surfacic components of the equations (4.5) write

$$\left\{\begin{array}{rcll} \mu\partial^2_{X_3} w^1_\sigma &=& 0 & \text{in} \quad \Omega, \\ \mu\partial_{X_3} w^1_\sigma &=& -\mu\theta_\sigma(\boldsymbol{z}^0) & \text{on} \quad \Gamma_\pm. \end{array}\right.$$

As before, we deduce that $w^1_\sigma = z^1_\sigma - X_3\theta_\sigma(\boldsymbol{z}^0)$ where z^1_σ does not depend on X_3 and defines a 1-form field on S_0. Finally we have the equations

$$\boldsymbol{w}^0 = \boldsymbol{z}^0 \quad \text{and} \quad \boldsymbol{w}^1 = \boldsymbol{z}^1 + \mathsf{V}^1\boldsymbol{z}^0, \tag{4.6}$$

where $\boldsymbol{z}^0$ and $\boldsymbol{z}^1$ are elements of the space $\Sigma(S_0)$, and $\mathsf{V}^1 : \Sigma(S_0) \to \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ is the operator defined by

$$\mathsf{V}^1_\sigma(\boldsymbol{z}) = -X_3\theta_\sigma(\boldsymbol{z}) \quad \text{and} \quad \mathsf{V}^1_3(\boldsymbol{z}) - X_3 p\gamma^\alpha_\alpha(\boldsymbol{z}). \tag{4.7}$$

The equation (4.6) show that the first terms of the formal series $\boldsymbol{w}[\varepsilon]$ solution of the problem (4.2) can be computed from the terms $\boldsymbol{z}^0$ and $\boldsymbol{z}^1$ and are polynomials in X_3.

4.B Formal series reduction

Let $\boldsymbol{w}[\varepsilon]$ be a formal series solution of the problem (4.2). The coefficients of this formal series satisfy the equations:

$$\forall k \in \mathbb{N}, \quad \left\{\begin{array}{rcll} \partial^2_{X_3}\boldsymbol{w}^k &=& -\sum_{\ell=1}^k \ell^{-1}\mathsf{L}^\ell\boldsymbol{w}^{k-\ell} - \ell^{-1}\boldsymbol{f}^{k-2} & \text{in} \quad \Omega \\ \partial_{X_3}\boldsymbol{w}^k &=& -\sum_{\ell=1}^k \ell^{-1}\mathsf{T}^\ell\boldsymbol{w}^{k-\ell} & \text{on} \quad \Gamma_\pm. \end{array}\right. \tag{4.8}$$

In the previous subsection, we showed that the two first terms of the formal series $\boldsymbol{w}[\varepsilon]$ depend on elements of $\Sigma(S_0)$. This dimension reduction constitute the main point of this section. We now resume the general facts.

Let k fixed, and let us suppose that there exist $k+1$ elements $\boldsymbol{z}^j \in \Sigma(S_0)$ for $j=0,\ldots,k$, such that for $j \leq k$, the terms $\boldsymbol{w}^j$ depend only on $\boldsymbol{z}^0,\ldots,\boldsymbol{z}^j$ and on $\boldsymbol{f}^0,\ldots,\boldsymbol{f}^{j-2}$. The equation (4.8) for $k+1$ show that $\boldsymbol{w}^{k+1}$ is a solution of a Neumann problem on I , which implies a compatibility condition on the right-hand side. This compatibility condition yields an equation for the terms $\boldsymbol{z}^j$ for $j=0,\ldots,k$ and $\boldsymbol{f}^j$ for $j=0,\ldots,k-1$. The term $\boldsymbol{w}^{k+1}$ is hence the sum of a term depending on the $\boldsymbol{z}^j$, for $j=0,\ldots,k$, a term depending on the $\boldsymbol{f}^j$ for $j=0,\ldots,k-1$, and of an element of the kernel of the Neumann problem. But this kernel turn to be precisely the space $\Sigma(S_0)$. We name this element $\boldsymbol{z}^{k+1}$, and this shows by induction that there exists a family $\{\boldsymbol{z}^k\}$ of elements of $\Sigma(S_0)$ such that for each k , the term $\boldsymbol{w}^k$ depends on the $\boldsymbol{z}^j$ for $j=0,\ldots,k$ and on the terms $\boldsymbol{f}^j$ for $j=0,\ldots,k-2$. Moreover, the terms $\{\boldsymbol{z}^k\}$ satisfy a collection of equations coming from the successive compatibility conditions.

The goal of this section is to show that the relation between the $\{\boldsymbol{w}^k\}$ and the two families $\{\boldsymbol{z}^k\}$ and $\{\boldsymbol{f}^k\}$ can be written as an equation in formal series: there exist formal series operators $\mathsf{V}[\varepsilon]$ and $\mathsf{Q}[\varepsilon]$ such that

$$\boldsymbol{w}[\varepsilon] = \mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{4.9}$$

where the coefficients V^k are operators acting on $\Sigma(S_0)$ and taking values in the 3D 1-form fields space. The coefficients Q^k are operators on the 3D 1-form fields. The computations made in the previous subsection show that we can take $\mathsf{V}^0 = \mathcal{I}$ and $\mathsf{V}^1 = \mathsf{V}^1$, and for Q^0 and Q^1 the null operator.

Moreover, the compatibility condition can also be written

$$\mathsf{A}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathsf{G}[\varepsilon]\boldsymbol{f}[\varepsilon] \tag{4.10}$$

where $\mathsf{A}[\varepsilon]$ is a formal series with 2D operator coefficients taking value in $\Sigma(S_0)$, and $\mathsf{G}[\varepsilon]$ is a formal series with coefficient operators on the 3D 1-form field space, taking value in $\Sigma(S_0)$. Hence, the formal series $\boldsymbol{w}[\varepsilon]$ is a solution of (4.2) if and only if there exists a formal series $\boldsymbol{z}[\varepsilon]$ with coefficients in $\Sigma(S_0)$ satisfying the equations (4.9) and (4.10).

In the previous computations, the compatibility conditions at the rank 0 and 1 where obviously satisfied. The operators A^n and G^n are determined from the compatibility conditions at the order $k \geq 2$. We now show the existence of the operators involved in the previous equations (4.9) and (4.10):

Theorem 4.1 (i) *For all $k \in \mathbb{N}$, there exist in a unique way:*

- *an operator $\mathsf{V}^k : \Sigma(S_0) \to \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ polynomial in X_3 with* 2D *operator coefficients, vanishing on S_0 for $k \geq 1$,*
- *a* 2D *operator $\mathsf{A}^k : \Sigma(S_0) \to \Sigma(S_0)$,*

such that the formal series $\mathsf{V}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \mathsf{V}^k$ *and* $\mathsf{A}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \mathsf{A}^k$ *satisfy the equations*

$$\begin{cases} \mathsf{L}[\varepsilon]\mathsf{V}[\varepsilon](\boldsymbol{z}) = -\mathcal{I} \circ \mathsf{A}[\varepsilon](\boldsymbol{z}), \\ \mathsf{T}[\varepsilon]\mathsf{V}[\varepsilon](\boldsymbol{z}) = 0, \end{cases} \tag{4.11}$$

for all $\boldsymbol{z} \in \Sigma(S_0)$ *. Moreover,* V^0 *is the canonical embedding* $\mathcal{I}$ *and* V^1 *is the operator defined by* (4.7)*.*

(ii) *For all* $k \in \mathbb{N}$ *, there exist in a unique way:*

- *an operator* $\mathsf{Q}^k : \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ *composition of* 2D *operator and integration with respect to* X_3 *, and vanishing on the mean surface,*
- *an operator* $\mathsf{G}^k : \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \Sigma(S_0)$ *composition of* 2D *operators and integration with respect to* X_3 *on* I *,*

such that the formal series $\mathsf{G}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \mathsf{G}^k$ *and* $\mathsf{Q}[\varepsilon] = \sum_{k\geq 2} \varepsilon^k \mathsf{Q}^k$ *satisfy the equations*

$$\begin{cases} \mathsf{L}[\varepsilon]\mathsf{Q}[\varepsilon](\boldsymbol{f}) = \mathcal{I} \circ \mathsf{G}[\varepsilon](\boldsymbol{f}) - \boldsymbol{f}, \\ \mathsf{T}[\varepsilon]\mathsf{Q}[\varepsilon](\boldsymbol{f}) = 0, \end{cases} \tag{4.12}$$

for all $\boldsymbol{f} \in \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ *. Moreover the operators* Q^0 *and* Q^1 *are the null operators, and we have*

$$\mathsf{G}^0(\boldsymbol{f}) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{f}(X_3)\,\mathrm{d}X_3. \tag{4.13}$$

Proof. We will show the existence of formal series $\mathsf{V}[\varepsilon]$, $\mathsf{A}[\varepsilon]$, $\mathsf{Q}[\varepsilon]$ and $\mathsf{G}[\varepsilon]$ satisfying the condition of the theorem, and such that the following equations are valid:

$$\begin{cases} \mathsf{L}[\varepsilon]\mathsf{V}[\varepsilon] = -\mathcal{I} \circ \mathsf{A}[\varepsilon], \\ \mathsf{T}[\varepsilon]\mathsf{V}[\varepsilon] = 0, \end{cases} \quad \text{and} \quad \begin{cases} \mathsf{L}[\varepsilon]\mathsf{Q}[\varepsilon] = \mathcal{I} \circ \mathsf{G}[\varepsilon] - \mathbf{Id}, \\ \mathsf{T}[\varepsilon]\mathsf{Q}[\varepsilon] = 0, \end{cases} \tag{4.14}$$

in the corresponding formal series spaces.

1. We first show the existence of the formal series $\mathsf{V}[\varepsilon]$ and $\mathsf{A}[\varepsilon]$ satisfying the first system in (4.14). This means that

$$\forall n \in \mathbb{N} \quad \begin{cases} \sum_{k=0}^{n} \mathsf{L}^k \mathsf{V}^{n-k} = -\mathcal{I}\mathsf{A}^{n-2}, \\ \sum_{k=0}^{n} \mathsf{T}^k \mathsf{V}^{n-k} = 0. \end{cases} \tag{4.15}$$

These equations write

$$\forall n \in \mathbb{N} \quad \begin{cases} \partial^2_{X_3} \mathsf{V}^n = -\sum_{k=1}^{n} \ell^{-1} \mathsf{L}^k \mathsf{V}^{n-k} - \ell^{-1}\mathcal{I}\mathsf{A}^{n-2}, \\ \partial_{X_3} \mathsf{V}^n = -\sum_{k=1}^{n} \ell^{-1} \mathsf{T}^k \mathsf{V}^{n-k}. \end{cases} \tag{4.16}$$

We set $\mathsf{A}^{-1} = \mathsf{A}^{-2} = 0$, and the previous subsection shows that (4.16) is satisfied for $n = 0$ et 1 with the operators V^0 et V^1 defined in the theorem.
Let $n \geq 0$, and suppose that the operators V^ℓ and $\mathsf{A}^{\ell-2}$ are determined for $\ell = 0, \dots, n$, such that the operators V^ℓ are polynomials in X_3 . let $\boldsymbol{z} \in \Sigma(S_0)$. Consider the equations

$$\left\{ \begin{array}{rcll} \partial^2_{X_3} \boldsymbol{v} & = & -\sum_{k=1}^{n+1} \ell^{-1} \mathsf{L}^k \mathsf{V}^{n+1-k} \boldsymbol{z} - \ell^{-1} \mathcal{I} \mathsf{A}^{n-1} \boldsymbol{z} & \text{in} \quad \Omega, \\ \partial_{X_3} \boldsymbol{v} & = & -\sum_{k=1}^{n+1} \ell^{-1} \mathsf{T}^k \mathsf{V}^{n+1-k} \boldsymbol{z} & \text{on} \quad \Gamma_\pm. \end{array} \right. \tag{4.17}$$

with unknowns $\boldsymbol{v}$ and $\mathsf{A}^{n-1}\boldsymbol{z}$. This is a Neumann problem on I , and the corresponding compatibility condition write

$$\int_{-1}^{1} \partial^2_{X_3} \boldsymbol{v} \, \mathrm{d}X_3 = \big[\partial_{X_3} \boldsymbol{v}\big]_{-1}^{+1},$$

which writes again

$$\begin{aligned} 2\mathsf{A}^{n-1}\boldsymbol{z} = -\textstyle\sum_{k=1}^{n+1} \int_{-1}^{1} \big(\mathsf{L}^k \mathsf{V}^{n+1-k} \boldsymbol{z}\big)(X_3) \, \mathrm{d}X_3 \\ + \textstyle\sum_{k=1}^{n+1} \big(\mathsf{T}^k \mathsf{V}^{n+1-k} \boldsymbol{z}\big)(+1) - \textstyle\sum_{k=1}^{n+1} \big(\mathsf{T}^k \mathsf{V}^{n+1-k} \boldsymbol{z}\big)(-1). \end{aligned} \tag{4.18}$$

This equation defines the operator A^{n-1} . The unique solution $\boldsymbol{v} = \mathsf{V}^{n+1}\boldsymbol{z}$ vanishing on the mean surface (for $X_3 = 0$) of the equation (4.17) yields the operator V^{n+1} . It is easy to verify that V^{n+1} is a polynomial in X_3 with 2D operators coefficients.
2. The second system in the equation (4.14) means that the formal series $\mathsf{Q}[\varepsilon]$ and $\mathsf{G}[\varepsilon]$ satisfy the following equations:

$$\forall n \in \mathbb{N}, \qquad \left\{ \begin{array}{l} \sum_{k=0}^{n} \mathsf{L}^{n-k} \mathsf{Q}^k = \mathcal{I} \mathsf{G}^{n-2} - \delta_n^2 \, \mathbf{Id}, \\ \sum_{k=0}^{n} \mathsf{T}^{n-k} \mathsf{Q}^k = 0, \end{array} \right. \tag{4.19}$$

where δ_n^2 is the Kronecker tensor. Setting $\mathsf{Q}^0 = \mathsf{Q}^1 = 0$ and $\mathsf{G}^{-2} = \mathsf{G}^{-1} = 0$, we see that these equations are satisfied for $n = 0$ et $n = 1$. Let $\boldsymbol{f} \in \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$. For $n = 2$, the problem is to find $\boldsymbol{q}$ and $\mathsf{G}^0 \boldsymbol{f}$ solution of

$$\left\{ \begin{array}{rcll} \partial^2_{X_3} \boldsymbol{q} & = & \ell^{-1} \mathcal{I} \mathsf{G}^0 \boldsymbol{f} - \ell^{-1} \boldsymbol{f} & \text{in} \quad \Omega, \\ \partial_{X_3} \boldsymbol{q} & = & 0 & \text{on} \quad \Gamma_\pm. \end{array} \right.$$

This Neumann problem has a solution if and only if a compatibility condition is satisfied, and we find the expression (4.13) for the operator G^0 . The operator Q^2 is thus the unique solution of the Neumann problem vanishing for $X_3 = 0$:

$$\mathsf{Q}^2 \boldsymbol{f} = \int_0^{X_3} \Big(\int_{-1}^{u} \ell^{-1} \big(\mathcal{I} \mathsf{G}^0 \boldsymbol{f} - \boldsymbol{f}\big)(t) \, \mathrm{d}t \Big) \mathrm{d}u. \tag{4.20}$$

Suppose now that the operators Q^ℓ and $\mathsf{G}^{\ell-2}$ are determined for $\ell = 0,\dots,n$, with $n \geq 3$. Let $\boldsymbol{f} \in \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ and consider the problem:

$$\begin{cases} \partial^2_{X_3}\boldsymbol{q} &= -\sum_{k=2}^{n+1} \ell^{-1}\mathsf{L}^{n+1-k}\mathsf{Q}^k\boldsymbol{f} + \ell^{-1}\mathcal{I}\mathsf{G}^{n-1}\boldsymbol{f} \quad \text{in} \quad \Omega, \\ \partial_{X_3}\boldsymbol{q} &= -\sum_{k=2}^{n+1} \ell^{-1}\mathsf{T}^{n+1-k}\mathsf{Q}^k\boldsymbol{f} \quad \text{on} \quad \Gamma_\pm, \end{cases} \tag{4.21}$$

with unknowns $\boldsymbol{q}$ and $\mathsf{G}^{n-1}\boldsymbol{f}$. Again, this problem has a solution if the compatibility condition

$$2\mathsf{G}^{n-1}\boldsymbol{f} = \sum_{k=2}^{n+1}\int_{-1}^{1}\big(\mathsf{L}^{n+1-k}\mathsf{Q}^k\boldsymbol{f}\big)(X_3)\,\mathrm{d}X_3 \\ - \sum_{k=2}^{n+1}\big(\mathsf{T}^{n+1-k}\mathsf{Q}^k\boldsymbol{f}\big)(+1) + \sum_{k=2}^{n+1}\big(\mathsf{T}^{n+1-k}\mathsf{Q}^k\boldsymbol{f}\big)(-1) \tag{4.22}$$

is satisfied. This equation defines the operator G^{n-1}, and the operator Q^{n+1} is the unique solution operator vanishing for $X_3 = 0$. ■

Let us consider a 2D operator acting on $\boldsymbol{z} = (z_\alpha, z_3) \in \Sigma(S_0)$. The notion of *surfacic* derivative order in z_α or z_3 is intrinsic. Hence if A is an operator taking values in $\Sigma(S_0)$, we set $a_i(j)$ the derivative order in z_j of the operator A_i, and we write

$$\deg A = \begin{pmatrix} a_\sigma(\alpha) & a_\sigma(3) \\ a_3(\alpha) & a_3(3) \end{pmatrix}.$$

Moreover, we define the order relation:

$$\deg A \leq \deg B \quad \Longleftrightarrow \quad \forall i,j \quad a_i(j) \leq b_i(j),$$

where the $b_i(j)$ are the orders of B. We also make the convention that an operator with negative order of derivative is the null operator, and that the null operator has the derivative degree $-\infty$. Note that this notation extends obviously to 2D operators acting on the space $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$. Using the previous theorem, we see that

$$\deg \mathsf{V}^0 = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix} \quad \text{and} \quad \deg \mathsf{V}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.23}$$

We now give the following result:

Proposition 4.2 *With the notations of Theorem* 4.1, *the operators* V^k *are polynomials of degree* k *in* X_3 *for all* $k \geq 0$. *Moreover, for all* $p \geq 0$ *we have the estimates*

$$\deg \mathsf{V}^{2p} \leq \begin{pmatrix} 2p & 2p-1 \\ 2p-1 & 2p \end{pmatrix} \quad \textit{and} \quad \deg \mathsf{V}^{2p+1} \leq \begin{pmatrix} 2p & 2p+1 \\ 2p+1 & 2p \end{pmatrix}, \tag{4.24}$$

$$\deg \mathsf{A}^{2p} \leq \begin{pmatrix} 2p+2 & 2p+1 \\ 2p+1 & 2p+2 \end{pmatrix} \quad \textit{and} \quad \deg \mathsf{A}^{2p+1} \leq \begin{pmatrix} 2p+2 & 2p+3 \\ 2p+3 & 2p+2 \end{pmatrix}. \tag{4.25}$$

