Perfect Simulation of a Stochastic Model for CDMA Coverage

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Abstract: The aim of this article is to present ways of getting realizations of the stochastic model for CDMA coverage introduced in [1]. This is a basic stochastic geometry model that represents the location of antennas as realizations of Poisson point processes in the plane and allows one to capture regions (CDMA cells) of the plane with *signal-to-noise ratio* SNR to/from a given antenna at a sufficient level. We describe an algorithm of perfect simulation of arbitrary close approximation of the the CDMA cell mosaic given some coverage constraints. This is an adaptation of the so called backward simulation of conditional coverage process developed in [4]. The approximation comes from the estimation of the SNR, that is modeled by Poisson shot-noise process, based on an influence window whose size is theoretically adjusted to meet the true value of SNR with an arbitrary precision. As a motivating example we estimate the distribution functions of the distance from a fixed location to regions under various given handoff conditions (so called contact distribution functions).

Key-words: Perfect Simulation, CDMA coverage, Conditionnal Simulation, Stochastic Model.

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Simulation exacte d'un modèle stochastique de couverture CDMA

Résumé : Cet article présente un moyen d'obtenir des réalisations du modèle stochastique de couverture CDMA introduit dans l'article [1]. Il s'agit d'un modèle assez élémentaire qui place les antennes selon la réalisation d'un processus ponctuel de Poisson sur le plan et nous permet de calculer les régions (cellules CDMA) du plan qui reçoivent un rapport signal sur bruit suffisant. Nous décrivons un algorithme de simulation exacte de l'ensemble des cellules CDMA avec des contraintes de couverture. Cet algorithme est une adaptation de l'algorithme de simulation exacte de schéma Booléen développé dans l'article [4]. Avec un tel outil de simulation, nous estimons la fonction de distribution de la distance à parcourir avant de changer de zone de réception.

Mot-clés : Simulation exacte, Simulation parfaite, Simulation conditionnelle, modèle stochastique, couverture CDMA.

1 Introduction

Most of current CDMA analysis is based on a simplified representation of the underlying geometry of network antennas by a finite regular pattern, which leads to the classical honeycomb model. This model is nevertheless acknowledged to be inadequate, as real patterns in fact contain a very large number of points with no planar regularity at all. Recently, in [1] a basic stochastic geometry model was introduced, that represents the locations of antennas as realizations of a stochastic (e.g. Poisson) point processes in the plane, and in this way takes into account the irregularities of antenna patterns in a statistical way. Together with classical models of propagation and fading, and with simple power control algorithm, this model allows one to capture regions of the plane with *signal-to-noise ratio* SNR to/from a given antenna at a sufficient level (called CDMA cells).

The aim of this article is to present ways of getting various conditional realizations of this stochastic model of CDMA coverage, assuming — possibly inhomogeneous — Poisson process of antennas. More precisely, we describe an algorithm of *perfect simulation of arbitrary close approximation* of the CDMA cells *conditioned given some coverage constraints.*

Unlike in standard "arbitrary long run and then stop" simulators, the *perfect simulation* achieves in a finite time an unbiased realization of the model under its stationary regime. Our perfect simulation, is an adaptation of the so called *backward simulation* (or *coupling from the past*) of conditional coverage process developed in [4].

The anavoidable *approximation* comes form the fact, that the Poisson model, if considered on the whole plane, assumes infinite number of antennas, which all add to the total noise modeled by the standard shot-noise construction. And since simulation of the Poisson process on the whole plane is unfeasible, we estimate the total noise (and thus the SNR), using an *influence window*, whose size is theoretically adjusted to meet the true value of the SNR in the observation window with arbitrary precision.

The *conditional* simulator is meant for getting stationary patterns of cells given each of a selected finite number of locations is covered by some prescribed number of cells. As a motivating example, we estimate the distribution functions of the distance from a fixed location to regions under various given handoff conditions (so called *contact distribution functions* in stochastic geometry).

The paper is organized as follows. In Section 2 we describe the simulated model. Section 3 deals with the Poisson shot-noise process. In particular, we show how to chose a simulation influence window, large enough to get en estimator of the total noise of the model in the observation window with arbitrary precision. In Section 4 we describe the unconditional evolution of the model, which paves the way for the main vector of the paper, Section 5 where we describe, step by step, the ideas of the perfect simulation of the model. Finally, in Section 6 we show some examples of characteristics of the model approximated by means of the described simulation. Section A serves as an appendix, to which we differed more tedious proofs of Section 3.

2 Description of the model

In this section we recall a basic stochastic geometry model, introduced in [1], that represents the locations of antennas as realizations of a stochastic (e.g. Poisson) point processes in the plane. Together with classical models of propagation and fading, and with simple power control algorithm, this model allows one to capture regions of the plain with *signal-to-noise ratio* SNR to/from a given antenna at a sufficient level (called CDMA cells). This is a simple model that lets us explain the main ideas of conditional simulation of the network. The simulation method itself, however, is fairly general and may by applied to more complex models.

We take into account the following elementary characteristics of each antenna: its location in the space $X \in \mathbb{R}^2$, emission power $S \in \mathbb{R}^+$ and the capacity of the link (signal to interference ratio threshold at a potential receptor) $T \in]0,1]$. The (random) network of antennas $\Phi = \{(X_i,(S_i,T_i))\}$ is assumed to form an independently marked Poisson point process on $\mathbb{R}^2$ with intensity (mean number of antennas per unit area) $\lambda \in \mathbb{R}^+$. We denote by $l : \mathbb{R} \to \mathbb{R}$ the power path loss function. Moreover, for each location $y \in \mathbb{R}^2$, we denote by $I_\Phi(y) = \sum_{i=0}^{\infty} S_i l(|y - X_i|)$ the total power received at y from all antennas of Φ. Thus, $I_\Phi(y)$ represents the noise created at y by the network. We allow also for an external noise $W \in \mathbb{R}^+$ that is added to $I_\Phi(y)$. The reception of antenna i is feasibly at $y \in \mathbb{R}^2$ if

$$\frac{S_i l(|y - X_i|)}{I_\Phi(y) + W} > T_i \,.$$

Thus the reception region of this antenna is

$$Cell_i(\Phi) = \left\{ y \in \mathbb{R}^2 \;\middle|\; \frac{S_i l(|y - X_i|)}{I_\Phi(y) + W} > T_i \right\} .$$

3 Simulation of the shot-noise process

The main difficulty when treating a particular realization of the geometrical model is the limitation of the observation window. In contrast to a Boolean Model, for which cells observed in a bounded set are not affected by points outside it, we cannot restrict ourself to an observation window because of the external noise term

$$I_\Phi(y) \;=\; \sum_{i=0}^{\infty} S_i l(|y - X_i|). \tag{1}$$

Looking at Figure 1, if we want to get a realization of the model within a grey window, we have to take into account also antennas that are possibly very far from it. And since simulation of the process on the whole plain is unfeasible, we consider another, larger *influence window.*

How large this window should be is a question that we address in this section. The proofs of the results of this section are deferred to the end of the article (see Section A).

3.1 Theoretical bounds on the influence window

The following proposition allows us to solve the problem of the size of the influence window for S of a general distribution.

Proposition 1 *Let $\Phi = \{X_i\}$ be the Poisson point process on $\mathbb{R}^2$ with intensity $\lambda \in \mathbb{R}$, $\{S_i\}$ a series of independent random variables, all distributed as S, and let l be a positive,*

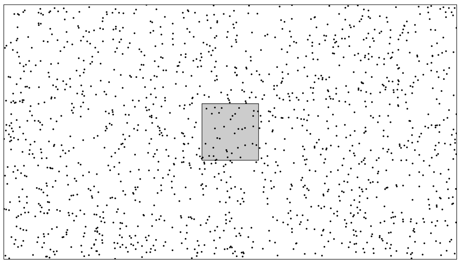

Figure 1: If we want to get a realization of the model within the gray observation window, we have to take into account also antennas outside it, in an *influence window*, in order to get close estimate of the noise term I_Φ inside observation window.

decreasing function, such that $l(x) < \frac{C}{x^\beta}$ *for* $x > 0$ *with* $\beta > 2$*. If there exists* $\delta \in]1, \beta/2[$ *such that* $\mathbb{E}\left(S^{\frac{1}{\beta/2-\delta}}\right) < \infty$ *then for any* $\varepsilon > 0$*,* $\alpha > 0$ *and* $R > 0$*, there exists* $R' > 0$ *such that:*

$$P\left(\sup_{|y|<R} \sum_{|X_i|>R'}^{\infty} S_i\, l(|y - X_i|) < \varepsilon\right) > 1 - \alpha\,. \qquad (2)$$

This proposition shows us how to calculate the noise term in an observation window $B(O, R)$, with a given precision, considering an influence window $B(O, R')$. In practice we often assume a log-normal distribution of S, as suggested e.g. in [6].

3.2 Practical limitations on the influence window

We will show now, for various distributions of S, how to choose in practice an influence window, assuming some ε, α and an observation window.

3.2.1 Exponential S

For given $\varepsilon, R, a > 0$, $\beta > 2$ and $\delta \in]1, \beta/2[$ we define auxiliary sequences of $n > 0$:

$$b_n = \frac{\varepsilon}{C(\lambda\pi)^{\beta/2}} \left(1 - R\sqrt{\frac{\lambda\pi}{n}}\right)^{\beta}, \qquad (3)$$

$$c_n = n^{\beta/2-\delta} b_n\,, \qquad (4)$$

$$\zeta(n, \delta) = 2\left(\frac{2}{ab_n}\right)^{\frac{1}{\beta/2-\delta}} \int_{(n-1)\left(\frac{ab_n}{2}\right)^{\frac{1}{\beta/2-\delta}}}^{\infty} \exp\left(-y^{\beta/2-\delta}\right) dy\,. \qquad (5)$$

Proposition 2 *Let* Φ*,* $\{S_i\}$*, and* $l(\cdot)$ *be as in Proposition* 1 *and assume that* S *is exponential with parameter* a*. If for given* $\varepsilon, R, a > 0$*,* $\beta > 2$ *and* $\delta \in]1, \beta/2[$*, the following three conditions are satisfied*

1. $1 < (\delta - 1)(n - 1)^{\delta-1}$
2. $\exp(-ac_n/2) < 0.75$

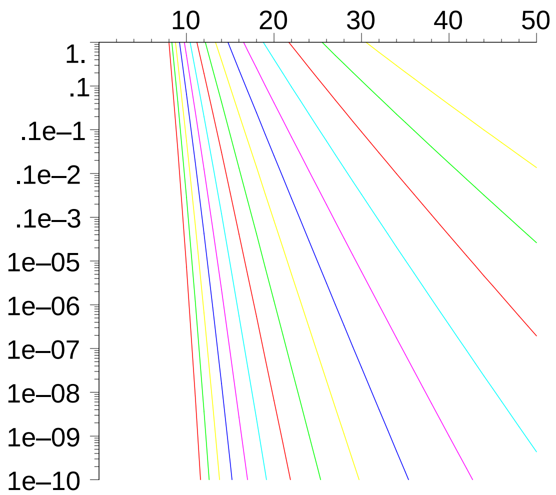

Figure 2: Value of α as the function of R' for $\varepsilon = \mathbb{E}(I_\Phi) * 10^{-n/10}$ with $n \in \{10, \dots, 25\}$ satisfying assumptions of Proposition 2 (and thus inequality (2)) for $C = 10^{-9}$, $\beta = 3$, $R = 5\text{km}^2$, $\lambda = 0.5\text{BS}/\text{km}^2$ et $a = 10^{-1}\text{dB}$.

3. $n > \lceil \lambda \pi R^2 \rceil$

then (2) *holds for* $R' = \sqrt{\frac{n}{\lambda\pi}}$ *and* $\alpha = \zeta(n, \delta)$.

