A Stochastic Particle Method with Random Weights for the Computation of Statistical Solutions of McKean-Vlasov Equations. Part II: Convergence Rate of The Method

Denis Talay, Olivier Vaillant

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Abstract: In the first part of this paper [7], we have proposed a stochastic particle method to compute statistical solutions of a McKean-Vlasov equation with random initial condition and we have empirically studied its convergence rate. In this second part, we estimate the convergence rate of our method in terms of the number of simulated particles and the time discretization step.

Key-words: stochastic particle system, McKean–Vlasov equations, statistical solutions

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Une méthode particulaire stochastique à poids aléatoires pour le calcul de solutions statistiques d'équations de McKean–Vlasov. Partie II : vitesse de convergence de la méthode.

Résumé : Dans la première partie de cet article [7], nous avons proposé une méthode particulaire stochastique pour calculer des solutions statistiques d'une équation de McKean-Vlasov à condition initiale aléatoire, et nous avons empiriquement étudié sa vitesse de convergence. Dans cette seconde partie, nous estimons la vitesse de convergence de notre méthode en fonction du nombre de particules simulées et du pas de discrétisation en temps.

Mots-clés : méthode particulaire stochastique, équations de McKean-Vlasov, solutions statistiques

1 Introduction

In the first part of this work, we have considered a McKean-Vlasov equation in $[0,T]\times\mathbb{R}$ with random initial condition:

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,x,\omega) & = -\dfrac{\partial}{\partial x}\left(u_b(t,x,\omega)p(t,x,\omega)\right)+\dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\left(\left(u_\sigma(t,x,\omega)\right)^2 p(t,x,\omega)\right), \\ p(0,x,\omega) & = p_0(x,\omega), \\ u_b(t,x,\omega) & := \int_{\mathbb{R}} b(x,y)p(t,y,\omega)dy, \\ u_\sigma(t,x,\omega) & := \int_{\mathbb{R}} \sigma(x,y)p(t,y,\omega)dy, \end{cases} \tag{1}$$

where b and σ are smooth and bounded functions from $\mathbb{R}^2$ to $\mathbb{R}$. For technical reasons we have supposed that the possible initial conditions of (1) are parametrized by realizations $\theta(\omega)$ of a real valued random variable θ with law ν concentrated on a closed interval of $\mathbb{R}$, say $[-1,1]$.
We have proposed a stochastic particle method to compute moments of the statistical solutions of Equation(6), that is, e.g.,

$$\langle M_1(t)\ ,\ f\rangle_{L^2(\mathbb{R})} := \mathbb{E}\int_{\mathbb{R}} p(t,x,\omega)f(x)dx, \tag{2}$$

where f is a square integrable, Lipschitz continuous and bounded function.
The objective of this second part is to estimate the convergence rate of this method, in terms of the number of simulated particles and the time discretization step.

Throughout this paper, we use the same notation as in [7]. We first recall the main result of [7] and the construction of the particle method, which is founded on the representation of $\langle M_1(t)\ ,\ f\rangle_{L^2(\mathbb{R})}$ as the expected value of a nonlinear stochastic process $X_\cdot$. The law of this process is characterized as follows:

Theorem 1.1. *Suppose that there exists $\varepsilon\in]0,1[$ and a strictly positive constant σ_* such that*

(i)
$$b\in C_b^{2+\varepsilon}\left(\mathbb{R}^2\right),\ \sigma\in C_b^{2+\varepsilon}\left(\mathbb{R}^2\right)\ \text{and, for any } (x,y)\in\mathbb{R}^2,\ \sigma(x,y)\geq\sigma_*>0. \tag{3}$$

In particular, b and σ are Lipschitz continuous in each of their variables with Lipschitz constants uniform over $\mathbb{R}^2$.

Suppose also that the function $\Phi a\in[-1,1]\mapsto p_0(\cdot,a)\in L^1(\mathbb{R})$ is a one to one application such that

(ii) $\sup_{a\in[-1,1]}\|p_0(.,a)\|_{W^{2,1}(\mathbb{R})}<+\infty$,

(iii) $\Phi([-1\ ,\ 1]) \subset C_b^{2+\varepsilon}(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ *and* $\Phi([-1\ ,\ 1])$ *is a set of probability density functions,*

(iv) Φ *is Lipschitz continuous for the norm in* $L^1(\mathbb{R})$.

Then the stochastic differential equation (SDE)

$$\begin{cases} dX_t & = \mathbb{E}\left[b\left(x, X_t\right) \mid \theta\right]|_{x=X_t}\, dt + \mathbb{E}\left[\sigma\left(x, X_t\right) \mid \theta\right]|_{x=X_t}\, dW_t,\ t \leq T, \\ (X_0, \theta) & \text{with law } \left[\Phi(a)\right](x)dx\ \nu(da), \\ \theta & \text{random variable independent of the Brownian motion } W_\cdot, \end{cases} \tag{4}$$

has a unique weak solution. The law of this solution is $\mathbb{P}_{X_\cdot(a)} \otimes \nu(da)$, *where* $X_\cdot(a)$ *is the weak solution of the SDE*

$$\begin{cases} dX_t(a) & = \mathbb{E}\left(b(x, X_t(a))\right)|_{x=X_t(a)}\, dt + \mathbb{E}\left(\sigma(x, X_t(a))\right)|_{x=X_t(a)}\, dW_t, \\ X_0(a) & \text{with law } \left[\Phi(a)\right](x)dx. \end{cases} \tag{5}$$

Under the hypotheses of Theorem 1.1 and for any $a \in [-1, 1]$, the McKean-Vlasov equation

$$\begin{cases} \dfrac{\partial p(t,x,a)}{\partial t} & = -\dfrac{\partial}{\partial x}\left(u_b(t,x,a)p(t,x,a)\right) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\Big(\left(u_\sigma(t,x,a)\right)^2 p(t,x,a)\Big), \\ p(0,x,a) & = p_0(x,a), \\ u_b(t,x,a) & := \displaystyle\int_{\mathbb{R}} b(x,y)p(t,y,a)dy, \\ u_\sigma(t,x,a) & := \displaystyle\int_{\mathbb{R}} \sigma(x,y)p(t,y,a)dy. \end{cases} \tag{6}$$

has a unique density solution in $C_b^{1,2+\varepsilon}([0,T] \times \mathbb{R}) \cap C\left([0,T], L^2(\mathbb{R})\right)$. Moreover, for any $t \in [0,T]$, $p(t,x,a)dx$ is the law of $X_t(a)$. Thus, in view of Theorem 1.1 and Definition (2),

$$u_b(t,x,a) = \mathbb{E}\left[b\left(x, X_t\right) \mid \theta = a\right], \quad u_\sigma(t,x,a) = \mathbb{E}\left[\sigma\left(x, X_t\right) \mid \theta = a\right] \tag{7}$$

and

$$\begin{aligned} \langle M_1(t)\ ,\ f\rangle_{L^2(\mathbb{R})} & = \int_{-1}^{1}\int_{\mathbb{R}} p(t,x,a)f(x)dx\ \nu(da), \\ & = \mathbb{E}^\nu\left[f\left(X_t\right)\right]. \end{aligned} \tag{8}$$

In order to develop a stochastic particle method to approximate $\mathbb{E}^\nu\left[f\left(X_t\right)\right]$, we approximate the conditional expectations in the right hand side of (4). To this end, we use nonparametric estimators [1] defined from N independent copies $\left(X^i,\ \theta^i\right)$, $1 \leq i \leq N$, of the process $(X_\cdot,\ \theta)$:

[1] For a discussion on the choice of these estimators, see Remark 4.1 in [7].

- if the measure ν is discrete, that is, $\nu = \sum_{\ell=1}^{M} p_\ell \delta_{a_\ell}$, we consider the regressogram estimator (see, e.g., Bouleau-Lépingle [1]):

$$\mathbb{E}\left[b\left(x,\ X_t\right) \mid \theta = a_\ell\right] \simeq \sum_{i=1}^{N} \frac{\mathbb{I}\left(\theta^i = a_\ell\right)}{\sum_{k=1}^{N} \mathbb{I}\left(\theta^k = a_\ell\right)} b\left(x, X_t^i\right). \tag{9}$$

- If the measure ν has a density, we consider the two following estimators:
  - the Nadaraya-Watson estimator (see, e.g., Hardle [3]):

$$\mathbb{E}\left[b\left(x,\ X_t\right) \mid \theta = a\right] \simeq \sum_{i=1}^{N} \frac{G\left((\theta^i - a)/h_N\right)}{\sum_{k=1}^{N} G\left((\theta^k - a)/h_N\right)} b\left(x, X_t^i\right), \tag{10}$$

    where $h_N > 0$ and G is, e.g., a Gaussian density,
  - the approximate regressogram estimator:

$$\mathbb{E}\left[b\left(x,\ X_t\right) \mid \theta = a\right] \simeq \sum_{i=1}^{N} \frac{\mathbb{I}\left(\tilde{\theta}^i = a\right)}{\sum_{k=1}^{N} \mathbb{I}\left(\tilde{\theta}^k = a\right)} b\left(x, \tilde{X}_t^i\right), \tag{11}$$

    where $\tilde{\nu}$ is a discrete probability measure approximating ν and the $\left(\tilde{X}^i, \tilde{\theta}^i\right)$, $1 \leq i \leq N$, are independent copies of the process $\left(\tilde{X}, \tilde{\theta}\right)$, weak solution of (4) with initial law $\left[\Phi(a)\right](x)dx\ \tilde{\nu}(da)$.