Proof. We prove this result by induction using the formulas in the proof of Theorem 4.1 and using the expressions of the operators L^k and T^k and their surfacic derivatives orders (see [14] for details). Note that we have similar estimates for the surfacic derivatives order of the operators Q^k and G^k (see [14]). ■

As corollary of Theorem 4.1, we have the following result:

Theorem 4.3 *The formal series* $\big(\mathsf{V}[\varepsilon], \mathsf{Q}[\varepsilon], \mathsf{A}[\varepsilon], \mathsf{G}[\varepsilon]\big)$ *have the following properties: Let* $\boldsymbol{f}[\varepsilon]$ *be a formal series with coefficients in* $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ *,*
(i) *If* $\boldsymbol{z}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{z}^k$ *is a formal series with coefficients in* $\Sigma(S_0)$ *satisfying the equation*

$$\mathsf{A}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathsf{G}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{4.26}$$

then the formal series $\boldsymbol{w}[\varepsilon]$ *defined by*

$$\boldsymbol{w}[\varepsilon] = \mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{4.27}$$

is a solution of the problem

$$\begin{aligned} \mathsf{L}[\varepsilon]\boldsymbol{w}[\varepsilon] &= -\boldsymbol{f}[\varepsilon] \quad \textit{in} \quad \Omega, \\ \mathsf{T}[\varepsilon]\boldsymbol{w}[\varepsilon] &= 0 \qquad\quad \textit{on} \quad \Gamma_\pm. \end{aligned} \tag{4.28}$$

(ii) *If* $\boldsymbol{w}[\varepsilon]$ *is a* 3D *formal series solution of the problem* (4.28)*, then the formal series* $\boldsymbol{z}[\varepsilon]$ *with coefficients in* $\Sigma(S_0)$ *defined by*

$$\boldsymbol{z}[\varepsilon] := \boldsymbol{w}[\varepsilon]\big|_{X_3=0}, \tag{4.29}$$

satisfies the equations (4.26) *and* (4.27)*.*

Proof. (i) Let $\boldsymbol{z}[\varepsilon]$ be a formal series with coefficients in $\Sigma(S_0)$. The equations (4.14) show that

$$\begin{cases} \mathsf{L}[\varepsilon]\mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] = -\mathcal{I}\mathsf{A}[\varepsilon]\boldsymbol{z}[\varepsilon], \\ \mathsf{T}[\varepsilon]\mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] = 0, \end{cases} \quad \text{and} \quad \begin{cases} \mathsf{L}[\varepsilon]\mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon] = \mathcal{I}\mathsf{G}[\varepsilon]\boldsymbol{f}[\varepsilon] - \boldsymbol{f}[\varepsilon], \\ \mathsf{T}[\varepsilon]\mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon] = 0. \end{cases}$$

By summing these equations, we see that the formal series $\boldsymbol{w}[\varepsilon] = \mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]$ is a solution of (4.28) if the condition (4.26) is satisfied.
(ii) Reciprocally, if $\boldsymbol{w}[\varepsilon]$ is a solution of (4.28), then we show by induction the existence of a formal series $\boldsymbol{z}[\varepsilon]$ with coefficients in $\Sigma(S_0)$ satisfying the equations (4.27) and (4.26) by using the solution operators of the theorem 4.1. The fact that the operators V^k for $k \geq 1$ and Q^k for ≥ 0 vanish on the mean surface shows that $\boldsymbol{z}[\varepsilon]$ is the restriction of $\boldsymbol{w}[\varepsilon]$ to S_0 . Thus the formal series $\boldsymbol{z}[\varepsilon] = \boldsymbol{w}[\varepsilon]\big|_{X_3=0}$ satisfies the equations (4.26) et (4.27), and this proves the result. ■

4.c Further terms

The following theorem gives the expression of the first terms of the formal series given in Theorem 4.1. In particular, the exact expression of the "bending" operator A^2 is given.

In the following we denote by M the *membrane* operator (see [3, 21, 23, 26]) defined by the equations

$$\begin{cases} \mathsf{M}_\sigma(\boldsymbol{z}) := -\widetilde{\lambda}\mathrm{D}_\sigma\gamma_\alpha^\alpha(\boldsymbol{z}) - 2\mu\mathrm{D}_\alpha\gamma_\sigma^\alpha(\boldsymbol{z}), \\ \mathsf{M}_3(\boldsymbol{z}) := -\widetilde{\lambda}b_\alpha^\alpha\gamma_\beta^\beta(\boldsymbol{z}) - 2\mu b_\beta^\alpha\gamma_\alpha^\beta(\boldsymbol{z}). \end{cases} \tag{4.30}$$

where $\widetilde{\lambda} = 2\lambda\mu(\lambda+2\mu)^{-1}$. This operator is associated with the bilinear form :

$$(\boldsymbol{z},\boldsymbol{z}') \mapsto \int_{S_0} M^{\alpha\beta\sigma\delta}\gamma_{\alpha\beta}(\boldsymbol{z})\gamma_{\sigma\delta}(\boldsymbol{z}')\mathrm{d}S_0,$$

where $M^{\alpha\beta\sigma\delta}$ is the tensor defined by $M^{\alpha\beta\sigma\delta} = \widetilde{\lambda}a^{\alpha\beta}a^{\sigma\delta} + \mu(a^{\alpha\sigma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\sigma})$. The expressions in the following theorem can be compared with those found in John [18].

Theorem 4.4 *We consider the formal series of Theorem 4.1. The formal series $\mathsf{A}[\varepsilon]$ has for first terms $\mathsf{A}^0 = \mathsf{M}$ the membrane operator defined by (4.30) and $\mathsf{A}^1 = 0$. The first terms of the formal series $\mathsf{V}[\varepsilon]$ write*

$$\mathsf{V}^0 = \mathcal{I}, \qquad \mathsf{V}_\sigma^1(\boldsymbol{z}) = -X_3\theta_\sigma(\boldsymbol{z}) \quad \textit{and} \quad \mathsf{V}_3^1(\boldsymbol{z}) = -X_3 p\gamma_\alpha^\alpha(\boldsymbol{z}).$$

Moreover we have

$$\mathsf{V}_\sigma^2(\boldsymbol{z}) = \tfrac{X_3^2}{2}p\mathrm{D}_\sigma\gamma_\alpha^\alpha(\boldsymbol{z}) \quad \textit{and} \quad \mathsf{V}_3^2(\boldsymbol{z}) = \tfrac{X_3^2}{2}p\big(\rho_\alpha^\alpha(\boldsymbol{z}) - pb_\alpha^\alpha\gamma_\beta^\beta(\boldsymbol{z}) - 2b_\alpha^\beta\gamma_\beta^\alpha(\boldsymbol{z})\big).$$

The first term of the formal series $\mathsf{G}[\varepsilon]$ writes $\mathsf{G}^0(\boldsymbol{f}) = \frac{1}{2}\int_{-1}^1 \boldsymbol{f}(X_3)\,\mathrm{d}X_3$. In the following we set $\mathsf{m}(\boldsymbol{f}) = \ell^{-1}\big(\mathcal{I}\mathsf{G}^0(\boldsymbol{f}) - \boldsymbol{f}\big)$ where $\ell(\boldsymbol{u}) = (\mu u_\sigma, (\lambda+2\mu)u_3)$. The operator G^1 then writes

$$\mathsf{G}_\sigma^1(\boldsymbol{f}) = \frac{1}{2}\int_{-1}^1 \Big(\int_{-1}^{X_3} (\lambda\mathrm{D}_\sigma\mathsf{m}_3 - \mu b_\alpha^\alpha\mathsf{m}_\sigma)(\boldsymbol{f})\mathrm{d}u\Big)\,\mathrm{d}X_3$$

and

$$\mathsf{G}_3^1(\boldsymbol{f}) = \frac{1}{2}\int_{-1}^1 \Big(\int_{-1}^{X_3} (\mu\gamma_\alpha^\alpha(\mathsf{m}) - \mu b_\alpha^\alpha\mathsf{m}_3)(\boldsymbol{f})\,\mathrm{d}u\Big)\,\mathrm{d}X_3.$$

Moreover, all the operators G^k for $k \geq 1$ are factorized through the operator m : there exists operators $\mathsf{R}^k : \mathcal{C}^\infty\big(I,\Sigma(S_0)\big) \to \Sigma(S_0)$ such that for $k \geq 1$ we have $\mathsf{G}^k = \mathsf{R}^k \circ \mathsf{m}$. The formal series $\mathsf{Q}[\varepsilon]$ has for first terms Q^0 and Q^1 the null operators, and Q^2 writes $\mathsf{Q}^2(\boldsymbol{f}) = \int_0^{X_3}\big(\int_{-1}^u \mathsf{m}(\boldsymbol{f})(t)\,\mathrm{d}t\big)\mathrm{d}u$. Like for the operators G^k , the operators Q^k are *factorized through* m *for all* $k \geq 2$ *. Finally, the operator* A^2 *is given by the formulas*

$$\begin{aligned}\mathsf{A}^2_\sigma &= -\tfrac{2}{3}\mu b^\nu_\nu \mathrm{D}_\alpha \rho^\alpha_\sigma - \tfrac{2}{3}\mu p b^\beta_\beta \mathrm{D}_\sigma \rho^\nu_\nu + \mu p^2 \mathrm{D}_\sigma (b^\alpha_\alpha \rho^\nu_\nu) + \tfrac{4}{3}\mu p \mathrm{D}_\sigma b^\alpha_\beta \rho^\beta_\alpha \\ &+ \tfrac{1}{3}\mu p \mathrm{D}_\alpha b^\alpha_\sigma \rho^\nu_\nu + \tfrac{2}{3}\mu b^\alpha_\beta \mathrm{D}_\alpha \rho^\beta_\sigma + \tfrac{2}{3}\mu \rho^\alpha_\beta \mathrm{D}_\alpha b^\beta_\sigma - \tfrac{1}{3}\mu \mathrm{D}_\alpha b^\beta_\sigma \mathrm{D}^\alpha \theta_\beta \\ &+ \tfrac{1}{3}\mu \mathrm{D}_\alpha b^\alpha_\beta \mathrm{D}^\beta \theta_\sigma - \tfrac{2}{3}\mu \mathrm{D}_\alpha b^\alpha_\nu \Lambda^\nu_{\cdot\,\sigma} - \tfrac{1}{3}\mu p^2 \mathrm{D}_\sigma \mathrm{D}^\alpha \mathrm{D}_\alpha \gamma^\nu_\nu - \tfrac{1}{3}\mu p \mathrm{D}_\alpha \mathrm{D}^\alpha \mathrm{D}_\sigma \gamma^\nu_\nu \\ &- 2\mu p^2 \mathrm{D}_\sigma (c^\alpha_\alpha \gamma^\nu_\nu) + \tfrac{2}{3}\mu p b^\beta_\beta \mathrm{D}_\alpha b^\alpha_\sigma \gamma^\nu_\nu - \tfrac{2}{3}\mu p b^\alpha_\beta \mathrm{D}_\alpha b^\beta_\sigma \gamma^\nu_\nu - \tfrac{2}{3}\mu p b^\alpha_\beta \gamma^\nu_\nu \mathrm{D}_\alpha b^\beta_\sigma \\ &+ \tfrac{2}{3}\mu b^\nu_\nu \mathrm{D}_\alpha b^\alpha_\beta \gamma^\beta_\sigma + \tfrac{2}{3}\mu p b^\beta_\beta \mathrm{D}_\sigma b^\alpha_\nu \gamma^\nu_\alpha - \mu p^2 \mathrm{D}_\sigma (b^\alpha_\alpha b^\beta_\nu \gamma^\nu_\beta) - \tfrac{10}{3}\mu p \mathrm{D}_\sigma (c^\alpha_\beta \gamma^\beta_\alpha) \\ &- \tfrac{1}{3}\mu p \mathrm{D}_\alpha b^\alpha_\sigma b^\beta_\nu \gamma^\nu_\beta - \tfrac{2}{3}\mu b^\alpha_\beta \mathrm{D}_\alpha b^\beta_\nu \gamma^\nu_\sigma - \tfrac{2}{3}\mu b^\nu_\beta \gamma^\alpha_\nu \mathrm{D}_\alpha b^\beta_\sigma - \tfrac{2}{3}\mu b^\alpha_\beta \mathrm{D}_\alpha b^\beta_\delta \gamma^\delta_\sigma \\ &- \tfrac{4}{3}\mu b^\delta_\nu \gamma^\alpha_\delta \mathrm{D}_\alpha b^\nu_\sigma + \tfrac{2}{3}\mu b^\alpha_\nu \gamma^\delta_\sigma \mathrm{D}_\alpha b^\nu_\delta + \tfrac{2}{3}\mu b^\beta_\beta \gamma^\alpha_\nu \mathrm{D}_\alpha b^\nu_\sigma + \tfrac{2}{3}\mu b^\nu_\nu b^\alpha_\beta \mathrm{D}_\alpha \gamma^\beta_\sigma \\ &+ \tfrac{1}{2} p^2 \mathrm{D}_\sigma (b^\alpha_\alpha \mathsf{M}_3) - \tfrac{1}{3} p b^\nu_\nu \mathrm{D}_\sigma \mathsf{M}_3 + \tfrac{1}{6} p \mathrm{D}_\alpha b^\alpha_\sigma \mathsf{M}_3.\end{aligned} \tag{4.31}$$

where M_3 *is given by* (4.30)*, and*

$$\begin{aligned}\mathsf{A}^2_3 &= \tfrac{2}{3}\mu p \mathrm{D}^\alpha \mathrm{D}_\alpha \rho^\nu_\nu + \tfrac{2}{3}\mu \mathrm{D}^\alpha \mathrm{D}_\nu \rho^\nu_\alpha + \mu p c^\alpha_\alpha \rho^\nu_\nu + \tfrac{4}{3}\mu c^\alpha_\beta \rho^\beta_\alpha + \tfrac{1}{3}\mu p(3p-2) b^\alpha_\alpha b^\beta_\beta \rho^\nu_\nu \\ &+ \tfrac{2}{3}\mu(2p-1) b^\alpha_\alpha b^\nu_\beta \rho^\beta_\nu - \tfrac{2}{3}\mu p \mathrm{D}^\alpha \mathrm{D}_\alpha b^\nu_\beta \gamma^\beta_\nu - \tfrac{1}{3}\mu \mathrm{D}^\sigma \mathrm{D}_\alpha b^\nu_\sigma \gamma^\alpha_\nu - \tfrac{1}{3}\mu \mathrm{D}^\sigma \mathrm{D}_\alpha b^\alpha_\nu \gamma^\nu_\sigma \\ &- \tfrac{1}{3}\mu p^2 b^\beta_\beta \mathrm{D}^\alpha \mathrm{D}_\alpha \gamma^\nu_\nu - \tfrac{2}{3}\mu p \mathrm{D}^\sigma \mathrm{D}_\alpha b^\alpha_\sigma \gamma^\nu_\nu - \tfrac{1}{3}\mu p b^\alpha_\beta \mathrm{D}^\beta \mathrm{D}_\alpha \gamma^\nu_\nu - \tfrac{2}{3}\mu \mathrm{D}^\sigma \gamma^\alpha_\nu \mathrm{D}_\alpha b^\nu_\sigma \\ &- \tfrac{2}{3}\mu \mathrm{D}^\sigma b^\alpha_\nu \mathrm{D}_\alpha \gamma^\nu_\sigma - 2\mu p d^\alpha_\alpha \gamma^\nu_\nu - \tfrac{10}{3}\mu d^\alpha_\beta \gamma^\beta_\alpha - \tfrac{1}{3}\mu p(3p-2) b^\alpha_\alpha b^\beta_\beta b^\sigma_\nu \gamma^\nu_\sigma \\ &- \mu p c^\alpha_\alpha b^\beta_\nu \gamma^\nu_\beta - \tfrac{2}{3}\mu p(3p-2) b^\nu_\nu c^\alpha_\alpha \gamma^\beta_\beta - \tfrac{2}{3}\mu(2p-1) b^\alpha_\alpha c^\nu_\beta \gamma^\beta_\nu - \tfrac{2}{3}\mu(3p-2) b^\alpha_\alpha c^\nu_\beta \gamma^\beta_\nu \\ &+ \tfrac{1}{3} p \mathrm{D}^\alpha \mathrm{D}_\alpha \mathsf{M}_3 + \tfrac{1}{6} p(3p-2) b^\nu_\nu b^\alpha_\alpha \mathsf{M}_3 + \tfrac{1}{2} p c^\alpha_\alpha \mathsf{M}_3.\end{aligned} \tag{4.32}$$

Proof. We first prove the results concerning the operators in $\boldsymbol{z}$.
1. We will first compute the operators A^0 and V^2 by using the formulas (4.18) and (4.16). Using the equations (3.15) and (4.7) we compute that if $\boldsymbol{z} \in \Sigma(S_0)$ we have

$$\begin{cases} \mathsf{L}^1_\sigma(\mathsf{V}^1 \boldsymbol{z}) &= \{\mu b^\alpha_\alpha \theta_\sigma - (\lambda+\mu) p \mathrm{D}_\sigma \gamma^\alpha_\alpha\}\boldsymbol{z}, \\ \mathsf{L}^1_3(\mathsf{V}^1 \boldsymbol{z}) &= \{(\lambda+2\mu) p b^\alpha_\alpha \gamma^\beta_\beta - (\lambda+\mu)\mathrm{D}^\alpha \theta_\alpha\}\boldsymbol{z}, \end{cases} \tag{4.33}$$

where we used the fact that $\gamma^\alpha_\alpha(\mathsf{V}^1 \boldsymbol{z}) = X_3\{b^\alpha_\alpha \gamma^\beta_\beta - \mathrm{D}^\alpha \theta_\alpha\}\boldsymbol{z}$. Moreover, using the equations (3.19) and (3.20) we have

$$\mathsf{T}^1_\sigma(\mathsf{V}^1 \boldsymbol{z}) = -X_3 \mu p \mathrm{D}_\sigma \gamma^\alpha_\alpha \boldsymbol{z} \quad \text{and} \quad \mathsf{T}^1_3(\mathsf{V}^1 \boldsymbol{z}) = X_3\{\lambda p b^\alpha_\alpha \gamma^\beta_\beta - \lambda \mathrm{D}^\alpha \theta_\alpha\}\boldsymbol{z}.$$