In order to make this proposition more useful we present some possible values of α, ε, R and R'. We chose reasonable values of ε considering mean value of I_ϕ with the attenuation function $l(x) = \frac{C}{(C^{\frac{1}{\beta}}+|x|)^\beta}$ that is

$$\mathbb{E}(I_\Phi) = \mathbb{E}(S) \int_{\mathbb{R}^2} l(r) \lambda r dr d\theta = \frac{2C^{2/\beta}\lambda\pi}{a(\beta-2)(\beta-1)} .$$

We fix $C = 10^{-9}$, $\beta = 3$, $R = 5\text{km}^2$, $\lambda = 0.5\text{BS}/\text{km}^2$ and $a = 10^{-1}$. This corresponds to the attenuation of 90dB at 1km, the density of antennas typical for urban areas and the emission power of order $10W$ making $\mathbb{E}(I_\Phi)$ of order $10^{-5}W$. Figure 2 page 6 presents various possible values of ε, R' and α.

3.2.2 S with density satisfying $f_S(s) < \frac{D}{(1+s)^\gamma}$

In this section we assume that the density of S satisfies

$$f_S(s) < \frac{D}{(1+s)^\gamma} \quad (s > 0) \tag{6}$$

for some $\gamma > 1$, $D > 0$. For a given $\eta > 0$ we define another auxiliary constant

$$E = \frac{D\eta}{(\gamma-1)(D-\eta)} \left(1 + e^{\frac{D}{\eta}-1}\right) . \tag{7}$$

Proposition 3 *Let* Φ, $\{S_i\}$, *and* $l(\cdot)$ *be as in Proposition* 1 *and assume that* S *has density satisfying* (6) *for some* $\gamma > 1$ *and* $D > 0$. *If for given* ε, R, $\beta > 2$, $\delta \in]1, \beta/2[$ *and* $\eta \in]0, \gamma - 1[$ *the following three conditions are satisfied*

1. $1 < (\delta - 1)(n-1)^{\delta - 1}$,
2. $2E < 0.7(1+c_n)^{\gamma-1-\eta}$, *where* c_n *is as in* (4),
3. $n > \lceil \lambda \pi R^2 \rceil$,

then (2) *holds for* $R' = \sqrt{\frac{n}{\lambda\pi}}$ *and*

$$\alpha = \frac{2E}{b_n^{\gamma-1-\eta}((\beta/2-\delta)(\gamma-1-\eta)-1)(n-1)^{(\beta/2-\delta)(\gamma-1-\eta)-1}}.$$

3.2.3 Log-normal S

Proposition 3 applies for S with a log-normal distribution too, but it gives R' value much larger than the minimal one satisfying (2). In the following we show a more efficient way of estimation of R' in this case. For this, lets assume that

$$S = \exp(m + \sigma Z), \tag{8}$$

where Z is the standard normal random variable.

Proposition 4 *Let* Φ, $\{S_i\}$, *and* $l(\cdot)$ *be as in Proposition* 1 *and assume that* S *is given by* (8). *If for given* $\varepsilon, R,\ \beta > 2\ \delta \in]1, \beta/2[$ *the following three conditions are satisfied*

1. $1 < (\delta - 1)(n-1)^{\delta - 1}$,
2. $\frac{2e^{\sqrt{2}-1}}{\sqrt{2}-1} < 0.7e^{(\ln c_n - m)^2/(4\sigma^2)}$, *where* c_n *is as in* (4),
3. $n > \lceil \lambda \pi R^2 \rceil$,

then (2) *holds for* $R' = \sqrt{\frac{n}{\lambda\pi}}$ *and*

$$\alpha = \frac{2e^{\sqrt{2}-1}}{\sqrt{2}-1} \int_{k-1}^{\infty} \exp\left(-\frac{(\ln b_{k-1} - m + (\beta/2-\delta)\ln x)^2}{4\sigma^2}\right) dx.$$

We give now some examples of (ϵ, R', α) satisfying (2) and obtained via Proposition 4. For S as in (8) and $l(x) = \frac{C}{(C^{\frac{1}{\beta}} + x)^\beta}$ we have

$$\mathbb{E}(I_\Phi) = \mathbb{E}(S) \int_{\mathbb{R}^2} l(r)\lambda r dr d\theta = \frac{2C^{2/\beta}\lambda\pi}{(\beta-2)(\beta-1)} e^{m+\frac{\sigma^2}{2}}.$$

As in Section 3.2.1 we take $C = 10^{-9}$, $\beta = 3$, $R = 5\text{km}^2$, $\lambda = 0.5\text{BS/km}^2$. For the log-normal S we assume $m = 7.0\ln(10)/10\text{dB}$ and $\sigma = 2.4\ln(10)/10\text{dB}$ which makes mean antenna power of order 5.8W and $\mathbb{E}(I_\Phi)$ of order 10^{-5}W. Figure 3 page 8 presents various possible values of ε, R' and α.

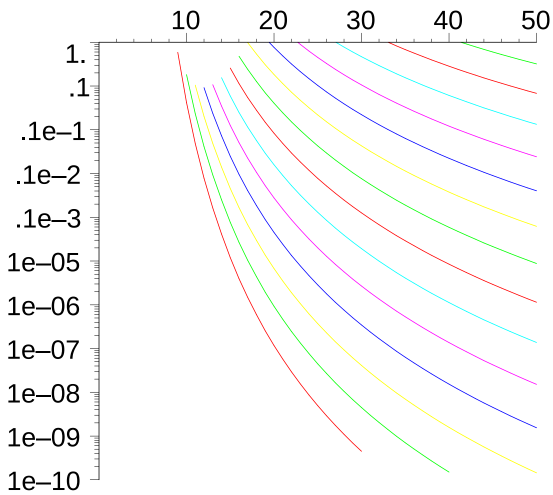

Figure 3: Value of α as the function of R' for $\varepsilon = \mathbb{E}(I_\Phi) * 10^{-n/10}$ with $n \in \{10, \ldots, 25\}$ satisfying assumptions of Proposition 4 (and thus inequality (2)) for $C = 10^{-9}$, $\beta = 3$, $R = 5\text{km}^2$, $\lambda = 0.5\text{BS/km}^2$, $m = 7.0 \ln(10)/10\text{dB}$ and $\sigma = 2.4 \ln(10)/10\text{dB}$

4 Unconditional evolution of the model, stationary distribution of a Markov process

In this part we present the distribution of the CDMA coverage model as the stationary distribution of a Markov process. This approach will be the basis for both conditional and, under some change of configuration, unconditional simulation of the model. Let B be the fixed simulation window and $\lambda(x)$ the density of the point process of antennas. We assume $\Lambda = \int_B \lambda$.

More specifically, we will consider a spatial birth-and-death process and will describe its evolution in two ways. In order to keep a general setting we will consider antennas as couples (X, M) with X being their positions and $M = (S, T)$ their marks.

4.1 Evolution with indistinguishable antennas.

Starting from a configuration of antennas $\Phi_t = \{(X_i, M_i)\}_{i \in \{1,\ldots,n\}}$, we let this configuration remain for a random time T that is exponentially distributed with parameter $\Lambda + n$ and then either add or remove one antenna. We make this choice “add or remove” (“birth or death”) randomly, such that the probability of adding antenna is $\frac{\Lambda}{\Lambda+n}$ and the probability of removing one is $\frac{n}{\Lambda+n}$. (This can be implemented by generating a random variable uniformly distributed on $[0, 1]$, indicating a birth if it is less than $\frac{\Lambda}{\Lambda+n}$ and a death elsewhere.) In the case of a birth we choose the location X of the new antenna randomly, according to the intensity function λ, and its mark M according to the mark distribution, independently of everything else. In the case of a death we pick uniformly one antenna from existing configuration Φ_t. Having completed this, we obtain a new configuration Φ_{t+T} valid from time $t + T$ on. Figure 4 illustrates this way of developing our spatial birth-and-death process. Obviously it is a Markov process and it admits our model distribution as its stationary regime [7, 5].

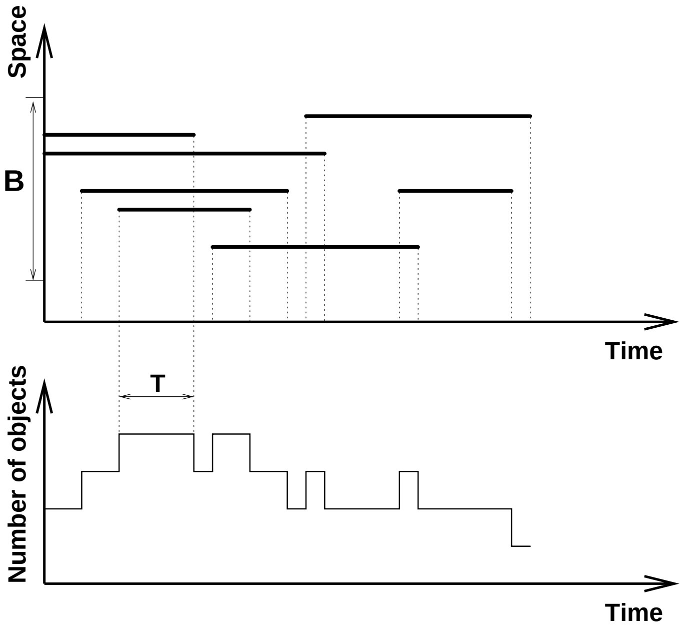

Figure 4: Example of the realization of the spatial birth-and-death process with its objects considered as indistinguishable in the sense that their life-times are *not* specified at their birth instants. We follow inter-instance (birth or death) times T.

4.2 Evolution with distinguishable antennas.

For this approach to the same spatial birth-and-death process, lets assume that each antenna has its life-time associated to it (which makes it distinguishable). We generate exponential, with parameter 1, life-time of each antenna at its birth instant and this makes its death-time known. As a consequence it suffices to specify inter-birth-times T' only, which we assume to be exponential with parameter Λ and independent of everything else. The evolution of the process is then the following: starting at time t from any configuration $\widetilde{\Phi}_t = \{(X_i, M_i, \tau_i)\}_{i\in\{1,\dots,n\}}$, where $\tau_i = \tau_i(t)$ is the *remaining* (or *residual*) *life-time* of the antenna i, we are going to add one antenna after random time T'. As previously, we will choose the position of the antenna according to the density function λ and its mark according to the appropriate distribution. In addition to this we generate its life-time according to the exponential law with parameter 1. Meanwhile, between instances t and and $t+T'$ we develop the process observing deaths of antennas, whose life-times expire. At the time $t+T'$ the new antenna is added and we continue from a new configuration $\widetilde{\Phi}_{t+T'}$ on. Figure 5 illustrates this approach.

The above two evolution mechanisms are equivalent. In fact, for a configuration with n antennas, which have the life-times $\tau_1,\dots,\tau_n$, we see that

$$P(\min(T',\tau_1,\dots,\tau_n) > t) = \exp(\Lambda+n)t = P(T>t)$$

and

$$P(T' > \min(\tau_1,\dots,\tau_n)) = \frac{n}{\Lambda+n}\,.$$

We will use this equivalence of approaches when describing perfect simulation of the conditional model.

