# A Stochastic Particle Method with Random Weights for the Computation of Statistical Solutions of McKean-Vlasov Equations. Part I: Foundation of the Method and Empirical Evidence

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**Abstract:** **Abstract** - We are interested in statistical solutions [28] of McKean-Vlasov equations. An example of motivation is the Navier-Stokes equation for the vorticity of a 2D incompressible fluid flow. We propose an original and efficient numerical method to compute moments of such solutions. It is a stochastic particle method with random weights. These weights are defined through nonparametric estimators of a regression function, and convey the uncertainty on the initial condition of the considered equation.

In this first part of our work, we prove an existence and uniqueness result for a class of nonlinear stochastic differential equations (SDE), and we study the relation between these nonlinear SDEs and statistical solutions of the corresponding McKean-Vlasov equations. This result founds our stochastic particle method, for which we show results of numerical experiments obtained for the Burgers and the 2D Navier-Stokes equation.

**Key-words:** stochastic particle system, McKean–Vlasov equations, statistical solutions

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# Une méthode particulaire stochastique à poids aléatoires pour le calcul de solutions statistiques d'équations de McKean–Vlasov. Partie I : Fondation de la méthode et validation empirique.

**Résumé :** Nous nous intéressons aux solutions statistiques [28] d'équations de McKean-Vlasov-Fokker-Planck. Un exemple de motivation est l'équation de la vorticité d'un fluide incompressible dans un écoulement plan. Nous proposons une méthode numérique originale et efficace pour approcher les moments de telles solutions. Il s'agit d'une méthode particulaire stochastique à poids aléatoires. Ces poids sont définis à l'aide d'estimateurs non paramétriques d'une fonction de régression et traduisent une incertitude sur la condition initiale de l'EDP.

Dans cette première partie de notre travail nous prouvons un résultat d'existence et unicité pour une classe d'équations différentielles stochastiques non linéaires au sens de McKean, et nous étudions les relations entre ces équations et les solutions statistiques des équations de McKean-Vlasov correspondantes. Ce résultat fonde notre méthode particulaire stochastique, pour laquelle nous montrons des résultats d'expérimentations numériques réalisées pour l'équation de Burgers et l'équation de Navier-Stokes en dimension 2.

**Mots-clés :** méthode particulaire stochastique, équations de McKean-Vlasov, solutions statistiques

# 1 Introduction

Partial differential equations (PDE) with random initial condition are possible models for some complex physical phenomena such as turbulence (see, e.g., Monin & Yaglom [23] and Vishik-Fursikov [28]). They also can express a lack of information on the initial state of a system, as in weather forecasting (see Chorin *et al.* [8]) where data are collected from a finite and relatively small number of meteorological stations. Of course, there are many functions fitting such a finite set of values therefore sparse data often lead to statistical models.
In both cases, one can only simulate mean quantities over the set of initial conditions of the model. However, it is often difficult to estimate the accuracy of usual related numerical methods such as closure models (see, e.g., Mohammadi & Pironneau [22], Vishik & Fursikov [28], Fox [14]). Following a quite different approach, we here propose an original stochastic particle method with random weights to compute

$$\langle M_1(t)\ ,\ f\rangle_{L^2(\mathbb{R})} := \mathbb{E}\int_{\mathbb{R}} p(t,x,\omega)f(x)dx, \tag{1}$$

where $f$ is a given test function and $p(t,x,\omega)$ is the solution of a McKean-Vlasov equation with random initial condition. Our motivation comes from the fact that the viscous Burgers equation and the 2D incompressible Navier-Stokes equation for the vorticity belong to the class of McKean-Vlasov equations, and thus have a probabilistic interpretation by mean of stochastic particle systems (see, e.g., Sznitman [25]).

