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Abstract: We analyze the behavior of Generalized Processor Sharing (GPS) queues with heavy tailed service times. The model consists of coupled queues : each one receives an arrival stream of customers with inter-arrival time that are i.i.d and with service times that are subexponential. We calculate the exact stationary workload asymptotic of an individual flow for this model.

Key-words: subexponential random variables, heavy tail, integrated tail, Veraverbeke's theorem, GPS, fluid limit.

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Comportement asymptotique de files d'attentes suivant une discipline de type Generalized Processor Sharing sous des hypothèses sous-exponentielles

Résumé : Nous analysons le comportement de files d'attentes suivant une discipline de type Generalized Processor Sharing (GPS). Les arrivées de clients pour les différentes files d'attente sont des processus de renouvellement indépendants et les temps de service sont sous-exponentiels. Nous calculons alors l'asymptotique exacte de la charge de chacune des files.

Mots-clés : distributions sous-exponentielles, queues lourdes, distribution d'excès, Theorème de Veraverbeke, GPS, limite fluide.

1 Introduction

Empirical evidence of the presence of heavy tails in network traffic have stimulated the analysis of subexponential queueing systems [20]. The importance of scheduling in the presence of heavy tails was first recognized in [1]. The present paper specifically examines the effectiveness of Generalized Processor Sharing (GPS). As a design paradigm, GPS is at the heart of commonly-used scheduling algorithms for high-speed switches, such as Weighted Fair Queueing, see for instance [18], [19].
A basic approach in the analysis of long-tailed traffic phenomena is the use of fluid models with long-tailed arrival processes. We refer to [11] for a comprehensive survey on fluid queues with long-tailed arrival processes. See also [16] for an extensive list of references on subexponential queueing models.
In the present paper, we consider the Generalized Processor Sharing (GPS) discipline with subexponential service times. We determine the asymptotic behavior of the stationary workload processes $\mathbb{P}(W^i > x)$ for large x.
In section 2, we investigate the case of a $G/G/1$ queue. We prove that under stationarity and ergodicity assumptions, we can define a mean service rate for the queue and we prove that the scaled workload converges in some sense to a fluid limit.
In section 3, we describe the GPS discipline and we show how the results of previous section remains true in this context : each queue receives a mean service rate that depends only on the average rate of the different inputs.
In section 4, we sum up the basic properties of subexponential distributions. We use results of the paper [5] to show how to calculate the exact asymptotic. The main argument here, is the decomposition over the typical event introduced in [5].
In section 5, we describe the model and give the stochastic assumptions (GI/GI input and subexponential distributions). We then show how to calculate some constants of the problem that will appear in the asymptotic. We construct in fact the fluid limit of the system.
In the last section, we give the exact asymptotic of an individual queue. Thanks to the results of section 3, we show that the use of fluid limit is justified (what was pointed out by [13]).
The same model has been studied in [9] with instantaneous or fluid input, in the case of intermediately regularly varying distributions. Here, we assume that we can compare the different distribution functions with one distribution belonging to the subexponential class (which contains the intermediately regularly varying class). In [9], the authors assume the existence of a mean rate at which a flow would receive service when some other are saturate. Here, we introduce the same condition (we observe the behavior of the system when some queues saturate) and we prove the existence of this mean service rate. Moreover, we give an explicit algorithm that calculates the constants of the exact asymptotic formula of section 6 which seems to be only available for the case of 2 queues with Poisson input and regularly varying distributions (see last section of [9]).

2 Basic facts about $G/G/1$ queue

We define now the $G/G/1/\infty$ queue following the presentation of [6].
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a measurable flow $\{\theta_t\}, t \in \mathbb{R}$ such that $(\mathbb{P}, \{\theta_t\})$ is ergodic. Let N be a point process defined on $(\Omega, \mathcal{F})$. Assume N is simple and compatible with $\{\theta_t\}$. It is called the arrival process and its n-th point T_n is interpreted as the arrival time of

customer n. Recall the convention $T_n < T_{n+1} (n \in \mathbb{Z})$ and $T_0 \leq 0 < T_1$. The inter-arrival time between customers n and $n+1$ is

$$\tau_n = T_{n+1} - T_n, \quad n \in \mathbb{Z}.$$

It will be assumed that the intensity of the arrival process is finite

$$\lambda = \mathbb{E}[N(0,1]] < \infty.$$

Customer n carries a service time denoted by $\sigma_n \geq 0$, where the sequence $\{\sigma_n\}$ is assumed to be a sequence of marks of the arrival process.
Letting $\mathbb{P}_N^0$ be the Palm probability associated with $\mathbb{P}$ and N, we define the traffic intensity ρ by

$$\rho = \lambda \mathbb{E}_N^0[\sigma_0].$$

The sequence $\{(T_n, \sigma_n)\}, n \in \mathbb{Z}$ describes the G/G input. In this paragraph, we suppose that this input feeds a $1/\infty$ queueing station (1 server, ∞ capacity) with capacity c.
$W_c(t)$ will denote the amount of service remaining to be done by the server at time t. For the $G/G/1/\infty$ queue, the evolution of $W_c(t)$ between two successive arrivals is described by Lindley's equation :

$$W_c(t) = (W_c(T_n-) + \sigma_n - c(t - T_n))^+, \quad t \in [T_n, T_{n+1}). \tag{1}$$

Property 1. *Under the stability condition*

$$\rho < c,$$

there exists a unique finite workload process $\{W_c^s(t)\}, t \in \mathbb{R}$, *compatible with the flow* $\{\theta_t\}$, *and satisfying equation (1) for all* $t \in \mathbb{R}$. *This process is such that*

$$W_c^s(0) = \sup_{n \leq 0}(cT_n + \sum_{i=0}^{n} \sigma_i)^+. \tag{2}$$

2.1 Existence of mean service rate

We consider the general case of stationary ergodic input with traffic intensity $0 < \rho < \infty$.
Denote by $A(s,t)$ the amount of traffic generated during the time interval $(s,t]$:

$$\begin{aligned} A(s,t) &= \sum_n \sigma_n \mathbb{1}_{\{T_n \in (s,t]\}} \quad \text{for} \quad s > 0, \text{ and we take the following convention :} \\ A(0,t) &= A_0 + \sum_n \sigma_n \mathbb{1}_{\{T_n \in (0,t]\}}, \end{aligned}$$

with A_0 a positive finite random variable (initial workload). We have then

$$W_c(t) = \sup_{0 \leq s \leq t} \left[A(s,t) - c(t-s) \right].$$

Define then, the output of the system :

$$B_c(s,t) = A(s,t) - W_c(t),$$

and look at the quantity

$$r_c(t) = \frac{B_c(0,t)}{A(0,t)}$$

which is the ratio of what goes out of the system over what enters the system. We then have :

- if $\rho < c$, then by property 1 and a coupling argument, we know that $W_c(t) < \infty$, hence we have $r_c(t) \xrightarrow{t\to\infty} 1$;
- if $\rho > c$, we have this time $r_c(t) \xrightarrow{t\to\infty} \frac{c}{\rho}$;
- if $\rho = c$, we have for any $c > \epsilon > 0$, $W_c(t) \le W_{c-\epsilon}(t)$, hence:

$$1 \ge r_c(t) \ge r_{c-\epsilon}(t), \text{ and}$$
$$r_{c-\epsilon}(t) \xrightarrow{t\to\infty} \frac{c-\epsilon}{\rho}.$$

  And we have : $r_c(t) \xrightarrow{t\to\infty} 1$.

Hence we can sum up this to

$$r_c(t) \xrightarrow{t\to\infty} 1 \wedge \frac{c}{\rho} \quad \mathbb{P}-a.s. \tag{3}$$

Define now

$$\hat{r}_c(t) = \frac{1}{t}\int_0^t c\mathbb{1}_{\{W_c(u)>0\}}du.$$

We have of course

$$B_c(s,t) = \int_s^t c\mathbb{1}_{\{W_c(u)>0\}}du,$$

hence

$$\hat{r}_c(t) = \frac{A(0,t)}{t} r_c(t).$$

And by ergodicity of the input process, we get

$$\hat{r}_c(t) = \frac{1}{t}\int_0^t c\mathbb{1}_{\{W_c(u)>0\}}du \xrightarrow{t\to\infty} \rho \wedge c \quad \mathbb{P}-a.s. \tag{4}$$

Moreover, it is easy to see that the limits (3) and (4) are uniform in the initial condition A_0 on a compact set.

2.2 Fluid limit

We note D the set of càdlàg functions on $\mathbb{R}_+$.
We define the following bijection :

$$\begin{array}{rccc} H: & D & \to & D \\ & f & \mapsto & Hf \end{array} \quad \text{with}$$

$$Hf(t) = \begin{cases} f(t) & t \leq 1, \\ \frac{f(t)}{t} & t > 1. \end{cases}$$

We define the following set of functions :

$$\begin{aligned} \mathcal{E}_l &= \left\{ f \in D,\ Hf(t) \xrightarrow{t\to\infty} l \right\}, \\ \mathcal{E} &= \bigcup_{l<\infty} \mathcal{E}_l. \end{aligned}$$

We have the natural norm on $\mathcal{E}$ defined as follows :

$$\begin{aligned} \|f\|_{\mathcal{E}} &= \|Hf\|_\infty \quad \text{with} \\ \|g\|_\infty &= \sup_{0\leq t} |g(t)|. \end{aligned}$$

We have of course for $f \in \mathcal{E}$: $\|f\|_{\mathcal{E}} < \infty$. Moreover $H^{-1}: (\mathcal{E}, \|\ \|_\infty) \to (\mathcal{E}, \|\ \|_{\mathcal{E}})$ is clearly continuous.
For any l the set $\mathcal{E}_l$ is a closed subset of D. If $f_n \underset{\|\ \|_{\mathcal{E}}}{\xrightarrow{n\to\infty}} f$ and $\forall n,\ f_n \in \mathcal{E}_l$, then $Hf(t) \xrightarrow{t\to\infty} l$.
Moreover $(D, \|\ \|_\infty)$ is a complete metric space and with the remark made before about H^{-1}, we see that any Cauchy sequence in $\mathcal{E}_l$ converges in D. Hence $(\mathcal{E}_l, \|\ \|_{\mathcal{E}})$ is a complete metric space.
Moreover, we have this very easy lemma :

Lemma 1. *The norm $\|\ \|_{\mathcal{E}}$ is continuous for the product.*

Proof :
Consider

$$\begin{aligned} f_n &\underset{\|\ \|_{\mathcal{E}}}{\xrightarrow{n\to\infty}} f, \\ g_n &\underset{\|\ \|_{\mathcal{E}}}{\xrightarrow{n\to\infty}} g. \end{aligned}$$

We have then :

$$\begin{aligned} \|f_n g_n - fg\|_{\mathcal{E}} &= \|f_n(g_n - g) + g(f_n - f)\|_{\mathcal{E}} \\ &\leq \|f_n(g_n - g)\|_{\mathcal{E}} + \|g(f_n - f)\|_{\mathcal{E}} \\ &= \|f_n\|_{\mathcal{E}}\|g_n - g\|_{\mathcal{E}} + \|g\|_{\mathcal{E}}\|f_n - f\|_{\mathcal{E}}. \end{aligned}$$

$\triangle$
Consider the following function:

$$\begin{array}{rccc} \phi: & (\mathcal{E}, \|\ \|_{\mathcal{E}}) & \to & (\mathcal{E}, \|\ \|_{\mathcal{E}}) \\ & f & \mapsto & \phi f \end{array} \quad \text{with}$$

$$\phi f(t) = \sup_{0\leq s\leq t} [f(t) - f(s) - c(t-s)].$$

Lemma 2. *The function ϕ is continuous.*

Proof :
We have :

$$\|f-g\|_{\mathcal{E}} \leq \eta \Leftrightarrow \begin{cases} t \leq 1 & |f(t)-g(t)| \leq \eta \\ t > 1 & |\frac{f(t)}{t} - \frac{g(t)}{t}| \leq \eta \end{cases}$$

Hence for such functions, we have the following inequalities :

$$\begin{array}{llll} s \leq t \leq 1 & |f(t)-f(s)-(g(t)-g(s))| & \leq 2\eta, & \\ 1 \leq s \leq t & |\frac{f(t)}{t} - \frac{f(s)}{t} - (\frac{g(t)}{t} - \frac{g(s)}{t})| & \leq |\frac{f(t)}{t} - \frac{g(t)}{t}| + |\frac{f(s)}{s} - \frac{g(s)}{s}| & \leq 2\eta, \\ s \leq 1 \leq t & |\frac{f(t)}{t} - \frac{f(s)}{t} - (\frac{g(t)}{t} - \frac{g(s)}{t})| & \leq |\frac{f(t)}{t} - \frac{g(t)}{t}| + \frac{1}{t}|f(s)-g(s)| & \leq 2\eta. \end{array}$$