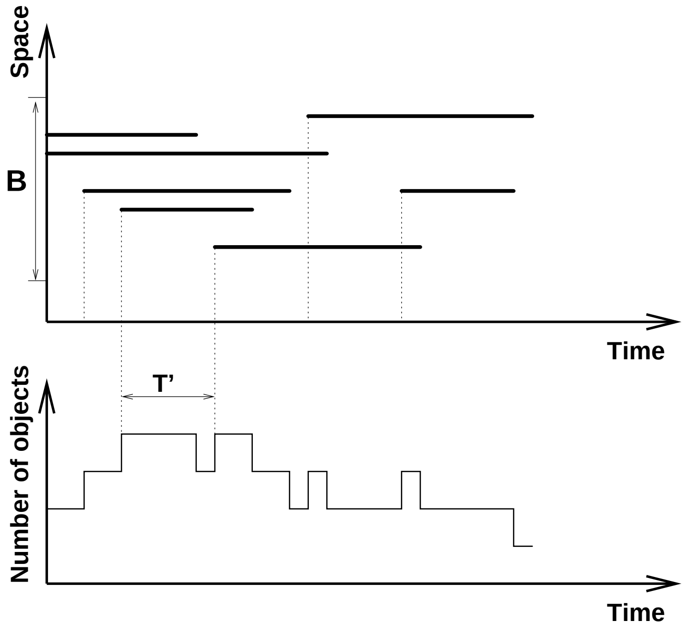

Figure 5: Example of the realization of a spatial birth-and-death process with its objects considered as distinguishable in the sense that their life-times are specified at their birth instants making death-times known. We follow inter-birth times T'.

5 Conditional simulation

In this part we explain how to get perfect simulation of the model under some conditions. We always perform our simulation in the window B and λ is the intensity of the Poisson p.p. of the antennas.

Let

$$\begin{cases} (\overline{z}_1, \overline{n}_1), \dots, (\overline{z}_p, \overline{n}_p) \in B \times \mathbb{R} \\ (\underline{z}_1, \underline{n}_1), \dots, (\underline{z}_q, \underline{n}_q) \in B \times \mathbb{R}. \end{cases}$$

We are looking for a realization of the model *given* the following two conditions:

$\overline{\mathsf{C}}$: points $\overline{z}_i$ are covered *at least* $\overline{n}_i$ times and

$\underline{\mathsf{C}}$: points $\underline{z}_j$ are covered *at most* $\underline{n}_j$ times

("n-time" coverage means coverage by n cells). Note that this kind of conditions allows us to model situations when points are covered by a given number of cells. In the following we will consider a spatial birth and death process whose stationary distribution satisfies these conditions almost surely.

5.1 Conditional evolution

In order to obtain the conditional realization of the model we will adopt the so-called backward coupling method proposed in [2, 3]. In particular we will develop the simulation similar to this proposed by Kendall in [4]. The first step of this approach is to specify the conditional evolution of the Markov process. We do it in the following two subsections making references to the unconditional evolution of section 4.

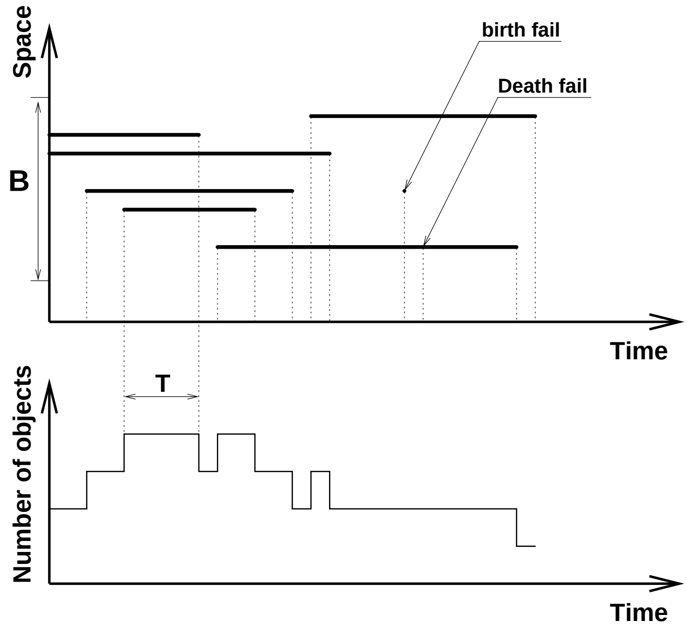

Figure 6: Example of the conditional realization of a spatial birth-and-death process with its objects considered as indistinguishable. Conditions are superimposed on the realization presented on Figure 4.

5.1.1 Conditional evolution with indistinguishable antennas

Starting from a configuration $\Phi_t = \{(X_i, M_i)\}_{i\in\{1,\dots,n\}}$ which satisfies the conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$, we let this configuration remain for a random time T that is exponentially distributed with parameter $\Lambda + n$ and then choose *potentially* whether to add or to remove one antenna with probabilities $\frac{\Lambda}{\Lambda+n}$ and $\frac{n}{\Lambda+n}$ respectively. In the case of a birth, we choose the location X of the new antenna randomly according to the density function λ and its mark M according to the mark distribution, independently of anything, as in the unconditional regime of section 4.1. However, the new antenna is added *only if it does not violate conditions* $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$. Similarly, in the case of a death, we pick uniformly one antenna from the existing configuration Φ_t and we remove it *only if it does not violate conditions* $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$. Having completed this, we obtain a configuration Φ_{t+T} valid from time $t + T$ on, which is the same as Φ_t if one of the two conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$ would be violated by adding or removing an antenna according to non-conditional evolution mechanism (we speak of *birth-* or *death-fail* then). Figure 6 illustrates the same realization of the model as presented on Figure 4 with superimposed conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$.

5.1.2 Conditional evolution with distinguishable antennas

As before, we take the evolution of the unconditional model described in section 4.2 and superimpose conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$. This time each antenna has its *potential* life-time τ (exponential with parameter 1) generated at its birth and this makes it potential death-time known. Inter-potential-birth-times are generated according to the exponential distribution with parameter 1. Potential events happen if they do not violate conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$.

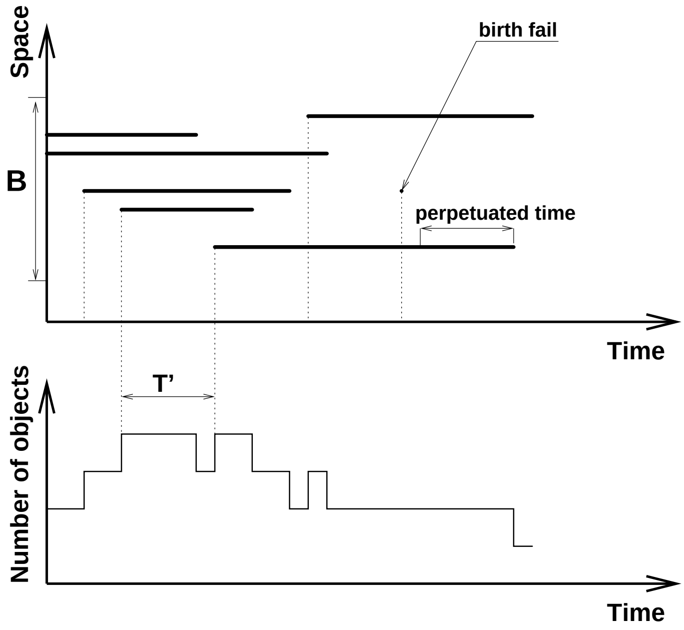

Figure 7: Example of the conditional realization of a spatial birth-and-death process with its objects considered as distinguishable. Conditions are superimposed on the realization presented on Figure 5.

More specifically, starting at time t from any configuration $\widetilde{\Phi}_t = \{(X_i, M_i, \tau_i)\}_{i \in \{1,\ldots,n\}}$ that satisfies the conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$, where τ_i are potential residual life-times, we are going to add one antenna after a random time T'. As previously, we will choose the position X' of the antenna according to the density function λ and its mark M' according to appropriate distribution. In addition to this we generate its potential life-time τ' according to the exponential law with parameter 1. Meanwhile, between instants t and and $t + T'$ we develop the process removing only these antennas, whose potential life-times expire while not violating condition $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$. The life of antennas whose death would violate this condition is extended (*perpetuated*) for another random time which has the same exponential distribution with parameter 1 and is independent of everything else. At time $t + T'$ the new antenna (X', M', τ') is added only if this does not violate condition $\underline{\mathsf{C}}$ and we continue from configuration $\widetilde{\Phi}_{t+T'}$ (that can by equal to $\widetilde{\Phi}_t$ if death- or birth-fail happened).

The above two equivalent mechanisms describe the evolution of the spatial Markov process that has for stationary regime the conditional law of our model.

5.2 Biased simulation

Having started from any configuration Φ_0 satisfying $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$ at time $t = 0$ and following the above conditional evolution we will approach the conditional regime of our model as time goes to infinity. In practice, infinite simulation is not possible, and thus stopping it after arbitrary, long enough, time, we will get a configuration close to the stationary one. The problem of good stoping time remains open.

5.3 Conditional evolution allowing for the coupling from the past

We now explain how to develop trajectories of the conditional simulation by extending their history in the past. The idea underlying is to observe at a given time, say $t = 0$, a given trajectory with longer and longer history, until any further extension of the past has no impact on the configuration at time $t = 0$. We understand then, that what is observed is a pattern that *"couples"* with this trajectory started off at $t = -\infty$, having thus infinitely long history, and so a pattern of the stationary regime. In order to be able to perform this task we have to solve the following two problems.

Note first, that when developing a trajectory according to the rules described in sections 5.1.1 and 5.1.2 on any interval $[t_1, t_2]$, starting off with some configuration Φ_{t_1}, we generate some "driving" random variables, as inter-potential-birth-times, potential lifeties and perpetuation-times, at each time $t \in [t_1, t_2]$ "on spot", depending on the current configuration Φ_t. If we want to be able to extend this trajectory to the past, starting it off at a time $t_0 < t_1$, we will possibly have to modify it within the interval $[t_1, t_2]$. And since we want it to be the same event, we have to use the same driving random variables. It is important then to generate "in advance" a rich enough set of these driving variables to be able to develop trajectory within $[t_1, t_2]$ meeting *any scenario*. We call this set $O(t_1, t_2)$. We explain in the following how to construct $O(t_1, t_2)$. Let us before mention only the second problem of the coupling from the past consists en recognizing the moment when any further extension of the past has no impact on the configuration at time $t = 0$; it will be the subject of the next section.

We will construct $O(t_1, t_2)$ allowing for development of the evolution $\mathcal{E}_{\overline{\mathsf{C}} \cup \underline{\mathsf{C}}}$ of the trajectory on the interval $[t_1, t_2]$ with distinguishable antennas, starting from any configuration $\widetilde{\Phi}_{t_1}$. We can see $\mathcal{E}_{\overline{\mathsf{C}} \cup \underline{\mathsf{C}}}$ as a function of $O(t_1, t_2)$ and $\widetilde{\Phi}_{t_1}$

$$\mathcal{E}_{\overline{\mathsf{C}} \cup \underline{\mathsf{C}}}(\widetilde{\Phi}_{t_1}, O(t_1, t_2)) = \widetilde{\Phi}_{t_2}$$

returning configuration $\widetilde{\Phi}_{t_2}$ at time t_2.

In order to define $O(t_1, t_2)$, one has to know that the number of antennas which might require perpetuation (in order to satisfy $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$) at a given time is bounded form above. We will denote this bound by ν; We will show in Section 5.3.3 how to calculate ν.