Finally we have using the equations (3.16)

$$\begin{cases} \mathsf{L}^2_\sigma(\mathsf{V}^0 \boldsymbol{z}) &= \{-\mu b^\alpha_\alpha \theta_\sigma + \lambda \mathrm{D}_\sigma \gamma^\alpha_\alpha + 2\mu \mathrm{D}_\alpha \gamma^\alpha_\sigma\}\boldsymbol{z}, \\ \mathsf{L}^2_3(\mathsf{V}^0 \boldsymbol{z}) &= \{(\lambda+2\mu) b^\beta_\alpha \gamma^\alpha_\beta + \mu \mathrm{D}^\alpha \theta_\alpha\}\boldsymbol{z}, \end{cases} \tag{4.34}$$

and similarly $\mathsf{T}^2_\sigma(\mathsf{V}^0\boldsymbol{z}) = 0$ and $\mathsf{T}^2_3(\mathsf{V}^0\boldsymbol{z}) = X_3\lambda b^\alpha_\beta\gamma^\beta_\alpha\boldsymbol{z}$. Collecting the previous computations, we see that the transverse component of the equation (4.16) writes

$$\left\{\begin{array}{lcll} \partial^2_{X_3}\mathsf{V}^2_3\boldsymbol{z} & = & \{-pb^\alpha_\alpha\gamma^\beta_\beta - b^\beta_\alpha\gamma^\alpha_\beta + p\mathrm{D}^\alpha\theta_\alpha\}\boldsymbol{z} - (\lambda+2\mu)^{-1}\mathsf{A}^0_3\boldsymbol{z} & \text{in} \quad \Omega, \\ \partial_{X_3}\mathsf{V}^2_3\boldsymbol{z} & = & X_3 p\{-pb^\alpha_\alpha\gamma^\beta_\beta + \mathrm{D}^\alpha\theta_\alpha - b^\alpha_\beta\gamma^\beta_\alpha\}\boldsymbol{z} & \text{on} \quad \Gamma_\pm. \end{array}\right. \tag{4.35}$$

The formula (4.18) written as a compatibility condition then shows that we have

$$2\mathsf{A}^0_3\boldsymbol{z} = 4\mu\{-pb^\alpha_\alpha\gamma^\beta_\beta - b^\beta_\alpha\gamma^\alpha_\beta\}\boldsymbol{z}.$$

But we have $2\mu p = \widetilde{\lambda}$, and thus it is clear that we have $\mathsf{A}^0_3 = \mathsf{M}_3$ (see (4.30)). Similarly, the surfacic components of (4.16) write

$$\left\{\begin{array}{lcll} \partial^2_{X_3}\mathsf{V}^2_\sigma\boldsymbol{z} & = & \{p\mathrm{D}_\sigma\gamma^\alpha_\alpha - \lambda(1-p)\mu^{-1}\mathrm{D}_\sigma\gamma^\alpha_\alpha - 2\mathrm{D}_\alpha\gamma^\alpha_\sigma\}\boldsymbol{z} - \mu^{-1}\mathsf{A}^0_\sigma\boldsymbol{z} & \text{in} \quad \Omega, \\ \partial_{X_3}\mathsf{V}^2_\sigma\boldsymbol{z} & = & X_3 p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z}) & \text{on} \quad \Gamma_\pm. \end{array}\right. \tag{4.36}$$

The compatibility condition then implies that

$$2\mathsf{A}^0_\sigma\boldsymbol{z} = \{-2\lambda(1-p)\mathrm{D}_\sigma\gamma^\alpha_\alpha - 4\mu\mathrm{D}_\alpha\gamma^\alpha_\sigma\}\boldsymbol{z}.$$

and using $\lambda(1-p) = \widetilde{\lambda}$ we easily get $\mathsf{A}^0_\sigma = \mathsf{M}_\sigma$ (see (4.30)).
Now by taking the integral of (4.35) from -1 to X_3 and using the boundary condition on Γ_-, we find (recall that $\mathsf{A}^0_3 = \mathsf{M}_3$):

$$\partial_{X_3}\mathsf{V}^2_3\boldsymbol{z} = (X_3+1)\{-pb^\alpha_\alpha\gamma^\beta_\beta - b^\beta_\alpha\gamma^\alpha_\beta + p\mathrm{D}^\alpha\theta_\alpha - (\lambda+2\mu)^{-1}\mathsf{M}_3\}\boldsymbol{z} + p\{pb^\alpha_\alpha\gamma^\beta_\beta - \mathrm{D}^\alpha\theta_\alpha + b^\alpha_\beta\gamma^\beta_\alpha\}\boldsymbol{z} \quad \text{in} \quad \Omega.$$

Replacing the expression of $\mathsf{M}_3\boldsymbol{z}$ we find $\partial_{X_3}\mathsf{V}^2_3\boldsymbol{z} = X_3 p\{\mathrm{D}^\alpha\theta_\alpha - pb^\alpha_\alpha\gamma^\beta_\beta - b^\alpha_\beta\gamma^\beta_\alpha\}\boldsymbol{z}$. As $\mathrm{D}^\alpha\theta_\alpha = \rho^\alpha_\alpha - b^\beta_\alpha\gamma^\alpha_\beta$ we get the expression of V^2_3 by simple integration.
Using the fact that $\mathsf{A}^0_\sigma = -\lambda(1-p)\mathrm{D}_\sigma\gamma^\alpha_\alpha - 2\mu\mathrm{D}_\alpha\gamma^\alpha_\sigma$, the equation (4.36) writes

$$\partial^2_{X_3}\mathsf{V}^2_\sigma\boldsymbol{z} = p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z}) \quad \text{in} \quad \Omega \quad \text{and} \quad \partial_{X_3}\mathsf{V}^2_\sigma\boldsymbol{z} = X_3 p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z}) \quad \text{on} \quad \Gamma_\pm.$$

Thus we easily find that $\partial_{X_3}\mathsf{V}^2_\sigma\boldsymbol{z} = X_3 p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z})$ and the result by integration.
The goal is now to compute the operators A^1 and A^2. The equation (3.14) shows that for $\boldsymbol{w} \in \mathcal{C}^\infty(I, \Sigma(S_0))$ we have $(\tilde{e}^\alpha_\sigma(\varepsilon)\boldsymbol{w})^1 = X_3 b^\alpha_\delta\gamma^\delta_\sigma(\boldsymbol{w}) + X_3\Lambda^\alpha_{\cdot\sigma}(\boldsymbol{w})$. For $\boldsymbol{z} \in \Sigma(S_0)$ and using the previous results, we compute successively that:

$$\mathsf{L}^1_\sigma(\mathsf{V}^2\boldsymbol{z}) = X_3\{-\mu pb^\alpha_\alpha\mathrm{D}_\sigma\gamma^\delta_\delta + (\lambda+\mu)p\mathrm{D}_\sigma\mathrm{D}^\alpha\theta_\alpha + (\lambda+\mu)\tfrac{p}{2\mu}\mathrm{D}_\sigma\mathsf{M}_3 - \mu pb^\alpha_\sigma\mathrm{D}_\alpha\gamma^\delta_\delta\}\boldsymbol{z},$$

$$\mathsf{L}^1_3(\mathsf{V}^2\boldsymbol{z}) = X_3\{-(\lambda+2\mu)pb^\alpha_\alpha\mathrm{D}^\delta\theta_\delta - (\lambda+2\mu)\tfrac{p}{2\mu}b^\alpha_\alpha\mathsf{M}_3 + (\lambda+\mu)p\mathrm{D}^\alpha\mathrm{D}_\alpha\gamma^\delta_\delta\}\boldsymbol{z},$$

and

$$\mathsf{T}^1_\sigma(\mathsf{V}^2\boldsymbol{z}) = \tfrac{X_3^2}{2}p\{\mu\mathrm{D}_\sigma\mathrm{D}^\alpha\theta_\alpha + \tfrac{1}{2}\mathrm{D}_\sigma\mathsf{M}_3 - \mu b^\alpha_\sigma\mathrm{D}_\alpha\gamma^\nu_\nu\}\boldsymbol{z},$$

$$\mathsf{T}^1_3(\mathsf{V}^2\boldsymbol{z}) = \tfrac{X_3^2}{2}\lambda p\{\mathrm{D}^\sigma\mathrm{D}_\sigma\gamma^\alpha_\alpha - b^\nu_\nu\mathrm{D}^\alpha\theta_\alpha - \tfrac{1}{2\mu}b^\alpha_\alpha\mathsf{M}_3\}\boldsymbol{z}.$$

Moreover we have

$$\mathsf{L}^2_\sigma(\mathsf{V}^1 z) = X_3\{\mu c_\alpha^\alpha\theta_\sigma - \mu b_\alpha^\alpha b_\sigma^\beta\theta_\beta + \mu p b_\alpha^\alpha \mathrm{D}_\sigma\gamma_\nu^\nu + \mu b_\beta^\beta b_\sigma^\alpha\theta_\alpha - \lambda\mathrm{D}_\sigma\mathrm{D}^\alpha\theta_\alpha + \lambda p\mathrm{D}_\sigma b_\alpha^\alpha\gamma_\nu^\nu - \mu\mathrm{D}_\alpha\mathrm{D}_\sigma\theta^\alpha - \mu\mathrm{D}_\alpha\mathrm{D}^\alpha\theta_\sigma + 2\mu p\mathrm{D}_\alpha b_\sigma^\alpha\gamma_\nu^\nu\}z$$

and

$$\mathsf{L}^2_3(\mathsf{V}^1 z) = X_3\{\mu p c_\alpha^\alpha\gamma_\nu^\nu - (2\lambda+3\mu)b_\alpha^\beta\mathrm{D}_\beta\theta^\alpha + (2\lambda+3\mu)pc_\alpha^\alpha\gamma_\nu^\nu - \mu p\mathrm{D}^\alpha\mathrm{D}_\alpha\gamma_\nu^\nu - \mu\mathrm{D}^\alpha b_\alpha^\nu\theta_\nu\}z,$$

while $\mathsf{T}^2_\sigma(\mathsf{V}^1 z) = 0$ and $\mathsf{T}^2_3(\mathsf{V}^1 z) = X_3^2\lambda\{-b_\beta^\alpha\mathrm{D}_\alpha\theta^\beta + pc_\alpha^\alpha\gamma_\nu^\nu\}z$. Finally we have

$$\mathsf{L}^3_\sigma(\mathsf{V}^0 z) = X_3\{-\mu c_\alpha^\alpha\theta_\sigma + \lambda\mathrm{D}_\sigma b_\beta^\alpha\gamma_\alpha^\beta + 2\mu\mathrm{D}_\alpha b_\delta^\alpha\gamma_\sigma^\delta + 2\mu\mathrm{D}_\alpha\Lambda^\alpha_{\cdot\sigma} + 2\mu\gamma_\nu^\alpha\mathrm{D}_\alpha b_\sigma^\nu - 2\mu\gamma_\sigma^\delta\mathrm{D}_\alpha b_\delta^\alpha\}z,$$

and

$$\mathsf{L}^3_3(\mathsf{V}^0 z) = X_3\{2(\lambda+2\mu)c_\beta^\alpha\gamma_\alpha^\beta + \mu\mathrm{D}^\alpha b_\alpha^\beta\theta_\beta + \mu b_\alpha^\delta\mathrm{D}^\alpha\theta_\delta\}z,$$

while $\mathsf{T}^3_\sigma(\mathsf{V}^0 z) = 0$ and $\mathsf{T}^3_3(\mathsf{V}^0 z) = X_3^2\lambda c_\alpha^\beta\gamma_\beta^\alpha(z)$.
Using the formula (4.18) for the operator A^1 we see easily that $\mathsf{A}^1 = 0$. As we will see, the computation of the operator A^2 via (4.18) only requires to compute the mean value $[\mathsf{V}^3 z]_{-1}^{+1}$. Using the previous equations and integrating the equation (4.16) for $n = 3$ from -1 to X_3 we find that

$$\partial_{X_3}\mathsf{V}^3_\sigma z = \tfrac{X_3^2}{2}p\{-\mathrm{D}_\sigma\mathrm{D}_\nu\theta^\nu + b_\sigma^\alpha\mathrm{D}_\alpha\gamma_\nu^\nu - \tfrac{1}{2\mu}\mathrm{D}_\sigma\mathsf{M}_3\}z + (\tfrac{X_3^2}{2} - \tfrac12)\{\mathrm{D}_\alpha\mathrm{D}_\sigma\theta^\alpha + \mathrm{D}_\alpha\mathrm{D}^\alpha\theta_\sigma + 2p\mathrm{D}_\sigma\mathrm{D}_\nu\theta^\nu + \tfrac{p}{\mu}\mathrm{D}_\sigma\mathsf{M}_3 - 2p\mathrm{D}_\alpha b_\sigma^\alpha\gamma_\nu^\nu - 2b_\nu^\alpha\mathrm{D}_\alpha\gamma_\sigma^\nu - 2\gamma_\nu^\alpha\mathrm{D}_\alpha b_\sigma^\nu - 2\mathrm{D}_\alpha\Lambda^\alpha_{\cdot\sigma}\}z.$$

But we compute that we have

$$\tfrac12(\mathrm{D}_\alpha\theta_\beta + \mathrm{D}_\beta\theta_\alpha) - \Lambda_{\alpha\beta} = \rho_{\alpha\beta} - b_\alpha^\sigma\gamma_{\beta\sigma}$$

and in particular $\mathrm{D}^\alpha\theta_\alpha = \rho_\alpha^\alpha - b_\alpha^\beta\gamma_\beta^\alpha$. Hence we deduce that we have

$$[\mathsf{V}^3_\sigma z]_{-1}^{+1} = \{-\tfrac53 p\mathrm{D}_\sigma\rho_\nu^\nu - \tfrac43\mathrm{D}_\alpha\rho_\sigma^\alpha - \tfrac{5p}{6\mu}\mathrm{D}_\sigma\mathsf{M}_3 + \tfrac53 p\mathrm{D}_\sigma b_\alpha^\beta\gamma_\beta^\alpha + \tfrac43\mathrm{D}_\alpha b_\beta^\alpha\gamma_\sigma^\beta + \tfrac{p}{3}b_\sigma^\alpha\mathrm{D}_\alpha\gamma_\nu^\nu + \tfrac43 p\mathrm{D}_\alpha b_\sigma^\alpha\gamma_\nu^\nu + \tfrac43 b_\nu^\alpha\mathrm{D}_\alpha\gamma_\sigma^\nu + \tfrac43\gamma_\nu^\alpha\mathrm{D}_\alpha b_\sigma^\nu\}z. \tag{4.37}$$

Similarly we have

$$\partial_{X_3}\mathsf{V}^3_3 z = \tfrac{X_3^2}{2}\{-p^2\mathrm{D}^\alpha\mathrm{D}_\alpha\gamma_\beta^\beta + p^2 b_\alpha^\alpha\mathrm{D}_\beta\theta^\beta + 2pb_\beta^\alpha\mathrm{D}_\alpha\theta^\beta - 2p^2c_\alpha^\alpha\gamma_\nu^\nu - 2pc_\beta^\alpha\gamma_\alpha^\beta + \tfrac{p^2}{2\mu}b_\alpha^\alpha\mathsf{M}_3\}z$$
$$(\tfrac{X_3^2}{2} - \tfrac12)(1-p)\{pb_\alpha^\alpha\mathrm{D}_\nu\theta^\nu + b_\beta^\alpha\mathrm{D}_\alpha\theta^\beta - 2pc_\alpha^\alpha\gamma_\nu^\nu - 2c_\beta^\alpha\gamma_\alpha^\beta + \tfrac{p}{2\mu}b_\alpha^\alpha\mathsf{M}_3\}z.$$

By integration we find using the fact that $b^{\alpha\beta}\Lambda_{\alpha\beta}=0$,

$$[\mathsf{V}^3_3 z]^{+1}_{-1} = \{-\tfrac{1}{3}p^2\mathrm{D}^\alpha\mathrm{D}_\alpha\gamma^\nu_\nu + \tfrac{p}{3}(3p-2)b^\alpha_\alpha\rho^\nu_\nu + \tfrac{2}{3}(2p-1)b^\alpha_\beta\rho^\beta_\alpha$$
$$- \tfrac{p}{3}(3p-2)b^\alpha_\alpha b^\nu_\beta\gamma^\beta_\nu - \tfrac{2}{3}p(3p-2)c^\alpha_\alpha\gamma^\nu_\nu - \tfrac{2}{3}(5p-3)c^\alpha_\beta\gamma^\beta_\alpha + \tfrac{p}{6\mu}(3p-2)b^\alpha_\alpha M_3\big](z). \quad (4.38)$$

Now using the equation (4.18) we have

$$2\mathsf{A}^2 z = -\int_{-1}^{1}\Big(\mathsf{L}^1\mathsf{V}^3 z + \mathsf{L}^2\mathsf{V}^2 z + \mathsf{L}^3\mathsf{V}^1 z + \mathsf{L}^4\mathsf{V}^0 z\Big)\mathrm{d}X_3$$
$$+ \Big[\mathsf{T}^1\mathsf{V}^3 z + \mathsf{T}^2\mathsf{V}^2 z + \mathsf{T}^3\mathsf{V}^1 z + \mathsf{T}^4\mathsf{V}^0 z\Big]^{+1}_{-1}. \quad (4.39)$$

We verify that only the term $[\mathsf{V}^3 z]^{+1}_{-1}$ is involved in this expression. This is due to the fact that we have

$$-\int_{-1}^{1}\mathsf{L}^1_\sigma(\mathsf{V}^3 z)\,\mathrm{d}X_3 + \Big[\mathsf{T}^1_\sigma(\mathsf{V}^3 z)\Big]^{+1}_{-1} = \mu b^\alpha_\alpha[\mathsf{V}^3_\sigma z]^{+1}_{-1} - \lambda\mathrm{D}_\sigma[\mathsf{V}^3_3 z]^{+1}_{-1}.$$

and

$$-\int_{-1}^{1}\mathsf{L}^1_3(\mathsf{V}^3 z)\,\mathrm{d}X_3 + \Big[\mathsf{T}^1_3(\mathsf{V}^3 z)\Big]^{+1}_{-1} = 2\mu b^\alpha_\alpha[\mathsf{V}^3_3 z]^{+1}_{-1} - \mu\mathrm{D}^\alpha[\mathsf{V}^3_\alpha z]^{+1}_{-1}.$$

The expression of A^2 comes from the computation of (4.39): see the details in [14].