The continuity of ϕ is then clear.
△
Define now the scaling function :

$$\begin{array}{rccc} S_z : & (\mathcal{E}, \| \, \|_{\mathcal{E}}) & \rightarrow & (\mathcal{E}, \| \, \|_{\mathcal{E}}) \\ & f & \mapsto & S_z f \end{array} \quad \text{with}$$

$$S_z f(t) = \frac{f(zt)}{z}.$$

Denote by e_l^x the following function belonging to $\mathcal{E}_l$:

$$\begin{array}{rcl} e_l^x(0) & = & x, \\ e_l^x(t) & = & lt \quad \text{for } t > 0. \end{array}$$

Then we have

Lemma 3. *Let $\{f_n\}$ be a sequence of functions in $\mathcal{E}_l$ such that :*

$$\begin{array}{rcl} f_n(t) & = & f(t) \in \mathcal{E}_l \quad \text{for } t > 0, \\ \lim_{n\to\infty} \frac{f_n(0)}{n} & = & f_0 < +\infty. \end{array}$$

Then the following convergence holds :

$$S_n(f_n) \xrightarrow[\| \, \|_{\mathcal{E}}]{n\to\infty} e_l^{f_0}.$$

Proof :
We have :

$$\begin{array}{rcl} \|S_n(f_n) - e_l^{f_0}\|_{\mathcal{E}} & = & \max\left[\left|\frac{f_n(0)}{n} - f_0\right| ; \sup_{t\leq 1}\left|\frac{f(nt)}{n} - lt\right| ; \sup_{t>1}\left|\frac{f(nt)}{nt} - l\right|\right] \\ & = & \max\left[\left|\frac{f_n(0)}{n} - f_0\right| ; \sup_{t\leq n}\frac{1}{n}\left|f(t) - lt\right| ; \sup_{t>n}\left|\frac{f(t)}{t} - l\right|\right]. \end{array}$$

Considering the two last terms of the max we have for any $T \leq n$:

$$\max\left[\sup_{t\leq n}\frac{1}{n}\left|f(t)-lt\right| ; \sup_{t>n}\left|\frac{f(t)}{t} - l\right|\right] \leq \max\left[\frac{1}{n}\sup_{t\leq T}\left|f(t)-lt\right| ; \sup_{t>T}\left|\frac{f(t)}{t} - l\right|\right].$$

Hence the three terms in the max can be as small as we want provided that n is large enough.
$\triangle$
Take for the function f :

$$\begin{aligned} f(0) &= -A_0, \\ f(t) &= \sum_n \sigma_n \mathbb{1}_{\{T_n \in (0,t]\}} = \lim_{s \to 0} A(s,t) \quad \text{for} \quad t > 0. \end{aligned}$$

Thanks to the ergodic theorem, we have :

$$\frac{f(t)}{t} \xrightarrow{t \to \infty} \rho \quad \mathbb{P}\text{-a.s.}$$

We have then :

$$W_c(t) = \sup_{0 \le s \le t} [f(t) - f(s) - c(t-s)] = \phi f(t).$$

Given any sequence of initial conditions $\{A_{0,n}\}$ such that

$$\frac{A_{0,n}}{n} \xrightarrow{n \to \infty} A_0 \quad \mathbb{P}\text{-a.s.,}$$

we can define the scaled workload of a $G/G/1$ queue:

$$W^s_{c,n}(t) = \frac{W_c(nt)}{n}.$$

We have then :

$$\begin{aligned} W^s_{c,n} &\xrightarrow[\| \|_\mathcal{E}]{n \to \infty} W^s_c \quad \mathbb{P}\text{-a.s. with} \\ & W^s_c(t) = (A_0 + (\rho - c)t)^+ . \end{aligned}$$

Moreover, if the stability condition holds $\rho < c$, then we have :

$$W^s_{c,n} \xrightarrow[\| \|_\infty]{n \to \infty} W^s_c \quad \mathbb{P}\text{-a.s.}$$

Note $T(n)$ the first positive time at which the queue empties. We have of course

$$\frac{T(n)}{n} \xrightarrow{n \to \infty} \frac{A_0}{c - \rho} \quad \mathbb{P}\text{-a.s.}$$

Moreover after $T(n)$ the workload of the queue remains in a compact set, hence :

$$\forall \epsilon > 0, \, \forall t \ge \frac{A_0}{c-\rho} + \epsilon, \quad \frac{W_c(nt)}{n} \le \frac{M}{n}.$$

2.3 More general results

In this section, we want to derive the same results as the both preceding sections but for a capacity varying queue.
Suppose that the capacity of the server is not constant but is a random process $c(t)$. Denote by $C(s,t)$ the cumulative capacity of the server on the time interval $(s,t]$, namely :

$$C(s,t) = \int_{(s,t]} c(u)du.$$

Assume that the process $c(u)$ is stationary (i.e. $c(u) = c(0) \circ \theta_u$). Since $(\mathbb{P}, \{\theta_t\})$ is ergodic, we can define

$$c = \lim_{t\to\infty} \frac{C(0,t)}{t} = \mathbb{E}[c(0)].$$

We have now

$$\begin{aligned} W_C(t) &= \sup_{0\le s\le t} \left[A(s,t) - C(s,t)\right], \quad \text{and we define as before} \\ B_C(s,t) &= A(s,t) - W_C(t), \\ r_C(t) &= \frac{S_C(0,t)}{A(0,t)}, \quad \text{and} \\ \hat{r}_C(t) &= \frac{1}{t}\int_0^t c(u)\mathbb{1}_{\{W_C(u)>0\}}du. \end{aligned}$$

And we have the same result as before. Consider the process:

$$A'(s,t) = \left[A(s,t) - C(s,t) + c(t-s)\right].$$

This process is stationary :

$$\begin{aligned} A'(s,t)\circ\theta_v &= \left[A(s,t)\circ\theta_v - C(s,t)\circ\theta_v + c(t-s)\right] \\ &= \left[A(s+v,t+v) - C(s+v,t+v) + c(t+v-(s+v))\right] \\ &= A'(s+v,t+v). \end{aligned}$$

and of intensity ρ,

$$\mathbb{E}[A'(0,1)] = \lim_{t\to\infty} \frac{A'(0,t)}{t} = \rho.$$

We have with $W_C(t) = \sup_{0\le s\le t}\left[A'(s,t) - c(t-s)\right]$, hence $r_C(t) \xrightarrow{t\to\infty} 1 \wedge \frac{c}{\rho}$.
We conclude as before that

$$\hat{r}_c(t) = \frac{1}{t}\int_0^t c(u)\mathbb{1}_{\{W_C(u)>0\}}du \xrightarrow{t\to\infty} \rho\wedge c \quad \mathbb{P}-a.s. \tag{5}$$

Moreover, defining as in previous section :

$$\begin{aligned} f(0) &= -A_0, \\ f(t) &= \lim_{s\to 0} A'(s,t) \quad \text{for} \quad t>0. \end{aligned}$$

We have $W_C(t) = \phi f(t)$. Hence, we proved

Property 2. *Consider a $1/\infty$ queueing station with capacity that is a random variable which couples with $C \circ \theta_t$ for some random variable C of mean $\mathbb{E}[C] = c < \infty$ and that is fed with an input process that couples with a G/G input process (stationary and ergodic) of intensity ρ, then there exists a mean service rate : $\gamma = \rho \wedge c$.*
Given any sequence of initial conditions $\{A_{0,n}\}$ such that

$$\frac{A_{0,n}}{n} \xrightarrow{n\to\infty} A_0 \quad \mathbb{P}\text{-}a.s.,$$

we can define the scaled workload :

$$W^s_{C,n}(t) \quad = \quad \frac{W_C(nt)}{n}.$$

We have then :

$$W^s_{C,n} \quad \xrightarrow[\|\ \|_{\mathcal{E}}]{n\to\infty} \quad W^s_C \quad \mathbb{P}\text{-}a.s. \quad with$$

$$W^s_C(t) = (A_0 + (\rho - c)t)^+ .$$

3 The GPS discipline : stability issues

Consider the following model of N coupled G/G/1 queues, $Q_1, \dots, Q_N$. Each queue is served in accordance with the Generalized Processor Sharing (GPS) discipline, which operates as follows. Queue j is assigned a weight ϕ^j, with $\sum_{j=1}^N \phi^j = 1$. If all the queues are backlogged, then queue j is served at speed ϕ^j. If some of the queues are empty, then the excess capacity is redistributed among the backlogged queues in proportion to their respective weights.
Denote by $A^j = \{T^j_n, \sigma^j_n\}$ the input process of queue j. We suppose that $(A^j, \theta_t, \mathbb{P})$ are N stationary point processes of finite intensity λ^j. Note that the A^j are jointly stationary, in the sense that their stationarity is relative to the same quadruplet $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$. Denote by $\rho^j = \lambda^j \mathbb{E}^0_j[\sigma^j_0]$ the traffic intensity of queue j.
We assume the queues are indexed in such a way that

$$\frac{\rho^1}{\phi^1} \leq \dots \leq \frac{\rho^N}{\phi^N}. \tag{6}$$

Define

$$\begin{aligned}
R_k &= \frac{1 - \sum_{j=1}^{k-1} \rho^j}{\sum_{j=k}^N \phi^j}, \\
K &= \max_{k=1,\dots,N} \left\{ \frac{\rho^k}{\phi^k} < R_k \right\}, \\
S &= \{1, \dots, K\}, \\
R &= \frac{1}{\sum_{j \notin S} \phi^j} \left(1 - \sum_{j \in S} \rho^j \right).
\end{aligned}$$

As we will see (property 3), S is the set of queues that are not asymptotically backlogged.
For any finite initial condition $\mathcal{Y} = (Y^1, \dots, Y^N)$, we can construct workload processes $W^j_{\mathcal{Y}}(t)$

, $j = 1, \dots, N$, that are compatible with θ_t for $t > 0$, fed by A^j and such that

$$\begin{aligned} W^j_{\mathcal{Y}}(0) &= Y^j \geq 0, \\ \frac{dW^j_{\mathcal{Y}}}{dt}(t) &= -1\!\!1_{\{W^j_{\mathcal{Y}}(t)>0\}} r^j_{\mathcal{Y}}(t) \quad \text{with} \\ I_{\mathcal{Y}}(t) &= \{i; W^i_{\mathcal{Y}}(t) = 0\}, \\ r^j_{\mathcal{Y}}(t) &= \begin{cases} \frac{\phi^j}{\sum_{k \notin I_{\mathcal{Y}}(t)} \phi^k} & j \in (I_{\mathcal{Y}}(t))^c \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Definition. *We will say that a queue j is stable if we can find a finite random variable M and a sequence of i.i.d random variables $\{X^j_i\}$ such that :*

$$\begin{aligned} \mathbb{E}[X^j_1] &< 0, \\ S^j_n &= \sum_{i=1}^{n} X^j_i, \quad S^j_0 = 0 \\ S^{j,*}_n &= \max_{i \leq n} S^j_i, \\ W^j_{\mathcal{Y}}(T^j_n) &\leq S^{j,*}_n \quad \forall n \geq M. \end{aligned}$$

Property 3. *For any finite initial condition $\mathcal{Y}$, the processes $\{W^j_{\mathcal{Y}}(t)\}, j \in S$ are stable. And we have the following limits uniformly in $\{Y^j, j \in S\}$ on a compact set :*

$$j \in S \;\Rightarrow\; \frac{1}{t}\int_0^t r^j_{\mathcal{Y}}(u)du \xrightarrow{t\to\infty} \rho^j \quad \mathbb{P}\text{-}a.s. \tag{7}$$

$$j \notin S \;\Rightarrow\; \frac{1}{t}\int_0^t r^j_{\mathcal{Y}}(u)du \xrightarrow{t\to\infty} \phi^j R \quad \mathbb{P}\text{-}a.s. \tag{8}$$

For the study of GPS discipline with heavy-tailed service times, we will need a little more general result. We must consider the system when some queues are backlogged for a very long period of time. Hence we study the following system :

Saturation of $\Delta \subset [1, N]$.