5.3.1 Driving element O

Let $\{\xi_i\}_{i \in \{1 \ldots m\}}$ be a realization of the Poisson p.p. with intensity Λ of potential birth-times on $[t_1, t_2]$ and $\{(X_i, M_i, \tau_i)\}_{i \in \{1 \ldots m\}}$ be the marks of the (distinguishable) antennas that potentially are to be born. Let

$$\left(\{\eta_i^1\}_{i \in \{1 \ldots m_1\}}, \ldots, \{\eta_i^\nu\}_{i \in \{1 \ldots m_\nu\}}\right)$$

be the instants of realizations of ν independent Poisson p.p's with intensity 1 on $[t_1, t_2]$. Finally, let $(\tau_1', \ldots, \tau_\nu')$ be a vector of realization of ν independent exponential random variables with parameter 1, and let $\eta_{m_j+1} = t_2 + \tau_j'$ (thus η_{m_j+1} is the first instant after t_2 of each Poisson p.p. $j = 1, \ldots, \nu$). We define $O(t_1, t_2)$ as the following set of independent random objects

$$\begin{aligned} O(t_1, t_2) = \quad & \big(\{(\xi_i, X_i, M_i, \tau_i)\}_{i \in \{1 \ldots m\}}, \\ & \{\eta_i^1\}_{i \in \{1 \ldots m_1+1\}}, \ldots, \{\eta_i^\nu\}_{i \in \{1 \ldots m_\nu+1\}}\big) \,. \end{aligned}$$

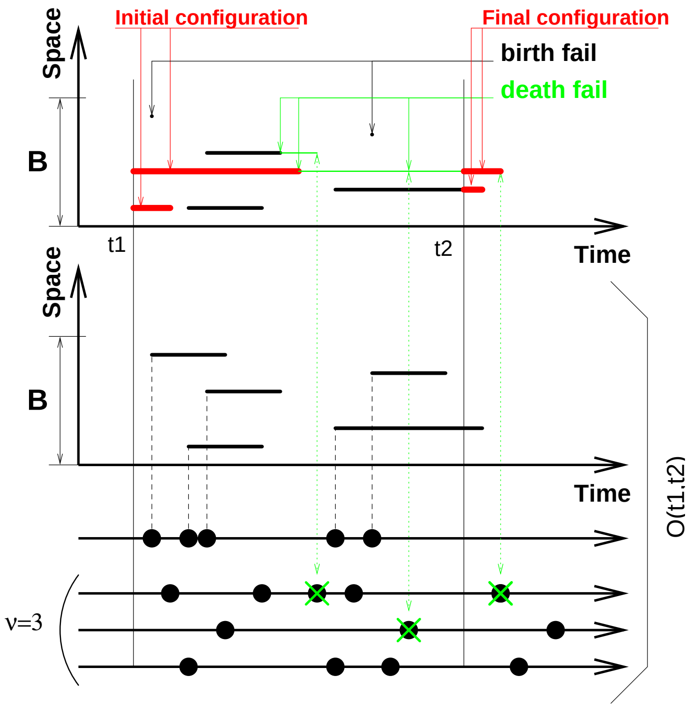

Figure 8: Example of the evolution $\mathcal{E}_{\overline{C} \cup \underline{C}}$ driven by the random object $O(t_1, t_2)$ generated "in advance".

The first random object Poisson marked p.p.
$\{(\xi_i, X_i, M_i, \tau_i)\}_{i \in \{1 \ldots m\}}$ gives us the instants and characteristics of antennas that potentially are to be added to the configuration. In particular τ_i is the residual life-time at the instant of birth (total life-time). Other Poisson p.p's serve for perpetuation of the antennas, which cannot be removed.

5.3.2 Procedure $\mathcal{E}_{\overline{C} \cup \underline{C}}$

We perform the evolution $\mathcal{E}_{\overline{C} \cup \underline{C}}$ starting from $\widetilde{\Phi}_{t_1}$ and using $O(t_1, t_2)$ in the following way. We append $\{(\xi_i, X_i, M_i, \tau_i)\}_{i \in \{1 \ldots m\}}$ to the configuration $\widetilde{\Phi}_{t_1}$ and observe on the time axis successive events (potential births and expirations of potential life-times). We add antennas only when condition $\overline{C} \cup \underline{C}$ is not violated. When a death-fail occurs at some t we perpetuate the life-lime of the antenna in question until the nearest (after t) available event of the composition of the ν Poisson p.p. $\{\eta_i^j\}_{i \in \{1 \ldots m_j+1\}}$, $j = 1, \ldots, \nu$. When we say "available" we understand only these instants that have not been used for this purpose yet. Formally, we perpetuate it to the time $\eta_{i_0}^{j_0}$ with

$$j_0 = \min_{j \in \{1, \ldots, \nu\}} \left\{ j \,\middle|\, \forall (\xi, X, M, \tau) \in \widetilde{\Phi}_t,\ \tau \neq \min\{\eta_i^j - t > 0\} \right\}$$

and $\eta_{i_0}^{j_0} = \min_{i \in \{1 \ldots m_j+1\}} \{\eta_i^{j_0} > t\}$. In other words, at the death-fail instant t, we assign to the antenna in question another potential residual life-time $\eta_{i_0}^{i_0} - t$. Following the above rules, we arrive at some configuration $\widetilde{\Phi}_{t_2}$ which describes locations of antennas at t_2 and their potential residual life-times. The Figure 8 illustrates this procedure.

Note that the *unconditional* evolution (see Section 4.2) on an interval (t_1, t_2) might also be seen as the appropriate function $\mathcal{E}_\emptyset$ of an initial configuration $\widetilde{\Phi}_{t_1}$ and the driving element $O(t_1, t_2)$. In fact the part of O which describes births and potential life-times is sufficient. We will use $\mathcal{E}_\emptyset$ in Section 5.5 to describe how to extend the history of the driving element O.

5.3.3 Calculation of ν

We will show now how to calculate an upper bound ν for the number of perpetuated antennas.

Lemma 1 *Suppose $T \geq T_{min}$, $S_{max} \geq S \geq S_{min}$, and the size of the influence window (maximal distance between antennas) is R'. Then, the number of antennas which might require perpetuation due to conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$ at any given time is not greater than*

$$\nu = \sum_{i=1}^{p} n_i + \frac{S_{max} l(2R')}{S_{min} l(R')} \sum_{i=1}^{q} \max(1 - n_i' T_{min}, 0)\,.$$

Proof: First note that that there are at most $\sum_{i=1}^{p} n_i$ antennas which are necessary to guarantee condition $\overline{\mathsf{C}}$. Moreover, for a given point $\underline{z}_i \in \underline{Z}$ we need at most $(1 - n_i' T_{min}) S_{max} l(2R')/(S_{min} l(R'))$ antennas (no matter where) to prevent coverage by more than $\underline{n}_i$ alls. Thus the maximal number of antennas necessary to guarantee $\underline{\mathsf{C}}$ is $S_{max} l(2R')/(S_{min} l(R')) \sum_{i=1}^{q} (1 - n_i' T_{min})$. This completes the proof. ■

5.4 Maximal and minimal process

In this section we define the so-called maximal and minimal processes whose coupling indicates coupling with infinitely long history.

5.4.1 Partial order on the space of antenna patterns with noise

We define first a partial order $\prec$ on the Cartesian product of the space of (indistinguishable) antenna patterns equipped with external noise. It is the order that will allow us to consider a crucial "sandwich relation": minimal process $\prec$ conditional evolution from the past $\prec$ maximal process.

For any (indistinguishable) antenna $A = (X, M)$, where $M = (S, T)$, and Q a set of antennas, we define a cell $\mathcal{C}(A, Q)$ associated to A by $\mathcal{C}(A, Q) = \{y \in \mathbb{R}^2 \mid Sl(|X - y|) \geq T(I_Q(y) + W)\}$. We say that two pairs of antenna patterns $P_j = (\{A_i^j\}_{i \in \{1,\ldots,n_j\}}, Q_j)$, $j = 1, 2$ are ordered by $\prec$ if

$$(P_1, Q_1) \prec (P_2, Q_2) \Leftrightarrow \begin{cases} I_{Q_2} \leq I_{Q_1} \\ \exists f \text{ injection} \mid \\ \quad \forall i\, \mathcal{C}(A_i^1, Q_1) \subset \mathcal{C}(A_{f(i)}^2, Q_2). \end{cases}$$

It is obvious that $\prec$ is a partial order. Note that the second antenna pattern Q_i in the couple (P_i, Q_i) is used to model the noise only.

We will also use the following notation:

$$\mathcal{C}(\{A_i\}_{i \in \{1,\ldots,n\}}, Q) = \{\mathcal{C}(A_i, Q)\}_{i \in \{1,\ldots,n\}}\,.$$

5.4.2 Definition of maximal and minimal processes

Let us now define two processes of antenna patterns: a maximal process $\widetilde{\Phi}_t^+$ and a minimal one $\widetilde{\Phi}_t^-$. They bound all possible conditional trajectories started off at the same initial configuration $\widetilde{\Phi}_{-T}$ and following the evolution mechanism $\mathcal{E}_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$ driven by the same random element $O(-T,0)$, as will be explained in Section 5.4.3.

For a given time interval $[t_1,t_2]$, initial configuration of antenas $\widetilde{\Phi}_{t_1}$ and driving random element $O(t_1,t_2)$, as in Section 5.3, we are going to define some initial values of two processess $(\widetilde{\Phi}_{t_1}^+,\widetilde{\Phi}_{t_1}^-)$ and two evolution procedures $(\mathcal{E}^+_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}},\mathcal{E}^-_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}})$ that will return configurations $(\widetilde{\Phi}_{t_2}^+,\widetilde{\Phi}_{t_2}^-)$. As for $\mathcal{E}_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$, we perform the evolutions $(\mathcal{E}^+_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}},\mathcal{E}^-_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}})$ starting from $(\widetilde{\Phi}_{t_1}^+,\widetilde{\Phi}_{t_1}^-)$ and using $O(t_1,t_2)$ in the following way. We append $\{(\xi_i,X_i,M_i,\tau_i)\}_{i\in\{1\ldots m\}}$ of $O(t_1,t_2)$ to each configuration $(\widetilde{\Phi}_{t_1}^+,\widetilde{\Phi}_{t_1}^-)$ and observe successive events on the time axis (potential births and expirations of potential life-times). The difference is that now we will use *another condition* $(\overline{\mathsf{C}}_\alpha^+\cup\underline{\mathsf{C}}_\alpha^+,\overline{\mathsf{C}}_\alpha^-\cup\underline{\mathsf{C}}_\alpha^-)$, $(\overline{\mathsf{C}}_\Omega^+\cup\underline{\mathsf{C}}_\Omega^+,\overline{\mathsf{C}}_\Omega^-\cup\underline{\mathsf{C}}_\Omega^-)$ (to be defined in the sequel) for potential events to happen. Namely, we add an antenna to $(\widetilde{\Phi}_t^+,\widetilde{\Phi}_t^-)$ only when conditions $(\overline{\mathsf{C}}_\alpha^+\cup\underline{\mathsf{C}}_\alpha^+,\overline{\mathsf{C}}_\alpha^-\cup\underline{\mathsf{C}}_\alpha^-)$, are not violated. Similarly, we will remove an antenna if conditions $(\overline{\mathsf{C}}_\Omega^+\cup\underline{\mathsf{C}}_\Omega^+,\overline{\mathsf{C}}_\Omega^-\cup\underline{\mathsf{C}}_\Omega^-)$, are not violated. When a death-fail occurs at some t we perpetuate the life-lime of the antenna in question until the nearest (after t) available event of the composition of the ν Poisson p.p. $\{\eta_i^j\}_{i\in\{1\ldots m_j+1\}}$, $j=1,\ldots,\nu$ of $O(t_1,t_2)$ exactly as for $\mathcal{E}_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$.