Thus, we get the particle system

$$1 \leq i \leq N, \left\{ \begin{array}{lcl} dX_t^{i,N} & = & \sum_{j=1}^{N} \alpha_{ij} b(X_t^{i,N}, X_t^{j,N})dt + \sum_{j=1}^{N} \alpha_{ij} \sigma(X_t^{i,N}, X_t^{j,N})dW_t^i, \\ X_t^{i,N}\mid_{t=0} & = & X_0^i, \end{array} \right. \tag{12}$$

where the $\left(X_0^i, \theta^i\right)$ are independent copies with common law $\left[\Phi(a)\right](x)dx\ \nu(da)$ and

$$\alpha_{ij} = \frac{\mathbb{I}(\theta^i = \theta^j)}{\sum_{k=1}^{N} \mathbb{I}(\theta^i = \theta^k)} \quad \text{(regressogram estimator)} \tag{13}$$

if ν is discrete, and

$$\alpha_{ij} = \frac{G\left((\theta^i - \theta^j)/h_N\right)}{\sum_{k=1}^{N} G\left((\theta^i - \theta^k)/h_N\right)} \quad \text{(Nadaraya-Watson estimator)} \tag{14}$$

or

$$\alpha_{ij} = \frac{\mathbb{I}(\tilde{\theta}^i = \tilde{\theta}^j)}{\sum_{k=1}^N \mathbb{I}(\tilde{\theta}^i = \tilde{\theta}^k)} \quad \text{(approximate regressogram estimator)} \tag{15}$$

if ν has a density.
Finally, the discretization of the SDE (12) by the Euler scheme with constant step $\Delta t = T/K$ ($t_k = k\Delta t,\ 0 \leq k \leq K$) leads to the simulated particle system

$$\begin{cases} \overline{X}^{i,N}_{t_{k+1}} &= \overline{X}^{i,N}_{t_k} + \sum_{j=1}^N \alpha_{ij} b(\overline{X}^{i,N}_{t_k}, \overline{X}^{j,N}_{t_k})\Delta t + \sum_{j=1}^N \alpha_{ij}\sigma(\overline{X}^{i,N}_{t_k}, \overline{X}^{j,N}_{t_k}) \left(W^i_{t_{k+1}} - W^i_{t_k}\right), \\ \overline{X}^{i,N}_0 &= X^i_0, \end{cases} \tag{16}$$

from which we deduce an approximation of the desired moments of the statistical solution

$$\langle M_1(T), f\rangle_{L^2(\mathbb{R})} = \mathbb{E}^\nu f(X_T) \simeq \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}^{i,N}_T\right). \tag{17}$$

We aim to estimate the accuracy of the particle method (17). To this end, we introduce the Euler scheme for the SDE (4)

$$\begin{cases} \overline{X}_{t_{k+1}} &= \overline{X}_{t_k} + u_b\left(t_k, \overline{X}_{t_k}, \theta\right)\Delta t + u_\sigma\left(t_k, \overline{X}_{t_k}, \theta\right)\left(W_{t_{k+1}} - W_{t_k}\right), \\ \overline{X}_0 &= X_0. \end{cases} \tag{18}$$

Considering N independent copies $\overline{X}^i_\cdot$, $1 \leq i \leq N$, of the process $\overline{X}_\cdot$, we split the convergence error of the particle method into three parts:

$$\begin{aligned} \langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}^{i,N}_T\right) &= \langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \mathbb{E}^\nu\left[f\left(\overline{X}_T\right)\right] \\ &\quad + \mathbb{E}^\nu\left[f\left(\overline{X}_T\right)\right] - \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}^i_T\right) \\ &\quad + \frac{1}{N}\sum_{i=1}^N \left(f\left(\overline{X}^i_T\right) - f\left(\overline{X}^{i,N}_T\right)\right). \end{aligned} \tag{19}$$

In view of Equation (8), the first term in the right hand side of (19) is a time discretization error. We estimate it in Section 2, owing to the results of Talay [6]. The second one is a statistical error. Indeed, in view of the Strong Law of Large Numbers,

$$\mathbb{E}^\nu \left| \mathbb{E}^\nu\left[f\left(\overline{X}_T\right)\right] - \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}^i_T\right)\right| \leq \frac{\|f\|_{L^\infty(\mathbb{R})}}{\sqrt{N}}. \tag{20}$$

The last term is an error related to the propagation of chaos of the particle system (12). In Sections 3 and 4, we estimate it successively for the three families of weights (13), (15) and (14).

2 Discretization of the limit process $X_{\cdot}$:

Proposition 2.1. *Suppose that the hypotheses of Theorem 1.1 hold. In addition, suppose that the functions* b *and* σ *are in* $C_b^{4+\varepsilon}\left(\mathbb{R}^2\right)$. *Then, for any test function* $f \in C_b^{4+\varepsilon}(\mathbb{R})$,

$$\left|\mathbb{E}^{\nu}\left[f\left(X_T\right)\right]-\mathbb{E}^{\nu}\left[f\left(\overline{X}_T\right)\right]\right| \leq C \Delta t.$$

The proof of Proposition 2.1 relies on the following technical lemma:

Lemma 2.2. *For any* $a \in [-1,1]$, *let* $\mathcal{L}_t^a$ *be the operator defined by*

$$\forall \phi \in C_b^2(\mathbb{R}), \ \mathcal{L}_t^a \phi(x)=\frac{1}{2} u_\sigma^2(t, x, a) \phi''(x)+u_b(t, x, a) \phi'(x).$$

Then, for any function $f \in C_b^{4+\varepsilon}(\mathbb{R})$, *the PDE*

$$\left\{\begin{array}{lll}\partial_t v(t, x, a)+\mathcal{L}_t^a v(t, x, a) & = & 0, \\ v(T, x, a) & = & f(x)\end{array}\right. \tag{21}$$

has a unique solution in the space $C_b^{2,4+\varepsilon}([0, T] \times \mathbb{R})$.
Moreover the mapping

$$a \in[-1,1] \mapsto \sup_{t \in[0, T]} \sum_{i=1}^4\left(\left\|\partial_x^{(i)} u_b(t, \cdot, a)\right\|_{L^{\infty}(\mathbb{R})}+\left\|\partial_x^{(i)} u_\sigma(t, \cdot, a)\right\|_{L^{\infty}(\mathbb{R})}+\left\|\partial_x^{(i)} v(t, \cdot, a)\right\|_{L^{\infty}(\mathbb{R})}\right)$$

is bounded.

Proof: we only sketch the proof detailed in Vaillant [8]. As the functions b et σ are in $C_b^{4+\varepsilon}\left(\mathbb{R}^2\right)$, one easily checks that, for any differentiation of order $q \leq 4$,

$$\begin{array}{lll}\sup\limits_{a \in[-1,1]} \sup\limits_{t \in[0, T]}\left(\left\|\partial_x^{(q)} u_b(t, \cdot, a)\right\|_{L^{\infty}(\mathbb{R})}+\left\|\partial_x^{(q)} u_\sigma(t, \cdot, a)\right\|_{L^{\infty}(\mathbb{R})}\right) & < & +\infty, \\ \sup\limits_{a \in[-1,1]}\left(\left\|u_b(t, \cdot, a)-u_b(s, \cdot, a)\right\|_{L^{\infty}(\mathbb{R})}+\left\|u_\sigma(t, \cdot, a)-u_\sigma(s, \cdot, a)\right\|_{L^{\infty}(\mathbb{R})}\right) & \leq & C \sqrt{t-s}.\end{array} \tag{22}$$

Lemma 2.2 is then a consequence of Theorem 5.1.9 in Lunardi [4]: Equation (21) has a unique solution $v(t, x, a)$ in $C_b^{2,4+\varepsilon}([0, T] \times \mathbb{R})$ and

$$\|v(., \cdot, a)\|_{C_b^{2,4+\varepsilon}([0, T] \times \mathbb{R})} \leq C\|f\|_{C_b^{4+\varepsilon}(\mathbb{R})}.$$

The constant C is uniform in a owing to Inequalities (22). □

Proof of Proposition 2.1: For any $a \in [-1,1]$, let $\left(\overline{X}_{t_k}(a)\right)_{k\leq K}$ the process defined by the Euler scheme, with constant discretization step $\Delta t = T/K$, applied to the SDE (5).The law of the process $\overline{X}.$ satisfies

$$\mathbb{P}_{\overline{X}.} = \mathbb{P}_{\overline{X}.(a)} \otimes \nu(da).$$

Consequently,

$$\mathbb{E}^{\nu}\left[f\left(X_T\right)\right] - \mathbb{E}^{\nu}\left[f\left(\overline{X}_T\right)\right] = \int_{-1}^{1}\left\{\mathbb{E}\left[f\left(X_T(a)\right)\right] - \mathbb{E}\left[f\left(\overline{X}_T(a)\right)\right]\right\}\nu(da). \tag{23}$$