We consider now that the queues $i \in \Delta$ claim continuously their full share of the bandwidth (they are always backlogged).
We suppose that $|\Delta| = N - n$ and we note $[1, N]\backslash\Delta = \{a_1, a_2, \dots, a_n\}$. Let $\mathcal{Y}^{(\Delta)} = (Y^{a_1}, \dots, Y^{a_n})$ be a vector with positive coordinates. We will denote for simplicity $\mathcal{Y} \equiv \mathcal{Y}^{(\Delta)}$. Of course thanks to our ordering (6), we have :

$$\frac{\rho^{a_1}}{\phi^{a_1}} \leq \dots \leq \frac{\rho^{a_n}}{\phi^{a_n}}.$$

We can construct the workload processes $W_{\mathcal{Y}}^{j,(\Delta)}(t)$ for $j \notin \Delta$ that are compatible with θ_t for $t > 0$, fed by A^j, and such that

$$\begin{aligned} W_{\mathcal{Y}}^{j,(\Delta)}(0) &= Y^j \geq 0, \\ \frac{dW_{\mathcal{Y}}^{j,(\Delta)}}{dt}(t) &= -r_{\mathcal{Y}}^{j,(\Delta)}(t) \quad \text{with} \\ I_{\mathcal{Y}}^{(\Delta)}(t) &= \{i \notin \Delta; W_{\mathcal{Y}}^{i,(\Delta)}(t) = 0\}, \\ r_{\mathcal{Y}}^{j,(\Delta)}(t) &= \begin{cases} \frac{\phi^j}{\sum_{k \notin I_{\mathcal{Y}}^{(\Delta)}(t)} \phi^k + \sum_{k\in\Delta} \phi^k} & j \in (I_{\mathcal{Y}}^{(\Delta)}(t))^c \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Remark 1. We extend the definition of $r_{\mathcal{Y}}^{j,(\Delta)}$ for $j \in \Delta$ by:

$$r_{\mathcal{Y}}^{j,(\Delta)}(t) = \frac{\phi^j}{\sum_{k \notin I_{\mathcal{Y}}^{(\Delta)}(t)} \phi^k + \sum_{k\in\Delta} \phi^k}.$$

We have of course:

$$\sum_i r_{\mathcal{Y}}^{i,(\Delta)}(t) = 1 \quad \forall t > 0.$$

♣

Define

$$\begin{aligned} R_k^{(\Delta)} &= \frac{1 - \sum_{j=1}^{k-1} \rho^j \mathbb{1}_{\{j\notin\Delta\}}}{\sum_{j=k}^N \phi^j \mathbb{1}_{\{j\notin\Delta\}} + \sum_{j\in\Delta} \phi^j}, \\ K^{(\Delta)} &= \max_{k=1,\dots,N} \left\{ \frac{\rho^k}{\phi^k} < R_k^{(\Delta)} \right\}, \\ S^{(\Delta)} &= \{1, \dots, K^{(\Delta)}\} \backslash \Delta, \\ U^{(\Delta)} &= \left(S^{(\Delta)}\right)^c \backslash \Delta, \\ R^{(\Delta)} &= \frac{1}{\sum_{j\notin S^{(\Delta)}} \phi^j} \left(1 - \sum_{j\in S^{(\Delta)}} \rho^j \right). \end{aligned}$$

Property 4. *For any finite initial condition $\mathcal{Y}$, the processes $\{W_{\mathcal{Y}}^{j,(\Delta)}(t)\}, j \in S^{(\Delta)}$ are stable, and we have :*

$$\forall j \in S^{(\Delta)}, \quad \frac{1}{t}\int_0^t r_{\mathcal{Y}}^{j,(\Delta)}(u)du \xrightarrow{t\to\infty} \rho^j \quad \mathbb{P}\text{-}a.s. \tag{9}$$

uniformly for $\{Y^j, j \in S^{(\Delta)}\}$ on a compact set.

Proof :
We have :

$$S^{(\Delta)} \neq \emptyset \Leftrightarrow \frac{\rho^{a_1}}{\phi^{a_1}} < R_{a_1}^{(\Delta)} = 1.$$

But we have $R_{a_1}^{(\Delta)} = 1$, hence :

$$\frac{\rho^{a_1}}{\phi^{a_1}} < 1. \tag{10}$$

Saturation of queues $i \neq a_1$.
Define the following $G/G/1$ workload process

$$\tilde{W}^{a_1,(\Delta)}(t) = (\tilde{W}^{a_1,(\Delta)}(T_n^{a_1}-) + \sigma_n^{a_1} - \phi^{a_1}(t - T_n^{a_1}))^+, \quad \text{for} \quad t \in [T_n^{a_1}, T_{n+1}^{a_1}). \tag{11}$$

Thanks to (10), we see that the stability condition holds for this process and we denote $\tilde{W}^{a_1,(\Delta)} \circ \theta_t$ the unique finite workload process compatible with the flow $\{\theta_t\}$ and satisfying equation (11). Moreover, we clearly have

$$W_{\mathcal{Y}}^{a_1,(\Delta)}(t) \leq \tilde{W}_{\mathcal{Y}}^{a_1,(\Delta)}(t),$$

and $\{\tilde{W}_{\mathcal{Y}}^{a_1,(\Delta)}(t)\}$ couples $\mathbb{P}$-almost surely with the stochastic process $\{\tilde{W}^{a_1,(\Delta)} \circ \theta_t\}$. Hence, we proved that $\{W_{\mathcal{Y}}^{a_1,(\Delta)}(t)\}$ is stable. Thanks to the results of previous section, we have (uniformly in Y^{a_1} on a compact set)

$$\frac{1}{t}\int_0^t r_{\mathcal{Y}}^{a_1,(\Delta)}(u)du \xrightarrow{t\to\infty} \rho^{a_1}.$$

Saturation of queues $i \notin \{a_1, a_2\}$.
Define the workload process

$$\tilde{W}^{a_2,(\Delta)}(t) = \left[\tilde{W}^{a_2,(\Delta)}(T_n^{a_2}-) + \sigma_n^{a_2} - \phi^{a_2}(t - T_n^{a_2}) - \frac{\phi^{a_1}\phi^{a_2}}{\sum_{j\neq a_1}\phi^j}\int_{[T_n^{a_2},t)} \mathbb{1}_{\{\tilde{W}_{\mathcal{Y}}^{a_1,(\Delta)}(u)=0\}}du\right]^+,$$
$$\text{for} \quad t \in [T_n^{a_2}, T_{n+1}^{a_2}). \tag{12}$$

There exists a random variable $T < \infty$ such that :

$$\forall u \geq T, \quad \tilde{W}_{\mathcal{Y}}^{a_1,(\Delta)}(u) = \tilde{W}^{a_1,(\Delta)} \circ \theta_u.$$

Hence if we only consider $t \geq T$, we can change $\tilde{W}_{\mathcal{Y}}^{a_1,(\Delta)}(u)$ into $\tilde{W}^{a_1,(\Delta)} \circ \theta_u$.
Of course T depends on Y^{a_1} and is an increasing function of Y^{a_1}. But as $Y^{a_1} \in K$, with K a compact set, we take T as the supremum of the coupling times over $Y^{a_1} \in K$.
We denote then

$$X_n^2 = \sigma_n^{a_2} - \phi^{a_2}(T_{n+1}^{a_2} - T_n^{a_2}) - \frac{\phi^{a_1}\phi^{a_2}}{\sum_{j\geq a_2}\phi^j}\int_{[T_n^{a_2},T_{n+1}^{a_2})} \mathbb{1}_{\{\tilde{W}^{a_1,(\Delta)}\circ\theta_u=0\}}du$$

and we have

$$\begin{aligned}
\mathbb{E}_{A^{a_2}}^0\left[X_n^2\right] &= b^{a_2} - \frac{\phi^{a_2}}{\lambda^{a_2}} - \frac{\phi^{a_1}\phi^{a_2}}{\sum_{j\neq a_1}\phi^j}\mathbb{E}_{A^{a_2}}^0\left[\int_{[0,T_1^{a_2})} \mathbb{1}_{\{\tilde{W}^{a_1,(\Delta)}(t)=0\}}dt\right] \\
&= b^{a_2} - \frac{\phi^{a_2}}{\lambda^{a_2}} - \frac{\phi^{a_2}}{\sum_{j\neq a_1}\phi^j}\frac{\phi^{a_1}}{\lambda^{a_2}}\mathbb{P}[\tilde{W}^{a_1,(\Delta)} = 0] \\
&= b^{a_2} - \frac{\phi^{a_2}}{\lambda^{a_2}} - \frac{\phi^{a_2}}{\sum_{j\neq a_1}\phi^j}\frac{\phi^{a_1} - \rho^{a_1}}{\lambda^{a_2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}\lambda^{a_2}\mathbb{E}^0_{A^{a_2}}\left[X_n^2\right] &= \rho^{a_2}-\phi^{a_2}-\frac{\phi^{a_2}}{\sum_{j\neq a_1}\phi^j}(\phi^{a_1}-\rho^{a_1})\\ &= \rho^{a_2}-\phi^{a_2}\frac{1-\rho^{a_1}}{\sum_{j\neq a_1}\phi^j},\end{aligned}$$

and if $a_2\in S^{(\Delta)}$, we have

$$\frac{\rho^{a_2}}{\phi^{a_2}}<\frac{1-\rho^{a_1}}{\sum_{j\neq a_1}\phi^j},$$

we have :

$$\lambda^{a_2}\mathbb{E}^0_{A^{a_2}}\left[X_n^2\right]<0.$$

Hence, the stability condition holds for this process too and we denote $\tilde{W}^{a_2,(\Delta)}\circ\theta_t$ the unique finite workload process compatible with the flow $\{\theta_t\}$ and satisfying equation (12). We have for sufficiently large t

$$W^{a_2,(\Delta)}_{\mathcal{Y}}(t)\quad\leq\quad\tilde{W}^{a_2,(\Delta)}_{\mathcal{Y}}(t),$$

and $\{\tilde{W}^{a_2,(\Delta)}_{\mathcal{Y}}(t)\}$ couples $\mathbb{P}$-almost surely with the stochastic process $\{\tilde{W}^{a_2,(\Delta)}\circ\theta_t\}$. Hence, we proved that $\{W^{a_2,(\Delta)}_{\mathcal{Y}}(t)\}$ is stable. And it is easy to see that uniformly for (Y^{a_1},Y^{a_2}) on a compact set

$$\frac{1}{t}\int_0^t r^{a_2,(\Delta)}_{\mathcal{Y}}(u)du\xrightarrow{t\to\infty}\rho^{a_2}.$$

An other way to define the process $\tilde{W}^{a_2,(\Delta)}(t)$ is as follow (we have shown that we can take $\tilde{W}^{i,(\Delta)}\circ\theta_t$ instead of $\tilde{W}^{i,(\Delta)}_{\mathcal{Y}}(t)$)

$$\begin{aligned}I_2^{(\Delta)}(t) &= \{i\in\{a_1\};\tilde{W}^{i,(\Delta)}\circ\theta_t=0\},\\ r_2^{a_2,(\Delta)}(t) &= \frac{\phi^{a_2}}{\sum_{j\notin I_2^{(\Delta)}(t)}\phi^j},\\ \tilde{W}^{a_2,(\Delta)}(t) &= \left[\tilde{W}^{a_2,(\Delta)}(T_n^{a_2}-)+\sigma_n^{a_2}-\int_{[T_n^{a_2},t)}r_2^{a_2,(\Delta)}(u)du\right]^+,\quad\text{for}\quad t\in[T_n^{a_2},T_{n+1}^{a_2}).\end{aligned}$$

And then we have :

$$\begin{aligned}X_n^2 &= \sigma_n^{a_2}-\int_{[T_n^{a_2},T_{n+1}^{a_2})}r_2^{a_2,(\Delta)}(u)du\\ \lambda^{a_2}\mathbb{E}^0_{A^{a_2}}\left[X_n^2\right] &= \rho^{a_2}-\frac{\phi^{a_2}}{\sum_{j\neq a_1}\phi^j}\mathbb{P}[\tilde{W}^{a_1,(\Delta)}\circ\theta_t=0]-\phi^{a_2}\mathbb{P}[\tilde{W}^{a_1,(\Delta)}\circ\theta_t>0]\\ &= \rho^{a_2}-\frac{\phi^{a_2}}{\sum_{j\neq a_1}\phi^j}(1-\frac{\rho^{a_1}}{\phi^{a_1}})-\phi^{a_2}\frac{\rho^{a_1}}{\phi^{a_1}}\\ &= \rho^{a_2}-\phi^{a_2}\frac{1-\rho^{a_1}}{1-\phi^{a_1}}.\end{aligned}$$

And we conclude as before.
We now prove the property by induction on k.
Let

$$H_{k-1} = \begin{cases} \text{For any finite initial condition } \mathcal{Y}, \text{ for any } j \in \{a_1, \ldots, a_{k-1}\} \quad \exists\{\tilde{W}^{j,(\Delta)}_{\mathcal{Y}}(t)\} \quad \text{such that} \\ W^{j,(\Delta)}_{\mathcal{Y}}(t) \leq \tilde{W}^{j,(\Delta)}_{\mathcal{Y}}(t) \quad \text{for sufficiently large } t, \text{ and,} \\ \tilde{W}^{j,(\Delta)}_{\mathcal{Y}}(t) \quad \text{couples } \mathbb{P}\text{-almost surely with a stochastic process } \{\tilde{W}^{j,(\Delta)} \circ \theta_t\} \text{ which is stable;} \\ \text{Moreover} \quad \mathbb{E}\left[\frac{1}{\sum_{j \notin I^{(\Delta)}_{k-1}(t)} \phi^j}\right] = R^{(\Delta)}_{a_{k-1}} \quad \text{with} \quad I^{(\Delta)}_{k-1}(t) = \{i \in \{a_1, a_2, \ldots, a_{k-2}\}; \tilde{W}^{i,(\Delta)} \circ \theta_t = 0\} \end{cases}$$