Before we define initial values of $\widetilde{\Phi}^+$ and $\widetilde{\Phi}^-$ and conditions $\overline{\mathsf{C}}_\alpha^+$, $\overline{\mathsf{C}}_\Omega^+$, $\overline{\mathsf{C}}_\alpha^-$, $\overline{\mathsf{C}}_\Omega^-$, $\underline{\mathsf{C}}_\alpha^+$, $\underline{\mathsf{C}}_\Omega^+$, $\underline{\mathsf{C}}_\alpha^-$, $\underline{\mathsf{C}}_\Omega^-$, we introduce some necessary notation. For two sets of antenna $P=\{A_k\}_{k\in\{1,\ldots,n\}}$ and Q, we define

$$\overline{N}_i(P,Q)=\#\{k\,|\,\overline{z}_i\in\mathcal{C}(A_k,Q)\}\,,$$
$$\underline{N}_i(P,Q)=\#\{k\,|\,\underline{z}_i\in\mathcal{C}(A_k,Q)\}\,.$$

Finally, let us denote by $\overline{Z}=\{\overline{z}_1,\ldots,\overline{z}_p\}$ and $\underline{Z}=\{\underline{z}_1,\ldots,\underline{z}_q\}$, the conditioning points constituting conditions $\overline{\mathsf{C}}\cup\underline{\mathsf{C}}$.

Starting $\widetilde{\Phi}^+$, and $\widetilde{\Phi}^-$ Let $\widetilde{\Phi}_{-T}$ satisfying $\overline{\mathsf{C}}\cup\underline{\mathsf{C}}$ be given. We define

$$\begin{cases}\widetilde{\Phi}^+_{-T}=\widetilde{\Phi}_{-T}\cup\{0,M_\infty,\tau_1^1\}\cup\cdots\cup\{0,M_\infty,\tau_1^\nu\}\\ \widetilde{\Phi}^-_{-T}=\emptyset\,,\end{cases}$$

where M_∞ is a special mark (here, of the antenna located at 0) guaranteeing that the cell $\mathcal{C}((0,M_\infty))$ covers the whole plane. (This can by done formally assuming, for example, that $M_\infty=(S_\infty,T_\infty)$ with either $S_\infty=S_{\max}$ or $T_\infty=0$.) Note that $\widetilde{\Phi}^+_{-T}$ is completely saturated, whereas $\widetilde{\Phi}^-_{-T}$ is void.

We now define now the conditions assuming configurations $\widetilde{\Phi}^+_{-T}$, $\widetilde{\Phi}^-_{-T}$ are given. For a given antenna A we add it or remove it from the configuration $\widetilde{\Phi}^+_{-T}$, $\widetilde{\Phi}^-_{-T}$ if the following conditions are satisfied respectively:

$\widetilde{\Phi}^+$ — birth condition We define the conditions

$\overline{\mathsf{C}}_\alpha^+$:

$$\overline{N}_i\left(\Phi^+_{t^-}\cup\{A\},\Phi^-_{t^-}\cup\{A\}\right)\geq\overline{n}_i\,,$$

$\underline{\mathsf{C}}_\alpha^+$:

$$\underline{N}_i\left(\Phi_{t-}^- \cup \{A\}, \Phi_{t-}^+ \cup \{A\}\right) \le \underline{n}_i\,.$$

$\widetilde{\Phi}^+$ — **death condition** Denote

$$\overline{Z}_* = \left\{\overline{z}_i \,\middle|\, \overline{N}_i\left(\Phi_{t-}^- \setminus \{A\}, \Phi_{t-}^+\right) \le \overline{n}_i\right\}.$$

Note that $\overline{Z}_*$ represents the subset of points of $\overline{Z}$, which are note covered by $\widetilde{\Phi}^-$ when we remove antenna A. Also for any $A' \in \Phi^-$ denote

$$\mathcal{C}_-(A') = \mathcal{C}(A', \Phi_{t-}^- \setminus A) \setminus \mathcal{C}(A', \Phi_{t-}^+).$$

We define the conditions

$\overline{\mathsf{C}}_\Omega^+$:

$$\overline{Z}_* \cap \mathcal{C}(A, \Phi_{t-}^+) = \emptyset\,,$$

$\underline{\mathsf{C}}_\Omega^+$:

$$\underline{Z} \cap \left(\bigcup_{A' \in \Phi_{t-}^+ \setminus A} \mathcal{C}_-(A')\right) = \emptyset\,.$$

$\widetilde{\Phi}^-$ — **birth condition** Denote

$$\underline{Z}_* = \left\{\underline{z}_i \,\middle|\, \underline{N}_i\left(\Phi_{t-}^+, \Phi_{t-}^-\right) \ge \underline{n}_i\right\}\,.$$

Also for any A' denote

$$\mathcal{C}_+(A') = \mathcal{C}(A', \Phi_{t-}^-) \setminus \mathcal{C}(A', \Phi_{t-}^+ \cup \{A\}).$$

We define the conditions

$\overline{\mathsf{C}}_\alpha^-$:

$$\overline{Z} \cap \left(\bigcup_{A' \in \Phi_{t-}^+} \mathcal{C}_+(A')\right) = \emptyset\,.$$

$\underline{\mathsf{C}}_\alpha^-$:

$$\underline{Z}_* \cap \mathcal{C}(A, \Phi_{t-}^-) = \emptyset\,.$$

$\widetilde{\Phi}^-$ — **death condition** We define the conditions

$\overline{\mathsf{C}}_\Omega^-$:

$$\underline{N}_i\left(\Phi_{t-}^- \setminus \{A\}, \Phi_{t-}^+ \setminus \{A\}\right) \le \underline{n}_i\,,$$

$\underline{\mathsf{C}}_\Omega^-$:

$$\overline{N}_i\left(\Phi_{t-}^+ \setminus \{A\}, \Phi_{t-}^- \setminus \{A\}\right) \ge \overline{n}_i\,.$$

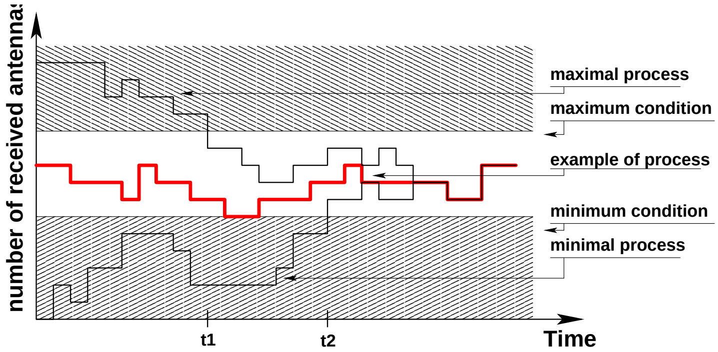

Figure 9: This figure shows the evolution of the number of antennas covering a given conditional point, in the three processes: the minimal, the maximal and the original conditional one. Note that the original process always verifies conditions $\overline{\mathsf{C}} \cup \underline{\mathsf{C}}$. The maximal process always verifies $\overline{\mathsf{C}}$ and $\underline{\mathsf{C}}$ only beginning from $t1$. Similarly, the minimal process always satisfies $\overline{\mathsf{C}}$ and $\underline{\mathsf{C}}$ only beginning form $t2$. The moments $t1$ and $t2$ are the first times that, respectively, the maximal process satisfies $\underline{\mathsf{C}}$ and the minimal one satisfies $\overline{\mathsf{C}}$.

5.4.3 Sandwich relation

Now, we give a crucial sandwich relation that connects maximal and minimal processes Φ^+, Φ^- and the original process Φ that is to be simulated. Since relation $\prec$ is defined for non-distinguishable antennas, we omit tildes in $\widetilde{\Phi}$'s in the sandwich formula.

Lemma 2 *Suppose three processes $\widetilde{\Phi}_t^+$, $\widetilde{\Phi}_t^-$ and $\widetilde{\Phi}_t$ are defined on $[t_1, t_2]$ by $\mathcal{E}^+_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$, $\mathcal{E}^-_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$ and $\mathcal{E}_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$ respectively, and the same random driving element $O(t_1, t_2)$. The following sandwich relation holds for all $t_1 \le t \le t_2$*

$$\left(\Phi_t^-, \Phi_t^+\right) \prec (\Phi_t, \Phi_t) \prec \left(\Phi_t^+, \Phi_t^-\right) . \tag{9}$$

(Note that location of the noise generating patterns Φ^- and Φ^+ is swapped in (9).)

Proof: For $t = t_1$ the relation is satisfied by the definition of the initial values. Since all three processes are unchanged in between potential instants (birth and death) it suffices to show that all potential event preserve the relation. Assume thus that at time $t-$ the relation (9) holds and at t a birth or death arrives. The configuration of the processes is possibly changed but the relation (9) remains to hold as a consequence of the following implications.

$$\overline{\mathsf{C}}_\alpha^- \Rightarrow \overline{\mathsf{C}} \Rightarrow \overline{\mathsf{C}}_\alpha^+ ,$$
$$\underline{\mathsf{C}}_\alpha^- \Rightarrow \underline{\mathsf{C}} \Rightarrow \underline{\mathsf{C}}_\alpha^+ ,$$
$$\overline{\mathsf{C}}_\Omega^- \Leftarrow \overline{\mathsf{C}} \Leftarrow \overline{\mathsf{C}}_\Omega^+ ,$$
$$\underline{\mathsf{C}}_\Omega^- \Leftarrow \underline{\mathsf{C}} \Leftarrow \underline{\mathsf{C}}_\Omega^+ ,$$

In order to demonstrate the first implication note if that the condition $\overline{\mathsf{C}}_\alpha^-$ is true for the minimal process, for all antenna $(X, (S, T)) \in \Phi_t$ we have for all antennas the relation

$$\frac{Sl(|X - \overline{z}_i|)}{T} \le I_{\Phi^-} + W$$

or

$$I_{\Phi^+\cup\{A\}} + W \le \frac{Sl(|X - \overline{z}_i|)}{T}\,.$$

Then the birth of the antenna A doesn't change the conditions for $\overline{Z}$, and $\overline{\mathsf{C}}$ is true for the intermediate process Φ_t.

If the condition $\overline{\mathsf{C}}$ is true for the intermediate process, then there are antennas in this process that verifie the condition for $\overline{Z}$. These antennas must verifie the same condition in the maximal process because we have the relation $I_\Phi \ge I_{\Phi^-}$.

If the condition $\underline{\mathsf{C}}_\alpha^-$is true, then for all antennas of the maximal process the birth of the antenna A doesn't change the condition for $\underline{Z}$. In fact $\mathcal{C}(A, \Phi^-)$ could only contain some conditionning points $\underline{z}_i$ that verifie $\underline{N}_i < \underline{n}_i$. Also, we have $I_{\Phi^+} \ge I_\Phi$ and then the condition $\underline{\mathsf{C}}$ is true for all intermediate process.

If the condition $\underline{\mathsf{C}}$ is true, then we have $\underline{N}_i(\Phi\cup\{A\}, \Phi\cup\{A\}) \le \underline{n}_i$ and the condition $\underline{\mathsf{C}}_\alpha^+$ is true.

We can demonstrate the remaining implications using similar arguements because the following relations hold:

$$\overline{N}_i(\Phi^-, \Phi^+) \le \overline{N}_i(\Phi, \Phi) \le \overline{N}_i(\Phi^+, \Phi^-)\,,$$

$$\overline{N}_i(\Phi, \Phi) \le \overline{n}_i\,,$$

$$\underline{N}_i(\Phi^-, \Phi^+) \le \underline{N}_i(\Phi, \Phi) \le \underline{N}_i(\Phi^+, \Phi^-)\,,$$

$$\underline{n}_i(\Phi^-, \Phi^+) \le \underline{N}_i(\Phi, \Phi)\,.$$

■

5.5 Unbiased simulation

We describe now the algorithm of perfect simulation of the stationary conditional pattern of antennas. Here are the main steps.

1. Take some initial duration time of simulation, say $T = 1$.

2. Generate initial unconditional configuration $\Phi_0' = \{(X_i, M_i)\}$ of indistinguishable antennas (see section 4.1). For each antenna X_i, generate two exponential life-times: its age τ_i, and residual (remaining) life-time τ_i'. (In other words $\{-\tau_i\}$ are birth times of the distinguishable pattern $\widetilde{\Phi}_0' = (\Phi_0{}', \{\tau_i'\}) = \{(X_i, M_i, \tau_i')\}$ observed at time 0.)