2. Now we investigate the computation of the operator G^1. Recall that the operators G^k are determined by the equation (4.22). We compute successively that

$$\mathsf{L}^1_\sigma(\mathsf{Q}^2 \boldsymbol{f}) = -\int_{-1}^{X_3}\mu b^\alpha_\alpha\mathsf{m}_\sigma(\boldsymbol{f})\,\mathrm{d}u + \int_{-1}^{X_3}(\lambda+\mu)\mathrm{D}_\sigma\mathsf{m}_3(\boldsymbol{f})\,\mathrm{d}u - X_3\mu b^\alpha_\sigma\mathsf{m}_\alpha(\boldsymbol{f}),$$

and

$$\mathsf{L}^1_3(\mathsf{Q}^2 \boldsymbol{f}) = -\int_{-1}^{X_3}\mu b^\alpha_\alpha\mathsf{m}_3(\boldsymbol{f})\,\mathrm{d}u + \int_{-1}^{X_3}(\lambda+\mu)\gamma^\alpha_\alpha\big(\mathsf{m}(\boldsymbol{f})\big)\,\mathrm{d}u,$$

while

$$\mathsf{T}^1_\sigma(\mathsf{Q}^2 \boldsymbol{f}) = \int_0^{X_3}\int_{-1}^{u}\mu\mathrm{D}_\sigma\mathsf{m}_3(\boldsymbol{f})\,\mathrm{d}t\,\mathrm{d}u + \int_0^{X_3}\int_{-1}^{u}\mu b^\alpha_\sigma\mathsf{m}_\alpha(\boldsymbol{f})\,\mathrm{d}t\,\mathrm{d}u - \mu X_3 b^\alpha_\sigma\int_{-1}^{x_3}\mathsf{m}_\alpha(\boldsymbol{f})\,\mathrm{d}u,$$

and $\mathsf{T}^1_3(\mathsf{Q}^2 \boldsymbol{f}) = \int_0^{X_3}\int_{-1}^{u}\lambda\gamma^\alpha_\alpha\big(\mathsf{m}(\boldsymbol{f})\big)\mathrm{d}t\,\mathrm{d}u$. Thus using the equation (4.22) we find

$$2\mathsf{G}^1_\sigma \boldsymbol{f} = -\int_{-1}^{1}\int_{-1}^{X_3}\mu b^\alpha_\alpha\mathsf{m}_\sigma(\boldsymbol{f})\,\mathrm{d}u\,\mathrm{d}X_3 + \int_{-1}^{1}\int_{-1}^{X_3}\lambda\mathrm{D}_\sigma\mathsf{m}_3(\boldsymbol{f})\,\mathrm{d}u\,\mathrm{d}X_3$$
$$-\int_{-1}^{1}X_3\mu b^\alpha_\sigma\mathsf{m}_\alpha(\boldsymbol{f})\,\mathrm{d}X_3 - \int_{-1}^{1}\int_{-1}^{u}\mu b^\alpha_\sigma\mathsf{m}_\alpha(\boldsymbol{f})\,\mathrm{d}t\,\mathrm{d}u + \int_{-1}^{1}\mu b^\alpha_\sigma\mathsf{m}_\alpha(\boldsymbol{f})\,\mathrm{d}u,$$

and we find the result after an integration by part. Similarly we have

$$2\mathsf{G}^1_3(\boldsymbol{f}) = -\int_{-1}^{1}\int_{-1}^{X_3}\mu b^\alpha_\alpha\mathsf{m}_3(\boldsymbol{f})\,\mathrm{d}u\,\mathrm{d}X_3 + \int_{-1}^{1}\int_{-1}^{X_3}(\lambda+\mu)\gamma^\alpha_\alpha\big(\mathsf{m}(\boldsymbol{f})\big)\,\mathrm{d}u\,\mathrm{d}X_3$$
$$-\int_0^{1}\int_{-1}^{u}\lambda\gamma^\alpha_\alpha\big(\mathsf{m}(\boldsymbol{f})\big)\mathrm{d}t\,\mathrm{d}u - \int_{-1}^{0}\int_{-1}^{u}\lambda\gamma^\alpha_\alpha\big(\mathsf{m}(\boldsymbol{f})\big)\mathrm{d}t\,\mathrm{d}u,$$

and this yields the result.

The factorization properties of the operators G^k and Q^k are easily shown by induction using the definition of the formal series $\mathsf{G}[\varepsilon]$ and $\mathsf{Q}[\varepsilon]$. The reader interested can find the expression of G^2 and Q^3 in [14]. ■

Now we give a result showing that the operator A^2 "contains" the classical bending operator. Recall that Koiter's bending operator B writes

$$\left\{\begin{array}{lcl} \mathsf{B}_\sigma & = & -\frac{2}{3}\mu p b_\sigma^\alpha \mathrm{D}_\alpha \rho_\nu^\nu - \frac{2}{3}\mu p \mathrm{D}_\alpha b_\sigma^\alpha \rho_\nu^\nu - \frac{2}{3}\mu b_\sigma^\alpha \mathrm{D}_\nu \rho_\alpha^\nu - \frac{2}{3}\mu \mathrm{D}_\nu b_\sigma^\alpha \rho_\alpha^\nu \\ \mathsf{B}_3 & = & \frac{2}{3}\mu p \mathrm{D}^\alpha \mathrm{D}_\alpha \rho_\nu^\nu + \frac{2}{3}\mu \mathrm{D}^\alpha \mathrm{D}_\nu \rho_\alpha^\nu - \frac{2}{3}\mu p c_\alpha^\alpha \rho_\nu^\nu - \frac{2}{3}\mu c_\alpha^\beta \rho_\beta^\alpha . \end{array}\right. \tag{4.40}$$

This operator is associated with the bilinear form :

$$(\boldsymbol{z}, \boldsymbol{z}') \mapsto \frac{1}{3}\int_{S_0} M^{\alpha\beta\sigma\delta} \rho_{\alpha\beta}(\boldsymbol{z}) \rho_{\sigma\delta}(\boldsymbol{z}') \mathrm{d}S_0 .$$

Recall that the notion of Sobolev space is consistent on manifolds, as the notion of derivative order. We denote by $\mathrm{H}^k(S_0)$ the Sobolev space of order k on S_0. For tensors, we easily can define the space $\mathbf{H}^k(S_0)$ corresponding to the type of the tensor. Hence, $\boldsymbol{z} \in \mathbf{H}^1 \times \mathrm{H}^2(S_0)$ means that the 1-form field (z_α) belongs to the space $\mathbf{H}^1(S_0)$ and z_3 belongs to $\mathrm{H}^2(S_0)$. We can also naturally define the space $\mathrm{H}_0^1(S_0)$ of functions vanishing on the boundary, and similarly $f \in \mathrm{H}_0^2(S_0)$ means that $f\big|_{\partial S_0} = \partial_r f\big|_{\partial S_0} = 0$ where ∂_r denote the reentrant derivative along the boundary of S_0. With these notations we have

Proposition 4.5 *Let* $\boldsymbol{z}$ *and* $\boldsymbol{\eta} \in \Sigma(S_0)$. *If* $\boldsymbol{\eta}$ *satisfies the boundary condition* $\boldsymbol{\eta}\big|_{\partial S_0} = 0$, *then we have*

$$\begin{aligned} \left|\left\langle (\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{\eta}\right\rangle_{\mathbf{L}^2(S_0)}\right| \leq C \Big(& \|\boldsymbol{\gamma}(\boldsymbol{z})\|_{\mathbf{H}^2(S_0)} \|\boldsymbol{\gamma}(\boldsymbol{\eta})\|_{\mathbf{L}^2(S_0)} \\ & + \|\boldsymbol{z}\|_{\mathbf{H}^1 \times \mathrm{H}^2(S_0)} \|\boldsymbol{\gamma}(\boldsymbol{\eta})\|_{\mathbf{L}^2(S_0)} + \|\boldsymbol{\gamma}(\boldsymbol{z})\|_{\mathbf{H}^1(S_0)} \|\boldsymbol{\eta}\|_{\mathbf{H}^1 \times \mathrm{H}^1(S_0)} \Big), \end{aligned} \tag{4.41}$$

where B *is the bending Koiter operator and* C *a constant depending only on* S_0.

The proof of this result is presented in the appendix B at the end of the paper. As corollary, the restriction of A^2 to the space of inextensional displacements coincides with the restriction of B : if $V_{\mathrm{B}} = \{\boldsymbol{z} \in \mathbf{H}_0^1 \times \mathrm{H}_0^2(S_0) | \gamma_{\alpha\beta}(\boldsymbol{z}) = 0\}$, then

$$\forall\, \boldsymbol{z}, \boldsymbol{\eta} \in V_{\mathrm{B}}, \quad \langle \mathsf{A}^2 \boldsymbol{z}, \boldsymbol{\eta}\rangle = \langle \mathsf{B}\boldsymbol{z}, \boldsymbol{\eta}\rangle .$$

This result is consistent with the convergence result in [3, 26].

Note that in the case where the boundary ∂S_0 is empty, we impose to the loading forces and to the displacement solution to be orthogonal to the rigid displacements in $\mathbb{R}^3$. We can expand the six rigid displacements of $\mathbb{R}^3$ in normal coordinates and hence to each rigid displacement R_i is associated a formal series $\mathsf{R}_i[\varepsilon]$ in powers of ε $(i = 1, \dots, 6)$. Thus the solution $\boldsymbol{z}[\varepsilon]$ has to fulfill a new condition due to the fact that the reconstructed displacement $\boldsymbol{w}[\varepsilon]$ is orthogonal to the formal series $\mathsf{R}_i[\varepsilon]$ for the L^2 scalar product in Ω.

5 Boundary layers

We consider the formal series $(\mathsf{V}[\varepsilon], \mathsf{Q}[\varepsilon], \mathsf{A}[\varepsilon], \mathsf{G}[\varepsilon])$ of Theorem 4.1. If $\boldsymbol{z}[\varepsilon]$ is a formal series with coefficients in $\Sigma(S_0)$, and if this formal series satisfies the equation (4.26), then the formal series $\boldsymbol{w}[\varepsilon]$ defined by the equation (4.27) is a solution of the problem (4.28). However, we can show that for all formal series $\boldsymbol{z}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{z}^k$, the trace

$$\boldsymbol{w}[\varepsilon]\big|_{\Gamma_0} = \big(\mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]\big)\big|_{\Gamma_0}$$

does not vanish in general and thus the problem (4.1) does not have a solution in general. In the following, r denotes the geodesic distance to the boundary ∂S_0 , and s denotes the arc-length along ∂S_0 . We can show that in this coordinate system, the metric $a_{\alpha\beta}$ satisfies $a_{rs} = 0$ and $a_{rr} = 1$ in a neighborhood of ∂S_0 , while $a_{ss} = 1$ on ∂S_0 .

The goal of this section is to show that under certain conditions there exist a formal series

$$\boldsymbol{\varphi}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{\varphi}^k(R, s, x_3)$$

with boundary layer coefficients $\boldsymbol{\varphi}^k(R, s, x_3)$ exponentially decreasing with $R = r/\varepsilon$, such that the equations

$$\begin{cases} \mathcal{L}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] &= 0, \\ \mathcal{T}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] &= 0, \\ \boldsymbol{\varphi}[\varepsilon]\big|_{R=0} + \big(\mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]\big)\big|_{\Gamma_0} &= 0, \end{cases} \tag{5.1}$$

are satisfied for given formal series $\boldsymbol{z}[\varepsilon]$ and $\boldsymbol{f}[\varepsilon]$, where the formal series $\mathcal{L}[\varepsilon]$ and $\mathcal{T}[\varepsilon]$ are obtained by doing the change of variables $(r, s, x_3) \mapsto (R, s, X_3)$ in the operators $\mathsf{L}(\varepsilon)$ et $\mathsf{T}(\varepsilon)$.

We then show that the existence of a formal series $\boldsymbol{\varphi}[\varepsilon]$ with exponentially decreasing coefficients satisfying the equations (5.1) relies upon the fact that the formal series $\boldsymbol{z}[\varepsilon]$ satisfies a condition written

$$\mathsf{d}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathsf{h}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{5.2}$$

where $\mathsf{d}[\varepsilon]$ is a formal series with coefficients operator taking values in $\mathcal{C}^\infty(\partial S_0)^4$, and such that

$$\mathsf{d}^0\boldsymbol{z} = (z_r, z_s, z_3, \partial_r z_3)\big|_{\partial S_0}.$$

Thus, if $\boldsymbol{z}[\varepsilon]$ satisfies the equations

$$\begin{cases} \mathsf{A}[\varepsilon]\boldsymbol{z}[\varepsilon] &= \mathsf{G}[\varepsilon]\boldsymbol{f}[\varepsilon], \\ \mathsf{d}[\varepsilon]\boldsymbol{z}[\varepsilon] &= \mathsf{h}[\varepsilon]\boldsymbol{f}[\varepsilon], \end{cases} \tag{5.3}$$

then there exists a formal series $\boldsymbol{\varphi}[\varepsilon]$ of boundary layer coefficients satisfying the equations (5.1), and moreover, the formal series $\boldsymbol{w}[\varepsilon]$ defined by (4.27) satisfies the system (4.28).

However, the problem (5.3) does not admit a solution in general. When it is possible, as in the case of plates, is gives an asymptotic expansion.

5.a Three-dimensional boundary layer operators

Consider the coordinate system (r, s, X_3) in a neighborhood of Γ_0 in Ω. We set

$$R = \frac{r}{\varepsilon} \qquad \text{and thus} \qquad \partial_r = \frac{1}{\varepsilon}\partial_R. \tag{5.4}$$

The coordinate system (R, X_3, s) is defined on the manifold $\Sigma^+ \times \partial S_0$ where $\Sigma^+ := \mathbb{R}^+ \times I \ni (R, X_3)$ is a semi-strip. The boundary of Σ^+ decomposes into a lateral boundary $\gamma_0 := \{R = 0\} \times I$ and the two half-lines $\gamma_\pm := \mathbb{R}^+ \times \{X_3 = \pm 1\}$. In coordinates (r, s, x_3), we write $(\mathsf{L}, \mathsf{T})(r, s, x_3; \partial_r, \partial_s, \partial_3)$ the 3D operators. For $\varepsilon \leq \varepsilon_0$, we define the operators $\big(\mathcal{L}(\varepsilon), \mathcal{T}(\varepsilon)\big)$ on $\Sigma^+ \times \partial S_0$ by the formulas

$$\begin{cases} \mathcal{L}(\varepsilon)(R, s, X_3; \partial_R, \partial_s, \partial_{X_3}) := \mathsf{L}(\varepsilon R, s, \varepsilon X_3; \varepsilon^{-1}\partial_R, \partial_s, \varepsilon^{-1}\partial_{X_3}) \quad \text{and} \\ \mathcal{T}(\varepsilon)(R, s, X_3; \partial_R, \partial_s, \partial_{X_3}) := \mathsf{T}(\varepsilon R, s, \varepsilon X_3; \varepsilon^{-1}\partial_R, \partial_s, \varepsilon^{-1}\partial_{X_3}). \end{cases} \tag{5.5}$$

The formal series $\big(\mathcal{L}[\varepsilon], \mathcal{T}[\varepsilon]\big)$ are then the formal series associated with these operators using the Taylor expansion in $R = 0$ and $X_3 = 0$ of the coefficients.

We then write

$$\mathcal{L}[\varepsilon] = \varepsilon^{-2} \sum_{k \geq 0} \varepsilon^k \mathcal{L}^k \quad \text{and} \quad \mathcal{T}[\varepsilon] = \varepsilon^{-1} \sum_{k \geq 0} \varepsilon^k \mathcal{T}^k,$$

where

$$\mathcal{L}^k : \mathcal{C}^\infty\big(\Sigma^+ \times \partial S_0\big)^3 \to \mathcal{C}^\infty\big(\Sigma^+ \times \partial S_0\big)^3 \quad \text{and} \quad \mathcal{T}^k : \mathcal{C}^\infty\big(\Sigma^+ \times \partial S_0\big)^3 \to \mathcal{C}^\infty\big(\gamma_\pm \times \partial S_0\big)^3$$

are operators of degree 2 polynomials in R and X_3.

We compute that the first term of the formal series $\mathcal{L}[\varepsilon]$ writes, using the fact that the metric tensor is the identity on ∂S_0:

$$\begin{aligned} \mathcal{L}^0_R(\boldsymbol{\psi}) &= \mu(\partial_R^2 \psi_R + \partial_{X_3}^2 \psi_R) + (\lambda + \mu)\partial_R(\partial_R \psi_R + \partial_{X_3}\psi_3), \\ \mathcal{L}^0_s(\boldsymbol{\psi}) &= \mu(\partial_R^2 \psi_s + \partial_{X_3}^2 \psi_s), \\ \mathcal{L}^0_3(\boldsymbol{\psi}) &= \mu(\partial_R^2 \psi_3 + \partial_{X_3}^2 \psi_3) + (\lambda + \mu)\partial_{X_3}(\partial_R \psi_R + \partial_{X_3}\psi_3). \end{aligned} \tag{5.6}$$

Remark that this operator is independent on s, and that it does not depend on the geometry of S_0. In particular, it is the same as for plates. Similarly, the first term of the formal series $\mathcal{T}[\varepsilon]$ writes

$$\begin{aligned} \mathcal{T}^0_R(\boldsymbol{\psi}) &= \mu(\partial_{X_3}\psi_R + \partial_R \psi_3), \\ \mathcal{T}^0_s(\boldsymbol{\psi}) &= \mu \partial_{X_3}\psi_s, \\ \mathcal{T}^0_3(\boldsymbol{\psi}) &= (\lambda + 2\mu)\partial_{X_3}\psi_3 + \lambda \partial_R \psi_R. \end{aligned} \tag{5.7}$$

As in [7], we introduce the following spaces: Let $\mathfrak{H}(\Sigma^+)$ be the space of $\mathcal{C}^\infty$ functions φ on the semi-strip Σ^+ except in the non regular points $(R=0, X_3=\pm 1)$, and such that φ is exponentially decreasing with R in the following sense:

$$\forall\, i,j,k \in \mathbb{N}, \qquad e^{\delta R}\, R^k\, \partial_R^i \partial_{X_3}^j \varphi \in \mathrm{L}^2(\Sigma^+),$$

where $\delta > 0$ is a real strictly less than the smallest Papkovich-Fadle exponent (see [17]). In the neighborhood of the two corners of the semi-strip, we impose the following: if ρ denote the distance in Σ^+ to a point $(R=0, X_3=\pm 1)$, we suppose that each φ in $\mathfrak{H}(\Sigma^+)$ satisfies

$$\forall\, i,j \in \mathbb{N},\ i+j \neq 0, \qquad \rho^{i+j-1}\, \partial_R^i \partial_{X_3}^j \varphi \in \mathrm{L}^2(\Sigma^+).$$

We then define the corresponding displacement space

$$\boldsymbol{\mathfrak{H}}(\Sigma^+) := \big\{ \boldsymbol{\varphi} = (\varphi_R, \varphi_s, \varphi_3) \in \mathfrak{H}(\Sigma^+)^3 \big\}.$$

As the arc-length appears as a parameter, the natural space in which the equations will be posed is hence $\mathcal{C}^\infty\big(\partial S_0, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$.

We now define the associated range spaces: We set $\mathfrak{K}(\Sigma^+)$ the space of $\psi \in \mathcal{C}^\infty(\Sigma^+)$ such that

$$\forall\, i,j,k \in \mathbb{N},\ e^{\delta R}\, R^k\, \partial_R^i \partial_{X_3}^j \psi \in \mathrm{L}^2(\Sigma^+) \quad \text{and} \quad \forall\, i,j \in \mathbb{N},\ \rho^{i+j+1}\, \partial_R^i \partial_{X_3}^j \psi \in \mathrm{L}^2(\Sigma^+)$$

with the same notations. Similarly, we introduce the same space corresponding to the trace operators on $\gamma_\pm$: let $\mathfrak{K}(\gamma_\pm)$ the space of couple of functions $\psi^\pm \in \mathcal{C}^\infty(\gamma_\pm)$ such that

$$\forall\, i,k \in \mathbb{N}, \quad e^{\delta R}\, R^k\, \partial_R^i \psi^\pm \in \mathrm{L}^2(\gamma_\pm) \quad \text{and} \quad \forall\, i,j \in \mathbb{N}, \quad \rho^{i+j+1/2}\, \partial_R^i \psi^\pm \in \mathrm{L}^2(\gamma_\pm).$$

We then define the spaces

$$\boldsymbol{\mathfrak{K}}(\Sigma^+) := \big\{ \boldsymbol{\psi} = (\psi_R, \psi_s, \psi_3) \in \mathfrak{K}(\Sigma^+)^3 \big\},$$

and

$$\boldsymbol{\mathfrak{K}}(\gamma_\pm) := \big\{ \boldsymbol{\psi}^\pm = (\psi_R^\pm, \psi_s^\pm, \psi_3^\pm) \in \mathfrak{K}(\gamma_\pm)^3 \big\}.$$

Thus the operators $\mathcal{L}^0$ et $\mathcal{T}^0$ act on the space $\mathcal{C}^\infty\big(\partial S_0, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$ and take values in $\mathcal{C}^\infty\big(\partial S_0, \boldsymbol{\mathfrak{K}}(\Sigma^+)\big)$ and $\mathcal{C}^\infty\big(\partial S_0, \boldsymbol{\mathfrak{K}}(\gamma_\pm)\big)$ respectively.