We show that

$$\begin{cases} H_{k-1} \\ a_k \in S^{(\Delta)} \end{cases} \Rightarrow H_k.$$

Saturation of queues $i \notin \{a_1, a_2, \ldots, a_k\}$.
Define the workload process as follow

$$\begin{aligned} I^{(\Delta)}_k(t) &= \{i \in \{a_1, a_2, \ldots, a_{k-1}\}; \tilde{W}^{i,(\Delta)} \circ \theta_t = 0\}, \\ r^{a_k,(\Delta)}_k(t) &= \frac{\phi^{a_k}}{\sum_{j \notin I^{(\Delta)}_k(t)} \phi^j}, \\ \tilde{W}^{a_k,(\Delta)}(t) &= \left[\tilde{W}^{a_k,(\Delta)}(T^{a_k}_n -) + \sigma^{a_k}_n - \int_{[T^{a_k}_n, t)} r^{a_k,(\Delta)}_k(u) du\right]^+, \quad \text{for} \quad t \in [T^{a_k}_n, T^{a_k}_{n+1}). \end{aligned} \tag{13}$$

We define

$$X^k_n = \sigma^{a_k}_n - \int_{[T^{a_k}_n, T^{a_k}_{n+1})} r^{a_k,(\Delta)}_k(u) du,$$

and we have to calculate :

$$\lambda^{a_k} \mathbb{E}^0_{A^k}[X^k_n] = \rho^{a_k} - \phi^{a_k} \mathbb{E}\left[\frac{1}{\sum_{j \notin I^{(\Delta)}_k(t)} \phi^j}\right].$$

But the following property is clear :

$$\underbrace{\mathbb{E}\left[\frac{\sum_{j \notin \{a_1, a_2, \ldots, a_{k-1}\}} \phi^j}{\sum_{j \notin I^{(\Delta)}_k(t)} \phi^j}\right]}_{\text{bandwidth of the saturated queues } + a_k} = \underbrace{\mathbb{E}\left[\frac{\sum_{j \notin \{a_1, a_2, \ldots, a_{k-1}\}} \phi^j}{\sum_{j \notin I^{(\Delta)}_{k-1}(t)} \phi^j}\right]}_{\text{bandwidth of the saturated queues at the preceding step}} + \underbrace{\mathbb{E}\left[\frac{\phi^{a_{k-1}}}{\sum_{j \notin I^{(\Delta)}_{k-1}(t)} \phi^j}\right] - \rho^{a_{k-1}}}_{\text{profit due to stable queue } a_{k-1}}.$$

Hence, we have

$$\begin{aligned} (1 - (\phi^{a_1} + \cdots + \phi^{a_{k-1}})) \mathbb{E}\left[\frac{1}{\sum_{j \notin I^{(\Delta)}_k(t)} \phi^j}\right] &= R^{(\Delta)}_{a_{k-1}} \left(\sum_{j \notin \{a_1, a_2, \ldots, a_{k-1}\}} \phi^j + \phi^{a_{k-1}}\right) - \rho^{a_{k-1}} \\ &= 1 - (\rho^{a_1} + \cdots + \rho^{a_k}). \end{aligned}$$

Then

$$\mathbb{E}\left[\frac{1}{\sum_{j\notin I_k^{(\Delta)}(t)}\phi^j}\right] = R_{a_k}^{(\Delta)},$$

and then the stability condition holds if $a_k < R_{a_k}^{(\Delta)} \Leftrightarrow a_k \in S^{(\Delta)}$. We denote $\tilde{W}^{a_k,(\Delta)} \circ \theta_t$ the unique finite workload process compatible with the flow $\{\theta_t\}$ and satisfying equation (13). We have for sufficiently large t

$$W_{\mathcal{Y}}^{a_k,(\Delta)}(t) \leq \tilde{W}_{\mathcal{Y}}^{a_k,(\Delta)}(t),$$

and $\{\tilde{W}_{\mathcal{Y}}^{a_k,(\Delta)}(t)\}$ couples $\mathbb{P}$-almost surely with the stochastic process $\{\tilde{W}^{a_k,(\Delta)} \circ \theta_t\}$. Hence, we proved that $\{W_{\mathcal{Y}}^{a_k,(\Delta)}(t)\}$ is stable. And it is easy to see that uniformly for $\mathcal{Y}$ on a compact set

$$\frac{1}{t}\int_0^t r_{\mathcal{Y}}^{a_k,(\Delta)}(u)du \xrightarrow{t\to\infty} \rho^{a_k}.$$

△

We can now prove

Property 5. *We have the following limits uniformly in $\{Y^j, j\in S^{(\Delta)}\}$ on a compact set :*

$$j \in S^{(\Delta)} \Rightarrow \frac{1}{t}\int_0^t r_{\mathcal{Y}}^{j,(\Delta)}(u)du \xrightarrow{t\to\infty} \rho^j \quad \mathbb{P}\text{-}a.s. \tag{14}$$

$$j \notin S^{(\Delta)} \Rightarrow \frac{1}{t}\int_0^t r_{\mathcal{Y}}^{j,(\Delta)}(u)du \xrightarrow{t\to\infty} \phi^j R^{(\Delta)} \quad \mathbb{P}\text{-}a.s. \tag{15}$$

Notation. We will denote $\gamma^{j,(\Delta)}$ this limit : the mean service rate of queue j when the set A is backlogged.

Proof :

We have only to prove (15). But since

$$\sum_i r_{\mathcal{Y}}^{i,(\Delta)}(t) = 1 \quad \forall t > 0,$$

we have

$$\begin{aligned}\lim_{t\to\infty}\left[\sum_{j\in S^{(\Delta)}}\frac{1}{t}\int_0^t r_{\mathcal{Y}}^{j,(\Delta)}(u)du + \sum_{j\notin S^{(\Delta)}}\frac{1}{t}\int_0^t r_{\mathcal{Y}}^{j,(\Delta)}(u)du\right] &= 1 \\ &= \sum_{j\in S^{(\Delta)}}\rho^j + \lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{j\notin S^{(\Delta)}} r_{\mathcal{Y}}^{j,(\Delta)}(u)du.\end{aligned}$$

Hence the limit $\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{j\notin S^{(\Delta)}} r_{\mathcal{Y}}^{j,(\Delta)}(u)du$ does not depend on $\mathcal{Y}$ as long as $\{Y^j, j\in S^{(\Delta)}\}$ stay on a compact set. We can take $\mathcal{Y}_\infty$ such that

$$\forall j \in S^{(\Delta)} \quad Y_\infty^j \leq M$$
$$\forall j \notin S^{(\Delta)} \quad Y_\infty^j = \infty;$$

hence the queue that are not in $S^{(\Delta)}$ are always backlogged. We have then

$$\sum_{j \notin S^{(\Delta)}} r^{j,(\Delta)}_{\mathcal{Y}_\infty}(u) = \frac{\sum_{j \notin S^{(\Delta)}} \phi^j}{\sum_{k \notin I^{(\Delta)}_{\mathcal{Y}_\infty}(u)} \phi^k + \sum_{k \in A} \phi^k},$$

hence

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\int_0^t \sum_{j \notin S^{(\Delta)}} r^{j,(\Delta)}_{\mathcal{Y}}(u)du &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \sum_{j \notin S^{(\Delta)}} r^{j,(\Delta)}_{\mathcal{Y}_\infty}(u)du \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\sum_{j \notin S^{(\Delta)}} \phi^j}{\sum_{k \notin I^{(\Delta)}_{\mathcal{Y}_\infty}(u)} \phi^k + \sum_{k \in A} \phi^k} du \\
&= \sum_{j \notin S^{(\Delta)}} \phi^j \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{1}{\sum_{k \notin I^{(\Delta)}_{\mathcal{Y}_\infty}(u)} \phi^k + \sum_{k \in A} \phi^k} du
\end{aligned}$$

and then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{1}{\sum_{k \notin I^{(\Delta)}_{\mathcal{Y}_\infty}(u)} \phi^k + \sum_{k \in A} \phi^k} du = R^{(\Delta)},$$

what implies directly for $j \notin S^{(\Delta)}$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t r^{j,(\Delta)}_{\mathcal{Y}}(u)du = \phi^j R^{(\Delta)}.$$

$\triangle$

We can now prove property 3.

Proof of property 3 :

First consider the case $\rho < 1$.

Denote $\{T_n, \sigma_n\}$ the superposition of the N input processes. $W(t)$ the unique stationary workload process of the GI/GI/1 queue with input process $\{T_n, \sigma_n\}$.

The point process R defined by

$$R(C) = \sum_{n \in \mathbb{Z}} \mathbb{1}_C(T_n)\mathbb{1}_{\{0\}}(W(T_n-)),$$

counts the construction points T_n that is the arrival times at which an arriving customer finds an empty queue. Clearly R is compatible with $\{\theta_t\}$. Let $\{U_n\}, n \in \mathbb{Z}$, be the sequence of points of R, with the usual convention

$$U_0 \leq 0 < U_1.$$

For each $n \in \mathbb{Z}$, let V_{n+1} be the first time t after U_n at which $W(t) = 0$. The interval $[U_n, U_{n+1})$ is called the n-nth cycle, $[U_n, V_{n+1})$ is the n-th busy period and $[V_{n+1}, U_{n+1})$ is the n-th idle period.

We then construct on each cycle the stationary workload process of queue j such that

$$\begin{aligned}
\frac{dW^j}{dt}(t) &= -\mathbb{1}_{\{W^j(t)>0\}} r^j(t) \quad \text{with} \\
I_t &= \{i; W^i(t) = 0\}, \\
r^j(t) &= \begin{cases} \frac{\phi^j}{\sum_{k \notin I_t} \phi^k} & j \in (I_t)^c \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The only thing to prove, is that each $W^j_{\mathcal{Y}}(t)$ couples with the stationary solution $W^j(t)$ for any finite initial condition $\mathcal{Y}$. But we know that $\sum_j W^j_{\mathcal{Y}}(t)$ couples with $W(t)$, says that for $t \geq T$, $\sum_j W^j_{\mathcal{Y}}(t) = W(t)$. T is finite, hence there exists $V_n \geq T$ and V_n is clearly a coupling time for each process $W^j_{\mathcal{Y}}(t)$.
Consider now the case $\rho \geq 1$.
We can take $\Delta = \emptyset$, and everything we told before is true, except :

$$\sum_i r^i_{\mathcal{Y}}(t) = \sum_i r^{i,(\emptyset)}_{\mathcal{Y}}(t) \neq 1 \quad \forall t > 0,$$

But thanks to previous section, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_i r^i_{\mathcal{Y}}(t) = 1.$$

Hence we can conclude as before.
△

Notation. We will denote $\gamma^j = \rho^j \wedge \phi^j R$ the mean service rate of queue j.

4 Subexponential distributions and asymptotic scale

4.1 Subexponential distributions

We only give here the definitions and notations for more general results see [15] or [17].

Definition. *Let F be a distribution function on $(0,\infty)$ such that $F(x) < 1$ for all $x > 0$. We say $F \in \mathcal{L}$ if*

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1, \quad \forall y > 0.$$

Definition. *A distribution F on $\mathbb{R}_+$ is called subexponential ($F \in \mathcal{S}$) if $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$.*

Let F be a distribution on $\mathbb{R}_+$ with positive and finite mean m. We use the notation F^S for the integrated tail distribution of F defined by

$$F^S(x) = \frac{1}{m} \int_0^x (1 - F(y))dy.$$

For simplicity of notation, we will denote $\overline{F}^S(x) = \overline{F^S}(x) = 1 - F^S(x)$.

Lemma 4. *If $F \in \mathcal{L}$, then $\overline{F}(x) = o(\overline{F}^S(x))$ as $x \to \infty$.*

Lemma 5. *There exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

- $g(x)\overline{F}(x) = o(\overline{F}^S(x))$ *as* $x \to \infty$,
- $g(x) \xrightarrow{x\to\infty} \infty$,

- $g(x) = o(x)$ *as* $x \to \infty$.

Proof :
Note $\epsilon(x) = \frac{\overline{F}(x)}{\overline{F^S}(x)}$. Thanks to lemma 4, we have : $\epsilon(x) \xrightarrow{x\to\infty} 0$.
We must find a function such that

1. $g(x)\epsilon(x) \xrightarrow{x\to\infty} 0$,
2. $g(x)/x \xrightarrow{x\to\infty} 0$,
3. $g(x) \xrightarrow{x\to\infty} \infty$.

But conditions 1 and 2 are redundant because $\epsilon(x)$and$1/x \xrightarrow{x\to\infty} 0$. We replace both conditions by the stronger one and we denote it : $g(x)f(x) \xrightarrow{x\to\infty} 0$ with $f(x) \xrightarrow{x\to\infty} 0$.
We now see that $g(x) = \frac{1}{\sqrt{f(x)}}$ fits the conditions.
△

Remark 2. If we note $N_x = \lfloor g(x) \rfloor$, we have

$$\begin{aligned} \sum_{n=0}^{N_x} \mathbb{P}(\xi > x + n\alpha) &\leq N_x \mathbb{P}(\xi > x) \\ &= N_x \overline{F}(x) \\ &= o(\overline{F^S}(x)). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \sum_n \mathbb{P}(\xi > x + n\alpha) &= \sum_n \int_{x+n\alpha}^{\infty} f(t)dt \quad \text{with} \quad \int_0^t f(u)du = \mathbb{P}(\xi < t). \\ &= \int_x^{\infty} \left\lfloor \frac{t-x}{\alpha} \right\rfloor f(t)dt \\ &\sim \frac{1}{\alpha} \int_x^{\infty} \mathbb{P}(\xi > t)dt. \end{aligned}$$

We have then :

$$\sum_n \mathbb{P}(\xi > x + n\alpha) \sim \sum_{n \geq N_x} \mathbb{P}(\xi > x + n\alpha) \sim \frac{1}{\alpha} \int_x^{\infty} \mathbb{P}(\xi > t)dt.$$

♣

4.2 Asymptotic scale

Let $(A^j, \theta_t, \mathbb{P})$ $(1 \leq j \leq N)$ be N independent stationary renewal processes with (finite) intensity λ^j. We denote T_n^j the n-th point of the process A^j interpreted as the arrival time of customer n of class j and τ_n^j the inter-arrival time. We denote by σ_n^j the service time of customer n of class j, where the sequence $\{\sigma_n^j\}$ is assumed to be a sequence of marks of the arrival process A^j. Let $(A, \theta_t, \mathbb{P})$ be the superposition of this point processes. We denote $A = \{T_n, \sigma_n, C_n\}$ with the following interpretation : T_n is the arrival time of customer n which carries a service time σ_n and which is of class C_n.