Thus we have to prove that, for any $a \in [-1,1]$,

$$\left|\mathbb{E}\left[f\left(X_T(a)\right)\right] - \mathbb{E}\left[f\left(\overline{X}_T(a)\right)\right]\right| \leq C\Delta t,$$

with a constant C independent of a.
By the Feynman-Kac formula, the solution of (21) defined by Lemma 2.2 is given by

$$v(t,x,a) = \mathbb{E}\left[f\left(X_T^{t,x}(a)\right)\right],$$

where $X_{\cdot}^{t,x}(a)$ is the Markov process whose generator is $\mathcal{L}_t^a$ and such that $X_t^{t,x}(a) = x$ a.s. Hence,

$$\begin{aligned}\mathbb{E}f\left(X_T(a)\right) - \mathbb{E}f\left(\overline{X}_T(a)\right) &= \mathbb{E}v\left(0, X_0(a), a\right) - \mathbb{E}v\left(T, \overline{X}_T(a), a\right) \\ &= \sum_{k=1}^{K}\left[\mathbb{E}v\left(t_{k-1}, \overline{X}_{t_{k-1}}(a), a\right) - \mathbb{E}v\left(t_k, \overline{X}_{t_k}(a), a\right)\right].\end{aligned}$$

From this representation of the discretization error, Talay [6] showed that

$$\mathbb{E}f\left(X_T(a)\right) - \mathbb{E}f\left(\overline{X}_T(a)\right) \leq \mathcal{C}(T,a)\Delta t,$$

where $a \mapsto \mathcal{C}(T,a)$ is a sum of terms of the type $\partial_x^{(i)}u_b\, \partial_x^{(j)}u_\sigma\, \partial_x^{(k)}v,\ i,j,k \leq 4$. We conclude by using our Lemma 2.2. Estimate (23) is thus proved. □

We now give estimates for the third term of (19), namely

$$\frac{1}{N}\sum_{i=1}^{N}\left(f\left(\overline{X}_T^i\right) - f\left(\overline{X}_T^{i,N}\right)\right), \tag{24}$$

depending on the choice of the random weights (13), (14) or (15).

3 Global error estimates for the stochastic particle system with the regressogram estimator.

We distinguish two cases: the probability measure ν is discrete, or it is absolutely continuous. First, suppose that the measure ν is discrete:

$$\nu = \sum_{\ell=1}^{M} p_\ell \delta_{a_\ell}, \quad a_\ell \in [-1,1] \ \ \forall \ell \leq M, \text{ and } a_\ell \neq a_m \text{ if } \ell \neq m, \tag{25}$$

and define the random weights α_{ij} by (13).

Proposition 3.1. *Suppose that the hypotheses of Theorem 1.1 hold. Moreover, suppose that the functions b and σ are in $C_b^{4+\varepsilon}\left(\mathbb{R}^2\right)$. Then*

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}^\nu \left|\overline{X}_T^i - \overline{X}_T^{i,N}\right|^2 \leq C\left(\frac{M}{N} + M\left(\Delta t\right)^2\right). \tag{26}$$

Proof. It is convenient to rewrite the left hand side of (26) after having gathered particles having the same initial law, that is, to split the particles system (12) into M independent subsystems. To this end, set

$$\mathcal{C}(M,N) := \left\{(c(1),\dots,c(N)) \in \mathbb{N}^N / \forall 1 \leq i \leq N, \ \ c(i) \leq M\right\}.$$

We identify an element of this set and a random choice according to the law ν. For any $c = (c(1),\dots,c(N)) \in \mathcal{C}(M,N)$, define the processes $\overline{X}^i(c)$ and $\overline{X}^{i,N}(c)$ by

$$\left\{\begin{array}{l} \overline{X}^i_{t_{k+1}}(c) = \overline{X}^i_{t_k}(c) + u_b\left(t_k, \overline{X}^i_{t_k}(c), a_{c(i)}\right)\Delta t + u_\sigma\left(t_k, \overline{X}^i_{t_k}(c), a_{c(i)}\right)\left(W^i_{t_{k+1}} - W^i_{t_k}\right), \\ \overline{X}^i_0(c) \text{ with law } \left[\Phi\left(a_{c(i)}\right)\right](x)dx, \\ \alpha_{ij}(c) = \dfrac{\mathbb{I}\left(c(i) = c(j)\right)}{\sum_{k=1}^N \mathbb{I}\left(c(k) = c(i)\right)}, \\ \overline{X}^{i,N}_{t_{k+1}}(c) = \overline{X}^{i,N}_{t_k}(c) + \Delta t \sum_{j=1}^N \alpha_{ij}(c) b\left(\overline{X}^{i,N}_{t_k}(c), \overline{X}^{j,N}_{t_k}(c)\right) \\ \qquad\qquad + \sum_{j=1}^N \alpha_{ij}(c)\sigma\left(\overline{X}^{i,N}_{t_k}(c), \overline{X}^{j,N}_{t_k}(c)\right)\left(W^i_{t_{k+1}} - W^i_{t_k}\right), \\ \overline{X}^{i,N}_0(c) = \overline{X}^i_0(c). \end{array}\right. \tag{27}$$

Setting $E_\ell(c) = \{j \in \mathbb{N}, 1 \leq j \leq N / c(j) = c(\ell)\}$, $\underline{\theta} = \left(\theta^1, \dots \theta^N\right)$ and $\underline{a}_c = \left(a_{c(1)}, \dots, a_{c(N)}\right)$, we observe that

$$\mathbb{E}^\nu\left\{\frac{1}{N}\sum_{i=1}^N \left|\overline{X}^i_{t_k} - \overline{X}^{i,N}_{t_k}\right|^2\right\} = \sum_{c \in \mathcal{C}(M,N)} \mathbb{P}\left(\underline{\theta} = \underline{a}_c\right)\left\{\frac{1}{N}\sum_{\ell=1}^M \sum_{j \in E_\ell(c)} \mathbb{E}\left|\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\right|^2\right\}. \tag{28}$$

Suppose to have proven that

$$\forall \ell \leq M,\ \forall j \in E_\ell(c), \quad \mathbb{E}\left[\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\right]^2 \leq C\left(\frac{1}{\sharp E_\ell(c)} + (\Delta t)^2\right). \tag{29}$$

Then

$$\begin{aligned}
\mathbb{E}^\nu\left\{\frac{1}{N}\sum_{i=1}^N \left|\overline{X}^i_{t_k} - \overline{X}^{i,N}_{t_k}\right|^2\right\} &\leq \sum_{c\in\mathcal{C}(M,N)} \mathbb{P}\left(\underline{\theta} = \underline{a}_c\right)\left\{\frac{C}{N}\sum_{\ell=1}^M \left(1 + \sharp E_\ell(c)\ (\Delta t)^2\right)\right\} \\
&\leq C\left\{\frac{M}{N} + M(\Delta t)^2\right\}.
\end{aligned}$$

Proof of Estimate (29).
For any $\ell \leq M$ and $j \in E_\ell(c)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\overline{X}^j_{t_{k+1}}(c) - \overline{X}^{j,N}_{t_{k+1}}(c)\right]^2 \\
&\quad = \mathbb{E}\left[\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\right]^2 \\
&\qquad + (\Delta t)^2\, \mathbb{E}\left[\tfrac{1}{\sharp E_\ell(c)}\textstyle\sum_{i\in E_\ell(c)} b\left(\overline{X}^{j,N}_{t_k}(c), \overline{X}^{i,N}_{t_k}(c)\right) - u_b(t_k, \overline{X}^j_{t_k}(c), a_{c(\ell)})\right]^2 \\
&\qquad + \Delta t\, \mathbb{E}\left[\tfrac{1}{\sharp E_\ell(c)}\textstyle\sum_{i\in E_\ell(c)} \sigma\left(\overline{X}^{j,N}_{t_k}(c), \overline{X}^{i,N}_{t_k}(c)\right) - u_\sigma(t_k, \overline{X}^j_{t_k}(c), a_{c(\ell)})\right]^2 \\
&\qquad + 2\Delta t\, \mathbb{E}\left[\left(\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\right)\left(\tfrac{1}{\sharp E_\ell(c)}\textstyle\sum_{i\in E_\ell(c)} b\left(\overline{X}^{j,N}_{k\Delta t}(c), \overline{X}^{i,N}_{t_k}(c)\right) - u_b(t_k, \overline{X}^j_{k\Delta t}(c), a_{c(\ell)})\right)\right].
\end{aligned} \tag{30}$$

Observe that, owing to Theorem 1.1 and Equalities (7),

$$u_b\left(t, x, a_{c(j)}\right) = \mathbb{E}\left[b\left(x, X_t\left(a_{c(j)}\right)\right)\right] \text{ and } u_\sigma\left(t, x, a_{c(j)}\right) = \mathbb{E}\left[\sigma\left(x, X_t\left(a_{c(j)}\right)\right)\right].$$