The properties of the operators $\mathcal{L}^0$ and $\mathcal{T}^0$ involve the rigid displacement space $\mathfrak{Z}$ spanned by the four following displacements, written in coordinates (R, s, X_3) (see [10]) :

$$\boldsymbol{\mathcal{Z}}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{\mathcal{Z}}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{\mathcal{Z}}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{\mathcal{Z}}^4 = \begin{pmatrix} -X_3 \\ 0 \\ R \end{pmatrix}. \tag{5.8}$$

These displacements are in the kernel of the operator $(\mathcal{L}^0, \mathcal{T}^0)$ without boundary condition on the lateral boundary. The operators $\mathcal{L}^0$ and $\mathcal{T}^0$ have the following property (see for example [10, section 5]):

Proposition 5.1 *Let* $\psi \in \mathfrak{K}(\Sigma^+)$, $\psi^\pm \in \mathfrak{K}(\gamma_\pm)$ *and* $\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\gamma}_0)^3$. *There exist a unique* $\varphi \in \mathfrak{H}(\Sigma^+)$ *and a unique* $\boldsymbol{\mathcal{Z}} \in \mathfrak{Z}$ *such that*

$$\left\{ \begin{array}{rcll} \mathcal{L}^0(\varphi - \boldsymbol{\mathcal{Z}}) & = & \psi & \textit{in} \quad \Sigma^\pm, \\ \mathcal{T}^0(\varphi - \boldsymbol{\mathcal{Z}}) & = & \psi^\pm & \textit{on} \quad \gamma_+ \times \gamma_-, \\ (\varphi - \boldsymbol{\mathcal{Z}})\big|_{R=0} + \boldsymbol{v}\big|_{\gamma_0} & = & 0. & \end{array} \right. \tag{5.9}$$

Remark that as $\boldsymbol{\mathcal{Z}} \in \mathfrak{Z}$ the left-hand sides of the two first equations of (5.9) are equals to $\mathcal{L}^0(\varphi)$ et $\mathcal{T}^0(\varphi)$. The following corollary is clear using the fact that the operator $(\mathcal{L}^0, \mathcal{T}^0)$ does not depend on s :

Corollary 5.2 *If we have* $\psi \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{K}(\Sigma^+)\big)$, $\psi^\pm \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{K}(\gamma_\pm)\big)$ *and* $\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Gamma}_0)^3$ *in the previous proposition, then the functions solution of* (5.9) *are in the spaces* $\varphi \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$ *and* $\boldsymbol{\mathcal{Z}} \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{Z}\big)$.

5.b Formal series solution

We now want to find a formal series $\varphi[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \varphi^k(R, s, X_3)$ with coefficients in the space $\mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$ satisfying the equation (5.1) If we set $\boldsymbol{w}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{w}^k = \mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]$, this formal series equation is equivalent to the following collection, for $k \geq 0$,

$$\left\{ \begin{array}{rcll} \mathcal{L}^0\varphi^k & = & -\sum_{\ell=1}^k \mathcal{L}^\ell \varphi^{k-\ell} & \text{in} \quad \partial S_0 \times \Sigma^+, \\ \mathcal{T}^0\varphi^k & = & -\sum_{\ell=1}^k \mathcal{T}^\ell \varphi^{k-\ell} & \text{on} \quad \partial S_0 \times \gamma_+ \times \gamma_-, \\ \varphi^k\big|_{R=0} + \boldsymbol{w}^k\big|_{\Gamma_0} & = & 0. & \end{array} \right.$$

Note that the sum $\varphi[\varepsilon]\big|_{R=0} + \boldsymbol{w}[\varepsilon]\big|_{\Gamma_0}$ only make sense on the boundary. Using Proposition 5.1 we prove the following theorem:

Theorem 5.3 *Let* $(\mathsf{V}[\varepsilon], \mathsf{Q}[\varepsilon], \mathsf{A}[\varepsilon], \mathsf{G}[\varepsilon])$ *be the formal series of Theorem* 4.1. *For all* $k \geq 0$ *there exist*

- *an operator* $\Psi^k : \Sigma(S_0) \to \mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$,
- *an operator* $\Theta^k : \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$,
- *an operator* $\mathfrak{d}^k : \Sigma(S_0) \to \mathcal{C}^\infty(\partial S_0, \mathfrak{Z})$,
- *an operator* $\mathfrak{h}^k : \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \mathcal{C}^\infty(\partial S_0, \mathfrak{Z})$,

such that if $\boldsymbol{z}[\varepsilon]$ *is a formal series with coefficients in* $\Sigma(S_0)$ *satisfying the relation*

$$\mathfrak{d}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathfrak{h}[\varepsilon]\boldsymbol{f}[\varepsilon],$$

where $\boldsymbol{f}[\varepsilon]$ *is a formal series with coefficients in* $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$, *then the formal series*

$$\varphi[\varepsilon] := \Psi[\varepsilon]\boldsymbol{z}[\varepsilon] + \Theta[\varepsilon]\boldsymbol{f}[\varepsilon]$$

is a solution of the formal series problem

$$\begin{cases} \mathcal{L}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] &= 0, \\ \mathcal{T}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] &= 0, \\ \boldsymbol{\varphi}[\varepsilon]\,\big|_{R=0} + \big(\mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]\big)\,\big|_{\Gamma_0} &= 0. \end{cases} \tag{5.10}$$

Moreover, Ψ^0 *,* Θ^0 *and* Θ^1 *,* $\mathfrak{h}^0$ *and* $\mathfrak{h}^1$ *are the null operators and we have that*

$$\mathfrak{d}^0\boldsymbol{z} = \big(z_r\,\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^1 + \big(z_s\,\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^2 + \big(z_3\,\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^3, \tag{5.11}$$

and

$$\mathfrak{d}^1\boldsymbol{z} = \big(c_1\gamma_\alpha^\alpha(\boldsymbol{z})\,\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^1 + \big(\theta_r(\boldsymbol{z})\,\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^4, \tag{5.12}$$

where c_1 *is a coefficients depending on* λ *and* μ *, and* Ψ^1 *is defined by*

$$\Psi_R^1\boldsymbol{z} = \big(p\gamma_\alpha^\alpha(\boldsymbol{z})\,\big|_{\partial S_0}\big)\overline{\varphi}_R^1, \quad \Psi_s^1\boldsymbol{z} = \big(\theta_s(\boldsymbol{z})\,\big|_{\partial S_0}\big)\overline{\varphi}_s^1, \quad \textit{and} \quad \Psi_3^1\boldsymbol{z} = \big(p\gamma_\alpha^\alpha(\boldsymbol{z})\,\big|_{\partial S_0}\big)\overline{\varphi}_3^1, \tag{5.13}$$

where $\overline{\boldsymbol{\varphi}}^1 = (\overline{\varphi}_R^1, \overline{\varphi}_s^1, \overline{\varphi}_3^1)$ *is an element of* $\mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$ *independent on* ε *.*

Proof. As for Theorems 4.1 and 4.3 we will show that the formal series satisfy in fact formal series functional equations. We will show separately the existence of the formal series $\Psi[\varepsilon]$ and $\mathfrak{d}[\varepsilon]$ acting on $\Sigma(S_0)$ and $\Theta[\varepsilon]$ and $\mathfrak{h}[\varepsilon]$ acting on the space $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ respectively. We conclude by summation.

1. We first show the existence of operator formal series $\Psi[\varepsilon] = \sum_{k\geq 0}\varepsilon^k\Psi^k$ and $\mathfrak{d}[\varepsilon] = \sum_{k\geq 0}\varepsilon^k\mathfrak{d}^k$ satisfying the equations

$$\begin{cases} \mathcal{L}[\varepsilon]\Psi[\varepsilon] &= 0, \\ \mathcal{T}[\varepsilon]\Psi[\varepsilon] &= 0, \\ \big(\Psi[\varepsilon] - \mathfrak{d}[\varepsilon]\big)\,\big|_{R=0} + \mathsf{V}[\varepsilon]\,\big|_{\Gamma_0} &= 0, \end{cases} \tag{5.14}$$

in the space of formal series with operators coefficients acting on $\Sigma(S_0)$. For $k = 0$ and for $\boldsymbol{z} \in \Sigma(S_0)$, the equations for Ψ^0 and $\mathfrak{d}^0$ write

$$\begin{cases} \mathcal{L}^0\Psi^0\boldsymbol{z} &= 0 \quad \text{in} \quad \partial S_0 \times \Sigma^+, \\ \mathcal{T}^0\Psi^0\boldsymbol{z} &= 0 \quad \text{on} \quad \partial S_0 \times \gamma_+ \times \gamma_-, \\ (\Psi^0\boldsymbol{z} - \mathfrak{d}^0\boldsymbol{z})\,\big|_{R=0} + \mathsf{V}^0\boldsymbol{z}\,\big|_{\Gamma_0} &= 0. \end{cases}$$

As $\mathsf{V}^0\boldsymbol{z} = \mathcal{I} \circ \boldsymbol{z}$, we see that $\Psi^0 = 0$ and $\mathfrak{d}^0$ given by (5.11) are solutions.

Suppose that Ψ^k and $\mathfrak{d}^k$ are constructed for $k \leq n$, where $n \in \mathbb{N}$, and let $\boldsymbol{z} \in \Sigma(S_0)$. We consider the equation in $\boldsymbol{\psi}$:

$$\left\{\begin{array}{rcll} \mathcal{L}^0\boldsymbol{\psi} & = & -\sum_{\ell=1}^{n+1} \mathcal{L}^\ell \Psi^{n+1-\ell}\boldsymbol{z} & \text{in} \quad \partial S_0 \times \Sigma^+, \\ \mathcal{T}^0\boldsymbol{\psi} & = & -\sum_{\ell=1}^{n+1} \mathcal{T}^\ell \Psi^{n+1-\ell}\boldsymbol{z} & \text{on} \quad \partial S_0 \times \gamma_+ \times \gamma_-, \\ \boldsymbol{\psi}\big|_{R=0} + \mathsf{V}^{n+1}\boldsymbol{z}\big|_{\Gamma_0} & = & 0. & \end{array}\right. \tag{5.15}$$

Using the properties of the operators $\mathcal{L}^\ell$ and $\mathcal{T}^\ell$, we see that the right-hand sides of the two first equations are in the spaces $\mathcal{C}^\infty\big(\partial S_0, \mathfrak{K}(\Sigma^+)\big)$ and $\mathcal{C}^\infty\big(\partial S_0, \mathfrak{K}(\gamma_\pm)\big)$ respectively. Corollary 5.2 then shows the existence of $\boldsymbol{\varphi} \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$ and $\boldsymbol{\mathcal{Z}} \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{Z}\big)$ such that $\boldsymbol{\psi} = \boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}}$ is solution of the system.
Setting $\Psi^{n+1}\boldsymbol{z} := \boldsymbol{\varphi}$ and $\mathfrak{d}^{n+1}\boldsymbol{z} := \boldsymbol{\mathcal{Z}}$, we obtain the existence of the operator at the rank $n+1$.
2. Similarly we show the existence of the formal series $\Theta[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \Theta^k$ and $\mathfrak{h}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \mathfrak{h}^k$ satisfying the equations:

$$\left\{\begin{array}{rcl} \mathcal{L}[\varepsilon]\Theta[\varepsilon] & = & 0, \\ \mathcal{T}[\varepsilon]\Theta[\varepsilon] & = & 0, \\ \big(\Theta[\varepsilon] + \mathfrak{h}[\varepsilon]\big)\big|_{R=0} + \mathsf{Q}[\varepsilon]\big|_{\Gamma_0} & = & 0, \end{array}\right. \tag{5.16}$$

in the space of formal series with operator coefficients acting on $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$. The fact that $\mathsf{Q}^0 = \mathsf{Q}^1 = 0$ shows that taking Θ^0, Θ^1, $\mathfrak{h}^0$ and $\mathfrak{h}^1$ as the null operators is a solution for $k = 0, 1$.
Suppose that the operators Θ^k and $\mathfrak{h}^k$ are constructed for $k \leq n$ where $n \in \mathbb{N}$. Let $\boldsymbol{f} \in \mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ and consider the equation in $\boldsymbol{\psi}$:

$$\left\{\begin{array}{rcll} \mathcal{L}^0\boldsymbol{\psi} & = & -\sum_{\ell=1}^{n+1} \mathcal{L}^\ell \Theta^{n+1-\ell}\boldsymbol{f} & \text{in} \quad \partial S_0 \times \Sigma^+, \\ \mathcal{T}^0\boldsymbol{\psi} & = & -\sum_{\ell=1}^{n+1} \mathcal{T}^\ell \Theta^{n+1-\ell}\boldsymbol{f} & \text{on} \quad \partial S_0 \times \gamma_+ \times \gamma_-, \\ \boldsymbol{\psi}\big|_{R=0} + \mathsf{Q}^{n+1}\boldsymbol{f}\big|_{\Gamma_0} & = & 0. & \end{array}\right.$$

Using the properties of $\mathcal{L}^\ell$ and $\mathcal{T}^\ell$, we see that the right-hand sides of the first two equations are in the spaces $\mathcal{C}^\infty\big(\partial S_0, \mathfrak{K}(\Sigma^+)\big)$ and $\mathcal{C}^\infty\big(\partial S_0, \mathfrak{K}(\gamma_\pm)\big)$. Corollary 5.1 then shows the existence of $\boldsymbol{\varphi} \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$ and $\boldsymbol{\mathcal{Z}} \in \mathcal{C}^\infty\big(\partial S_0, \mathfrak{Z}\big)$ such that $\boldsymbol{\psi} = \boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}}$ is solution of the system. Setting $\Theta^{n+1}\boldsymbol{z} := \boldsymbol{\varphi}$ and $\mathfrak{h}^{n+1}\boldsymbol{z} := -\boldsymbol{\mathcal{Z}}$ proves the existence of the operators at the order $n+1$.
3. Now let $\boldsymbol{z}[\varepsilon]$ is a formal series with coefficients in $\Sigma(S_0)$ and $\boldsymbol{f}[\varepsilon]$ a formal series with coefficients in $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$. By summing the equations (5.14) applied to $\boldsymbol{z}[\varepsilon]$ and the equations (5.16) applied to $\boldsymbol{f}[\varepsilon]$, we see that $\boldsymbol{\varphi}[\varepsilon] := \Psi[\varepsilon]\boldsymbol{z}[\varepsilon] + \Theta[\varepsilon]\boldsymbol{f}[\varepsilon]$ satisfies the equation $\big(\mathcal{L}[\varepsilon], \mathcal{T}[\varepsilon]\big)\boldsymbol{\varphi}[\varepsilon] = 0$ with the boundary equation

$$\boldsymbol{\varphi}[\varepsilon]\big|_{R=0} + \big(\mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]\big)\big|_{\Gamma_0} + \big(-\mathfrak{d}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathfrak{h}[\varepsilon]\boldsymbol{f}[\varepsilon]\big)\big|_{R=0} = 0.$$

We deduce the result from these equations.
4. The equations satisfied by Ψ^1 and $\mathfrak{d}^1$ write, for $z \in \Sigma(S_0)$,

$$(\mathcal{L}^0, \mathcal{T}^0)\Psi^1 z = 0 \quad \text{and} \quad (\Psi^1 z - \mathfrak{d}^1 z)\big|_{R=0} + \mathsf{V}^1 z\big|_{\partial S_0} = 0.$$

In coordinates (r, s, x_3) the components of V^1 are $\mathsf{V}^1_r(z) = -X_3\theta_r(z)$, $\mathsf{V}^1_s(z) = -X_3\theta_s(z)$ and $\mathsf{V}^1_3(z) = -X_3 p\gamma^\alpha_\alpha(z)$. Note that the operator $(\mathcal{L}^0, \mathcal{T}^0)$ does not depend on s and does not contain derivatives of s. As only $\mathsf{V}^1\big|_{\partial S_0}$ is involved in the equations and thanks to the linearity of the operator we only have to look for the solution of the similar problem with right-hand sides $-X_3$ in each component.
(a) We first consider the solution $(\boldsymbol{\varphi}, \boldsymbol{\mathcal{Z}})$ for the problem

$$(\mathcal{L}^0, \mathcal{T}^0)\boldsymbol{\varphi} = 0 \quad \text{and} \quad (\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}})\big|_{R=0} + (-X_3, 0, 0)\big|_{\partial S_0} = 0.$$

Recall that $\boldsymbol{\mathcal{Z}}^4 = (-X_3, 0, R)$ and that this element is in the kernel of $(\mathcal{L}^0, \mathcal{T}^0)$. Hence the solution of the previous problem is simply $\boldsymbol{\varphi} = 0$ and $\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{Z}}^4$. But as

$$(\mathcal{L}^0, \mathcal{T}^0)\big(\theta_r(z)\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^4 = \big(\theta_r(z)\big|_{\partial S_0}\big)(\mathcal{L}^0, \mathcal{T}^0)\boldsymbol{\mathcal{Z}}^4 = 0$$

we deduce that the couple $\boldsymbol{\varphi} = 0$ and $\boldsymbol{\mathcal{Z}} = \big(\theta_r(z)\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^4$ satisfies the equations

$$(\mathcal{L}^0, \mathcal{T}^0)\boldsymbol{\varphi} = 0 \quad \text{and} \quad (\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}})\big|_{R=0} + (\mathsf{V}^1_r(z), 0, 0)\big|_{\partial S_0} = 0.$$

(b) Consider now the equation

$$(\mathcal{L}^0, \mathcal{T}^0)\boldsymbol{\varphi} = 0 \quad \text{and} \quad (\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}})\big|_{R=0} + (0, -X_3, 0)\big|_{\partial S_0} = 0. \tag{5.17}$$

We note that the operators $\mathcal{L}^0$ and $\mathcal{T}^0$ decouples into two parts: the operators $\mathcal{L}^0_s$ and $\mathcal{T}^0_s$ acting on φ_s and the operators $(\mathcal{L}^0_R, \mathcal{L}^0_3)$ and $(\mathcal{T}^0_R, \mathcal{T}^0_3)$ acting on (φ_R, φ_3) respectively. In particular, the components φ_R and φ_3 of $\boldsymbol{\varphi}$ equal to zero, and the components of $\boldsymbol{\mathcal{Z}}$ on the vectors $\boldsymbol{\mathcal{Z}}^1$, $\boldsymbol{\mathcal{Z}}^3$ and $\boldsymbol{\mathcal{Z}}^4$ are zero.
Moreover, Proposition 5.4 and Lemma 5.5 of [10] yield that there exists a unique non zero function $\overline{\varphi}^1_s$ of the space $\mathfrak{H}(\Sigma^+)$ such that

$$(\mathcal{L}^0_s, \mathcal{T}^0_s)\overline{\varphi}^1_s = 0 \quad \text{and} \quad \overline{\varphi}^1_s\big|_{R=0} = X_3.$$

The terms $\boldsymbol{\varphi} = (0, \overline{\varphi}^1_s, 0)$ and $\boldsymbol{\mathcal{Z}} = 0$ are solution of (5.17). We hence verify that for $z \in \Sigma(S_0)$ the elements

$$\boldsymbol{\mathcal{Z}} = 0 \quad \text{et} \quad \boldsymbol{\varphi} = \big(\theta_s(z)\big|_{\partial S_0}\big)(0, \overline{\varphi}^1_s, 0)$$

are solution of the equations

$$(\mathcal{L}^0, \mathcal{T}^0)\boldsymbol{\varphi} = 0 \quad \text{and} \quad (\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}})\big|_{R=0} + (0, \mathsf{V}^1_s(z), 0)\big|_{\partial S_0} = 0. \tag{5.18}$$

(c) Finally we consider the equations

$$(\mathcal{L}^0, \mathcal{T}^0)\boldsymbol{\varphi} = 0 \quad \text{and} \quad (\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}})\big|_{R=0} + (0, 0, -X_3)\big|_{\partial S_0} = 0. \tag{5.19}$$

The splitting of the operator $(\mathcal{L}^0, \mathcal{T}^0)$ in components s and (R, X_3) shows that the components s of the elements $\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{\varphi}$ solutions of (5.19) are zero.