We now fix some notation, and consider the McKean-Vlasov equation

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,x,\omega) & = -\dfrac{\partial}{\partial x}\left(u_b(t,x,\omega)p(t,x,\omega)\right) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}\left(\left(u_\sigma(t,x,\omega)\right)^2 p(t,x,\omega)\right), \\ p(0,x,\omega) & = p_0(x,\omega), \\ u_b(t,x,\omega) & := \int_{\mathbb{R}} b(x,y)p(t,y,\omega)dy, \\ u_\sigma(t,x,\omega) & := \int_{\mathbb{R}} \sigma(x,y)p(t,y,\omega)dy, \end{cases} \tag{2}$$

where $b$ and $\sigma$ are smooth and bounded functions from $\mathbb{R}^2$ to $\mathbb{R}$. For technical reasons we thereafter suppose that the possible initial conditions of (2) are parametrized by realizations $\theta(\omega)$ of a real valued random variable $\theta$ with law $\nu$ concentrated on a closed interval of $\mathbb{R}$, say $[-1,1]$.

The paper is organized as follows:

- First, we briefly outline the theory of *statistical solutions* of an evolution problem, especially the notion of moments of a statistical solution, and we present some known results on the 2D incompressible Navier-Stokes equation. We then study statistical solutions of the model problem (2) and their moments. In particular, we show that Identity (1) defines the first moment of the statistical solution of (2).

- Second, we prove an original probabilistic interpretation of the moments. To this end, we prove the following result which is interesting in itself. Consider the nonlinear stochastic differential equation

$$\left\{\begin{array}{lcl} dX_t & = & \mathbb{E}\left[b\left(x, X_t\right) \mid \theta\right]|_{x=X_t}\, dt + \mathbb{E}\left[\sigma\left(x, X_t\right) \mid \theta\right]|_{x=X_t}\, dW_t,\ t \leq T, \\ (X_0, \theta) & & \text{with law } \left[\Phi(a)\right](x)dx\ \nu(da), \\ \theta & & \text{random variable independent of } W_{\cdot} \end{array}\right.$$

  Under appropriate hypotheses on the kernels $b$ and $\sigma$, we show that this equation has a unique weak solution and we describe the relation between this solution and the moment defined in (1).

- Third, we develop the following stochastic particle method:

$$\left\{\begin{array}{lcl} \overline{X}^{i,N}_{(k+1)\Delta t} & = & \overline{X}^{i,N}_{k\Delta t} + \sum_{j=1}^{N} \alpha_{ij} b(\overline{X}^{i,N}_{k\Delta t}, \overline{X}^{j,N}_{k\Delta t})\Delta t + \sum_{j=1}^{N} \alpha_{ij}\sigma(\overline{X}^{i,N}_{k\Delta t}, \overline{X}^{j,N}_{k\Delta t}) \left(W^i_{(k+1)\Delta t} - W^i_{k\Delta t}\right), \\ \overline{X}^{i,N}_0 & = & X^i_0, \end{array}\right.$$

  where the $W^i_{\cdot}$, $1 \leq i \leq N$ are independent real Brownian motions. The weights $\alpha_{ij}$ are defined from nonparametric estimators of the functions $u_b$ and $u_\sigma$. We show that

$$\langle M_1(t)\ ,\ f\rangle_{L^2(\mathbb{R})} \simeq \frac{1}{N}\sum_{i=1}^{N} f\left(\overline{X}^{i,N}_t\right).$$

- Finally, we test the numerical performances of the particle method by computing statistical solutions of the Burgers and the Navier-Stokes equations. Admittedly, the method is not rigorously justified in these cases, as our technical assumptions on the functions $b$ and $\sigma$ exclude the singular interaction kernels corresponding to the Burgers and the Navier-Stokes equations. This point will hopefully be treated in the future.

In the second part of this work [26] we prove estimates on the convergence rate of the above approximation in terms of the number $N$ of simulated particles and the time discretization step $\Delta t$.