Then, inserting $\mathbb{E}\left[b(x, \overline{X}^j_{t_k}(c))\right]\,|_{x=\overline{X}^j_{t_k}(c)}$ and $\mathbb{E}\left[\sigma(x, \overline{X}^j_{t_k}(c))\right]\,|_{x=\overline{X}^j_{t_k}(c)}$ in Equality (30), it comes:

$$\begin{aligned}
\mathbb{E}\left[\overline{X}^j_{t_{k+1}}(c) - \overline{X}^{j,N}_{t_{k+1}}(c)\right]^2 &\leq (1 + C\Delta t)\,\mathbb{E}\left[\overline{X}^j_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\right]^2 \\
&\quad + C\Delta t A_1(j, l, c, t_k) + C\Delta t A_2(j, l, c, t_k),
\end{aligned} \tag{31}$$

with

$$
\begin{aligned}
A_1(j,l,c,t_k) &= \mathbb{E}\left[\frac{1}{\sharp E_\ell(c)}\sum_{i\in E_\ell(c)} b\left(\overline{X}^{j,N}_{t_k}(c),\overline{X}^{i,N}_{t_k}(c)\right) - \mathbb{E}\left[b(x,\overline{X}^{j}_{t_k}(c))\right]|_{x=\overline{X}^{j}_{t_k}(c)}\right]^2 \\
&+\mathbb{E}\left[\frac{1}{\sharp E_\ell(c)}\sum_{i\in E_\ell(c)} \sigma\left(\overline{X}^{j,N}_{t_k}(c),\overline{X}^{i,N}_{t_k}(c)\right) - \mathbb{E}\left[\sigma(x,\overline{X}^{j}_{t_k}(c))\right]|_{x=\overline{X}^{j}_{t_k}(c)}\right]^2, \\
A_2(j,l,c,t_k) &= \left|\mathbb{E}\left[b\left(x,X_{t_k}\left(a_{c(j)}\right)\right)\right]|_{x=\overline{X}^{j}_{t_k}(c)} - \mathbb{E}\left[b(x,\overline{X}^{j}_{t_k}(c))\right]|_{x=\overline{X}^{j}_{t_k}(c)}\right|^2 \\
&+\left|\mathbb{E}\left[\sigma\left(x,X_{t_k}\left(a_{c(j)}\right)\right)\right]|_{x=\overline{X}^{j}_{t_k}(c)} - \mathbb{E}\left[\sigma(x,\overline{X}^{j}_{t_k}(c))\right]|_{x=\overline{X}^{j}_{t_k}(c)}\right|^2.
\end{aligned}
$$

In order to estimate $A_1(j,l,c,t_k)$, observe that the particles $\left(\overline{X}^{i,N}_{\cdot}\right)_{i\in E_\ell(c)}$ have the same weight $1/\left(\sharp E_\ell(c)\right)$ and the same initial law $p_0\left(x,a_{c(\ell)}\right)dx$. Thus, using the symetry of the particle system $\left(\overline{X}^{i,N}_{\cdot}\right)_{i\in E_\ell(c)}$ and the Lipschitz property of functions b and σ, one shows that (see, e.g., Sznitman [5])

$$
A_1(j,l,c,t_k) \le C\left(\mathbb{E}\left[\overline{X}^{j}_{t_k}(c) - \overline{X}^{j,N}_{t_k}(c)\right]^2 + \frac{1}{\sharp E_\ell(c)}\right). \tag{32}
$$

Furthermore, in view of Proposition 2.1,

$$
A_2(j,l,c,t_k) \le C(\Delta t)^2. \tag{33}
$$

Owing to Inequalities (31), (32) and (33), we deduce Estimate (29) by induction. □

We are now in a position to estimate the accuracy of the particle method. This is a straightforward consequence of (19), (20) and Propositions 2.1 and 3.1:

Theorem 3.2. (**Discrete case**) *Suppose that the probability measure ν is of the form (25) and that the hypotheses of Theorem 1.1 hold. In addition, suppose that the functions b and σ are in $C_b^{4+\varepsilon}\left(\mathbb{R}^2\right)$. Consider the particle system (16) with weights (13). Then, for any function $f \in C_b^{4+\varepsilon}(\mathbb{R})$,*

$$
\mathbb{E}^\nu\left|\langle M_1(T),f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}^{i,N}_T\right)\right|^2 \le C\left(\frac{M}{N} + M(\Delta t)^2\right)
$$

We now study the convergence rate of the particle method with weights (15).

Theorem 3.3. *Let ν a probability measure on $[-1,1]$ with density q and distribution function V. We define the weights of the particle method by*

$$\alpha_{ij} = \frac{\mathbb{I}\left(\tilde{\theta}^i = \tilde{\theta}^j\right)}{\sum_{k=1}^{N} \mathbb{I}\left(\tilde{\theta}^i = \tilde{\theta}^k\right)},$$

where the independent random variables $\tilde{\theta}^i$, $1 \le i \le N$, have common law

$$\nu_M = \frac{1}{M} \sum_{\ell=1}^{M} \delta_{V^{-1}(\ell/M)}, \quad M \in \mathbb{N}.$$

Suppose that

(i) *The hypotheses of Theorem 3.2 hold,*

(ii) *There exists a strictly positive constant q_* such that*

$$\forall a \in [-1,1], \;\; q(a) \ge q_* > 0.$$

Then, for any test function $f \in C_b^{4+\varepsilon}(\mathbb{R})$,

$$\mathbb{E}^{\nu_M} \left| \langle M_1(T), f \rangle_{L^2(\mathbb{R})} - \frac{1}{N} \sum_{i=1}^{N} f(\overline{X}_T^{i,N}) \right|^2 \le C \left(\frac{M}{N} + M(\Delta t)^2 + \frac{1}{M^2} \right).$$

Proof. Owing to Theorem 3.2, we already know that

$$\mathbb{E}^{\nu_M} \left| \mathbb{E}^{\nu_M} f(X_T) - \frac{1}{N} \sum_{i=1}^{N} f(\overline{X}_T^{i,N}) \right|^2 \le C \left(\frac{M}{N} + M(\Delta t)^2 \right).$$

We thus have to prove that

$$\left| \langle M_1(T), f \rangle_{L^2(\mathbb{R})} - \mathbb{E}^{\nu_M} f(X_T) \right|^2 \le \frac{C}{M^2}$$

that is,

$$\left| \mathbb{E}^{\nu} f(X_T) - \mathbb{E}^{\nu_M} f(X_T) \right|^2 \le \frac{C}{M^2}. \tag{34}$$

Let $X_{\cdot}\left(V^{-1}(y)\right)$ denote the solution of the SDE (5) with initial law $\left[\Phi\left(V^{-1}(y)\right)\right](x)dx$. Set

$$F_T(f,\cdot)\, y \in [0,1] \mapsto \mathbb{E}\left[f\left(X_T\left(V^{-1}(y)\right)\right)\right].$$

As V is the distribution function of the measure ν, one has

$$\mathbb{E}^{\nu} f\left(X_T\right) - \mathbb{E}^{\nu_M} f\left(X_T\right) \quad = \quad \int_0^1 F_Y(f,y)dy - \frac{1}{M}\sum_{\ell=1}^{M} F_T(f,\ell/M)$$

Observe that, for any $(y_1, y_2) \in [0,1]^2$,

$$F_T\left(f,y_1\right) - F_T\left(f,y_2\right) \quad = \quad \int_{\mathbb{R}} f(x)\left(p\left(t,x,V^{-1}(y_1)\right) - p\left(t,x,V^{-1}(y_2)\right)\right)dx,$$

where $p\left(t,x,V^{-1}(y_i)\right)$ is solution of the PDE (6) with initial condition $\Phi\left(V^{-1}(y_i)\right)$. In view of Proposition 2.2 of [7], we know that the mapping

$$a \in [-1,1] \mapsto p(t,\cdot,a) \in L^1\left(\mathbb{R}\right)$$

is Lipschitz continuous. Thus we deduce that

$$\left|F_T\left(f,y_1\right) - F_T\left(f,y_2\right)\right| \leq C\left|V^{-1}(y_1) - V^{-1}(y_2)\right|.$$

In addition, the mapping V^{-1} is Lipschitz continuous since

$$\sup_{y\in[0,1]}\left|\frac{d}{dy}V^{-1}(y)\right| = \sup_{y\in[0,1]}\left|\frac{1}{q(V^{-1}(y))}\right| \leq 1/q_*.$$

Consequently,

$$\left|F_T\left(f,y_1\right) - F_T\left(f,y_2\right)\right| \leq C\left|y_1 - y_2\right|. \tag{35}$$

Thus Estimate (34) readily follows from (35).

Remark 3.4. *We can deduce from Theorem 3.3 that, in this case, the particle method converges if*

$$\lim_{N\to+\infty}\frac{M}{N} = 0 \quad \text{and} \quad \lim_{\Delta t\to 0} M(\Delta t)^2 = 0.$$

The first condition is natural: the measure ν_M, concentrated in M points, can be well approximated by the empirical measure of $\left(\theta^1,\ldots,\theta^N\right)$ only if $N \geq\geq M$. The second condition is a relationship between the time and space discretization steps, Δt and $1/M$. It is implied by $M\Delta t$ = constant, which is a C.F.L. condition, implying the stability of the numerical scheme.

4 Global error estimates for the particle system with the Nadaraya-Watson estimator.

In this section, we suppose that the probability measure ν is supported in $[-1,1]$, and has a strictly positive Lipschitz continuous density q. The weights α_{ij} of the particle method are defined by (14):

$$\alpha_{ij} = \frac{G_N\left(\theta^i - \theta^j\right)}{\sum_{k=1}^N G_N\left(\theta^i - \theta^k\right)},$$

where $G_N(.) := \frac{1}{h_N} G\left(./h_N\right)$, $h_N > 0$ and G is a Gaussian density on $\mathbb{R}$.

We gather estimates for G_N is the following lemma.