Moreover the equations (6.4) and (6.5) of [10], using the parities of the operators $\mathcal{L}^0$ and $\mathcal{T}^0$, show that there exists a unique element $(\overline{\varphi}^1_R, \overline{\varphi}^1_3)$ of the space $\mathfrak{H}(\Sigma^+)^2$ and a unique constant c_1 depending only on λ and μ, such that

$$\begin{cases} (\mathcal{L}^0_R, \mathcal{L}^0_3)(\overline{\varphi}^1_R, \overline{\varphi}^1_3) &= 0 \quad \text{in} \quad \Sigma^+, \\ (\mathcal{T}^0_R, \mathcal{T}^0_3)(\overline{\varphi}^1_R, \overline{\varphi}^1_3) &= 0 \quad \text{on} \quad \gamma_+ \times \gamma_-, \\ \overline{\varphi}^1_R\big|_{R=0} - c_1 &= 0, \\ \overline{\varphi}^1_3\big|_{R=0} &= X_3. \end{cases}$$

The couple $\boldsymbol{\varphi} = (\overline{\varphi}^1_R, 0, \overline{\varphi}^1_3)$ and $\boldsymbol{\mathcal{Z}} = c_1 \boldsymbol{\mathcal{Z}}^1$ is then a solution of (5.19). Thus the elements

$$\boldsymbol{\mathcal{Z}} = \big(c_1 p\gamma^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S_0}\big)\boldsymbol{\mathcal{Z}}^1 \quad \text{and} \quad \boldsymbol{\varphi} = \big(p\gamma^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S_0}\big)(\overline{\varphi}^1_R, 0, \overline{\varphi}^1_3)$$

are solution of the system

$$(\mathcal{L}^0, \mathcal{T}^0)\boldsymbol{\varphi} = 0 \quad \text{and} \quad (\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}})\big|_{R=0} + (0, 0, \mathsf{V}^1_3(\boldsymbol{z}))\big|_{\partial S_0} = 0. \tag{5.20}$$

The previous equations show the theorem. ■

For all k, the operators $\mathfrak{d}^k$ decompose into 4 operators $\mathfrak{d}^k_i : \Sigma(S_0) \to \mathcal{C}^\infty(\partial S_0)$ where $\mathfrak{d}^k_i \boldsymbol{z}$ is the component of $\mathfrak{d}^k \boldsymbol{z}$ along $\boldsymbol{\mathcal{Z}}^i$. We define also similarly the operators $\mathfrak{h}^k_i : \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \mathcal{C}^\infty(\partial S_0)$. Theorem 5.3 then shows that $\mathfrak{d}^0_1 \boldsymbol{z} = z_r\big|_{\partial S_0}$, $\mathfrak{d}^0_2 \boldsymbol{z} = z_s\big|_{\partial S_0}$, $\mathfrak{d}^0_3 \boldsymbol{z} = z_3\big|_{\partial S_0}$, $\mathfrak{d}^0_4 \boldsymbol{z} = 0$ and moreover $\mathfrak{d}^1_4 \boldsymbol{z} = (\partial_r z_3 + b^r_r z_r + b^s_r z_s)\big|_{\partial S_0}$. More generally, we can prove the following result (see [14]):

Proposition 5.4 *Let $\Psi[\varepsilon]$ and $\mathfrak{d}[\varepsilon]$ be the formal series of Theorem* 5.3. *For all k there exist a finite subset F_k of $\mathbb{N}$ such that*

- *for all $j \in F_k$, there exist functions $\boldsymbol{\varphi}^{k,j}$ of $\mathcal{C}^\infty\big(\partial S_0, \mathfrak{H}(\Sigma^+)\big)$, depending only on S_0, λ and μ,*
- *for all $j \in F_k$, there exist* 2D *operators P^k_j with scalar values, with degree of derivative at most k,*
- *for all $i = 1, 2, 3, 4$, there exist* 2D *operators $\mathfrak{D}^k_i$ with scalar values with degree of derivative at most k,*

such that for $k \geq 0$ and for $\boldsymbol{z} \in \Sigma(S_0)$, we have

$$\Psi^k \boldsymbol{z} = \sum_{j \in F_k} (P^k_j \boldsymbol{z})\big|_{\partial S_0} \boldsymbol{\varphi}^{k,j} \quad \textit{and} \quad \mathfrak{d}^k \boldsymbol{z} = \sum_{i=1}^{4} (\mathfrak{D}^k_i \boldsymbol{z})\big|_{\partial S_0} \boldsymbol{\mathcal{Z}}^i. \tag{5.21}$$

6 Conclusion

In order to obtain an equation of the form (5.2), we transform the equation $\mathfrak{d}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathfrak{h}[\varepsilon]\boldsymbol{f}[\varepsilon]$ in the following way: We define the formal series $\mathsf{d}[\varepsilon]$ and $\mathsf{h}[\varepsilon]$ with coefficients $\mathsf{d}^k : \Sigma(S_0) \to \mathcal{C}^\infty(\partial S_0)^4$, and $\mathsf{h}^k : \mathcal{C}^\infty\big(I, \Sigma(S_0)\big) \to \mathcal{C}^\infty(\partial S_0)^4$, as

$$\mathsf{d}[\varepsilon] = \big(\mathfrak{d}_1[\varepsilon], \mathfrak{d}_2[\varepsilon], \mathfrak{d}_3[\varepsilon], \varepsilon^{-1}\mathfrak{d}_4[\varepsilon] - b^r_r \mathfrak{d}_1[\varepsilon] - b^s_r \mathfrak{d}_2[\varepsilon]\big) \tag{6.1}$$

and

$$\mathsf{h}[\varepsilon] = \big(\mathfrak{h}_1[\varepsilon], \mathfrak{h}_2[\varepsilon], \mathfrak{h}_3[\varepsilon], \varepsilon^{-1}\mathfrak{h}_4[\varepsilon] - b_r^r\mathfrak{h}_1[\varepsilon] - b_r^s\mathfrak{h}_2[\varepsilon]\big), \tag{6.2}$$

where b_r^r and b_r^s are the components $b_r^r(r,s)$ and $b_r^s(r,s)$ evaluated on ∂S_0 We see that h^0 is the null operator and

$$\mathsf{d}^0\boldsymbol{z} = (z_r, z_s, z_3, \partial_r z_3)\,\big|_{\partial S_0}, \tag{6.3}$$

and it is clear that the formal series equations $\mathfrak{d}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathfrak{h}[\varepsilon]\boldsymbol{f}[\varepsilon]$ and $\mathsf{d}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathsf{h}[\varepsilon]\boldsymbol{f}[\varepsilon]$ are equivalent. The final result then states:

Theorem 6.1 *Let* $(\mathsf{V}[\varepsilon], \mathsf{Q}[\varepsilon], \mathsf{A}[\varepsilon], \mathsf{G}[\varepsilon])$ *the formal series given by Theorem* 4.1 *and let* $(\Psi[\varepsilon], \Theta[\varepsilon], \mathsf{d}[\varepsilon], \mathsf{h}[\varepsilon])$ *the formal series of Theorem* 5.3 *and the equations* (6.1),(6.2). *If* $\boldsymbol{f}[\varepsilon]$ *is a formal series with coefficient in* $\mathcal{C}^\infty\big(I, \Sigma(S_0)\big)$ *and if* $\boldsymbol{z}[\varepsilon]$ *is a formal series with coefficients in* $\Sigma(S_0)$ *, such that the equation*

$$\left\{\begin{array}{rcl} \mathsf{A}[\varepsilon]\boldsymbol{z}[\varepsilon] &=& \mathsf{G}[\varepsilon]\boldsymbol{f}[\varepsilon],\\ \mathsf{d}[\varepsilon]\boldsymbol{z}[\varepsilon] &=& \mathsf{h}[\varepsilon]\boldsymbol{f}[\varepsilon], \end{array}\right. \tag{6.4}$$

are satisfied, then the formal series $\boldsymbol{w}[\varepsilon] := \mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]$ *and* $\boldsymbol{\varphi}[\varepsilon] := \Psi[\varepsilon]\boldsymbol{z}[\varepsilon] + \Theta[\varepsilon]\boldsymbol{f}[\varepsilon]$ *are solutions of the equations*

$$\left\{\begin{array}{rcl} \mathsf{L}[\varepsilon]\boldsymbol{w}[\varepsilon] &=& -\boldsymbol{f}[\varepsilon],\\ \mathsf{T}[\varepsilon]\boldsymbol{w}[\varepsilon] &=& 0, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcl} \mathcal{L}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] &=& 0,\\ \mathcal{T}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] &=& 0, \end{array}\right.$$

with the boundary condition

$$\boldsymbol{w}[\varepsilon]\,\big|_{\Gamma_0} + \boldsymbol{\varphi}[\varepsilon]\,\big|_{R=0} = 0.$$

Appendix A: Proof of Theorem 3.3

(a) We first consider the surfacic components of the operator. Using the equation (3.12) and the fact that $\tilde{e}_\ell^\ell(\boldsymbol{w})$ is a function, we have in normal coordinates on the manifold Ω^ε that

$$\mathsf{L}_\sigma(\boldsymbol{w}) = \lambda \mathrm{D}_\sigma\big(\tilde{e}_\alpha^\alpha(\boldsymbol{w}) + \tilde{e}_3^3(\boldsymbol{w})\big) + 2\mu\big(\nabla_\alpha \tilde{e}_\sigma^\alpha(\boldsymbol{w}) + \nabla_3 \tilde{e}_\sigma^3(\boldsymbol{w})\big).$$

In this expression, we have

$$\begin{aligned}\nabla_\alpha \tilde{e}_\sigma^\alpha(\boldsymbol{w}) = \partial_\alpha \tilde{e}_\sigma^\alpha(\boldsymbol{w}) &+ \Gamma_{\alpha\beta}^\alpha(x_3)\tilde{e}_\sigma^\beta(\boldsymbol{w})\\ &+ \Gamma_{\alpha 3}^\alpha(x_3)\tilde{e}_\sigma^3(\boldsymbol{w}) - \Gamma_{\alpha\sigma}^3(x_3)\tilde{e}_3^\alpha(\boldsymbol{w}) - \Gamma_{\alpha\sigma}^\beta(x_3)\tilde{e}_\beta^\alpha(\boldsymbol{w}).\end{aligned}$$

As the terms $\Gamma_{\alpha\sigma}^\beta(x_3)$ are the Christoffel symbols of the connexion $\mathrm{D}_\sigma^{x_3}$ on S_{x_3} , and using the equation (3.4) we have

$$\nabla_\alpha \tilde{e}_\sigma^\alpha(\boldsymbol{w}) = \mathrm{D}_\alpha^{x_3}\tilde{e}_\sigma^\alpha(\boldsymbol{w}) - b_\alpha^\delta(\mu^{-1})_\delta^\alpha(x_3)\tilde{e}_\sigma^3(\boldsymbol{w}) - b_\alpha^\delta\mu_{\delta\sigma}(x_3)\tilde{e}_3^\alpha(\boldsymbol{w}), \tag{6.5}$$

where we have using (3.6)

$$\begin{aligned} \mathrm{D}^{x_3}_\alpha \tilde{e}^\alpha_\sigma(\boldsymbol{w}) &= \partial_\alpha \tilde{e}^\alpha_\sigma(\boldsymbol{w}) + \Gamma^\alpha_{\alpha\beta}(x_3)\tilde{e}^\beta_\sigma(\boldsymbol{w}) - \Gamma^\beta_{\alpha\sigma}(x_3)\tilde{e}^\alpha_\beta(\boldsymbol{w}) \\ &= \mathrm{D}_\alpha \tilde{e}^\alpha_\sigma(\boldsymbol{w}) - x_3(\mu^{-1})^\alpha_\delta(x_3)\tilde{e}^\beta_\sigma(\boldsymbol{w})\mathrm{D}_\alpha b^\delta_\beta + x_3(\mu^{-1})^\beta_\delta(x_3)\tilde{e}^\alpha_\beta(\boldsymbol{w})\mathrm{D}_\alpha b^\delta_\sigma. \end{aligned} \tag{6.6}$$

Moreover, we have using (3.4)

$$\nabla_3 \tilde{e}^3_\sigma(\boldsymbol{w}) = \partial_3 \tilde{e}^3_\sigma(\boldsymbol{w}) - \Gamma^\beta_{3\sigma}(x_3)\tilde{e}^3_\beta(\boldsymbol{w}) = \partial_3 \tilde{e}^3_\sigma(\boldsymbol{w}) + b^\delta_\sigma(\mu^{-1})^\beta_\delta(x_3)\tilde{e}^3_\beta(\boldsymbol{w}). \tag{6.7}$$

But we have $\tilde{e}^\alpha_3(\boldsymbol{w}) = g^{\alpha\beta}(x_3)g_{33}(x_3)\tilde{e}^3_\beta(\boldsymbol{w})$. As $g^{\alpha\beta}(x_3) = (\mu^{-1})^\alpha_\sigma(x_3)(\mu^{-1})^\beta_\delta(x_3)a^{\sigma\delta}$, we see that

$$b^\delta_\alpha \mu_{\delta\sigma}(x_3)\tilde{e}^\alpha_3(\boldsymbol{w}) = b^\delta_\alpha \mu_{\delta\sigma}(x_3)(\mu^{-1})^\alpha_\nu(x_3)(\mu^{-1})^\beta_\lambda(x_3)a^{\nu\lambda}\tilde{e}^3_\beta(\boldsymbol{w}).$$

As we have $b^\delta_\alpha \mu_{\delta\sigma}(x_3) = b^\delta_\sigma \mu_{\delta\alpha}(x_3)$ we thus have

$$b^\delta_\sigma(\mu^{-1})^\beta_\delta(x_3)\tilde{e}^3_\beta(\boldsymbol{w}) = b^\delta_\alpha \mu_{\delta\sigma}(x_3)\tilde{e}^\alpha_3(\boldsymbol{w}).$$

Hence, by summing the equations (6.5) and (6.7) we get

$$\nabla_\alpha \tilde{e}^\alpha_\sigma(\boldsymbol{w}) + \nabla_3 \tilde{e}^3_\sigma(\boldsymbol{w}) = \partial_3 \tilde{e}^3_\sigma(\boldsymbol{w}) + \mathrm{D}^{x_3}_\alpha \tilde{e}^\alpha_\sigma(\boldsymbol{w}) - b^\delta_\alpha(\mu^{-1})^\alpha_\delta(x_3)\tilde{e}^3_\sigma(\boldsymbol{w})$$

and as $\tilde{e}^3_3(\boldsymbol{w}) = \partial_3 w_3$ we get

$$\begin{aligned} \mathsf{L}_\sigma(\boldsymbol{w}) = \lambda \mathrm{D}_\sigma \tilde{e}^\alpha_\alpha(\boldsymbol{w}) + \lambda\partial_3 \mathrm{D}_\sigma w_3 + 2\mu\partial_3\tilde{e}^3_\sigma(\boldsymbol{w}) \\ + 2\mu \mathrm{D}^{x_3}_\alpha \tilde{e}^\alpha_\sigma(\boldsymbol{w}) - 2\mu b^\delta_\alpha(\mu^{-1})^\alpha_\delta(x_3)\tilde{e}^3_\sigma(\boldsymbol{w}). \end{aligned} \tag{6.8}$$

We denote by $\mathrm{D}_\alpha(\varepsilon)$ the connexion $\mathrm{D}^{x_3}_\alpha$ after the scaling, viewed as an operator on S_0 . After the scaling, the previous equation then writes

$$\begin{aligned} \mathsf{L}_\sigma(\varepsilon)(\boldsymbol{w}) = \lambda \mathrm{D}_\sigma \tilde{e}^\alpha_\alpha(\varepsilon)\boldsymbol{w} + \lambda\varepsilon^{-1}\partial_{X_3}\mathrm{D}_\sigma w_3 + 2\mu \mathrm{D}_\alpha(\varepsilon)\tilde{e}^\alpha_\sigma(\varepsilon)\boldsymbol{w} \\ + 2\mu\varepsilon^{-1}\partial_{X_3}\tilde{e}^3_\sigma(\varepsilon)\boldsymbol{w} - 2\mu b^\delta_\alpha(\mu^{-1})^\alpha_\delta(\varepsilon X_3)\tilde{e}^3_\sigma(\varepsilon)\boldsymbol{w}. \end{aligned} \tag{6.9}$$

We compute successively the expansions of the terms in this equation. Using the equation (3.14) and the fact that $(b^k)^\beta_\alpha \Lambda^{\alpha\,\cdot}_{\cdot\,\beta} = 0$ for $k \geq 0$ we find that

$$\lambda \mathrm{D}_\sigma \tilde{e}^\alpha_\alpha(\varepsilon)\boldsymbol{w} = \lambda \mathrm{D}_\sigma \gamma^\alpha_\alpha(\boldsymbol{w}) + \sum_{n=1}^{\infty} \varepsilon^n \mathrm{D}_\sigma(b^n)^\beta_\alpha \gamma^\alpha_\beta(X_3^n \boldsymbol{w}).$$

Using (6.6), we compute that

$$2\mu \mathrm{D}_\alpha(\varepsilon)\tilde{e}^\alpha_\sigma(\varepsilon)\boldsymbol{w} = \sum_{n=0}^{\infty} \varepsilon^n 2\mu\big(\mathrm{D}_\alpha(\varepsilon)\tilde{e}^\alpha_\sigma(\varepsilon)\boldsymbol{w}\big)^n$$

with

$$\big(\mathrm{D}_\alpha(\varepsilon)\tilde{e}^\alpha_\sigma(\varepsilon)\boldsymbol{w}\big)^n = \mathrm{D}_\alpha\big(\tilde{e}^\alpha_\sigma(\varepsilon)\boldsymbol{w}\big)^n - \textstyle\sum_{k=1}^n x_3^k(b^{k-1})^\alpha_\nu\big(\tilde{e}^\delta_\sigma(\varepsilon)\boldsymbol{w}\big)^{n-k}\mathrm{D}_\alpha b^\nu_\delta \\ + \textstyle\sum_{k=1}^n x_3^k(b^{k-1})^\delta_\nu\big(\tilde{e}^\alpha_\delta(\varepsilon)\boldsymbol{w}\big)^{n-k}\mathrm{D}_\alpha b^\nu_\sigma. \tag{6.10}$$

Using the equation (3.10) and the fact that $g^{\alpha 3} = 0$, we see that we have

$$2\mu\tilde{e}^3_\sigma(\varepsilon)\boldsymbol{w} = \varepsilon^{-1}\mu\partial_{X_3}w_\sigma + \mu\theta_\sigma(\boldsymbol{w}) - \mu X_3 b^\alpha_\sigma\partial_{X_3}w_\alpha. \tag{6.11}$$

We thus have that

$$2\mu\varepsilon^{-1}\partial_{X_3}\tilde{e}^3_\sigma(\varepsilon)\boldsymbol{w} = \varepsilon^{-2}\mu\partial^2_{X_3}w_\sigma + \mu\varepsilon^{-1}\partial_{X_3}\theta_\sigma(\boldsymbol{w}) - \mu\varepsilon^{-1}\partial_{X_3}(X_3 b^\alpha_\sigma\partial_3 w_\alpha),$$

and

$$2\mu b^\delta_\alpha(\mu^{-1})^\alpha_\delta(\varepsilon X_3)\tilde{e}^3_\sigma(\varepsilon)\boldsymbol{w} = \\ \varepsilon^{-1}\mu b^\beta_\beta\partial_{X_3}w_\sigma + \mu X_3 c^\beta_\beta\partial_{X_3}w_\sigma - \mu b^\beta_\beta X_3 b^\beta_\sigma\partial_{X_3}w_\beta + \mu b^\beta_\beta\theta_\sigma(\boldsymbol{w}) \\ + \textstyle\sum_{n=1}^\infty \varepsilon^n\big\{\mu X_3^{n+1}(b^{n+2})^\alpha_\alpha\partial_{X_3}w_\sigma + \mu X_3^n(b^{n+1})^\alpha_\alpha\theta_\sigma(\boldsymbol{w}) - \mu X_3^{n+1}(b^{n+1})^\alpha_\alpha b^\beta_\sigma\partial_{X_3}w_\beta\big\}. \tag{6.12}$$

By using (6.9) and summing the previous equations, we get the expressions claimed in Theorem 3.3 after identifying the powers of ε .