3. Generate the driving element $O' = O(0, T)$ (section 5.3.1).

4. Generate the unconditional evolution $\widetilde{\Phi}_t' = \mathcal{E}_\emptyset(\widetilde{\Phi}_0', O(0,t))$ for $0 \le t \le T$ (section 5.3); note $\widetilde{\Phi}_t'$ is defined on the positive half-line.

5. Reflect the time of all processes w.r.t. point $t = 0$. That is, define $O(-T, 0)$ and $\widetilde{\Phi}_t$ $(-T \le t \le 0)$ as the reflection of, respectively, O' and $\widetilde{\Phi}_t'$. Birth-times of $O', \widetilde{\Phi}'$ are death-times of $O, \widetilde{\Phi}$ and vice-versa. In particular $\mathcal{E}_\emptyset(\widetilde{\Phi}_{-T}, O(-T, 0)) = (\Phi_0{}', \{\tau_i\}) = \{(X_i, M_i, \tau_i)\}$.

6. Define $\widetilde{\Phi}^{+}_{-T}, \widetilde{\Phi}^{-}_{-T}$ with respect to $\widetilde{\Phi}_{-T}$ and run the evolutions $\mathcal{E}^{-}_{\overline{C}\cup\underline{C}}$ and $\mathcal{E}^{+}_{\overline{C}\cup\underline{C}}$ driven by $O(-T,0)$ (section 5.4.2).

7. Check if $\Phi^{-}_{0} = \Phi^{+}_{0}$. If so, then stop the simulation taking the common pattern as the realization of the stationary conditional distribution of the pattern of antennas.

8. Increase the simulation time, say up to $2T$ (The algorithm has a complexity $\mathcal{O}(n \log n)$). This means, generate the respective Poisson p.p.'s of $O'(0,2T)$ only on $[T,2T]$ and append them to the right-hand-side of the previously generated realizations $O(O,T)$.

9. Substitute $T \leftarrow 2T$ and goto step 4.

Lemma 3 *Suppose there is no contradiction in the assumptions (i.e. for all $z \in \overline{Z} \cap \underline{Z}$ the coverage constraints $\overline{n}, \underline{n}$ of $\overline{C} \cup \underline{C}$ are ordered $\underline{n} \le \overline{n}$, and for an $\overline{z} \in \overline{Z}$ $\overline{n} \le 1/T_{min}$). Then, the algorithm stops almost surely in a finite time.*

Proof: Fix $\delta > 0$. For any initial configuration of antennas $\widetilde{\Phi}$ and any time t, the probability of finding $O(t, t+\delta)$ such that the respective minimal and maximal processes couples within $(t, t+\delta)$ is positive and bounded away from 0. The standard Borel-Cantelli lemma argument completes the proof. ■

5.5.1 Coupling form the past

Here is the essential result for the coupling from the past.

Proposition 5 *Suppose for some $T > 0$, some configuration $\widetilde{\Phi}_{-T}$ and $O(-T,0)$, we have coupling at time 0 of the maximal and minimal processes*

$$\begin{aligned}\widetilde{\Phi}^{-}_{0} &= \mathcal{E}^{-}_{\overline{C}\cup\underline{C}}(\widetilde{\Phi}^{-}_{-T}, O(-T,0)) \\ &= \mathcal{E}^{+}_{\overline{C}\cup\underline{C}}(\widetilde{\Phi}^{+}_{-T}, O(-T,0)).\end{aligned}$$

Then there exist a realization $\widetilde{\Phi}''_t$ that satisfies $\overline{C} \cup \underline{C}$ for $t \in (-\infty, 0)$, and such that for some $t_0 \in [-T, 0)$

$$\widetilde{\Phi}''_t = \mathcal{E}_{\overline{C}\cup\underline{C}}\Big(\mathcal{E}_{\overline{C}\cup\underline{C}}(\widetilde{\Phi}_{-T}, O(-T,t_0)), O(t_0,t)\Big)$$

for all $t_0 < t \le 0$. In particular $\widetilde{\Phi}''_0 = \widetilde{\Phi}^{-}_{0}$.

Proof: The configuration $\widetilde{\Phi}_{-T}$ does not satisfy necessarily $\overline{C}\cup\underline{C}$, but $\mathcal{E}_{\overline{C}\cup\underline{C}}(\widetilde{\Phi}_{-T}, O(-T,0)) = \widetilde{\Phi}^{-}_{0}$ does verify, because the maximal and minimal processes couple. Moreover, there exists t_0, $-T \le t_0 < 0$, such that $\mathcal{E}_{\overline{C}\cup\underline{C}}(\widetilde{\Phi}_{-T}, O(-T,t))$ satisfy $\overline{C} \cup \underline{C}$ for $t \in [t_0, 0]$. Thus it suffices to define $\widetilde{\Phi}''_t = \mathcal{E}_{\overline{C}\cup\underline{C}}(\widetilde{\Phi}_{-T}, O(-T,t))$ for $t \in [t_0, 0]$ and extend its history for $t < t_0$, which is possible because the process is reversible. ■

6 Results

6.1 Contact Distribution

For a point x which is covered by exactly k cells, we denote

- R the largest random variable such the ball $B(x, R)$ is covered by exactly k cells too (namely the largest ball where the reception conditions do not change)
- L the longest segment with extremity x such that the whole segment is covered by exactly k cells.

These questions are related to the so called spherical and linear contact distribution. Analytical answers to this question are only available in the Boolean case and for $k = 0$. Thanks to the stationarity and the isotropy of the model, we can choose $x = O$, without loss of generality. We concentrate on simulation results all based on the conditional simulation algorithm described before. In order to obtain samples with the right condition (namely that the origin is covered exactly k times), it is enough to take $p = q = 1$ and $\overline{z}_1 = \underline{z}_1 = 0$ and $\overline{n}_1 = \underline{n}_1 = k$ in this algorithm. All figures bear on an observation window $[-5\text{km}, 5\text{km}]^2$ and are based on a circular influence window with radius $R' = 40\text{km}$.

Figure 10 focuses on conditional samples allowing one to estimate the fluctuations of L and R.

Figures 11, 12 13 and 14 give the histograms of the random variables L and R as obtained from the simulator. Table 15 gives more global results such as the mean of L and R. For instance, for this set of values, if there is no reception at a given location, one must move away 400 meters away in mean to recover good reception again.

The numbers of samples on which these histograms are based are 24000 for $k = 0$, 13000 for $k = 1$, 19000 for $k = 2$ and 15000 for $k = 3$. We limited ourselves to testing the hypothesis of Gamma distributions. The Kolmogorov-Smirnov test was applied using the R package. The hypothesis was accepted for the linear case when $k = 0$, and rejected in all other cases.

Notice that for all quantities pertaining to spatial joint distributions (such as the probability that all points of a given segment or of a given ball have soft handoff level with a prescribed value), the difference between simplified fading and point dependent fading actually matters. In that, the distributions which are studied in this section are only approximations of the quantities of practical interest for mobile communications, since the distributions in question bear on the smoothed versions of the cells only.

7 Conclusion

In this paper, we demonstrate how one can obtain realizations of a CDMA coverage model conditioned by some coverage constraints. The model represents the locations of antennas as realizations of Poisson point processes in the plane, and in this way takes into account the irregularities of antenna patterns in a statistical way.

Since the goal of the paper is to describe the simulation method itself, in order to make the presentation more transparent, we assumed simple classical model of propagation and fading, and a very simple power control algorithm. The presented simulation method, however, is fairly general and may by applied to more complex models.

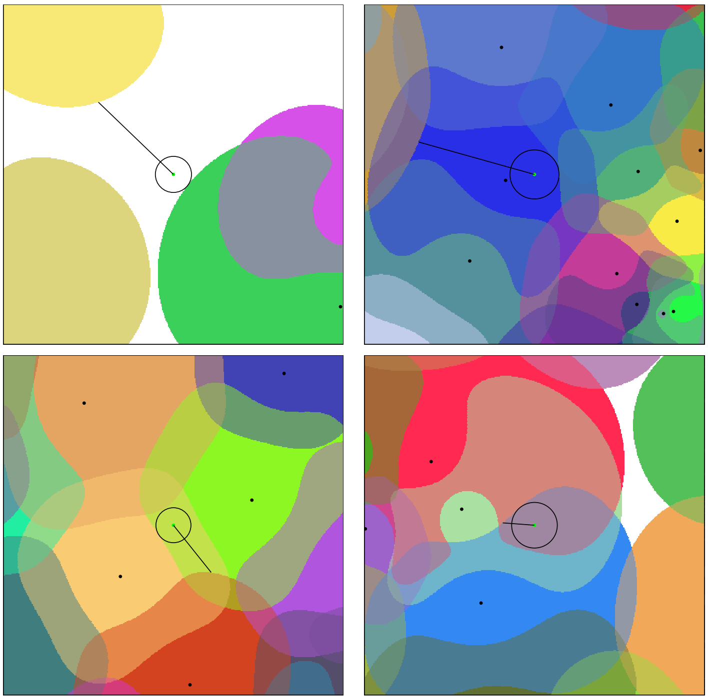

Figure 10: Conditional sample of Model 3. Point O is covered 0,1,2 or 3 times. The corresponding samples of L and R are also given.

We consider Poisson process as an extremal model, in which patterns perhaps exhibit "to much randomness" comparing to real patterns of BS's. In this way our approach is a complement of the typical regular (hexagonal) grid modeling that is in practice rarely possible too. With our simulation tool we can numerically estimate many geometrical characteristics that cannot be achieved in an analytical way and compare them with both real observations and values predicted by regular grid models.

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References

[1] Baccelli, F. and Błaszczyszyn, B. (2001), On a coverage process ranging from the Boolean model to the Poisson Voronoi tessellation, with applications to wireless communications. *Adv. in Appl. Probab. (SGSA)* **33**, 293-323.

[2] Prop, J.G. and Wilson, D.B. *Exact Sampling with Coupled Markov Chains and Applications to Statistical Mechanics*, Random Structures an Algorithms, 1996.

[3] Foss, S.G. and Tweedie, R.L. *Perfect Simulation and Backward Coupling*, 1997.

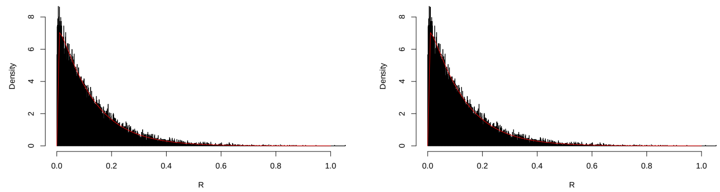

Figure 11: Histogram of L (on the right) and R (left) given O is covered 0 times.

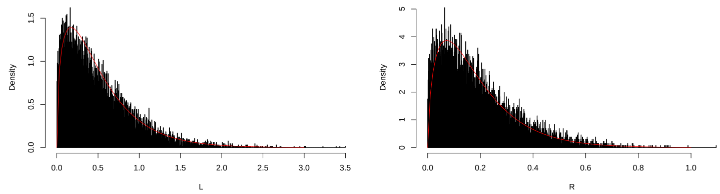

Figure 12: Histogram of L (on the right) and R (left) given O is covered 1 times.

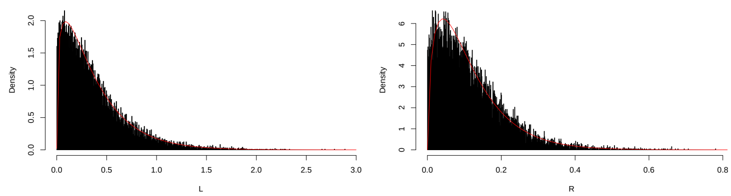

Figure 13: Histogram of L (on the right) and R (left) given O is covered 2 times.