Lemma 4.1. *One has*

$$\forall N \in \mathbb{N},\ \forall x \in \mathbb{R},\ \mathbb{E}^\nu G_N(\theta - x) \leq \|q\|_{L^\infty(\mathbb{R})}, \tag{36}$$

and

$$\forall x \in \mathbb{R},\ \mathbb{E}^\nu G_N(x - \theta) \geq q_* \int_0^1 G(z)dz. \tag{37}$$

Moreover, for any Lipschitz continuous bounded real function ϕ on $[-1,1]$ with Lipschitz constant L_ϕ, one has

$$\int_{-1}^1 \left| \int_{\mathbb{R}} G_N(z-x)\phi(z)dz - \phi(x) \right|^2 dx \leq C(L_\phi^2 h_N^2 + \|\phi\|_\infty^2 T_G(N)), \tag{38}$$

where

$$T_G(N) := \int_{-1}^1 \left[\int_{-\infty}^{\frac{x-1}{h_N}} \left(G(z) + G(z)^2\right) dz + \int_{\frac{x+1}{h_N}}^{+\infty} \left(G(z) + G(z)^2\right) dz \right]^2 dx. \tag{39}$$

In addition,

$$T_G(N) \leq C h_N \ \textit{and} \ \int_{-1}^1 \left| \int_{\mathbb{R}} G_N(x-z)\phi(z)dz - \phi(x) \right|^2 dx \leq C\ h_N. \tag{40}$$

Proof. Inequality (36) results from

$$\begin{aligned} \mathbb{E}^\nu G_N(\theta - x) &= \int_{-1}^1 \frac{1}{h_N} G\left(\frac{x-z}{h_N}\right) \phi(z)dz \\ &= \int_{\frac{x-1}{h_N}}^{\frac{x+1}{h_N}} G(z)\phi(x - zh_N)dz \\ &\leq \int_{\mathbb{R}} G(z)\phi(x - zh_N)dz. \end{aligned}$$

Inequality (37) results from $h_N \leq 1$.

Now observe:

$$\int_{\mathbb{R}} G_N(x-z)\phi(z)dz - \phi(x) = \int_{\frac{x-1}{h_N}}^{\frac{x+1}{h_N}} G\left(\frac{z}{h_N}\right)\phi(x-z)dz - \int_{\mathbb{R}} G(z)\phi(x)dz$$
$$= \int_{\frac{x-1}{h_N}}^{\frac{x+1}{h_N}} G(z)(\phi(x-zh_N)-\phi(x))dz - \phi(x)\left[\int_{-\infty}^{\frac{x-1}{h_N}} G(z)dz + \int_{\frac{x+1}{h_N}}^{+\infty} G(z)dz\right].$$

Inequality (38) readily follows.

Finally, Inequality (40) follows from

$$\forall z>0,\quad \int_z^{+\infty} \exp(-x^2)dx \le C\exp(-z^2).$$

□

Our next statement provides a propagation of chaos type estimate.

Proposition 4.2. *Suppose that*

(i) *The hypotheses of Theorem 1.1 hold,*

(ii) *In addition, the interaction kernels b and σ are in $C_b^{4+\varepsilon}\left(\mathbb{R}^2\right)$, $0<\varepsilon<1$,*

(iii) *The sequence (h_N) tends to 0 and $\lim_{N\to+\infty} \log(N)/(Nh_N^2)=0$,*

(iv) $\sup_{a\in[-1,1]} \int_{\mathbb{R}} x^4 p_0(x,a)dx < +\infty$,

(v) *The probability measure ν has a strictly positive Lipschitz continuous density q on $[-1,1]$.*

Then there exist a strictly positive constant C independent of N and Δt, and an integer N_0 such that, for any $N \ge N_0$,

$$\frac{1}{N}\sum_{i=1}^N \left(\mathbb{E}^\nu \left|\overline{X}_T^i - \overline{X}_T^{i,N}\right|^2\right) \quad \le \quad C\left(\frac{1}{\sqrt{Nh_N^2}} + \sqrt{h_N} + (\Delta t)^2\right).$$

Proof. Similarly to what we got in the proof of Proposition 3.1, for all indices $i\le N$ and $k\le K$ we have

$$\mathbb{E}^\nu \left|\overline{X}_{t_{k+1}}^i - \overline{X}_{t_{k+1}}^{i,N}\right|^2 \le (1+C\Delta t)\,\mathbb{E}^\nu\left|\overline{X}_{t_k}^i - \overline{X}_{t_k}^{i,N}\right|^2 + C\Delta t\left(A_1(i,t_k)+A_2(i,t_k)\right), \quad (41)$$

$$A_1(i,t_k) = \mathbb{E}^\nu\left[\sum_{j=1}^N \alpha_{ij} b\left(\overline{X}^{i,N}_{t_k}, \overline{X}^{j,N}_{t_k}\right) - \mathbb{E}^\nu\left[b(x,\overline{X}^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})}\right]^2 + \mathbb{E}^\nu\left[\sum_{j=1}^N \alpha_{ij} \sigma\left(\overline{X}^{i,N}_{t_k}, \overline{X}^{j,N}_{t_k}\right) - \mathbb{E}^\nu\left[\sigma(x,\overline{X}^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})}\right]^2,$$

where

$$A_2(i,t_k) = \left|\mathbb{E}^\nu\left[b(x,X^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})} - \mathbb{E}^\nu\left[b(x,\overline{X}^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})}\right|^2 + \left|\mathbb{E}^\nu\left[\sigma(x,X^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})} - \mathbb{E}^\nu\left[\sigma(x,\overline{X}^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})}\right|^2.$$

In view of Proposition 2.1,

$$A_2(i,t_k) \le C(\Delta t)^2. \tag{42}$$

We first consider $A_1(i,t_k)$. We insert $b\left(\overline{X}^i_{t_k}, \overline{X}^j_{t_k}\right)$ and use the Lipschitz property of b. As

$$\forall 1 \le i,\ j \le N,\ \ \alpha_{ij} > 0 \ \text{ and } \sum_{j=1}^N \alpha_{ij} = 1, \tag{43}$$

Jensen's inequality and easy computations lead to

$$\begin{aligned} A_1(i,t_k) \le\ & C\left(\mathbb{E}^\nu\left|\overline{X}^{i,N}_{t_k} - \overline{X}^i_{t_k}\right|^2 + \mathbb{E}^\nu\left[\sum_{j=1}^N \alpha_{ij}\left|\overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k}\right|^2\right]\right) \\ & + C\left(\mathbb{E}^\nu\left[\alpha_{ii}\ b\left(\overline{X}^i_{t_k}, \overline{X}^i_{t_k}\right)\right]^2 + \mathbb{E}^\nu\left[\alpha_{ii}\ \sigma\left(\overline{X}^i_{t_k}, \overline{X}^i_{t_k}\right)\right]^2\right) \\ & + C\mathbb{E}^\nu\left[\sum_{\substack{j=1\\ j\neq i}}^N \alpha_{ij}\ b\left(\overline{X}^i_{t_k}, \overline{X}^j_{t_k}\right) - \mathbb{E}^\nu\left[b(x,\overline{X}^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})}\right]^2 \\ & + C\mathbb{E}^\nu\left[\sum_{\substack{j=1\\ j\neq i}}^N \alpha_{ij}\ \sigma\left(\overline{X}^i_{t_k}, \overline{X}^j_{t_k}\right) - \mathbb{E}^\nu\left[\sigma(x,\overline{X}^i_{t_k}) \mid \theta^i\right] |_{x=\overline{X}^i_{t_k})}\right]^2. \end{aligned} \tag{44}$$

Set

$$
\begin{aligned}
S_N(t_k) &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}^{\nu}\left|\overline{X}_{t_k}^{i,N} - \overline{X}_{t_k}^{i}\right|^2, \\
\delta_0(t_k) &= \frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}^{\nu}\left[\alpha_{ii}\; b\left(\overline{X}_{t_k}^{i}, \overline{X}_{t_k}^{i}\right)\right]^2 + \mathbb{E}^{\nu}\left[\alpha_{ii}\;\sigma\left(\overline{X}_{t_k}^{i}, \overline{X}_{t_k}^{i}\right)\right]^2\right), \\
\Delta_b(t_k) &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}^{\nu}\left[\sum_{\substack{j=1\\ j\neq i}}^{N}\alpha_{ij}\; b\left(\overline{X}_{t_k}^{i}, \overline{X}_{t_k}^{j}\right) - \mathbb{E}^{\nu}\left[b(x, \overline{X}_{t_k}^{i}) \mid \theta^i\right]|_{x=\overline{X}_{t_k}^{i})}\right]^2, \\
\Delta_\sigma(t_k) &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}^{\nu}\left[\sum_{\substack{j=1\\ j\neq i}}^{N}\alpha_{ij}\;\sigma\left(\overline{X}_{t_k}^{i}, \overline{X}_{t_k}^{j}\right) - \mathbb{E}^{\nu}\left[\sigma(x, \overline{X}_{t_k}^{i}) \mid \theta^i\right]|_{x=\overline{X}_{t_k}^{i})}\right]^2.
\end{aligned}
$$