We assume that the stability condition holds : $\rho < 1$. Hence we can consider the stationary workload :

$$W = \sup_{n \leq 0}(T_n + \sum_{i=0}^{n} \sigma_i)^+.$$

We assume that the following assumption holds :

(SE). *The required amounts of service* $\{\sigma_n^j\}$ *are i.i.d random variables* ($\mathbb{E}_j^0[\sigma_0^j] = b^j$). *There exists a distribution function* F *on* $\mathbb{R}^+$ *such that :*

- $F \in \mathcal{S}$, *with finite first moment* $m = \int_0^\infty \overline{F}(u)du$;
- *the integrated distribution of* F *:*

$$F^S(x) = \frac{1}{m}\int_0^x \overline{F}(u)du$$

  is subexponential;
- *the following equivalence holds when* x *tends to* ∞ *:*

$$\mathbb{P}_j^0(\sigma_0^j > x) \sim d^j \overline{F}(x),$$

  with $\sum_j d^j > 0$ *and where* $\mathbb{P}_j^0$ *is the Palm probability of point process* A^j.

We then have :

$$\mathbb{P}_A^0[\sigma > x] = \sum_j \frac{\lambda^j d^j}{\lambda} \overline{F}(x).$$

We take the same notation as in [5] and we define the typical event for a $G/G/1$ queue as follows :
put $\xi_n = \sigma_n - \tau_n$, $S_{-n} = \sum_{-n}^{-1} \xi_i$, $S_0 = 0$, $S_{-\infty} = -\infty$. Put $S_{-n}^* = \max_{0 \leq j \leq n} S_{-j}$. Let N_x be defined as in previous paragraph.

Property 6. *We define the following event :*

$$\begin{aligned} H_{x,n} &= \left\{ \left| \frac{S_{-l}}{l} - \frac{\rho - 1}{\lambda} \right| \leq \epsilon_l,\ N_x \leq l \leq n-1 \right\}; \\ A_{x,n} &= \left\{ \sigma_{-n} > x + n(\frac{1-\rho}{\lambda} + \epsilon_n) \right\} \cap H_{x,n}; \\ A_x &= \bigcup_{n \geq N_x} A_{x,n}. \end{aligned}$$

Under **(SE)**, *there exists a sequence* ϵ_n *such that* $\epsilon_n \downarrow 0$*and* $n\epsilon_n \uparrow \infty$, *and such that*

$$\mathbb{P}(W > x, A_x) \sim \mathbb{P}(W > x) \sim \frac{\rho}{1-\rho} \overline{F}^S(x). \tag{16}$$

Proof :
see [3] and [5].
△
Let X be a random variable such that : $X \leq W$. We have :

$$\begin{aligned}\mathbb{P}(X > x) &= \mathbb{P}(X > x, A_x) + \mathbb{P}(X > x, W > x, A_x^c) \\ &\leq \mathbb{P}(X > x, A_x) + \mathbb{P}(W > x, A_x^c).\end{aligned}$$

Thanks to (16), we have :

$$\mathbb{P}(X > x, A_x) \leq \mathbb{P}(X > x) \leq \mathbb{P}(X > x, A_x) + o(\overline{F}^S(x)).$$

If $\mathbb{P}(X > x, A_x) = O(\overline{F}^S(x))$, we have the exact asymptotic for $\mathbb{P}(X > x)$. Hence, we have the following property :

Property 7. *Take :*

$$\begin{aligned}\sigma_{x,n} &= x + n\frac{1-\rho}{\lambda}, \\ B_{x,n} &= \{\sigma_{-n} > \sigma_{x,n}\}.\end{aligned}$$

We have :

If $\sum_{n \geq N_x} \mathbb{P}(X > x, B_{x,n}) = O(\overline{F}^S(x))$ *then* $\mathbb{P}(X > x) \sim \sum_{n \geq N_x} \mathbb{P}(X > x, B_{x,n})$.
If $\sum_{n \geq N_x} \mathbb{P}(X > x, B_{x,n}) = o(\overline{F}^S(x))$ *then* $\mathbb{P}(X > x) = o(\overline{F}^S(x))$.

Proof :
We have of course $\mathbb{P}(X > x, A_{x,n}) \leq \mathbb{P}(X > x, B_{x,n})$. Hence, if $\sum_{n \geq N_x} \mathbb{P}(X > x, B_{x,n}) = o(\overline{F}^S(x))$, we have :

$$\begin{aligned}\mathbb{P}(X > x) &\leq \mathbb{P}(X > x, A_x) + o(\overline{F}^S(x)) \\ &= \sum_{n \geq N_x} \mathbb{P}(X > x, A_{x,n}) + o(\overline{F}^S(x)) \\ &\leq \sum_{n \geq N_x} \mathbb{P}(X > x, B_{x,n}) + o(\overline{F}^S(x)) \\ &= o(\overline{F}^S(x)).\end{aligned}$$

In the other case, we have :

$$\begin{aligned}\mathbb{P}(X > x, \sigma_{-n} > \sigma_{x,n} + n\epsilon_n) &= \mathbb{P}(X > x, \sigma_{-n} > \sigma_{x,n} + n\epsilon_n, H_{x,n}) + \mathbb{P}(X > x, \sigma_{-n} > \sigma_{x,n} + n\epsilon_n, H_{x,n}^c) \\ &\leq \mathbb{P}(X > x, A_{x,n}) + \mathbb{P}(\sigma_{-n} > \sigma_{x,n} + n\epsilon_n)\mathbb{P}(H_{x,n}^c),\end{aligned}$$

because σ_{-n} and $H_{x,n}$ are independent. Hence, we have :

$$\begin{aligned}\sum_{n \geq N_x} \mathbb{P}(X > x, \sigma_{-n} > \sigma_{x,n} + n\epsilon_n) &\leq \sum_{n \geq N_x} \mathbb{P}(X > x, A_{x,n}) + o(\overline{F}^S(x)) \\ &= \mathbb{P}(X > x, A_x) + o(\overline{F}^S(x)).\end{aligned}$$

Hence, if we prove that :

$$\sum_{n\geq N_x}\mathbb{P}(X>x,B_{x,n})\leq\sum_{n\geq N_x}\mathbb{P}(X>x,\sigma_{-n}>\sigma_{x,n}+n\epsilon_n)+o(\overline{F}^S(x)),$$

we have :

$$\begin{aligned}\sum_{n\geq N_x}\mathbb{P}(X>x,B_{x,n}) &\leq \mathbb{P}(X>x,A_x)+o(\overline{F}^S(x)),\quad\text{and,}\\ \mathbb{P}(X>x,A_x) &\leq \mathbb{P}(X>x),\quad\text{and,}\\ \mathbb{P}(X>x) &\leq \sum_{n\geq N_x}\mathbb{P}(X>x,B_{x,n})+o(\overline{F}^S(x)).\end{aligned}$$

And the property is proved.
But, we have :

$$\begin{aligned}\mathbb{P}(X>x,B_{x,n}) &= \mathbb{P}(X>x,\sigma_{-n}>\sigma_{x,n}+n\epsilon_n)+\mathbb{P}(X>x,\sigma_{x,n}+n\epsilon_n>\sigma_{-n}>\sigma_{x,n})\\ &\leq \mathbb{P}(X>x,\sigma_{-n}>\sigma_{x,n}+n\epsilon_n)+\mathbb{P}(\sigma_{x,n}+n\epsilon_n>\sigma_{-n}>\sigma_{x,n}).\end{aligned}$$

Since $\sum_{n\geq N_x}\mathbb{P}(\sigma>\sigma_{x,n})\sim\sum_{n\geq N_x}\mathbb{P}(\sigma>\sigma_{x,n}+n\epsilon_n)\sim\frac{\rho}{1-\rho}\overline{F}^S(x)$, we have

$$\sum_{n\geq N_x}\mathbb{P}(X>x,B_{x,n})\leq\sum_{n\geq N_x}\mathbb{P}(X>x,\sigma_{-n}>\sigma_{x,n}+n\epsilon_n)+o(\overline{F}^S(x)).$$

$\triangle$

5 Generalized Processor Sharing queues with heavy tailed service times

5.1 Model description

We consider the following model of N coupled GI/GI/1 queues, $Q_1,\dots,Q_N$. Each queue is served in accordance with the Generalized Processor Sharing (GPS) discipline.
Denote $\{T_n^j,\sigma_n^j\}$ the input process of queue j. Denote by λ^j the intensity of this arrival process, by $\rho^j=\lambda^j\mathbb{E}_j^0[\sigma_0^j]$ the traffic intensity of queue j. We suppose that assumption **(SE)** is satisfied.
We suppose moreover that

$$\rho=\rho^1+\cdots+\rho^N<1.$$

Hence, the framework described before applies. Denote $\{T_n,\sigma_n,C_n\}$ the superposition of the N input processes. $C_n\in\{1,\dots,N\}$ is the class of customer n, $W\circ\theta_t$ the unique stationary workload process of the GI/GI/1 queue with input process $\{T_n,\sigma_n\}$and $W^j\circ\theta_t$ the unique stationary workload process for queue j (see section 3).
We assume the queues are indexed in such a way that

$$\frac{\rho^1}{\phi^1}\leq\cdots\leq\frac{\rho^N}{\phi^N}.$$

We end this section with an easy lemma and some remarks (recall that we give a superscript $.^{(\Delta)}$ when we consider the system with queues in Δ that are backlogged) :

Lemma 6. *If $\Delta \subset \Gamma$, then $S^{(\Gamma)} \subset S^{(\Delta)}$ and $R^{(\Delta)} \geq R^{(\Gamma)}$ and $S^{(\Delta)} \neq S^{(\Gamma)} \Rightarrow R^{(\Delta)} > R^{(\Gamma)}$.*

Remark 3. Consider the case $\Delta = \{k\}$. Instead of $(\{k\})$, we will use the notation (k).

- If $\rho^k \geq \phi^k$ then $K^{(k)} = N$.
- If i is an unstable queue then $i \geq k$.
- We have $R^{(k)}\phi^k > \rho^k$.

♣

5.2 A typical busy period : deterministic calculus.

We consider a typical busy period induced by a very big service of type k. We will give a superscript $.^{\{k\}}$ to the constants that are calculated in this busy period.
We denote

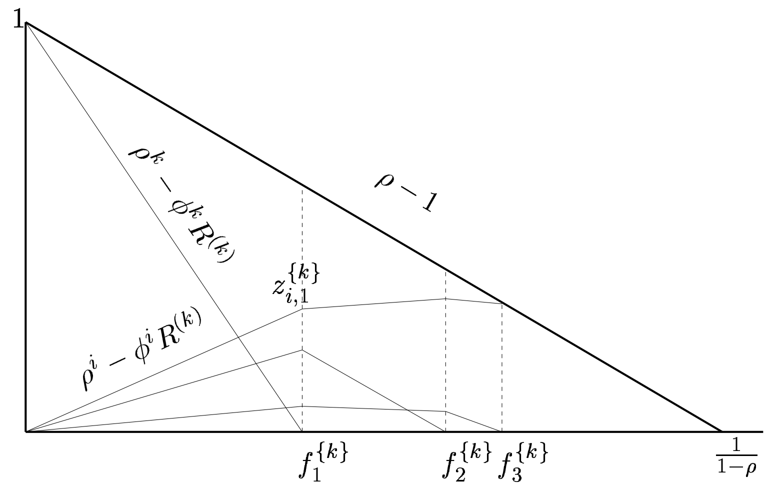
$$\begin{aligned}
S^{(k)} &= \left\{1, \ldots, K^{(k)}\right\} \setminus \{k\}, \\
U^{(k)} &= \left\{K^{(k)} + 1, \ldots, N\right\}, \\
i_1^{\{k\}} &= k, \\
f_1^{\{k\}} &= \frac{1}{\phi^k R^{(k)} - \rho^k}, \\
z_1^{i,\{k\}} &= \left(f_1^{\{k\}}(\rho^i - \phi^i R^{(k)}\right)^+, \\
I_1 &= U^{(k)}, \\
N^{\{k\}} &= |U^{(k)}| = N - K^{(k)}.
\end{aligned}$$