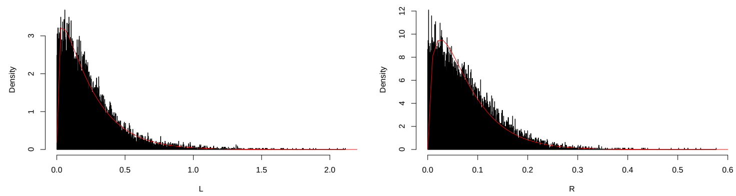

Figure 14: Histogram of L (on the right) and R (left) given O is covered 3 times.

handoff level	EL	$varL$	ER	$varR$
0	0.423 km	0.191 km^2	0.121 km	0.013 km^2
1	0.521 km	0.182 km^2	0.186 km	0.021 km^2
2	0.375 km	0.107 km^2	0.116 km	0.008 km^2
3	0.239 km	0.047 km^2	0.075 km	0.003 km^2

Figure 15: Mean and variance.

[4] Kendall, W.S. and Thönnes, E. (1999), Perfect Simulation in Stochastic Geometry, *Pattern Recognition*, 32(9), pp. 1569-1586, special issue on random sets.

[5] Lantuéjoul, C. (1996), Conditional simulation of object-based models, *Proceedings of the international symposium on advances in theory and applications of random sets*, Fontaineblueau, pp. 271-288, D. Jeulin Ed., World Scientific Publishing Company.

[6] Viterbi, A. and Viterbi, A. (1993), Erlang Capacity of a Power Controlled CDMA

System, IEEE On Selected Areas in Communications, pp. 892-900, vol. 11, no. 6, August 1993.

[7] Matheron, G. (1967), Elément pour une théorie des milieux poreux, Masson & Cie.

A Proofs

We first state and prove the following lemma, that will be used in the proof of Proposition 1 page 4.

Lemma 4 *Let N be a Poisson random variable with mean λ and let S be a generic random variable for the sequence $\{S_i\}_{i\in\mathbb{N}}$ of independent identically distributed random variables. For a given $\alpha \in \mathbb{R}$ the following equivalence holds*

$$\mathbb{E}\left(\left(\sum_{i=0}^{N} S_i\right)^{\alpha}\right) < \infty \Longleftrightarrow \mathbb{E}\left(S^{\alpha}\right) < \infty\,.$$

Proof:

$\Longrightarrow$ **:** $\mathbb{E}((\sum_{i=0}^{\infty} S_i)^{\alpha}) = \sum_{n=0}^{\infty} P(N=n)\cdot\mathbb{E}((\sum_{i=0}^{n} S_i)^{\alpha})$ We find the term $\mathbb{E}(S^{\alpha})$ in the last expansion and this shows the implication.

$\Longleftarrow$ **:** $\mathbb{E}((\sum_{i=0}^{\infty} S_i)^{\alpha}) = \sum_{n=0}^{\infty} P(N=n)\mathbb{E}((\sum_{i=0}^{n} S_i)^{\alpha})$ Now, we use inequality $(\sum_{i=0}^{n} S_i)^{\alpha} < \prod_{i=0}^{n}(1+S_i)^{\alpha}$, and by independence of S_i we get

$$\begin{array}{rcl}\mathbb{E}((\sum_{i=0}^{\infty} S_i)^{\alpha}) & < & \sum_{n=0}^{\infty} P(N=n)(\mathbb{E}((1+S)^{\alpha}))^{n} \\ & \leq & e^{\lambda(\mathbb{E}((1+S)^{\alpha})-1)}\end{array}.$$

This justifies the implication.

■

Now we are ready to prove Proposition 1 page 4.

Proof of Proposition 1: Let $\varepsilon > 0$, $\alpha > 0$ et $R > 0$. For $n \in \mathbb{N}$, define the following constants, sets and random variables for $\in \mathbb{N}$

$$\begin{array}{rcl}R_n & = & \sqrt{\dfrac{n}{\lambda\pi}}\,, \\ D_n & = & \{x \in \mathbb{R}^2 \mid R_n \leq x < R_{n+1}\}\,, \\ N_n & = & \#\{X_i \in D_n\}\,.\end{array}$$

By the construction $\{N_n\}$ are independent Poisson random variables with intensity 1. There exists $n_0 \in \mathbb{N}$ such that $R_{n_0} > R$. Thus, for $n > n_0$, we have the following inequality

$$\sum_{X_i\in D_n} S_i\, l(|y-X_i|) < \frac{C}{(R_n-R)^{\beta}}\sum_{i=1}^{N_n} S_i \qquad \forall y \in B(0,R).$$

Lets take $\delta \in]1, \beta/2[$ for which $\mathbb{E}(S^{\frac{1}{\beta/2-\delta}}) < \infty$ and choose $n_1 \geq n_0$ such that for $n > n_1$ $R_n < 2(R_n - R)$ we get for $n > n_1$

$$\begin{array}{l}P\left(\sup_{|y|<R}\sum_{X_i\in D_n} S_i\, l(|y-X_i|) < \frac{\varepsilon}{n^{\delta}}\right) \\ \qquad\qquad > P\left(\sum_{i=1}^{N_n} S_i < \frac{\varepsilon n^{\beta/2-\delta}}{C 2^{\beta}(\lambda\pi)^{\beta/2}}\right)\,.\end{array}$$

Using Lemma 4 we know that $\mathbb{E}\left(\left(\sum_{i=1}^{N_n} S_i\right)^{\frac{1}{\beta/2-\delta}}\right) < \infty$ for $n \geq n_1$, and thus the following series is convergent

$$\sum_{n=n_1}^{\infty} P\left(\left(\frac{C2^\beta(\lambda\pi)^{\beta/2}}{\varepsilon}\right)^{\frac{1}{\beta/2-\delta}}\left(\sum_{i=1}^{N_n} S_i\right)^{\frac{1}{\beta/2-\delta}} > n\right) < \infty\,.$$

It follows that there exists $n_2 > n_1$ such that

$$\prod_{n=n_2}^{\infty}\left(1 - P\left(\sum_{i=1}^{N_n} S_i > \frac{\varepsilon n^{\beta/2-\delta}}{C2^\beta(\lambda\pi)^{\beta/2}}\right)\right) > 1-\alpha\,.$$

Noting that $\delta > 1$, there exists another $n_3 \geq n_2$ such that $\sum_{n \geq n_3} n^{-\delta} < 1$ and thus 2 holds for $R' = R_{n_3}$. This completes the proof. ∎

We now prove Proposition 2 page 5.

Proof of Proposition 2: Let $\varepsilon, \alpha, R > 0$, as well as R_n, D_n, N_n be such as in the proof of Proposition 1. Taking $n_0 = \lceil \lambda\pi R^2 \rceil$ we get $R_{n_0} > R$. Assuming $\delta \in]1, \beta/2[$, for all $n \geq n_0$ we easily get

$$\begin{aligned} &P\left(\sup_{|y|<R} \textstyle\sum_{X_i \in D_n} S_i\, l(|y - X_i|) < \frac{\varepsilon}{n^\delta}\right) \\ &\qquad > P\left(\textstyle\sum_{i=1}^{N_n} S_i < \frac{\varepsilon\left(\sqrt{n}-\sqrt{\lambda\pi}\,R\right)^\beta}{C(\lambda\pi)^{\beta/2}\, n^\delta}\right) \\ &\qquad > P\left(\textstyle\sum_{i=1}^{N_n} S_i < c_n\right) \end{aligned}$$

where c_n is as in (4). Denoting $R_k(x) = \sum_{i=0}^{k-1} \frac{a^i x^i}{i!}$ and knowing that $P(S > x) = e^{-ax}$ we can express $P(\sum_{i=1}^{k} S_i > x) = R_k(x)e^{-ax}$. Indeed, it is evident for $k = 1$ and for any $k \geq 1$ follows by the induction

$$\begin{aligned} P\left(\textstyle\sum_{i=1}^{k+1} S_i > x\right) &= \textstyle\int_0^x aR_k(x-y)e^{-a(x-y)}e^{-ay}dy \\ &\quad + e^{-ax} \\ &= R_{k+1}(x)e^{-ax}\,. \end{aligned}$$

Using this representation we have

$$\begin{aligned} P\left(\textstyle\sum_{i=1}^{N_n} S_i > x\right) &= e^{-ax-1}\textstyle\sum_{k=1}^{\infty} \frac{R_k(x)}{k!} \\ &= e^{-ax-1}\textstyle\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} \frac{a^i x^i}{i!k!} \\ &= e^{-ax-1}\textstyle\sum_{i=0}^{\infty} \frac{a^i x^i}{i!}\sum_{k=i+1}^{\infty} \frac{1}{k!} \end{aligned}$$

and an upper bound

$$\begin{aligned} P\left(\textstyle\sum_{i=1}^{N_n} S_i > x\right) &< e^{-ax-1}\textstyle\sum_{i=0}^{\infty} \frac{a^i x^i}{i!(i+1)!}\sum_{k=0}^{\infty} \frac{1}{k!} \\ &\leq e^{-ax}\textstyle\sum_{i=0}^{\infty} \frac{a^i x^i}{i!2^i} \\ &\leq e^{-ax/2}\,. \end{aligned}$$

Denote now

$$\psi(n) = P\left(\sup_{|y|<R} \sum_{|X_i|>R_n}^{\infty} S_i\, l(|y - X_i|) < \varepsilon\right)\,.$$

If we take $n_1 \geq n_0$ such that $\sum_{n=n_1}^{\infty} n^{-\delta} < 1$, that is $1 < (\delta-1)(n_1-1)^{\delta-1}$, then we get a first lower bound $\forall k \geq n_1$

$$\psi(k) > \prod_{n=k}^{\infty} P\left(\sup_{|y|<R} \sum_{X_i \in D_n} S_i\, l(|y-X_i|) < \frac{\varepsilon}{n^\delta}\right).$$

From this we obtain $\forall k \geq n_1$

$$\ln \psi(k) > \sum_{n=k}^{\infty} \ln(1-\exp(-ac_n/2)).$$

Note now that there exists $n_2 \geq n_1$ such that $\ln(1-\exp(-ac_n/2)) > -2\exp(-ac_n/2)$ — it suffices to take $\exp(-ac_n/2) < 0.7$. We have thus $b_k n^{\beta/2-\delta} < c_n$ for $n > k$, such that $\forall k \geq n_2$

$$\ln \psi(k) > -2 \sum_{n=k}^{\infty} \exp\left(-\frac{1}{2} a b_k n^{\beta/2-\delta}\right).$$

As the result we get $\forall k \geq n_2$

$$\psi(k) > 1 - 2\left(\frac{2}{ab_k}\right)^{\frac{1}{\beta/2-\delta}} \int_{(k-1)\left(\frac{ab_k}{2}\right)^{\frac{1}{\beta/2-\delta}}}^{\infty} \exp\left(-y^{\beta/2-\delta}\right) dy\,,$$

which completes the proof. ■

Now we will prove Proposition 3 page 6. The following Lemma will be used.

Lemma 5 *Suppose density f_S of S satisfies (6) and N is Poisson random variable with mean 1. Then for any $\eta \in]0, \gamma-1[$*

$$P\left(\sum_{i=1}^{N} S_i > x\right) \leq \frac{E}{(1+x)^{\gamma-1-\eta}}.$$

Proof: Lets define

$$\begin{cases} d_1 = \frac{D}{\gamma-1} \\ d_{k+1} = \frac{Dd_k}{\eta} + \frac{D}{(\gamma-1)} \end{cases}$$

We can prove by induction with respect to k that

$$(P_k): \qquad P\left(\sum_{i=1}^{k} S_i > x\right) < \frac{d_k}{(1+x)^{\gamma-1-\eta}} \ \forall x \in \mathbb{R}^+$$

Indeed,

for $k=1$: $P(S_1 > x) < \int_x^\infty \frac{D}{(1+s)^\gamma} ds \geq \frac{D}{(\gamma-1)(1+x)^{\gamma-1-\eta}}$, so (P_1) holds.