In view of (41), (42) and (44) we finally get

$$
\begin{aligned}
S_N(t_{k+1}) \leq\; & (1 + C\Delta t)\, S_N(t_k) \\
& + C\Delta t\,(\Delta t + \delta_0(t_k) + \Delta_b(t_k) + \Delta_\sigma(t_k)) \\
& + \frac{C\Delta t}{N}\sum_{j=1}^{N}\mathbb{E}^{\nu}\left(\left|\overline{X}_{t_k}^{j,N} - \overline{X}_{t_k}^{j}\right|^2 \sum_{i=1}^{N}\alpha_{ij}\right).
\end{aligned} \tag{45}
$$

The terms $\Delta_b(t_k)$ and $\Delta_\sigma(t_k)$ characterize the accuracy of the approximation of the regression functions $\mathbb{E}^{\nu}\left[b(x, \overline{X}_{t_k}^{i}) \mid \theta^i = a\right]$ and $\mathbb{E}^{\nu}\left[\sigma(x, \overline{X}_{t_k}^{i}) \mid \theta^i = a\right]$ by the Nadaraya-Watson estimator (10). Indeed, using results of Collomb [2] (we refer to Vaillant [8, Sec.4.3.3] for the easy and lengthy modification of Collomb's calculation[2]) and (38) one can check that

$$
\Delta_b(t_k) + \Delta_\sigma(t_k) \leq C\left(\frac{1}{Nh_N} + h_N^2 + T_G(N)\right). \tag{46}
$$

In view of (40) we deduce

$$
\Delta_b(t_k) + \Delta_\sigma(t_k) \leq C\left(\frac{1}{Nh_N} + h_N\right). \tag{47}
$$

The term

$$
\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}^{\nu}\left(\left|\overline{X}_{t_k}^{j,N} - \overline{X}_{t_k}^{j}\right|^2 \sum_{i=1}^{N}\alpha_{ij}\right)
$$

[2] Here the assumption that q is Lipschitz is used in force.

characterizes the uncertainty on the initial condition of the PDE (6). Indeed, if the measure ν were a Dirac mass, all the weights α_{ij} would be equal to $1/N$ and

$$\frac{1}{N}\sum_{j=1}^{N} \mathbb{E}^{\nu}\left(\left|\overline{X}_{t_k}^{j,N} - \overline{X}_{t_k}^{j}\right|^2 \sum_{i=1}^{N} \alpha_{ij}\right) = S_N\left(t_k\right).$$

Suppose that we have shown the lemma 4.3 below. Owing to estimates (47) and (48), Proposition 4.2 then follows from the induction (45). □

Lemma 4.3. *Suppose that the hypotheses of Proposition 4.2 hold. Then there exists a strictly positive constant C, independent of N and Δt, and an integer N_0 such that, for any $N \geq N_0$,*

$$\delta_0\left(t_k\right) + \frac{1}{N}\sum_{j=1}^{N} \mathbb{E}^{\nu}\left(\sum_{i=1}^{N} \alpha_{ij}\left|\overline{X}_{t_k}^{j,N} - \overline{X}_{t_k}^{j}\right|^2\right) \leq C\left(S_N\left(t_k\right) + \frac{1}{\sqrt{N h_N^2}} + \sqrt{h_N}\right). \tag{48}$$

Proof. As noticed before, if the measure ν were a Dirac mass, all the weights α_{ij} would be equal to $1/N$. We thus naturally found useful to rewrite $\sum_{i=1}^{N}\alpha_{ij}$ in order to separately estimate the different sources of fluctuation around the value 1:

$$\begin{aligned} \sum_{i=1}^{N} \alpha_{ij} &= \sum_{i=1}^{N} \frac{G_N(\theta^i - \theta^j)}{\frac{G(0)}{h_N} + \sum_{\substack{k=1\\k\neq i}}^{N} G_N(\theta^i - \theta^k)}, \\ &=: 1 + A_1(j) + A_2(j) + A_3(j) + A_4(j) + A_5(j) + A_6(j), \end{aligned} \tag{49}$$

with

$$
\begin{aligned}
A_1(j) &= \mathbb{E}^{\nu}\left(\frac{G_N(\theta^i - x)}{q(\theta^i)}\right)\Big|_{x=\theta^j} - 1,\\
A_2(j) &= \mathbb{E}^{\nu}\left(\frac{G_N(\theta^i - x)}{\frac{G(0)}{(N-1)h_N} + q(\theta^i)}\right)\Bigg|_{x=\theta^j} - \mathbb{E}^{\nu}\left(\frac{G_N(\theta^i - x)}{q(\theta^i)}\right)\Big|_{x=\theta^j},\\
A_3(j) &= \frac{1}{N-1}\sum_{\substack{i=1\\ i\neq j}}^{N}\frac{G_N(\theta^i - \theta^j)}{\frac{G(0)}{(N-1)h_N} + q(\theta^i)} - \mathbb{E}^{\nu}\left(\frac{G_N(\theta^i - x)}{\frac{G(0)}{(N-1)h_N} + q(\theta^i)}\right)\Bigg|_{x=\theta^j},\\
A_4(j) &= \frac{1}{N-1}\sum_{\substack{i=1\\ i\neq j}}^{N}\left(\frac{G_N(\theta^i - \theta^j)}{\frac{G(0)}{(N-1)h_N} + \mathbb{E}^{\nu}G_N(x-\theta)\mid_{x=\theta^i}} - \frac{G_N(\theta^i - \theta^j)}{\frac{G(0)}{(N-1)h_N} + q(\theta^i)}\right),\\
A_5(j) &= \frac{1}{N-1}\sum_{\substack{i=1\\ i\neq j}}^{N}\left(\frac{G_N(\theta^i - \theta^j)}{\frac{G(0)}{(N-1)h_N} + \frac{1}{N-1}\sum_{\substack{k=1\\ k\neq i}}^{N} G_N(\theta^i - \theta^k)} - \frac{G_N(\theta^i - \theta^j)}{\frac{G(0)}{(N-1)h_N} + \mathbb{E}^{\nu}G_N(x-\theta)\mid_{x=\theta^i}}\right),\\
A_6(j) &= \frac{1}{N-1}\left(\frac{G_N(0)}{\frac{G(0)}{(N-1)h_N} + \frac{1}{N-1}\sum_{\substack{k=1\\ k\neq j}}^{N} G_N(\theta^j - \theta^k)} - \frac{G_N(0)}{\frac{G(0)}{(N-1)h_N} + \mathbb{E}^{\nu}G_N(x-\theta)\mid_{x=\theta^j}}\right)\\
&\quad + \frac{1}{N-1}\left(\frac{G_N(0)}{\frac{G(0)}{(N-1)h_N} + \mathbb{E}^{\nu}G_N(x-\theta)\mid_{x=\theta^j}} - \frac{G_N(0)}{\frac{G(0)}{(N-1)h_N} + q(\theta^j)}\right)\\
&\quad + \frac{1}{N-1}\frac{G_N(0)}{\frac{G(0)}{(N-1)h_N} + q(\theta^j)}.
\end{aligned}
$$

Under hypothesis **(iv)** of Proposition 4.2, the random variables X_0^i, $1 \le i \le N$ have moments up to order 4. Hence, since the functions b and σ are bounded and the weights α_{ij} satisfy (43),

$$
\sup_{t_k\in[0,T]}\ \sup_{1\le i\le N}\left(\mathbb{E}^{\nu}\left[\overline{X}^i_{t_k}\right]^4 + \mathbb{E}^{\nu}\left[\overline{X}^{i,N}_{t_k}\right]^4\right) < +\infty.
$$

Hence we have

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}^{\nu}\left\{\left(\sum_{i=1}^{N}\alpha_{ij}\right)\left|\overline{X}^{j,N}_{t_k}-\overline{X}^{j}_{t_k}\right|^2\right\}\\
&\qquad = S_N(t_k)\\
&\qquad\quad +\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}^{\nu}\left(\sum_{k=1}^{6}A_k(j)\left|\overline{X}^{j,N}_{t_k}-\overline{X}^{j}_{t_k}\right|^2\right),\\
&\qquad \leq S_N(t_k)\\
&\qquad\quad +C\frac{1}{N}\sum_{j=1}^{N}\left(\mathbb{E}^{\nu}\left|A_4(j)\right|+\sqrt{\mathbb{E}^{\nu}\left|\overline{X}^{j,N}_{t_k}-\overline{X}^{j}_{t_k}\right|^4}\left(\sum_{\substack{k=1\\k\neq 4}}^{6}\sqrt{\mathbb{E}^{\nu}A_k(j)^2}\right)\right).
\end{aligned}$$

Estimate (48) then results from estimates (50), (51), (52), (54), (59) and (61) below. □

Estimate for the second moment of $A_1(j)$. We have

$$\begin{aligned}
\mathbb{E}^{\nu}\mid A_1(j)\mid^2 &= \int_{-1}^{1}\mid \mathbb{E}^{\nu}\left(\frac{G_N(\theta-x)}{q(\theta)}\right)-1\mid^2 q(x)dx\\
&= \int_{-1}^{1}\mid\int_{\mathbb{R}}G_N(x-z)\mathbb{I}_{[-1,1]}(z)dz-\mathbb{I}_{[-1,1]}(x)\mid^2 q(x)dx\\
&\leq \|q\|_{L^{\infty}(\mathbb{R})}\int_{-1}^{1}\mid\int_{\mathbb{R}}G_N(x-z)\mathbb{I}_{[-1,1]}(z)dz-\mathbb{I}_{[-1,1]}(x)\mid^2 dx.
\end{aligned}$$