We have then $S^{(I_1)} = \{1, \ldots, K_1\}$ with $K_1 = K^{(k)}$ because

$$\begin{aligned}
\frac{1 - \sum_{j=1}^{K^{(k)}-1} \rho^j \mathbb{1}_{\{j \notin I_1\}}}{\sum_{j=K^{(k)}}^{N} \phi^j \mathbb{1}_{\{j \notin I_1\}} + \sum_{j \in I_1} \phi^j} &\geq \frac{\sum_{j=K^{(k)}}^{N} \rho^j}{\sum_{j=K^{(k)}}^{N} \phi^j} \\
&\geq \frac{\rho^{K^{(k)}}}{\phi^{K^{(k)}}}.
\end{aligned}$$

Then, $S^{(I_1)} = S^{(k)} \cup \{k\}$ and then by lemma 6, $R^{(I_1)} > R^{(k)}$.
We denote

$$\begin{aligned}
\left\{i_2^{\{k\}}\right\} &= \arg\min_{i \in_1^S} \left\{\frac{z_{i,1}^{\{k\}}}{\phi^i R^{(I_1)} - \rho^i}\right\}, \\
f_2^{\{k\}} &= \inf_{i \in E_1^S} \left\{\frac{z_{i,1}^{\{k\}}}{\phi^i R^{(I_1)} - \rho^i}\right\} + f_1^{\{k\}}, \\
z_2^{i,\{k\}} &= \left(z_1^{i,\{k\}} + f_2^{\{k\}}(\rho^i - \phi^i R^{(I_1)})\right)^+.
\end{aligned}$$



We define recursively

$$
\begin{aligned}
f_{j+1}^{\{k\}} &= \inf_{i \in E_j^S} \left\{ \frac{z_{i,j}^{\{k\}}}{\phi^i R^{(I_j)} - \rho^i} \right\} + f_j^{\{k\}}, \\
\left\{ i_{j+1}^{\{k\}} \right\} &= \arg\min_{i \in I_j^S} \left\{ \frac{z_{i,j}^{\{k\}}}{\phi^i R^{(I_j)} - \rho^i} \right\}, \\
z_{j+1}^{i,\{k\}} &= \left(z_j^{i,\{k\}} + (f_{j+1}^{\{k\}} - f_j^{\{k\}})(\rho^i - \phi^i R^{(I_j)}) \right)^+, \\
I_{j+1} &= I_j \backslash \left\{ i_j^{\{k\}} \right\}, \\
S^{(I_{j+1})} &= \{1, \dots, K_{j+1}\} \backslash I_{j+1}.
\end{aligned}
$$

We now consider a particular queue i such that $\rho^i > \phi^i R^{(k)}$.
There exists an integer j such that $i_j^{\{k\}} = i$, we will denote it $j^{\{k\}}(i)$.
We define the following function :

$$
\begin{array}{rccc} w^{i,\{k\}} : & \mathbb{R}_+ & \to & \mathbb{R}_+ \\ & t & \mapsto & w^{i,\{k\}}(t) \end{array} \text{ with,}
$$

$$
w^{i,\{k\}}(t) = \sum_{0 \le j} \left(z_j^{i,\{k\}} + (\rho^i - \phi^i R^{(I_j)})(t - f_j^{\{k\}}) \right)^+ \mathbb{1}_{\{t \in [f_j^{\{k\}}, f_{j+1}^{\{k\}})\}}, \tag{17}
$$

$$
\text{with} \quad z_0^{i,\{k\}} = 0 \quad f_0^{\{k\}} = 0.
$$

In fact we sum only over $j \le j^{\{k\}}(i)$. We will denote $f^{i,\{k\}} = f_{j^{\{k\}}(i)}^{\{k\}}$.
For any x, we define the following domain of $\mathbb{R}_+^2$:

$$
D^{i,\{k\}}(x) = \left\{ (t,z), \quad w^{i,\{k\}}(\frac{t}{z}) > \frac{x}{z} \right\} \tag{18}
$$

Observe that the function $w^{i,\{k\}}$ is concave, then :

Lemma 7. *We have the following properties for the domain :*

- *The set* $D^{i,\{k\}}(x)$ *is a convex subset of* $\{(t,z),\ z \ge \frac{t}{f^{i,\{k\}}}\}$;

- *For any $\mu > 0$, we have :* $D^{i,\{k\}}(\mu x) = \mu D^{i,\{k\}}(x)$*;*
- *There exists a piecewise linear function $b^{i,\{k\}}$ such that*

$$D^{i,\{k\}}(x) = \bigcup_{t \geq t^i(x)} [b^{i,\{k\}}(t), +\infty),$$

and for the function $b^{i,\{k\}}$, there exist some constants $\alpha_j^{i,\{k\}}, \beta_j^{i,\{k\}}$ and $\gamma_j^{i,\{k\}}$ depending only on ρ^i and ϕ^i, such that

$$\forall 1 \leq j \leq j^{\{k\}}(i) - 1, \quad \text{for} \quad x\alpha_j^{i,\{k\}} \leq t < x\alpha_{j+1}^{i,\{k\}}, \quad b^{i,\{k\}}(t) = x\beta_j^{i,\{k\}} + \gamma_j^{i,\{k\}} t.$$

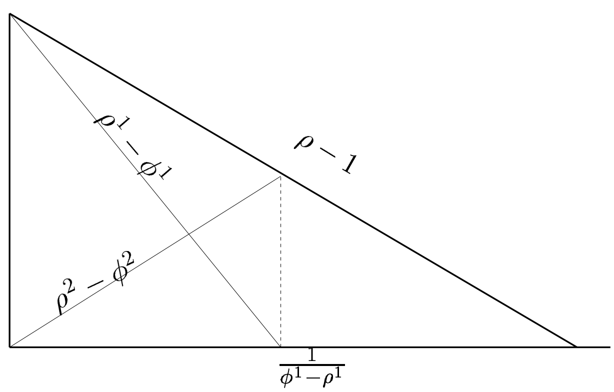
Examples :

The case N=2

We suppose that

$$\frac{\rho^1}{\phi^1} < \frac{\rho^2}{\phi^2}.$$

We have $\frac{\rho^1}{\phi^1} < 1$, if not $\rho^1 + \rho^2 \geq \phi^1 + \phi^2 = 1$. Hence, queue 1 is always stable.
We study the only interesting case, when queue 1 receives a very big service (that induces a busy period for this queue) and when $\frac{\rho^2}{\phi^2} > 1$.
We have $R^{(1)} = 1$.

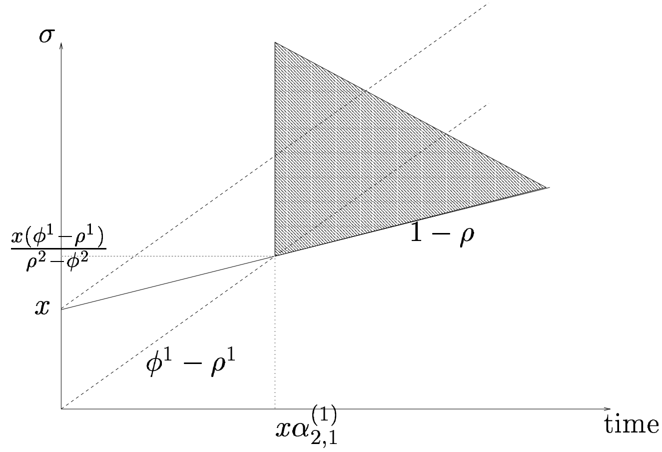


We have for $D^{2,(1)}$:

$$\begin{aligned}
\alpha_1^{2,(1)} &= \frac{1}{\rho^2 - \phi^2}, \\
\beta_1^{2,(1)} &= 1, \\
\gamma_1^{2,(1)} &= 1 - \rho.
\end{aligned}$$

And then :

$$\begin{aligned}
D^{1,\{1\}}(x) &= \left\{(t,z), \quad z > x + (\phi^1 - \rho^1)t\right\}, \\
D^{2,\{1\}}(x) &= \left\{(t,z), \quad t > \frac{x}{\rho^2 - \phi^2}, \quad z > x + (1-\rho)t\right\}.
\end{aligned}$$



The case N=3

In the previous case, if you fix the traffic intensities ρ^1 and ρ^2, the behavior of the system depends only on one parameter : $\frac{\phi^1}{\phi^2}$. But, in the case $N=3$, we have a lot of different cases, so we take one of this case and compute the algorithm with the following parameters :

$$\rho_1 = 0.2, \phi_1 = 0.55; \quad \rho_2 = 0.5, \phi_2 = 0.4; \quad \rho_3 = 0.1, \phi_3 = 0.05.$$

Here we have :

$$\frac{\rho_1}{\phi_1} < 1 < \frac{\rho_2}{\phi_2} < \frac{\rho_3}{\phi_3}.$$

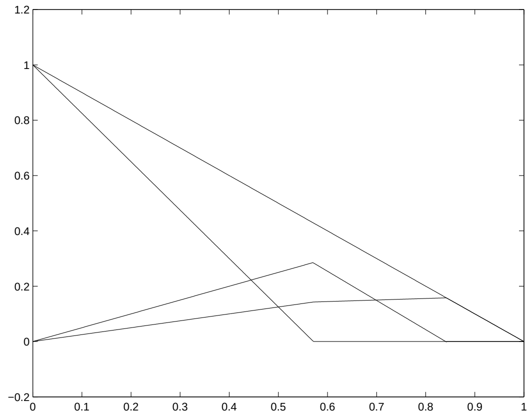
Thanks to remark 3, the only case to study is the typical busy period induced by a very big service of type 1. We then have for the first step of the algorithm :

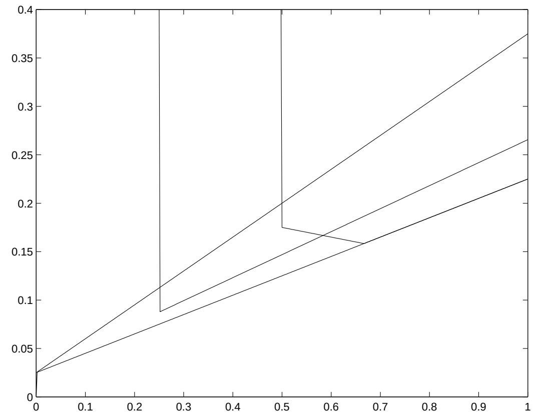
$$\begin{aligned} S^{(1)} &= \emptyset, \\ U^{(1)} &= \{2,3\}, \\ i_1^{\{1\}} &= 1, \\ f_1^{\{1\}} &= \frac{1}{\phi^1-\rho^1}, \\ z_1^{2,\{1\}} &= \frac{\rho^2-\phi^2}{\phi^1-\rho^1}, \\ z_1^{3,\{1\}} &= \frac{\rho^3-\phi^3}{\phi^1-\rho^1}, \\ I_1 &= \{2,3\}. \end{aligned}$$

Here we have $z_1^{2,\{1\}} > z_1^{3,\{1\}}$: when queue 1 is backlogged, queue 2 growths faster as queue 3. For the next steps, we have :

$$\begin{aligned} R^{(I_1)} &= \frac{1-\rho^1}{\phi^2+\phi^3}, \\ i_2^{\{1\}} &= 2, \\ I_2 &= \{3\}, \\ R^{(I_2)} &= \frac{1-\rho^1-\rho^2}{\phi^3}. \end{aligned}$$

This correspond to the following figures :





In this configuration, we have for the domains :

$$\begin{aligned}
D^{1,\{1\}}(x) &= \left\{(t,z), \quad z > x + (\phi^1 - \rho^1)t\right\}, \\
D^{2,\{1\}}(x) &= \left\{(t,z), \quad t > \frac{x}{\rho^2 - \phi^2}, \quad z > x\frac{\phi^2+\phi^3}{\phi^2} + t\left[1 - \rho^1 - \rho^2\left(1 + \frac{\phi^3}{\phi^2}\right)\right]\right\}, \\
D^{3,\{1\}}(x) &= \left\{(t,z), \quad \frac{x\phi^2}{\rho^3\phi^2 - \rho^2\phi^3} \geq t > \frac{x}{\rho^2 - \phi^2}, \quad z > x\frac{\phi^2+\phi^3}{\phi^3} + t\left(1 - \rho^1 - \rho^3\left(1 + \frac{\phi^2}{\phi^3}\right)\right)\right\} \\
&\quad \cup \left\{(t,z), \quad t > \frac{x\phi^2}{\rho^3\phi^2 - \rho^2\phi^3}, \quad z > x + (1-\rho)t\right\}.
\end{aligned}$$

6 A typical busy period : probabilistic calculus.

6.1 Stable queues

We consider a queue j such that $j \in S^{(k)}$ so that,

$$\frac{\rho^j}{\phi^j} < R_j^{(k)}.$$

Hence, we have

Lemma 8. *There exists $1 - \rho > \epsilon > 0$ such that when replacing ρ^j by $\rho^j_\epsilon = \rho^j + \epsilon$, queue j remains stable under condition (k) (saturation of queue k).*