(b) In normal coordinates on Ω^ε , the transverse component of L writes

$$\mathsf{L}_3(\boldsymbol{w}) = \lambda\nabla_3\big(\tilde{e}^\alpha_\alpha(\boldsymbol{w}) + \tilde{e}^3_3(\boldsymbol{w})\big) + 2\mu\big(\nabla_\alpha\tilde{e}^\alpha_3(\boldsymbol{w}) + \nabla_3\tilde{e}^3_3(\boldsymbol{w})\big) \\ = (\lambda+2\mu)\partial_{33}w_3 + \lambda\partial_3\tilde{e}^\alpha_\alpha(\boldsymbol{w}) + 2\mu\big(\mathrm{D}^{x_3}_\alpha\tilde{e}^\alpha_3(\boldsymbol{w}) + \Gamma^\alpha_{\alpha 3}(x_3)\partial_3 w_3 - \Gamma^\beta_{\alpha 3}(x_3)\tilde{e}^\alpha_\beta(\boldsymbol{w})\big).$$

Thus after the scaling we have

$$\mathsf{L}_3(\varepsilon)\boldsymbol{w} = \varepsilon^{-2}(\lambda+2\mu)\partial^2_{X_3}w_3 + \varepsilon^{-1}\lambda\partial_{X_3}\tilde{e}^\alpha_\alpha(\varepsilon)\boldsymbol{w} \\ + 2\mu\mathrm{D}_\alpha(\varepsilon)\tilde{e}^\alpha_3(\varepsilon)\boldsymbol{w} + 2\mu\varepsilon^{-1}\Gamma^\alpha_{\alpha 3}(\varepsilon X_3)\partial_{X_3}w_3 - 2\mu\Gamma^\beta_{\alpha 3}(\varepsilon X_3)\tilde{e}^\alpha_\beta(\varepsilon)\boldsymbol{w}$$

But using (3.4) we compute that

$$\Gamma^\alpha_{\alpha 3}(\varepsilon X_3)\partial_{X_3}w_3 = -\sum_{n=0}^\infty \varepsilon^n X_3^n(b^{n+1})^\alpha_\alpha\partial_{X_3}w_3 \tag{6.13}$$

and similarly by (3.14)

$$\partial_{X_3}\tilde{e}^\alpha_\alpha(\varepsilon)\boldsymbol{w} = \sum_{n=0}^\infty \varepsilon^n(b^n)^\alpha_\beta\gamma^\beta_\alpha\big(\partial_{X_3}(X_3^n\boldsymbol{w})\big). \tag{6.14}$$

Moreover, we have

$$\Gamma^{\beta}_{\alpha 3}(\varepsilon X_3)\tilde{e}^{\alpha}_{\beta}(\varepsilon)\boldsymbol{w} = -\sum_{n=0}^{\infty}\varepsilon^n\sum_{k=0}^{n}X_3^k(b^{k+1})^{\alpha}_{\beta}\big(\tilde{e}^{\beta}_{\alpha}(\varepsilon)\boldsymbol{w}\big)^{n-k}.$$

Again using the fact that $(b^k)^{\beta}_{\alpha}\Lambda^{\alpha\,\cdot}_{\cdot\,\beta} = 0$ for $k \geq 0$ we find by the equation (3.14) that

$$\Gamma^{\beta}_{\alpha 3}(\varepsilon X_3)\tilde{e}^{\alpha}_{\beta}(\varepsilon)\boldsymbol{w} = \sum_{n=0}^{\infty}(n+1)\varepsilon^n X_3^n(b^{n+1})^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}(\boldsymbol{w}). \tag{6.15}$$

To show the result, it remains to find the expansion of $\mathrm{D}_\alpha(\varepsilon)\tilde{e}^{\alpha}_{3}(\varepsilon)\boldsymbol{w}$. As $g_{\alpha\beta}(\varepsilon X_3)$ is the metric on the surface $S_{\varepsilon X_3}$, this tensor commutes with the covariant derivative $\mathrm{D}_\alpha(\varepsilon)$. Hence we have

$$\mathrm{D}_\alpha(\varepsilon)\tilde{e}^{\alpha}_{3}(\varepsilon)\boldsymbol{w} = g^{\alpha\beta}(\varepsilon X_3)\mathrm{D}_\alpha(\varepsilon)\tilde{e}_{\beta 3}(\varepsilon)\boldsymbol{w}.$$

But we have using (3.6)

$$\mathrm{D}_\alpha(\varepsilon)\tilde{e}_{\beta 3}(\varepsilon)\boldsymbol{w} = \mathrm{D}_\alpha\tilde{e}_{\beta 3}(\varepsilon)\boldsymbol{w} + \varepsilon X_3(\mu^{-1})^{\sigma}_{\delta}(\varepsilon X_3)\tilde{e}_{3\sigma}(\varepsilon)\boldsymbol{w}\,\mathrm{D}_\alpha b^{\delta}_{\beta}.$$

As the equation (6.11) can be written $2\tilde{e}_{\beta 3}(\varepsilon)\boldsymbol{w} = \varepsilon^{-1}\mu^{\alpha}_{\beta}(\varepsilon X_3)\partial_{X_3}w_\alpha + \theta_\beta(\boldsymbol{w})$, we find:

$$\begin{aligned}2\mathrm{D}_\alpha(\varepsilon)\tilde{e}_{\beta 3}(\varepsilon)\boldsymbol{w} &= \varepsilon^{-1}\mathrm{D}_\alpha\mu^{\sigma}_{\beta}(\varepsilon X_3)\partial_{X_3}w_\sigma + \mathrm{D}_\alpha\theta_\beta(\boldsymbol{w})\\ &\qquad + X_3(\partial_{X_3}w_\delta)\mathrm{D}_\alpha b^{\delta}_{\beta} + \varepsilon X_3(\mu^{-1})^{\sigma}_{\delta}(\varepsilon X_3)\theta_\sigma(\boldsymbol{w})\mathrm{D}_\alpha b^{\delta}_{\beta},\end{aligned}$$

and thus

$$\begin{aligned}2\mathrm{D}_\alpha(\varepsilon)\tilde{e}_{\beta 3}(\varepsilon)\boldsymbol{w} &= \varepsilon^{-1}\mathrm{D}_\alpha\partial_{X_3}w_\beta - X_3 b^{\sigma}_{\beta}\mathrm{D}_\alpha w_\sigma\\ &\qquad + \mathrm{D}_\alpha\theta_\beta(\boldsymbol{w}) + \sum_{n=1}^{\infty}\varepsilon^n X_3^n(b^{n-1})^{\sigma}_{\delta}\theta_\sigma(\boldsymbol{w})\mathrm{D}_\alpha b^{\delta}_{\beta}.\end{aligned}$$

Using the expansion (3.3) of the inverse of the metric tensor we obtain the expansion of the term $\mathrm{D}_\alpha(\varepsilon)\tilde{e}^{\alpha}_{3}(\varepsilon)\boldsymbol{w}$. Grouping these expansions, we get the expressions in Theorem 3.3 after tedious computations (see [14] for details).

(c) The equation (3.19) and (3.20) are consequences of the formulas for the expansions of the tensors $\tilde{e}^{j}_{i}(\boldsymbol{w})$ and the fact that

$$\mathsf{T}_\sigma(\boldsymbol{w}) = 2\mu\tilde{e}^{3}_{\sigma}(\boldsymbol{w}) \quad\text{and}\quad \mathsf{T}_3(\boldsymbol{w}) = \lambda\tilde{e}^{\alpha}_{\alpha}(\boldsymbol{w}) + (\lambda+2\mu)\tilde{e}^{3}_{3}(\boldsymbol{w}).$$

The expressions (3.10) and (3.11) yield the result after the scaling.

Appendix B: Proof of Proposition 4.5

Using the equation (4.40) and the expressions (4.31), (4.32) for the operator A^2 we compute that we have

$$
\begin{aligned}
A^2_\sigma - B_\sigma = {} & \tfrac{2}{3}\mu p b^\alpha_\sigma \mathrm{D}_\alpha \rho^\nu_\nu + \tfrac{2}{3}\mu p \mathrm{D}_\alpha b^\alpha_\sigma \rho^\nu_\nu + \tfrac{2}{3}\mu b^\alpha_\sigma \mathrm{D}_\nu \rho^\nu_\alpha + \tfrac{2}{3}\mu \mathrm{D}_\nu b^\alpha_\sigma \rho^\nu_\alpha \\
& - \tfrac{2}{3}\mu b^\nu_\nu \mathrm{D}_\alpha \rho^\alpha_\sigma - \tfrac{2}{3}\mu p b^\beta_\beta \mathrm{D}_\sigma \rho^\nu_\nu + \mu p^2 \mathrm{D}_\sigma (b^\alpha_\alpha \rho^\nu_\nu) + \tfrac{4}{3}\mu p \mathrm{D}_\sigma b^\alpha_\beta \rho^\beta_\alpha \\
& + \tfrac{1}{3}\mu p \mathrm{D}_\alpha b^\alpha_\sigma \rho^\nu_\nu + \tfrac{2}{3}\mu b^\alpha_\beta \mathrm{D}_\alpha \rho^\beta_\sigma + \tfrac{2}{3}\mu \rho^\alpha_\beta \mathrm{D}_\alpha b^\beta_\sigma - \tfrac{1}{3}\mu \mathrm{D}_\alpha b^\beta_\sigma \mathrm{D}^\alpha \theta_\beta \\
& + \tfrac{1}{3}\mu \mathrm{D}_\alpha b^\alpha_\beta \mathrm{D}^\beta \theta_\sigma - \tfrac{2}{3}\mu \mathrm{D}_\alpha b^\alpha_\nu \Lambda^{\nu\,\cdot}_{\,\cdot\,\sigma} - \tfrac{1}{3}\mu p^2 \mathrm{D}_\sigma \mathrm{D}^\alpha \mathrm{D}_\alpha \gamma^\nu_\nu - \tfrac{1}{3}\mu p \mathrm{D}_\alpha \mathrm{D}^\alpha \mathrm{D}_\sigma \gamma^\nu_\nu \\
& - 2\mu p^2 \mathrm{D}_\sigma (c^\alpha_\alpha \gamma^\nu_\nu) + \tfrac{2}{3}\mu p b^\beta_\beta \mathrm{D}_\alpha b^\alpha_\sigma \gamma^\nu_\nu - \tfrac{2}{3}\mu p b^\alpha_\beta \mathrm{D}_\alpha b^\beta_\sigma \gamma^\nu_\nu - \tfrac{2}{3}\mu p b^\alpha_\beta \gamma^\nu_\nu \mathrm{D}_\alpha b^\beta_\sigma \\
& + \tfrac{2}{3}\mu b^\nu_\nu \mathrm{D}_\alpha b^\alpha_\beta \gamma^\beta_\sigma + \tfrac{2}{3}\mu p b^\beta_\beta \mathrm{D}_\sigma b^\alpha_\nu \gamma^\nu_\alpha - \mu p^2 \mathrm{D}_\sigma (b^\alpha_\alpha b^\beta_\nu \gamma^\nu_\beta) - \tfrac{10}{3}\mu p \mathrm{D}_\sigma (c^\alpha_\beta \gamma^\beta_\alpha) \\
& - \tfrac{1}{3}\mu p \mathrm{D}_\alpha b^\alpha_\sigma b^\beta_\nu \gamma^\nu_\beta - \tfrac{2}{3}\mu b^\alpha_\beta \mathrm{D}_\alpha b^\beta_\nu \gamma^\nu_\sigma - \tfrac{2}{3}\mu b^\nu_\beta \gamma^\alpha_\nu \mathrm{D}_\alpha b^\beta_\sigma - \tfrac{2}{3}\mu b^\alpha_\beta \mathrm{D}_\alpha b^\beta_\delta \gamma^\delta_\sigma \\
& - \tfrac{4}{3}\mu b^\delta_\nu \gamma^\alpha_\delta \mathrm{D}_\alpha b^\nu_\sigma + \tfrac{2}{3}\mu b^\alpha_\nu \gamma^\delta_\sigma \mathrm{D}_\alpha b^\nu_\delta + \tfrac{2}{3}\mu b^\beta_\beta \gamma^\alpha_\nu \mathrm{D}_\alpha b^\nu_\sigma + \tfrac{2}{3}\mu b^\nu_\nu b^\alpha_\beta \mathrm{D}_\alpha \gamma^\beta_\sigma \\
& + \tfrac{1}{2}p^2 \mathrm{D}_\sigma (b^\alpha_\alpha M_3) - \tfrac{1}{3}p b^\nu_\nu \mathrm{D}_\sigma M_3 + \tfrac{1}{6}p \mathrm{D}_\alpha b^\alpha_\sigma M_3.
\end{aligned}
$$

and

$$
\begin{aligned}
A^2_3 - B_3 = {} & \tfrac{5}{3}\mu p c^\alpha_\alpha \rho^\nu_\nu + 2\mu c^\alpha_\beta \rho^\beta_\alpha + \tfrac{1}{3}\mu p(3p-2) b^\alpha_\alpha b^\beta_\beta \rho^\nu_\nu \\
& + \tfrac{2}{3}\mu(2p-1) b^\alpha_\alpha b^\nu_\beta \rho^\beta_\nu - \tfrac{2}{3}\mu p \mathrm{D}^\alpha \mathrm{D}_\alpha b^\nu_\beta \gamma^\beta_\nu - \tfrac{1}{3}\mu \mathrm{D}^\sigma \mathrm{D}_\alpha b^\nu_\sigma \gamma^\alpha_\nu - \tfrac{1}{3}\mu \mathrm{D}^\sigma \mathrm{D}_\alpha b^\alpha_\nu \gamma^\nu_\sigma \\
& - \tfrac{1}{3}\mu p^2 b^\beta_\beta \mathrm{D}^\alpha \mathrm{D}_\alpha \gamma^\nu_\nu - \tfrac{2}{3}\mu p \mathrm{D}^\sigma \mathrm{D}_\alpha b^\alpha_\sigma \gamma^\nu_\nu - \tfrac{1}{3}\mu p b^\alpha_\beta \mathrm{D}^\beta \mathrm{D}_\alpha \gamma^\nu_\nu - \tfrac{2}{3}\mu \mathrm{D}^\sigma \gamma^\alpha_\nu \mathrm{D}_\alpha b^\nu_\sigma \\
& - \tfrac{2}{3}\mu \mathrm{D}^\sigma b^\alpha_\nu \mathrm{D}_\alpha \gamma^\nu_\sigma - 2\mu p d^\alpha_\alpha \gamma^\nu_\nu - \tfrac{10}{3}\mu d^\alpha_\beta \gamma^\beta_\alpha - \tfrac{1}{3}\mu p(3p-2) b^\alpha_\alpha b^\beta_\beta b^\sigma_\nu \gamma^\nu_\sigma \\
& - \mu p c^\alpha_\alpha b^\beta_\nu \gamma^\nu_\beta - \tfrac{2}{3}\mu p(3p-2) b^\nu_\nu c^\alpha_\alpha \gamma^\beta_\beta - \tfrac{2}{3}\mu(2p-1) b^\alpha_\alpha c^\nu_\beta \gamma^\beta_\nu - \tfrac{2}{3}\mu(3p-2) b^\alpha_\alpha c^\nu_\beta \gamma^\beta_\nu \\
& + \tfrac{1}{3}p \mathrm{D}^\alpha \mathrm{D}_\alpha M_3 + \tfrac{1}{6}p(3p-2) b^\nu_\nu b^\alpha_\alpha M_3 + \tfrac{1}{2}p c^\alpha_\alpha M_3.
\end{aligned}
$$