In view of Lemma 4.1 we conclude

$$\mathbb{E}^{\nu}\mid A_1(j)\mid^2\leq C\ h_N. \tag{50}$$

Estimate for the second moment of $A_2(j)$. As q is a strictly positive continuous function on the compact set $[-1,1]$, there exists a strictly positive constant q_* such that, for any $y\in[-1,1]$, $q(y)\geq q_*>0$. Fix $x\in[-1,1]$. We have

$$\begin{aligned}
\mathbb{E}^{\nu}\left(\frac{G_N(\theta-x)}{q(\theta)}\right)-\mathbb{E}^{\nu}\left(\frac{G_N(\theta-x)}{\frac{G(0)}{(N-1)h_N}+q(\theta)}\right) &= \mathbb{E}^{\nu}\left(\frac{G_N(\theta-x)}{q(\theta)\left(\frac{G(0)}{(N-1)h_N}+q(\theta)\right)}\right)\frac{G(0)}{(N-1)h_N},\\
&\leq \frac{\mathbb{E}^{\nu}G_N(\theta-x)}{q_*^2}\frac{G(0)}{(N-1)h_N}.
\end{aligned}$$

In view of Lemma 4.1 we finally get

$$\begin{aligned} \mathbb{E}^\nu \mid A_2(j) \mid^2 &= \int_{-1}^1 \left| \mathbb{E}^\nu \left(\frac{G_N(\theta - x)}{q(\theta)} \right) - \mathbb{E}^\nu \left(\frac{G_N(\theta - x)}{\frac{G(0)}{(N-1)h_N} + q(\theta)} \right) \right|^2 q(x)dx \\ &\leq \frac{C}{[(N-1)h_N]^2}. \end{aligned} \tag{51}$$

Estimate for the second moment of $A_3(j)$. As the random variables θ^i, $1 \leq i \leq N$, are independent and identically distributed, we have

$$\begin{aligned} \mathbb{E}^\nu \mid A_3(j) \mid^2 &= \int_{-1}^1 \mathbb{E}^\nu \left| \frac{1}{N-1} \sum_{\substack{i=1 \\ i \neq j}}^N \frac{G_N(x - \theta^i)}{q(\theta^i) + \frac{G(0)}{(N-1)h_N}} - \mathbb{E}^\nu \left(\frac{G_N(x - \theta^i)}{q(\theta^i) + \frac{G(0)}{(N-1)h_N}} \right) \right|^2 q(x)dx, \\ &\leq \int_{-1}^1 \frac{1}{N-1} \mathbb{E}^\nu \left(\frac{G_N(x - \theta)}{q(\theta) + \frac{G(0)}{(N-1)h_N}} \right)^2 q(x)dx. \end{aligned}$$

For any $x \in [-1, 1]$, one has

$$\begin{aligned} \mathbb{E}^\nu \left(\frac{G_N(x - \theta)}{q(\theta) + \frac{G(0)}{(N-1)h_N}} \right)^2 &\leq \mathbb{E}^\nu \left(\frac{G_N(x - \theta)}{q(\theta)} \right)^2 \\ &= \int_{-1}^1 \frac{1}{h_N^2} \frac{G^2(\frac{x-y}{h_N})}{q^2(y)} q(y)dy, \\ &= \frac{1}{h_N} \int_{\frac{x-1}{h_N}}^{\frac{x+1}{h_N}} \frac{G^2(z)}{q(x - zh_N)} dz \\ &\leq \frac{\|G\|^2_{L^2(\mathbb{R})}}{q_*} \cdot \frac{1}{h_N}. \end{aligned}$$

Consequently,

$$\mathbb{E}^\nu \mid A_3(j) \mid^2 \leq \frac{C}{(N-1)h_N}. \tag{52}$$

Contribution of $A_4(j)$ to the convergence error. Setting $\underline{\theta} = (\theta^1, \ldots, \theta^N)$, we have

$$\mathbb{E}^\nu \left(|A_4(j)| \mid \overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k} \mid^2 \right) = \mathbb{E}^\nu \left(|A_4(j)| \, \mathbb{E}^\nu \left[\mid \overline{X}^{j,N}_{t_k} - \overline{X}^j_{t_k} \mid^2 \middle| \underline{\theta} \right] \right),$$

since $A_4(j)$ is $\sigma\left(\underline{\theta}\right)$-mesurable. Moreover, owing to the boundedness of functions b and σ, hypothesis **(iv)** of Proposition 4.2 and (43), there exists a strictly positive constant C such that

$$\forall N \geq 1,\ \forall j = 1,\dots,N,\ \forall k \leq K,\ \ \mathbb{E}\left[\,|\,\overline{X}^{j,N}_{t_k} - \overline{X}^{j}_{t_k}\,|^2 \middle| \underline{\theta}\right] \leq C\ a.s.$$

Hence we have

$$\mathbb{E}^{\nu}\left(|A_4(j)|\,|\,\overline{X}^{j,N}_{t_k} - \overline{X}^{j}_{t_k}\,|^2\right) \leq C\mathbb{E}^{\nu}\,|A_4(j)|\,. \tag{53}$$

Then, as the random variables $\theta^i,\ 1 \leq i \leq N$ are i.i.d,

$$\begin{aligned}
\mathbb{E}^{\nu}\,|A_4(j)| &\leq C\mathbb{E}^{\nu}\left[\frac{1}{N-1}\sum_{\substack{i=1\\ i\neq j}}\frac{G_N\left(\theta^i-\theta^j\right)}{q\left(\theta^i\right)\left[\mathbb{E}^{\nu}G_N\left(x-\theta^j\right)\right]|_{x=\theta^i}}\left|\left[\mathbb{E}^{\nu}G_N\left(x-\theta^j\right)\right]|_{x=\theta^i} - q\left(\theta^i\right)\right|\right]\\
&\leq C\mathbb{E}^{\nu}\left[\frac{G_N\left(\theta^1-\theta^2\right)}{q\left(\theta^1\right)\left[\mathbb{E}^{\nu}G_N\left(x-\theta^2\right)\right]|_{x=\theta^1}}\left|\left[\mathbb{E}^{\nu}G_N\left(x-\theta^2\right)\right]|_{x=\theta^1} - q\left(\theta^1\right)\right|\right].
\end{aligned}$$

In view of (37) it comes

$$\begin{aligned}
\mathbb{E}^{\nu}\,|A_4(j)| &\leq \left((q_*)^2\int_0^1 G(z)dz\right)^{-1}\mathbb{E}^{\nu}\left[G_N\left(\theta^1-\theta^2\right)\left|\left[\mathbb{E}^{\nu}G_N\left(x-\theta^2\right)\right]|_{x=\theta^1} - q\left(\theta^1\right)\right|\right],\\
&= \left((q_*)^2\int_0^1 G(z)dz\right)^{-1}\mathbb{E}^{\nu}\left\{\mathbb{E}^{\nu}\left[G_N\left(\theta^1-\theta^2\right)\middle|\theta^1\right]\left|\left[\mathbb{E}^{\nu}G_N\left(x-\theta^2\right)\right]|_{x=\theta^1} - q\left(\theta^1\right)\right|\right\},\\
&\leq C\mathbb{E}^{\nu}\left|\left[\mathbb{E}^{\nu}G_N\left(x-\theta^2\right)\right]|_{x=\theta^1} - q\left(\theta^1\right)\right| \ \text{ owing to Inequality (36) },\\
&\leq C\sqrt{h_N}\ \text{ owing to Lemma 4.1.}
\end{aligned}$$

Therefore, in view of Inequality (53), it holds that

$$\mathbb{E}\left|A_4(j)\left(X^{j,N}_t - X^j_t\right)^2\right| \leq C\sqrt{h_N}. \tag{54}$$

Estimate for the second moment of $A_5(j)$. The term $A_5(j)$ measures the convergence rate of the denominator of α_{ij} towards its mean value:

$$\begin{aligned}
&\frac{G_N(\theta^i-\theta^j)}{\frac{G(0)}{(N-1)h_N} + \frac{1}{(N-1)h_N}\sum_{\substack{k=1\\ k\neq i}}^N G_N(\theta^i-\theta^k)} - \frac{G_N(\theta^i-\theta^j)}{\frac{G(0)}{(N-1)h_N} + \mathbb{E}^{\nu}G_N(x-\theta)\,|_{x=\theta^i}}\\
&= \frac{G_N(\theta^i-\theta^j)}{D_1(N).D_2(N)}\left(\mathbb{E}^{\nu}G_N(x-\theta)\,|_{x=\theta^i} - \frac{1}{(N-1)h_N}\sum_{\substack{k=1\\ k\neq i}}^N G_N(\theta^i-\theta^k)\right),
\end{aligned}$$

where

$$D_1(N) \quad = \quad \frac{G(0)}{(N-1)h_N} + \frac{1}{(N-1)h_N}\sum_{\substack{k=1\\k\neq i}}^{N} G_N(\theta^i-\theta^k),$$

$$D_2(N) \quad = \quad \frac{G(0)}{(N-1)h_N} + \mathbb{E}^\nu G_N(x-\theta)\mid_{x=\theta^i}.$$