And we conclude (recall that C_n is the class of customer n, see section 5.1) :

Property 8. *We have for $j \in S$:*

$$\sum_{n \geq N_x} \mathbb{P}[W^j > x, \sigma_{-n} > \sigma_{x,n}, C_{-n} = k] = o(\overline{F^S}(x)).$$

Proof :
Consider the workload process of queue j when saturating queue k and with input $\{T_n^j, \sigma_n^j + \epsilon/\lambda_j\}$. This queue is stable and then :

$$\frac{1}{x}\int_0^x r_\epsilon^{j,(k)}(t)dt \quad \xrightarrow{x\to\infty} \quad \rho^j_\epsilon = \rho^j + \epsilon.$$

We define $\tilde{W}^j$ the workload process of queue j with input $\{T_n^j, \sigma_n^j\}$ and service rate $r_\epsilon^{j,(k)}(t)$. We have for t such that $r^j(t) > 0$, $r^j(t) \geq r_\epsilon^{j,(k)}(t)$. Hence we have

$$W^j(t) \leq W^j(0) + \tilde{W}^j(t).$$

We have

$$\begin{aligned}
\mathbb{P}[W^j(0) > x, \sigma_{-n} > \sigma_{x,n}, C_{-n} = k] &= \mathbb{P}[W^j(-T_{-n}) > x, \sigma_0 > \sigma_{x,n}, C_0 = k] \\
&\leq \mathbb{P}[W^j(0) + \tilde{W}^j(-T_{-n}) > x, \sigma_0 > \sigma_{x,n}, C_0 = k] \\
&= \mathbb{P}[W^j(0) + \tilde{W}^j(-T_{-n}) > x]\mathbb{P}[\sigma_0 > \sigma_{x,n}, C_0 = k]
\end{aligned}$$

because for $t > 0$ $(W^j(0), \tilde{W}^j(t))$ and $(\sigma_0, C_0 = k)$ are independent. Moreover as $\tilde{W}^j(t)$ has negative drift, $\mathbb{P}[W^j(0) + \tilde{W}^j(-T_{-n}) > x] < \delta_x$ and then we have

$$\begin{aligned}
\sum_{n \geq N_x} \mathbb{P}[W^j > x, \sigma_{-n} > \sigma_{x,n}, C_{-n} = k] &= \delta_x \sum_{n \geq N_x} \mathbb{P}[\sigma_0 > \sigma_{x,n}, C_0 = k] \\
&= o(\overline{F^S}(x)).
\end{aligned}$$

△

6.2 Unstable queues

We must keep in mind the very basic result that follows.

Remark 4. For any function f such that $f(0) = 0$, $f(T) = 0$, $\forall t \in [0, T], f(t) \geq 0$, $\forall t \geq s > T, f(t) \leq f(s)$, we have :

$$\sup_{0 \leq s \leq t} [f(t) - f(s)] = f(t \wedge T).$$

In particular, a convex function such that $f(0) = f(T) = 0$ satisfies the previous equality. ♣

We denote

$$\begin{aligned}
X^i(t) &= \frac{1}{\sigma_0} W^i(t\sigma_0), \\
A_{T,\epsilon}^{\{k\}} &= \left\{ \max_{i \in U^{(k)}} \sup_{0 \leq s \leq T} |X^i(s) - w^{i,\{k\}}(s)| \leq \epsilon \right\},
\end{aligned}$$

where the function $w^{i,\{k\}}$ is the one defined in (17).

Property 9. *We define the function $\Psi^{(k)}$ (which is not random) as follows*

$$\Psi^{\{k\}}(\sigma) = \mathbb{P}\left[A_{T,\epsilon}^{\{k\}} \middle| \sigma_0 = \sigma, C_0 = k \right].$$

We have

$$\forall T \leq \infty, \ \forall \epsilon, \quad \lim_{z \to \infty} \Psi^{\{k\}}(z) = 1.$$

Proof :
For simplicity of notation, we assume that $T_0^k = 0$.
We consider the processes $\{W_{\mathcal{Y}}^{j,\{k\}},\ j \in [1,N]\}$, where $W_{\mathcal{Y}}^{j,\{k\}}$ is the workload process of queue j with initial condition Y^j and fed by A^j except for $j = k$ for which instead of A^k we take the following input process :

$$\tilde{A}^k = \{T_n^k, \tilde{\sigma}_n^k\} \text{ with } \begin{cases} \tilde{\sigma}_0^k & = \sigma, \\ \tilde{\sigma}_n^k & = \sigma_n^k \text{ for } k \geq 1. \end{cases}$$

If we take for initial conditions the stationary workload at time 0, $Y^j = W^j$, we have of course :

$$\mathbb{P}\left[\forall j,\ \forall t > 0,\ W_{\mathcal{Y}}^{j,\{k\}}(t) = W^j(t) \middle| \sigma_0 = \sigma, C_0 = k\right] = 1.$$

We omit the subscript $\mathcal{Y}$ and we denote

$$X_\sigma^{i,\{k\}}(t) = \frac{1}{\sigma} W^{i,\{k\}}(t\sigma).$$

We denote $F_1(\sigma)$ the end of the busy period of queue k containing 0. In section 5.2, we defined a sequence of indices $\{i_j^{\{k\}}\}$ which correspond to the queues that are not stable when queue k is saturated. Moreover this sequence induces an order on these queues, namely we know that in the deterministic calculus, queue $i_j^{\{k\}}$ will empty before queue $i_{j+1}^{\{k\}}$. Denote then by $F_j(\sigma)$ the end of the busy period containing $F_1(\sigma)$ of queue $i_j^{\{k\}}$. Admit first that $0 < F_1(\sigma) < \cdots < F_{N^{\{k\}}}(\sigma)$, then, with the same notation as in section 5.2, we have :

$$\begin{array}{cc} \forall t \in [0, F_1(\sigma)] & r^i(t) = r^{i,(k)}(t), \\ \forall t \in [F_1(\sigma), F_2(\sigma)] & r^i(t) = r^{i,(E_1)}(t), \\ \vdots & \vdots \\ \forall t \in [F_j(\sigma), F_{j+1}(\sigma)] & r^i(t) = r^{i,(E_j)}(t), \\ \vdots & \vdots \\ \forall t \in [F_{N^{\{k\}}-1}(\sigma), F_{N^{\{k\}}}(\sigma)] & r^i(t) = r^{i,(i_{N^{\{k\}}}^{\{k\}})}(t). \end{array}$$

We see that we cannot apply directly the result of section 2, because the assumptions of Lemma 3, namely,

$$\begin{aligned} f_n(t) &= f(t) \in \mathcal{E}_l \quad \text{for } t > 0, \\ \lim_{n \to \infty} \frac{f_n(0)}{n} &= f_0. \end{aligned}$$

are not true here since $f_n(t)$ depends on the scaling factor $n (= \sigma$ here) for $t > 0$ through the varying capacity. We then need the following lemma,

Lemma 9. *Let $\{f_n\}$ be a sequence of functions such that there exists k functions $(g_1(t),\dots g_k(t))$ and a sequence $(t_n^0,\dots,t_n^k)_{n\in\mathbb{N}}$ such that,*

$$\begin{aligned} g_i \in \mathcal{E}_{l_i} \quad &\text{and} \quad t_n^0 = 0 < t_n^1 < \cdots < t_n^k, \\ \lim_{n\to\infty}\frac{t_n^i}{n} &= t^i, \\ f_n(t) &= \sum_{i=1}^k \mathbb{1}_{\{t_n^{i-1}<t\}} g_i(t\wedge t_n^i) \quad \text{for } t>0, \\ \lim_{n\to\infty}\frac{f_n(0)}{n} &= f_0. \end{aligned}$$

Then the following convergence holds :

$$S_n(f_n) \xrightarrow[\|\,\|_{\mathcal{E}}]{n\to\infty} f^s \text{ with}$$

$$f^s(t) = \sum_{i=1}^k \mathbb{1}_{\{t^{i-1}<t\}} e_{l_i}^{f_0}(t\wedge t^i).$$

Proof

Consider the following functions :

$$\begin{aligned} f_n^j(0) &= f_n(0), \\ f_n^j(t) &= \sum_{i=1}^{j-1} \mathbb{1}_{\{t_n^{i-1}<t\}} g_i(t\wedge t_n^i) + \mathbb{1}_{\{t_n^{j-1}<t\}} g_j(t). \end{aligned}$$

We have of course $f_n^k = f_n$. Moreover, we have $f_n^1 = g_1$ and thanks to the result of section 2, we have:

$$S_n(f_n^1) \xrightarrow[\|\,\|_{\mathcal{E}}]{n\to\infty} e_{l_1}^{f_0}.$$

Then we have :

$$\begin{aligned} f_n^2(t) &= f_n^1(t) + \mathbb{1}_{\{t_n^1<t\}} g_2(t), \\ S_n(f_n^2)(t) &= S_n(f_n^1)(t) + \mathbb{1}_{\{t_n^1<nt\}} S_n(g_2)(t). \end{aligned}$$

And it is easy to see that $S_n(f_n^1) \xrightarrow[\|\,\|_{\mathcal{E}}]{n\to\infty} e_{l_1}^{f_0} + \mathbb{1}_{\{t^1<t\}} e_{l_2}^{f_0}$. And finally that the lemma follows.

△

We now prove property 9. First consider queue k. Take :

$$\begin{aligned} \tilde{A}^k(s,t) &= \sum_n \tilde{\sigma}_n^k \mathbb{1}_{\{T_n^k\in(s,t]\}} \quad \text{for} \quad s>0, \text{ and,} \\ \tilde{A}^k(0,t) &= \tilde{\sigma}_0^k + \sum_n \tilde{\sigma}_n^k \mathbb{1}_{\{T_n\in(0,t]\}}, \\ \tilde{W}^{k,\{k\}}(t) &= \sup_{0\le s\le t}\left[\tilde{A}^k(s,t) - \int_{(s,t]} r^{k,(k)}(u)du\right], \\ \tilde{X}_\sigma^{k,\{k\}}(t) &= \frac{1}{\sigma}\tilde{W}^{k,\{k\}}(t\sigma). \end{aligned}$$

Thanks to the result of section 2, we have :

$$\tilde{X}_\sigma^{k,\{k\}} \underset{\| \|_\mathcal{E}}{\xrightarrow{\sigma\to\infty}} w^{k,\{k\}} \quad \mathbb{P}\text{-a.s.}$$

But we have $\tilde{X}_\sigma^{k,\{k\}}(t) = X_\sigma^{k,\{k\}}(t)$ for $t \leq F_1(\sigma)$, and by definition $F_1(\sigma)$ is the first time after 0 at which queue k empties. Hence we have shown :

$$\frac{F_1(\sigma)}{\sigma} \xrightarrow{\sigma\to\infty} f_1^{\{k\}} \quad \mathbb{P}\text{-a.s.}$$

We can now apply previous lemma to see that :

$$X_\sigma^{k,\{k\}} \underset{\| \|_\mathcal{E}}{\xrightarrow{\sigma\to\infty}} w^{k,\{k\}} \quad \mathbb{P}\text{-a.s.}$$

We consider then $F(\sigma) = \min(F_2(\sigma), \ldots, F_{N\{k\}}(\sigma))$, on $[F_1(\sigma), F(\sigma)]$, we have : $r^i(t) = r^{i,(E_1)}(t)$. Hence we can show that for sufficiently large σ, we have $F(\sigma) = F_2(\sigma)$ and that :

$$\frac{F_2(\sigma)}{\sigma} \xrightarrow{\sigma\to\infty} f_2^{\{k\}} \; \mathbb{P}\text{-a.s.}$$

Thanks to the previous lemma, we see that ;

$$X_\sigma^{i_2^{\{k\}},\{k\}} \underset{\| \|_\mathcal{E}}{\xrightarrow{\sigma\to\infty}} w^{i_2^{\{k\}},\{k\}} \; \mathbb{P}\text{-a.s.}$$

We conclude the case $T < \infty$ with the same type of arguments.
Now to see that the limit holds for $T = \infty$, we use the same argument as in section 2 (the workload remains in a compact set after $f_{N\{k\}}^k$).
△
Consider now a queue $j \in U^{(k)} \cup \{k\}$. Instead of $\{\sigma_0 = z, C_0 = k\}$, we will note $\{\sigma_{\{k\}} = z\}$. Then we derive from the previous property the following corollary :

Corollary 1. *There exists $\epsilon > 0$, $\eta(z) \xrightarrow{z\to\infty} 0$ and a finite random variable M independent of $\{\sigma_{\{k\}} = z\}$ such that*

$$H_z \quad = \quad \left\{ \sup_{0\leq s \leq f^{j,\{k\}}+\epsilon} \left| X^j(s) - w^{j,\{k\}}(s) \right| \leq \eta(z), \; \forall s > f^{j,\{k\}} + \epsilon, \; X^j(s) \leq \frac{M}{z} \wedge \eta(z) \right\},$$

$$\mathbb{P}\left[H_z | \, \sigma_{\{k\}} = z \right] \quad \xrightarrow{z\to\infty} \quad 1.$$

We then have the following property :

Property 10. *We have for $j \in U^{(k)} \cup \{k\}$ such that $\rho^j \neq \phi^j R^{(k)}$,*

$$\sum_{n\geq N_x} \mathbb{P}[W^j > x, \sigma_{-n} > \sigma_{x,n}, C_{-n} = k] \quad \sim \quad \lambda^k \sum_{i=1}^{j^{(k)}(j)} \int_{x\alpha_i^{j,\{k\}}}^{x\alpha_{i+1}^{j,\{k\}}} \mathbb{P}[\sigma_{\{k\}} > x\beta_i^{j,\{k\}} + \gamma_i^{j,\{k\}} t] dt.$$

where the constants $\alpha_i^{j,\{k\}}, \beta_i^{j,\{k\}}$ and $\gamma_i^{j,\{k\}}$ were defined in section 5.2, see (18).