Now let $\boldsymbol{\eta}$ satisfying $\boldsymbol{\eta}\big|_{\partial S_0} = 0$. Then when evaluating the scalar product

$$
\big\langle (\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{\eta} \big\rangle_{\mathbf{L}^2(S_0)}
$$

it is possible to integrate by parts one time. We have

$$
\big\langle (\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{\eta} \big\rangle_{\mathbf{L}^2(S_0)} = \int_{S_0} \eta^i (A^2_i - B_i)(\boldsymbol{z})\, \mathrm{d}S_0,
$$

and after integration by part, we see that :

$$
\big| \big\langle (\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{\eta} \big\rangle_{\mathbf{L}^2(S_0)} \big| \leq \int_{S_0} |\mathcal{J}(\boldsymbol{z}, \boldsymbol{\eta})|\, \mathrm{d}S_0 + C \|\boldsymbol{\gamma}(\boldsymbol{z})\|_{\mathbf{H}^1(S_0)} \|\boldsymbol{\eta}\|_{\mathbf{H}^1 \times \mathrm{H}^1(S_0)}. \tag{6.16}
$$

where

$$\begin{aligned}\mathcal{J}(\boldsymbol{z},\boldsymbol{\eta}) = & -\tfrac{2}{3}\mu p(\mathrm{D}_\alpha\eta^\sigma b_\sigma^\alpha)\rho_\nu^\nu(\boldsymbol{z}) - \tfrac{2}{3}\mu p(\mathrm{D}_\alpha\eta^\sigma)b_\sigma^\alpha\rho_\nu^\nu(\boldsymbol{z}) - \tfrac{2}{3}\mu(\mathrm{D}_\nu\eta^\sigma b_\sigma^\alpha)\rho_\alpha^\nu(\boldsymbol{z})\\ & - \tfrac{2}{3}\mu(\mathrm{D}_\nu\eta^\sigma)b_\sigma^\alpha\rho_\alpha^\nu(\boldsymbol{z}) + \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma b_\nu^\nu)\rho_\sigma^\alpha(\boldsymbol{z}) + \tfrac{2}{3}\mu p(\mathrm{D}_\sigma\eta^\sigma b_\beta^\beta)\rho_\nu^\nu(\boldsymbol{z})\\ & - \mu p^2(\mathrm{D}_\sigma\eta^\sigma)b_\alpha^\alpha\rho_\nu^\nu(\boldsymbol{z}) - \tfrac{4}{3}\mu p(\mathrm{D}_\sigma\eta^\sigma)b_\beta^\alpha\rho_\alpha^\beta(\boldsymbol{z}) - \tfrac{1}{3}\mu p(\mathrm{D}_\alpha\eta^\sigma)b_\sigma^\alpha\rho_\nu^\nu(\boldsymbol{z})\\ & - \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma b_\beta^\alpha)\rho_\sigma^\beta(\boldsymbol{z}) + \tfrac{2}{3}\mu\eta^\sigma\rho_\beta^\alpha(\boldsymbol{z})\mathrm{D}_\alpha b_\sigma^\beta + \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b_\sigma^\beta\mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\\ & - \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b_\beta^\alpha\mathrm{D}^\beta\theta_\sigma(\boldsymbol{z}) + \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b_\nu^\alpha\Lambda^\nu{}_{\cdot\sigma}(\boldsymbol{z}) + \tfrac{1}{3}\mu p^2(\mathrm{D}_\sigma\eta^\sigma)\mathrm{D}^\alpha\mathrm{D}_\alpha\gamma_\nu^\nu(\boldsymbol{z})\\ & + \tfrac{1}{3}\mu p(\mathrm{D}_\alpha\eta^\sigma)\mathrm{D}^\alpha\mathrm{D}_\sigma\gamma_\nu^\nu(\boldsymbol{z}) + \tfrac{5}{3}\mu p\eta^3c_\alpha^\alpha\rho_\nu^\nu(\boldsymbol{z}) + 2\mu\eta^3c_\beta^\alpha\rho_\alpha^\beta(\boldsymbol{z})\\ & + \tfrac{1}{3}\mu p(3p-2)b_\alpha^\alpha\eta^3b_\beta^\beta\rho_\nu^\nu(\boldsymbol{z}) + \tfrac{2}{3}\mu(2p-1)b_\alpha^\alpha b_\beta^\nu\eta^3\rho_\nu^\beta(\boldsymbol{z}) - \tfrac{1}{3}\mu p^2b_\beta^\beta\eta^3\mathrm{D}^\alpha\mathrm{D}_\alpha\gamma_\nu^\nu(\boldsymbol{z})\\ & - \tfrac{1}{3}\mu pb_\beta^\alpha\eta^3\mathrm{D}^\beta\mathrm{D}_\alpha\gamma_\nu^\nu(\boldsymbol{z}).\end{aligned}$$

Hence we have :

$$\begin{aligned}\mathcal{J}(\boldsymbol{z},\boldsymbol{\eta}) = & \,\rho_\nu^\nu(\boldsymbol{z})\big[-\tfrac{2}{3}\mu p\mathrm{D}_\alpha\eta^\sigma b_\sigma^\alpha - \tfrac{2}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{2}{3}\mu p\mathrm{D}_\sigma\eta^\sigma b_\beta^\beta\\ & - \mu p^2b_\alpha^\alpha\mathrm{D}_\sigma\eta^\sigma - \tfrac{1}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{5}{3}\mu p\eta^3c_\alpha^\alpha + \tfrac{1}{3}\mu p(3p-2)b_\alpha^\alpha\eta^3b_\beta^\beta\big]\\ & + \rho_\alpha^\nu(\boldsymbol{z})\big[-\tfrac{2}{3}\mu\mathrm{D}_\nu\eta^\sigma b_\sigma^\alpha - \tfrac{2}{3}\mu b_\sigma^\alpha\mathrm{D}_\nu\eta^\sigma + \tfrac{2}{3}\mu\mathrm{D}_\nu\eta^\alpha b_\beta^\beta\\ & - \tfrac{4}{3}\mu pb_\nu^\alpha\mathrm{D}_\sigma\eta^\sigma - \tfrac{2}{3}\mu\mathrm{D}_\beta\eta^\alpha b_\nu^\beta + \tfrac{2}{3}\mu\eta^\sigma\mathrm{D}_\nu b_\sigma^\alpha + 2\mu\eta^3c_\nu^\alpha + \tfrac{2}{3}\mu(2p-1)b_\delta^\delta b_\nu^\alpha\eta^3\big]\\ & + \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b_\sigma^\beta\mathrm{D}^\alpha\theta_\beta(\boldsymbol{z}) - \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b_\beta^\alpha\mathrm{D}^\beta\theta_\sigma(\boldsymbol{z}) + \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b_\nu^\alpha\Lambda^\nu{}_{\cdot\sigma}(\boldsymbol{z})\\ & + \tfrac{1}{3}\mu p^2\gamma_\beta^\beta(\boldsymbol{\eta})\mathrm{D}^\alpha\mathrm{D}_\alpha\gamma_\nu^\nu(\boldsymbol{z}) + \tfrac{1}{3}\mu p\gamma_\beta^\alpha(\boldsymbol{\eta})\mathrm{D}^\beta\mathrm{D}_\alpha\gamma_\nu^\nu(\boldsymbol{z}).\end{aligned} \tag{6.17}$$

The term multiplied by $\rho_\nu^\nu(\boldsymbol{z})$ writes

$$\begin{aligned} & -\tfrac{2}{3}\mu p\mathrm{D}_\alpha\eta^\sigma b_\sigma^\alpha - \tfrac{2}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{2}{3}\mu p\mathrm{D}_\sigma\eta^\sigma b_\beta^\beta - \mu p^2b_\alpha^\alpha\mathrm{D}_\sigma\eta^\sigma\\ & \quad - \tfrac{1}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{5}{3}\mu p\eta^3c_\alpha^\alpha + \tfrac{1}{3}\mu p(3p-2)b_\alpha^\alpha\eta^3b_\beta^\beta\\ & = -\tfrac{2}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma - \tfrac{2}{3}\mu p\eta^\sigma\mathrm{D}_\alpha b_\sigma^\alpha - \tfrac{2}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{2}{3}\mu p\eta^\sigma\mathrm{D}_\sigma b_\beta^\beta\\ & \quad + \tfrac{2}{3}\mu pb_\beta^\beta\mathrm{D}_\sigma\eta^\sigma - \mu p^2b_\alpha^\alpha\mathrm{D}_\sigma\eta^\sigma - \tfrac{1}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{5}{3}\mu p\eta^3c_\alpha^\alpha + \tfrac{1}{3}\mu p(3p-2)b_\alpha^\alpha\eta^3b_\beta^\beta.\end{aligned}$$

This terms writes also

$$-\tfrac{4}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{2}{3}\mu pb_\beta^\beta\gamma_\sigma^\sigma(\boldsymbol{\eta}) - \tfrac{1}{3}\mu pb_\sigma^\alpha\mathrm{D}_\alpha\eta^\sigma + \tfrac{5}{3}\mu pc_\alpha^\alpha\eta^3 - \mu p^2b_\beta^\beta\gamma_\alpha^\alpha(\boldsymbol{\eta}),$$

or

$$-\tfrac{1}{3}\mu p(3p-2)b_\beta^\beta\gamma_\alpha^\alpha(\boldsymbol{\eta}) - \tfrac{5}{3}\mu pb_\sigma^\alpha\gamma_\alpha^\sigma(\boldsymbol{\eta}).$$

In the same way, the term multiplied by $\rho^\nu_\alpha(z)$ in the equation (6.17) writes

$$\begin{aligned}
&-\tfrac{2}{3}\mu \mathrm{D}_\nu\eta^\sigma b^\alpha_\sigma - \tfrac{2}{3}\mu b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma + \tfrac{2}{3}\mu \mathrm{D}_\nu\eta^\alpha b^\beta_\beta - \tfrac{4}{3}\mu p b^\alpha_\nu \mathrm{D}_\sigma\eta^\sigma \\
&\qquad - \tfrac{2}{3}\mu \mathrm{D}_\beta\eta^\alpha b^\beta_\nu + \tfrac{2}{3}\mu\eta^\sigma \mathrm{D}_\nu b^\alpha_\sigma + 2\mu\eta^3 c^\alpha_\nu + \tfrac{2}{3}\mu(2p-1)b^\delta_\delta b^\alpha_\nu\eta^3 \\
&= -\tfrac{2}{3}\mu b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu\eta^\sigma \mathrm{D}_\nu b^\alpha_\sigma - \tfrac{2}{3}\mu b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma + \tfrac{2}{3}\mu b^\beta_\beta \mathrm{D}_\nu\eta^\alpha + \tfrac{2}{3}\mu\eta^\alpha \mathrm{D}_\nu b^\beta_\beta \\
&\qquad - \tfrac{4}{3}\mu p b^\alpha_\nu \mathrm{D}_\sigma\eta^\sigma - \tfrac{2}{3}\mu b^\beta_\nu \mathrm{D}_\beta\eta^\alpha - \tfrac{2}{3}\mu\eta^\alpha \mathrm{D}_\beta b^\beta_\nu + \tfrac{2}{3}\mu\eta^\sigma \mathrm{D}_\nu b^\alpha_\sigma + 2\mu c^\alpha_\nu\eta^3 \\
&\qquad + \tfrac{4}{3}\mu p b^\delta_\delta b^\alpha_\nu\eta^3 - \tfrac{2}{3}\mu b^\delta_\delta b^\alpha_\nu\eta^3.
\end{aligned}$$

This last term also writes

$$\tfrac{2}{3}\mu b^\beta_\beta\gamma^\alpha_\nu(\boldsymbol{\eta}) - \tfrac{4}{3}\mu p b^\alpha_\nu\gamma^\sigma_\sigma(\boldsymbol{\eta}) - \tfrac{2}{3}\mu b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu b^\beta_\nu \mathrm{D}_\beta\eta^\alpha + 2\mu c^\alpha_\nu\eta^3.$$

Thus we have

$$\begin{aligned}
\mathcal{J}(\boldsymbol{z},\boldsymbol{\eta}) &= -\tfrac{1}{3}\mu p(3p-2)\rho^\nu_\nu(\boldsymbol{z})b^\beta_\beta\gamma^\alpha_\alpha(\boldsymbol{\eta}) - \tfrac{5}{3}\mu p\rho^\nu_\nu(\boldsymbol{z})b^\alpha_\sigma\gamma^\sigma_\alpha(\boldsymbol{\eta}). + \tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\beta_\beta\gamma^\alpha_\nu(\boldsymbol{\eta}) \\
&\quad - \tfrac{4}{3}\mu p\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\nu\gamma^\sigma_\sigma(\boldsymbol{\eta}) - \tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\beta_\nu \mathrm{D}_\beta\eta^\alpha \\
&\quad + 2\mu\rho^\nu_\alpha(\boldsymbol{z})\eta^3 c^\alpha_\nu. + \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z}) - \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\alpha_\beta \mathrm{D}^\beta\theta_\sigma(\boldsymbol{z}) \\
&\quad + \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\alpha_\nu\Lambda^\nu_{\cdot\sigma}(\boldsymbol{z}) + \tfrac{1}{3}\mu p^2\gamma^\beta_\beta(\boldsymbol{\eta})\mathrm{D}^\alpha \mathrm{D}_\alpha\gamma^\nu_\nu(\boldsymbol{z}) + \tfrac{1}{3}\mu p\gamma^\alpha_\beta(\boldsymbol{\eta})\mathrm{D}^\beta \mathrm{D}_\alpha\gamma^\nu_\nu(\boldsymbol{z}).
\end{aligned} \tag{6.18}$$

Now we focus our attention to the terms that are not multiplied by the tensor in $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ in the equation (6.18). These terms write :

$$\begin{aligned}
\Delta(\boldsymbol{z},\boldsymbol{\eta}) := &-\tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\beta_\nu \mathrm{D}_\beta\eta^\alpha + 2\mu\rho^\nu_\alpha(\boldsymbol{z})c^\alpha_\nu\eta^3. \\
&+ \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z}) - \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\alpha_\beta \mathrm{D}^\beta\theta_\sigma(\boldsymbol{z}) + \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\alpha_\nu\Lambda^\nu_{\cdot\sigma}(\boldsymbol{z}).
\end{aligned}$$

Using the fact that :

$$\Lambda_{\alpha\beta} = \tfrac{1}{2}(\mathrm{D}_\alpha\theta_\beta + \mathrm{D}_\beta\theta_\alpha) - \rho_{\alpha\beta} + b^\sigma_\alpha\gamma_{\beta\sigma}$$

we have

$$\begin{aligned}
\Delta(\boldsymbol{z},\boldsymbol{\eta}) &= -\tfrac{4}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\beta_\nu \mathrm{D}_\beta\eta^\alpha + 2\mu\rho^\nu_\alpha(\boldsymbol{z})c^\alpha_\nu\eta^3 \\
&\quad + \tfrac{1}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})(\mathrm{D}_\alpha\eta^\sigma) - \tfrac{1}{3}\mu b^\alpha_\beta \mathrm{D}^\beta\theta_\sigma(\boldsymbol{z})(\mathrm{D}_\alpha\eta^\sigma) + \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\alpha_\nu \mathrm{D}_\sigma\theta^\nu(\boldsymbol{z}) \\
&\quad + \tfrac{1}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\alpha_\nu \mathrm{D}^\nu\theta_\sigma(\boldsymbol{z}) - \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)b^\alpha_\nu\rho^\nu_\sigma(\boldsymbol{z}) + \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)c^\alpha_\nu\gamma^\nu_\sigma(\boldsymbol{z}).
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta(\boldsymbol{z},\boldsymbol{\eta}) &= -\tfrac{4}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma - \tfrac{4}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\beta_\nu \mathrm{D}_\beta\eta^\alpha + 2\mu\rho^\nu_\alpha(\boldsymbol{z})c^\alpha_\nu\eta^3 \\
&\quad + \tfrac{2}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\tfrac{1}{2}(\mathrm{D}_\alpha\eta^\sigma + \mathrm{D}^\sigma\eta_\alpha) + \tfrac{2}{3}\mu(\mathrm{D}_\alpha\eta^\sigma)c^\alpha_\nu\gamma^\nu_\sigma(\boldsymbol{z}).
\end{aligned}$$

But we have

$$\begin{aligned}\tfrac{2}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\tfrac{1}{2}(\mathrm{D}_\alpha\eta^\sigma+\mathrm{D}^\sigma\eta_\alpha) &= \tfrac{2}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\gamma^\sigma_\alpha(\boldsymbol{\eta})+\tfrac{2}{3}\mu c^\beta_\alpha \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\eta^3\\ &= \tfrac{2}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\gamma^\sigma_\alpha(\boldsymbol{\eta})+\tfrac{2}{3}\mu c^\beta_\alpha\rho^\alpha_\beta(\boldsymbol{z})\eta^3-\tfrac{2}{3}\mu d^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{z})\eta^3.\end{aligned}$$

Hence we have

$$\begin{aligned}\Delta(\boldsymbol{z},\boldsymbol{\eta}) &= -\tfrac{4}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma \mathrm{D}_\nu\eta^\sigma-\tfrac{4}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\beta_\nu \mathrm{D}_\beta\eta^\alpha+\tfrac{8}{3}\mu c^\beta_\alpha\rho^\alpha_\beta(\boldsymbol{z})\eta^3\\ &\quad+\tfrac{2}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\gamma^\sigma_\alpha(\boldsymbol{\eta})-\tfrac{2}{3}\mu d^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{z})\eta^3.+\tfrac{2}{3}\mu c^\alpha_\nu\gamma^\nu_\sigma(\boldsymbol{z})(\mathrm{D}_\alpha\eta^\sigma).\end{aligned}$$

or

$$\Delta(\boldsymbol{z},\boldsymbol{\eta}) = -\tfrac{8}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma\gamma^\sigma_\nu(\boldsymbol{\eta})+\tfrac{2}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\gamma^\sigma_\alpha(\boldsymbol{\eta})-\tfrac{2}{3}\mu d^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{z})\eta^3+\tfrac{2}{3}\mu c^\alpha_\nu\gamma^\nu_\sigma(\boldsymbol{z})(\mathrm{D}_\alpha\eta^\sigma).$$

The equation (6.18) then writes :

$$\begin{aligned}\mathcal{I}(\boldsymbol{z},\boldsymbol{\eta}) &= -\tfrac{1}{3}\mu p(3p-2)\rho^\nu_\nu(\boldsymbol{z})b^\beta_\beta\gamma^\alpha_\alpha(\boldsymbol{\eta})-\tfrac{5}{3}\mu p\rho^\nu_\nu(\boldsymbol{z})b^\alpha_\sigma\gamma^\sigma_\alpha(\boldsymbol{\eta}).+\tfrac{2}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\beta_\beta\gamma^\alpha_\nu(\boldsymbol{\eta})\\ &\quad-\tfrac{4}{3}\mu p\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\nu\gamma^\sigma_\sigma(\boldsymbol{\eta})-\tfrac{8}{3}\mu\rho^\nu_\alpha(\boldsymbol{z})b^\alpha_\sigma\gamma^\sigma_\nu(\boldsymbol{\eta})+\tfrac{2}{3}\mu b^\beta_\sigma \mathrm{D}^\alpha\theta_\beta(\boldsymbol{z})\gamma^\sigma_\alpha(\boldsymbol{\eta})\\ &\quad-\tfrac{2}{3}\mu d^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{z})\eta^3+\tfrac{2}{3}\mu c^\alpha_\nu\gamma^\nu_\sigma(\boldsymbol{z})(\mathrm{D}_\alpha\eta^\sigma)+\tfrac{1}{3}\mu p^2\big(\mathrm{D}^\alpha \mathrm{D}_\alpha\gamma^\nu_\nu(\boldsymbol{z})\big)\gamma^\beta_\beta(\boldsymbol{\eta})\\ &\quad+\tfrac{1}{3}\mu p\big(\mathrm{D}^\beta \mathrm{D}_\alpha\gamma^\nu_\nu(\boldsymbol{z})\big)\gamma^\alpha_\beta(\boldsymbol{\eta}).\end{aligned}$$

This result and the equation (6.16) then yield the result.

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