Owing to the lower bound (37), we see that $D_2(N)$ is bounded from below by a strictly positive constant independent of N. This property does not hold for $D_1(N)$. We thus use a localization argument by introducing the event

$$\left[\left|\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^{N} G_N(x-\theta^k) - \mathbb{E}^\nu G_N(x-\theta)\right| \geq \eta(N,a)\right].$$

We start by showing that there exists $\eta(N,a)>0$ such that

$$\mathbb{P}^\nu\left(\left|\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^{N} G_N(x-\theta^k) - \mathbb{E}^\nu G_N(x-\theta)\right| \geq \eta(N,a)\right) \leq \frac{1}{N^a}, \tag{55}$$

for all $(N,a)\in \mathbb{N}\times\mathbb{R}_*^+$, and that

$$\lim_{N\to+\infty}\eta(N,a)=0 \ \text{ if } \ \lim_{N\to+\infty}\frac{\log(N)}{Nh_N^2}=0. \tag{56}$$

Indeed, the random variables

$$Y_k(N,x) := h_N G_N(x-\theta^k) - \mathbb{E}^\nu G_n(x-\theta^k),\ 1\leq k\leq N,$$

are i.i.d. and bounded by $2\,\|G\|_{L^\infty(\mathbb{R})}$. Then Hoeffding's inequality implies

$$\mathbb{P}^\nu\left(\left|\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^{N} G_N(x-\theta^k) - \mathbb{E}^\nu G_N(x-\theta)\right| \geq \eta(N,a)\right)$$
$$= \mathbb{P}^\nu\left(\left|\sum_{k=1}^{N-1} Y_k(N,x)\right| \geq (N-1)\eta(N,a)h_N\right),$$
$$\leq \exp\left(-\frac{(N-1)(\eta(N,a)h_N)^2}{2\|G\|^2_{L^\infty(\mathbb{R})}}\right).$$

Hence Inequality (55) and limits in (56) hold for

$$\eta(N,a)=\sqrt{2\|G\|^2_{L^\infty(\mathbb{R})}a\frac{\log N}{(N-1)h_N^2}}.$$

We are now in a position to estimate the second moment of $A_5(j)$. For $x\in[-1,1]$ and $a>0$, set

$$\begin{aligned}
E(x,N,a) &= \left\{\omega/\mid \tfrac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^N G_N(x-\theta^k(\omega))-\mathbb{E}^\nu G_N(x-\theta)\mid\leq\eta(N,a)\right\},\\
A_5(j,x) &= \frac{G_N(x-\theta^j)}{\frac{G(0)}{(N-1)h_N}+\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^N G_N(x-\theta^k)}-\frac{G_N(x-\theta^j)}{\frac{G(0)}{(N-1)h_N}+\mathbb{E}^\nu G_N(x-\theta)}.
\end{aligned}$$

As $\eta(N,a)$ tends to 0 when N tends to infinity, we have, for N large enough,

$$\eta(N,a)\leq\frac{1}{2}q_*\int_0^1 G(z)dz.$$

Thus, in view of (37) we have

$$\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^N G_N(x-\theta^k)\geq\frac{1}{2}q_*\int_0^1 G(z)dz$$

on the event $E(x,N,a)$. Therefore,

$$\begin{aligned}
&\mathbb{E}^\nu\left\{\mid A_5(j,x)\mid^2\mathbb{I}(E(x,N,a))\right\}\\
&\qquad\leq C\mathbb{E}^\nu\left\{G_N(x-\theta^j)\left|\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^N G_N(x-\theta^k)-\mathbb{E}^\nu G_N(x-\theta^i)\right|\right\}^2.
\end{aligned}\tag{57}$$

We then distinguish two cases:

- $j=i$: as the random variables θ^k are independent and the sum only concerns subscripts different from i, the random variables $G_N(x-\theta^i)$ and $\frac{1}{N-1}\sum_{\substack{k=1\\k\neq i}}^N G_N(x-\theta^k)-$

$\mathbb{E}^\nu [G_N(x-\theta)]$ are independent. Thus,

$$\begin{aligned}\mathbb{E}^\nu \left\{ \mid A_5(i,x) \mid^2 \mathbb{I}(E(x,\eta,N)) \right\} &\leq \mathbb{E}^\nu \left[G_N^2(x-\theta^i) \right] \\ &\quad \mathbb{E}^\nu \left| \frac{1}{N-1} \sum_{\substack{k=1\\k\neq i}}^{N} G_N(x-\theta^k) - \mathbb{E}^\nu \left[G_N(x-\theta) \right] \right|^2 \\ &\leq C\mathbb{E}^\nu \left[G_N^2(x-\theta^i) \right] . \frac{1}{N-1} \mathbb{E}^\nu \left[G_N^2(x-\theta) \right], \\ &\leq \frac{C}{(N-1)h_N^2}.\end{aligned}$$

- $j \neq i$, we isolate the term $G_N(x-\theta^i)$ from the rest of the sum. A computation similar to the case $j=i$ leads to

$$\mathbb{E}^\nu \left\{ \mid A_5(j,x) \mid^2 \mathbb{I}(E(x,\eta,N)) \right\} \leq C \left(\frac{1}{Nh_N^2} + \frac{1}{(N-1)^2 h_N^2} \right).$$

Finally, it comes

$$\mathbb{E}^\nu \left\{ \mid A_5(j,x) \mid^2 \mathbb{I}(E(x,N,a)) \right\} \leq \frac{C}{Nh_N^2}. \tag{58}$$

On the other hand, roughly bounding G_N by $1/h_N$ and then $A_5(j,x)$ by CN and using (55), we have

$$\mathbb{E}^\nu \mid A_5(j,x) \mid^2 \leq \frac{C}{Nh_N^2}$$

for a and $\eta(N,a)$ suitably chosen and any $j \leq N$.

Observing that the constant C does not depend on $x \in [-1,1]$, we conclude

$$\mathbb{E}^\nu \mid A_5(j) \mid^2 = \frac{1}{N-1} \sum_{\substack{i=1\\i\neq j}}^{N} \int_{-1}^{1} \mathbb{E}^\nu \mid A_5(j,x) \mid^2 q(x)dx \leq \frac{C}{Nh_N^2}. \tag{59}$$

Estimate for $\mathbb{E}^\nu [A_6(j)]^2$ and $\delta_0(t_k)$. These two terms concern the interaction of a particle with itself. For the first term of $A_6(j)$ we observe that it looks like $A_5(j)$ except that the numerator is constant and of order $1/h_N$; for the two last terms, we use Lemma 4.1 and get

$$\mathbb{E}^\nu \mid A_6(j) \mid^2 \leq C \left(\frac{1}{N^2 h_N^3} + \frac{1}{N^2 h_N^2} \right) \leq 2C \left(\frac{1}{N^2 h_N^3} \right). \tag{60}$$

Moreover, as the random variables $\left(\overline{X}^i_{t_k}, \theta^i\right)$, $1 \le i \le N$, are i.i.d.,

$$\begin{aligned}
\delta_0(t_k) &= \frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}^\nu\left[\alpha_{ii}\, b\left(\overline{X}^i_{t_k}, \overline{X}^i_{t_k}\right)\right]^2 + \mathbb{E}^\nu\left[\alpha_{ii}\,\sigma\left(\overline{X}^i_{t_k}, \overline{X}^i_{t_k}\right)\right]^2\right), \\
&= \mathbb{E}^\nu\left[\alpha_{11} b(\overline{X}^1_{t_k}, \overline{X}^1_{t_k})\right]^2 + \mathbb{E}^\nu\left[\alpha_{11}\sigma(\overline{X^1}_{t_k}, \overline{X}^1_{t_k})\right]^2, \\
&\le \left(\|b\|^2_{L^\infty(\mathbb{R}^2)} + \|\sigma\|^2_{L^\infty(\mathbb{R}^2)}\right)\mathbb{E}^\nu\left[\alpha_{11}\right], \\
&\le C\mathbb{E}^\nu\left|\alpha_{11} - \frac{1}{N-1}\right| + \frac{1}{N-1}.
\end{aligned}$$

Proceeding as in the preceding steps we get

$$\delta_0(t_k) \le C\left(\frac{1}{N}\left(\frac{1}{\sqrt{Nh_N^2}} + \sqrt{h_N}\right) + \frac{1}{N-1}\right). \tag{61}$$

We can finally estimate the accuracy of the particle method. This is a straightforward consequence of (19), (20), (47), Proposition 2.1 and Lemma 4.3:

Theorem 4.4. *Suppose that the hypotheses of Proposition 4.2 hold.*

Then there exists an integer N_0 such that, for any $N \ge N_0$ and any test function $f \in C_b^{4+\varepsilon}(\mathbb{R})$, $0 < \varepsilon < 1$,

$$\mathbb{E}^\nu\left|\langle M_1(T), f\rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^{N} f\left(\overline{X}^{i,N}_T\right)\right|^2 \le C\left(\frac{1}{\sqrt{Nh_N^2}} + \sqrt{h_N} + (\Delta t)^2\right).$$

5 Conclusion.

We have constructed an original and efficient stochastic method to compute moments of statistical solutions of McKean-Vlasov equations, and we have analyzed the convergence rate of the method. Several extensions should be studied in the future, for example: first, the cases of non smooth interaction kernels and, in particular, the cases of Burgers and Navier-Stokes equations; second, the use of random weights other than ours, for example weights resulting from conditional expectation estimators using wavelets.

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Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
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