$k \to k+1$: Now assume (P_k) holds.

$$\begin{aligned} &P\left(\sum_{i=1}^{k+1} S_i > x\right) \\ &\quad < \int_0^\infty \frac{D}{(1+s)^\gamma} P\left(\sum_{i=1}^{k} S_i > x-s\right) ds \\ &\quad < \int_0^x \frac{D}{(1+s)^\gamma} \frac{d_k}{(1+x-s)^{\gamma-1}} ds + \int_x^\infty \frac{D}{(1+s)^\gamma} ds \end{aligned}$$

Dominating $((1+s)(1+x-s))^{\gamma-1-\eta}$ par $(1+x)^{\gamma-1-\eta}$, we obtain:

$$\begin{aligned} & P\left(\sum_{i=1}^{k+1} S_i > x\right) \\ & \quad < \frac{Dd_k}{(1+x)^{\gamma-1-\eta}} \int_0^\infty \frac{1}{(1+s)^{1+\eta}} ds + \frac{D}{(\gamma-1)(1+x)^{\gamma-1-\eta}} \\ & \quad < \left(\frac{Dd_k}{\eta} + \frac{D}{(\gamma-1)}\right) \frac{1}{(1+x)^{\gamma-1-\eta}} , \end{aligned}$$

which means (P_{k+1}) holds.

Note that d_k can be expressed in closed form

$$d_k = \left(\frac{D}{\eta}\right)^{k-1} \frac{D^2}{(\gamma-1)(D-\eta)} + \frac{D\eta}{(\gamma-1)(D-\eta)} .$$

Now, if N is Poisson random variable with parameter 1, then

$$\begin{aligned} P\left(\sum_{i=1}^{N} S_i > x\right) & < \sum_{n=0}^\infty \frac{e^{-1}}{n!} \frac{d_n}{(1+x)^{\gamma-1-\eta}} \\ & < \frac{D\eta}{(\gamma-1)(D-\eta)(1+x)^{\gamma-1-\eta}} \left(1 + e^{\frac{D}{\eta}-1}\right) . \end{aligned}$$

Substitution of E given by (7) completes the proof of the lemma. ■

Proof of Proposition 3: Let $\varepsilon, \alpha, R > 0$, as well as R_n, D_n, N_n and n_0 be such as in the proof of Proposition 1. Following the same arguments we get for $n \geq n_0$

$$\begin{aligned} & P\left(\sup_{|y|<R} \sum_{X_i \in D_n} S_i\, l(|y-X_i|) < \frac{\varepsilon}{n^\delta}\right) \\ & \geq \; P\left(\sum_{i=1}^{N_n} S_i < \frac{\varepsilon n^{\beta/2-\delta}}{C(\lambda\pi)^{\beta/2}} \left(1 - R\sqrt{\frac{\lambda\pi}{n}}\right)^\beta\right) \\ & \geq \; P\left(\sum_{i=1}^{N_n} S_i < c_n\right) , \end{aligned}$$

where c_n is as in (4). Lets denote now

$$\psi(n) = P\left(\sup_{|y|<R} \sum_{|X_i|>R_n}^{\infty} S_i\, l(|y-X_i|) < \varepsilon\right) .$$

If we take $n_1 \geq n_0$, such that $\sum_{n=n_1}^\infty n^{-\delta} < 1$ — that is $1 < (\delta-1)(n_1-1)^{\delta-1}$ — then

$$\psi(k) > \prod_{n=k}^\infty P\left(\sup_{|y|<R} \sum_{X_i \in D_n} S_i\, l(|y-X_i|) < \frac{\varepsilon}{n^\delta}\right) \quad \forall k \geq n_1 .$$

Using Lemma 5 we get

$$\ln\psi(k) > \sum_{n=k}^\infty \ln\left(1 - \frac{E}{(1+c_n)^{\gamma-1-\eta}}\right) \;\; \forall k \geq n_1 .$$

Now, there exists $n_2 \geq n_1$ such that $\ln\left(1-\frac{E}{(1+c_n)^{\gamma-1-\eta}}\right) > -2\frac{E}{(1+c_n)^{\gamma-1-\eta}}$ — it suffices that $2\frac{E}{(1+c_n)^{\gamma-1-\eta}} < 0.7$ — which gives

$$\psi(k) > 1 - 2E\sum_{n=k}^{\infty}\frac{1}{(1+c_n)^{\gamma-1-\eta}} \qquad \forall k \geq n_2 .$$

From this

$$\psi(n) > 1 - \frac{2E}{b_n^{\gamma-1-\eta}((\beta/2-\delta)(\gamma-1-\eta)-1)(n-1)^{(\beta/2-\delta)(\gamma-1-\eta)-1}}$$

follows immediately, which completes the proof. ■

Now we will prove Proposition 4 page 7.

Proof of Proposition 4: We continue with the same notation and will show first by the induction with respect to k that for log-normal S

$$P\left(\sum_{i=1}^{k} S_i > x\right) \leq \frac{\sqrt{2}^k - 1}{\sqrt{2}-1}\exp\left(-\frac{(\ln x - m)^2}{4\sigma^2}\right) . \tag{10}$$

Indeed,

<u>for $k=1$</u> : the result holds because $\frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-y^2/2}dy < e^{-x^2/4}$.

<u>$k \to k+1$</u> **:** Assume now (10) for some k. Then

$$P\left(\sum_{i=1}^{k+1} S_i > x\right) < P(S > x) + \int_0^x f_S(s) P\left(\sum_{i=1}^{k} S_i > x - s\right) ds$$

Recall that $f_S(x) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left(-\frac{(\ln x - m)^2}{2\sigma^2}\right)$ and denoting $b = \min_{s\in[0,x]}\{(\ln s - m)^2 + (\ln(x-s) - m)^2\}$, we see that

$$P\left(\sum_{i=1}^{k+1} S_i > x\right) < P(S > x) + \frac{\sqrt{2}^k - 1}{\sqrt{2}-1} e^{-b/(4\sigma^2)} \int_0^x \frac{1}{\sqrt{2\pi}} e^{-s^2/4} ds .$$

We know that $\int_0^x \frac{1}{\sqrt{2\pi}} e^{-s^2/4} ds = \sqrt{2}$ and that b is attained for $s = x/2$. Finally, we can easily verify that $b \geq (\ln x - m)^2$. This proves (10) for $k+1$ and completes the induction.

Now, using (10)

$$P\left(\sum_{i=1}^{N_n} S_i > x\right) \leq \frac{e^{\sqrt{2}-1}}{\sqrt{2}-1}\exp\left(-\frac{(\ln x - m)^2}{4\sigma^2}\right) .$$

Following the same steps as in the proof of Proposition 3 we obtain

$$\psi(k) > 1 - 2\frac{e^{\sqrt{2}-1}}{\sqrt{2}-1}\sum_{n=k}^{\infty}\exp\left(-\frac{(\ln c_n - m)^2}{4\sigma^2}\right) ,$$

which completes the proof of Proposition 4. ■

B Notation

X	Position of the antenna in $\mathbb{R}^2$.
S	Power of the antenna.
T	Reception quality of the antenna.
M	Mark of the antenna $M = (S, T)$.
l	Attenuation function.
I_Φ	For $y \in \mathbb{R}^2$, $I_\Phi(y) = \sum_{(X,(S,T))\in\Phi} Sl(\|y - X\|)$.
Φ_t	Processus with indistinguishables antennas at the instant t, $\Phi_t = \{(X_i, M_i)\}_{i\in\{1,\dots,n\}}$.
$\widetilde{\Phi}_t$	Processus with distinguishables antennas at the instant t, $\Phi_t = \{(X_i, M_i, \tau_i)\}_{i\in\{1,\dots,n\}}$, $\tau_i(t)$ is the residual life time of the antenna (X_i, M_i).
$\overline{z}_i$	Conditionning points, $i \in \{1, \dots, p\}$, $\overline{z}_i$ must be covered at least $\overline{n}_i$ times.
$\underline{z}_i$	Conditionning points, $i \in \{1, \dots, q\}$, $\underline{z}_i$ must be covered at most $\underline{n}_i$ times.
$\overline{\mathsf{C}}$	Condition for the points $\{\overline{z}_i\}_{i\in\{1,\dots,p\}}$.
$\underline{\mathsf{C}}$	Condition for the points $\{\underline{z}_i\}_{i\in\{1,\dots,q\}}$.
$O(t_1, t_2)$	Driving element between t_1 and t_2.
$\mathcal{E}_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$	Evolution function with conditions $\overline{\mathsf{C}}\cup\underline{\mathsf{C}}$, $\mathcal{E}_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}(\widetilde{\Phi}_{t_1}, O(t_1, t_2)) = \widetilde{\Phi}_{t_2}$.
$\mathcal{E}_\emptyset$	Evolution function without conditions, $\mathcal{E}_\emptyset(\widetilde{\Phi}_{t_1}, O(t_1, t_2)) = \widetilde{\Phi}_{t_2}$.
Φ_t^+	Maximal process with indistinguishables antennas.
Φ_t^-	Minimal process with indistinguishables antennas.
$\widetilde{\Phi}_t^+$	Maximal process with distinguishables antennas.
$\widetilde{\Phi}_t^-$	Minimal process with distinguishables antennas.
$\overline{\mathsf{C}}_\alpha^+$	Birth condition for the maximal process for the conditionning points $\overline{z}_i$.
$\overline{\mathsf{C}}_\alpha^-$	Birth condition for the minimal process for the conditionning points $\overline{z}_i$.
$\overline{\mathsf{C}}_\Omega^+$	Death condition for the maximal process for the conditionning points $\overline{z}_i$.
$\overline{\mathsf{C}}_\Omega^-$	Death condition for the minimal process for the conditionning points $\overline{z}_i$.
$\underline{\mathsf{C}}_\alpha^+$	Birth condition for the maximal process for the conditionning points $\underline{z}_i$.
$\underline{\mathsf{C}}_\alpha^-$	Birth condition for the minimal process for the conditionning points $\underline{z}_i$.
$\underline{\mathsf{C}}_\Omega^+$	Death condition for the maximal process for the conditionning points $\underline{z}_i$.

$\underline{\mathsf{C}}_\Omega^-$	Death condition for the minimal process for the conditionning points $\underline{z}_i$.
$\mathcal{E}^+_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$	Evolution function for the maximal process with conditions $\overline{\mathsf{C}}_\alpha^+ \cup \underline{\mathsf{C}}_\alpha^+$ for the births, $\overline{\mathsf{C}}_\Omega^+ \cup \underline{\mathsf{C}}_\Omega^+$ for the Deaths, $\mathcal{E}^+_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}(\widetilde{\Phi}^+_{t_1}, O(t_1,t_2)) = \widetilde{\Phi}^+_{t_2}$.
$\mathcal{E}^-_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}$	Evolution function for the maximal process with conditions $\overline{\mathsf{C}}_\alpha^- \cup \underline{\mathsf{C}}_\alpha^-$ for the births, $\overline{\mathsf{C}}_\Omega^- \cup \underline{\mathsf{C}}_\Omega^-$ for the Deaths, $\mathcal{E}^-_{\overline{\mathsf{C}}\cup\underline{\mathsf{C}}}(\widetilde{\Phi}^-_{t_1}, O(t_1,t_2)) = \widetilde{\Phi}^-_{t_2}$.

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