Proof :
We have

$$\begin{aligned}\mathbb{P}[W^j(0)>x,\sigma_{-n}>\sigma_{x,n},C_{-n}=k] &= \mathbb{P}[W^j(-T_{-n})>x,\sigma_0>\sigma_{x,n},C_0=k]\\ &= \mathbb{P}(C_0=k)\underbrace{\int_{z\geq\sigma_{x,n}}\int_{\mathbb{R}_+}\mathbb{P}[W^j(t)>x|\sigma_0=z,C_0=k]f_k(z)g_n(t)dzdt}_{I_n}.\end{aligned}$$

with f_k the density function of $\sigma_{\{k\}}$ and g_n the density function of $-T_{-n}$. We denote Δ_x the domain of integration and $\mu_n(dz,dt)=f_k(z)g_n(t)dzdt$. Hence we have :

$$\begin{aligned}\mathcal{I}_n &= \int_{\Delta_x}\mathbb{P}\left[X^j\left(\frac{t}{z}\right)>\frac{x}{z}\middle|\sigma_{\{k\}}=z\right]\mu_n(dz,dt)\\ &= \underbrace{\int_{\Delta_x}\mathbb{P}\left[X^j\left(\frac{t}{z}\right)>\frac{x}{z},\ H_z\middle|\sigma_{\{k\}}=z\right]\mu_n(dz,dt)}_{\mathcal{I}_n^1}+\underbrace{\int_{\Delta_x}\mathbb{P}\left[X^j\left(\frac{t}{z}\right)>\frac{x}{z},\ H_z^c\middle|\sigma_{\{k\}}=z\right]\mu_n(dz,dt)}_{\mathcal{I}_n^2}.\end{aligned}$$

First examine the second term. Thanks to last corollary, there exists a function $\delta(z)\xrightarrow{z\to\infty}0$, such that

$$\mathbb{P}\left[X^j\left(\frac{t}{z}\right)>\frac{x}{z},\ H_z^c\middle|\sigma_{\{k\}}=z\right]\leq\delta(z).$$

But on Δ_x, we have $z\geq x$, hence we have $\delta(z)\leq\delta(x)$, and

$$\begin{aligned}\mathcal{I}_n^2 &\leq \delta(x)\int_{\Delta_x}\mu_n(dz,dt)\\ &= \delta(x)\mathbb{P}[\sigma_{\{k\}}>\sigma_{x,n}].\end{aligned}$$

Hence, we have

$$\sum_n\mathcal{I}_n^2=o(\overline{F}^S(x)).$$

We now study the first term :

$$\begin{aligned}\mathcal{I}_n^1 &= \int_{\Delta_x\cap\left\{\frac{t}{z}>f^{j,\{k\}}+\epsilon\right\}}\mathbb{P}\left[.|.\right]\mu_n(dz,dt)+\int_{\Delta_x\cap\left\{\frac{t}{z}\leq f^{j,\{k\}}+\epsilon\right\}}\mathbb{P}\left[.|.\right]\mu_n(dz,dt)\\ &= A_n+B_n.\end{aligned}$$

On H_z, we know that

$$\forall\frac{t}{z}>f^{j,\{k\}}+\epsilon,\ X^j\left(\frac{t}{z}\right)\leq\frac{M}{z}\wedge\epsilon(z).$$

Hence, we have :

$$\mathbb{P}\left[X^j\left(\frac{t}{z}\right)>\frac{x}{z},\ H_z\middle|\sigma_{\{k\}}=z\right]\leq\mathbb{P}[M>x],$$

and we see that

$$\sum_n A_n = o(\overline{F}^S(x)).$$

Let study the second term. On H_z, we have

$$\forall \frac{t}{z} \leq f^{j,\{k\}} + \epsilon,\ \left(w^{j,\{k\}}\left(\frac{t}{z}\right) - \eta(z)\right)^+ \leq X^j\left(\frac{t}{z}\right) \leq w^{j,\{k\}}\left(\frac{t}{z}\right) + \eta(z).$$

From this we get (with $\Delta'_x = \Delta_x \cap \left\{\frac{t}{z} \leq f^{j,\{k\}} + \epsilon\right\}$)

$$\int_{\Delta'_x} \mathbb{1}_{\{D^{j,\{k\}}(x+z\eta(z))\}} \mu_n(dz,dt) \leq B_n \leq \int_{\Delta'_x} \mathbb{1}_{\{D^{j,\{k\}}(x-z\eta(z))\}} \mu_n(dz,dt)$$

Hence, we see that

$$B_n = (1+o(1)) \int_{D^{i,\{k\}}(x)} \mu_n(dz,dt).$$

Hence, we have :

$$\begin{aligned}\sum_n \mathcal{I}^n &= (1+o(1)) \sum_n \int_{D^{j,\{k\}}(x)} \mu_n(dz,dt) \\ &= (1+o(1))\lambda^k \sum_{i=1}^{j^{(k)}(j)} \int_{x\alpha_i^{j,\{k\}}}^{x\alpha_{i+1}^{j,\{k\}}} \mathbb{P}[\sigma_{\{k\}} > x\beta_i^{j,\{k\}} + \gamma_i^{j,\{k\}} t]dt.\end{aligned}$$

For simplicity of notations, we note $\mathbb{P}$ instead of the Palm probability.
$\triangle$

6.3 General result

We consider the model of section 5 with the following assumptions :

$$\forall j,k,\ \rho^j \neq \phi^j R^{(k)}.$$

We take the same notations as in the previous paragraph.
Recall that :

$$\mathbb{P}(\sigma_0^j > x) \sim d^j \overline{F}(x).$$

Property 11. *The asymptotic of the stationary workload of an individual queue j behaves the following manner :*

- *if $d^j > 0$, then :*

$$\begin{aligned}\mathbb{P}[W^j > x] &= \frac{\lambda_j}{\phi^j R^{(j)} - \rho_j} \int_x^\infty \mathbb{P}[\sigma_{\{j\}} > t]dt + \\ &\sum_{\{k: j \in U^{(k)}\}} \lambda^k \sum_{i=1}^{j^{(k)}(j)} \int_{x\alpha_i^{j,\{k\}}}^{x\alpha_{i+1}^{j,\{k\}}} \mathbb{P}[\sigma_{\{k\}} > x\beta_i^{j,\{k\}} + \gamma_i^{j,\{k\}} t]dt + o(\overline{F^S}(x)).\end{aligned} \tag{19}$$

- *if* $d^j = 0$ *and* $\sum_{k:j\in U^{(k)}} d^k > 0$ *and* $\overline{F^S}$ *is dominatedly varying, that is :*

$$\liminf_{x\to\infty} \frac{\overline{F^S}(ax)}{\overline{F^S}(x)} > 0 \quad \textit{for some (equivalently, all) } a > 1. \tag{20}$$

then :

$$\mathbb{P}[W^j > x] = \sum_{\{k:j\in U^{(k)}\}} \lambda^k \sum_{i=1}^{j^{(k)}(j)} \int_{x\alpha_i^{j,\{k\}}}^{x\alpha_{i+1}^{j,\{k\}}} \mathbb{P}[\sigma_{\{k\}} > x\beta_i^{j,\{k\}} + \gamma_i^{j,\{k\}} t]dt + o(\overline{F^S}(x)). \tag{21}$$

- *else :*

$$\mathbb{P}[W^j > x] = o(\overline{F^S}(x)). \tag{22}$$

Remark 5. • As told in section 4, we are able to determine if the asymptotic scale is $O(\overline{F^S}(x))$ or not. And we can calculate the exact asymptotic only in the first case : equations (19) and (21). The additional assumption (20) ensures that the asymptotic is $O(\overline{F^S}(x))$.

- In the paper [17], the class of dominatedly varying function is called dominated-variation distributions and denoted $\mathcal{D}$. It is shown that $F^S \in \mathcal{D}$ if and only if $F^S \in \mathcal{D}\cap\mathcal{L}$. And $\mathcal{D}\cap\mathcal{L}$ is a subclass of $\mathcal{S}$. Moreover, if $F \in \mathcal{D}\cap\mathcal{L}$ has finite expectation, then $F^S \in \mathcal{D}\cap\mathcal{L}$. Hence in these conditions, every heavy tails assumptions that are needed for this paper are satisfied. For more details concerning relations between the classes $\mathcal{D}, \mathcal{S}, \mathcal{L}$, see [14] (but as pointed out in [17], $F^S \in \mathcal{D}$ does not imply $F \in \mathcal{D}$ and the remark (i) on p. 84 of [14] is not true).

Examples :

The case N=2

We take the same conditions as in section 5 :

$$\frac{\rho^1}{\phi^1} < 1 < \frac{\rho^2}{\phi^2}.$$

Hence the technical assumption is satisfied and we have :

$$\begin{aligned}
\mathbb{P}(W^1 > x) &\sim \frac{d^1\lambda^1 m}{\phi^1-\rho^1}\overline{F^S}(x),\\
\mathbb{P}(W^2 > x) &\sim \frac{d^1\lambda^1 m}{1-\rho}\overline{F^S}\left(x\frac{\phi^1-\rho^1}{\rho^2-\phi^2}\right) + \frac{d^2\lambda^2 m}{\phi^2-\rho^2}\overline{F^S}(x).
\end{aligned}$$

This must be understood of the following manner :

- if $d^1 = 0$ then

$$\begin{aligned}
\mathbb{P}(W^1 > x) &= o(\overline{F^S}(x)),\\
\mathbb{P}(W^2 > x) &= \frac{d^2\lambda^2 m}{\phi^2-\rho^2}\overline{F^S}(x) + o(\overline{F^S}(x)).
\end{aligned}$$

- if $d^2 = 0$ then

$$\mathbb{P}(W^1 > x) = \frac{d^1 \lambda^1 m}{\phi^1 - \rho^1} \overline{F^S}(x) + o(\overline{F^S}(x)),$$
$$\mathbb{P}(W^2 > x) = \frac{d^1 \lambda^1 m}{1 - \rho} \overline{F^S} \left(x \frac{\phi^1 - \rho^1}{\rho^2 - \phi^2} \right) + o(\overline{F^S}(x)).$$

- else

$$\mathbb{P}(W^1 > x) = \frac{d^1 \lambda^1 m}{\phi^1 - \rho^1} \overline{F^S}(x) + o(\overline{F^S}(x)),$$
$$\mathbb{P}(W^2 > x) = \frac{d^1 \lambda^1 m}{1 - \rho} \overline{F^S} \left(x \frac{\phi^1 - \rho^1}{\rho^2 - \phi^2} \right) + \frac{d^2 \lambda^2 m}{\phi^2 - \rho^2} \overline{F^S}(x) + o(\overline{F^S}(x)).$$

The case N=3

With the same parameters as in section 5.2 , we have :

$$\begin{aligned}\mathbb{P}(W_3 > x) \sim\ & \lambda^1 \int_{\frac{x}{\rho^2 - \phi^2}}^{\frac{x\phi^2}{\rho^3\phi^2 - \rho^2\phi^3}} \mathbb{P}\left[\sigma_0^1 > x \frac{\phi^2 + \phi^3}{\phi^3} + t\left(1 - \rho^1 - \rho^3 \left(1 + \frac{\phi^2}{\phi^3}\right)\right)\right] dt \\ + & \lambda^1 \int_{\frac{x\phi^2}{\rho^3\phi^2 - \rho^2\phi^3}}^{\infty} \mathbb{P}\left[\sigma_0^1 > x + (1-\rho)t\right] dt \\ + & \frac{\lambda^3}{\frac{\phi^3(1-\rho^1)}{\phi^2+\phi^3} - \rho^3} \int_x^{\infty} \mathbb{P}\left[\sigma_0^3 > t\right] dt.\end{aligned}$$

Here, we see that if the service times of class 3 are light-tailed and the service times of class 1 are heavy-tailed ($d^1 > 0$), then the stationary workload of queue 3 behaves like $O(\overline{F^S}(x))$ and we have the exact asymptotic.

7 Conclusion

We analyzed the behavior of Generalized Processor Sharing (GPS) queues with heavy tailed service times. We showed that the qualitative behavior of the individual queues depends on the relative values of the weight parameters and of the traffic intensities.
We can determine if the asymptotic of the stationary workload of an individual queue is $O(\overline{F^S}(x))$ or not. In the first case we calculated the exact asymptotic.

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