## Notation:

- For $k \in \mathbb{N}$, $C^k_b\left(\mathbb{R}^n\right)$ is the set of functions from $\mathbb{R}^n$ to $\mathbb{R}$ whose partial derivatives up to order $k$ are continuous and uniformly bounded over $\mathbb{R}^n$,
- $\mathbb{E}$ is the expectation operator under the law $\mathbb{P}_W$ of a real valued Brownian motion $W_{\cdot}$ and, for any probability measure $\nu$ on $[-1, 1]$, $\mathbb{E}^\nu$ is the expectation operator under the product measure $\mathbb{P}_W \otimes \nu$,
- $C$, $C(T)$ are strictly positive real constants which can change from line to line.

## 2 Statistical solution of a Cauchy problem. Application to the model equation (2).

In this section, we define the notion of statistical solution and of moments of such a solution. We give assumptions under which a statistical solution and moments exist for Equation (2).

The notion of statistical solution was first proposed by Hopf [18] in order to describe turbulence. This approach has then been studied by several authors, in particular Foias [11], Foias-Temam [12], [13], and Vishik-Fursikov [28]. A somewhat different notion of statistical solution has been studied by Carraro and Duchon [6] (see also the references therein) for the inviscid Burgers equation.
Consider an evolution equation on a strip $[0,T] \times \mathbb{R}^n$, $n \leq 3$:

$$\begin{cases} \dfrac{du}{dt} + Au &= 0, \\ u\,|_{t=0} &= u_0 \in L^2(\mathbb{R}^n), \end{cases} \tag{3}$$

where $u_0$ is a random variable with law $\mu$.

**Definition 2.1.** *(Vishik-Fursikov [28], p 87) The Cauchy problem (3) is said to have a 'low Reynolds number', by analogy with Fluid Mechanics if, for each initial condition $u_0$ in the support of $\mu$, Equation (3) has a unique solution $Su_0 \in C\left([0,T], L^2(\mathbb{R}^n)\right)$.*
*In this case, the statistical solution of (3) with initial condition $\mu$ is the probability measure on*
*$L^2\left(0,T; L^2(\mathbb{R}^n)\right)$ defined by*

$$m := \mu \circ S^{-1}.$$

**Remark 2.2.** *The notion of statistical solution may be defined in a more general setting (see [28, Definition 1.1, p.122]). Indeed, the Cauchy problem (3) may have a statistical solution even if it does not have a low Reynolds number, but we do not consider this situation here.*

Suppose now that (3) has a statistical solution $m$ whose marginal, or *spatial statistical solution*, at time $t$ satisfies

$$\text{for some integer } k \geq 1, \quad \int_{L^2(\mathbb{R}^n)} \|u\|^k_{L^2(\mathbb{R}^n)}\, m_t(du) \;< +\infty. \tag{4}$$

In particular, if $k = 1$, we assume that the mean energy at time $t$ is finite. For any $p \leq k$, consider the linear form on $L^2(p) := \overset{p}{\otimes} L^2(\mathbb{R}^n)$ defined by

$$F_p : \phi \in L^2(p) \quad \mapsto \quad \int_{L^2(\mathbb{R}^n)} \left\langle \overset{p}{\otimes} u, \phi \right\rangle_{L^2(p)} m_t(du).$$

By the Cauchy-Schwartz inequality and Assumption (4), the application $F_p$ is continuous. Hence, by the Riesz representation theorem, there exists a unique element $M_p(t) \in L^2(p)$ such that

$$\forall p \leq k,\ \forall \phi \in L^2(p), \quad \langle M_p(t), \phi \rangle_{L^2(p)} = \int_{L^2(\mathbb{R}^n)} \left\langle \overset{p}{\otimes} u\ ,\ \phi \right\rangle_{L^2(p)} m_t(du). \tag{5}$$

By definition, $M_p(t)$ is the $p$-th *moment* of the measure $m_t$.
We now illustrate these notions in the particular case of the Navier-Stokes equation:

$$\begin{cases} \partial_t \omega(t,x,\omega_0) &=& -\nabla.\left(\mathbf{u}(t,x,\omega_0)\omega(t,x,\omega_0)\right) + \dfrac{\sigma^2}{2}\Delta\omega(t,x,\omega_0), \\ \mathbf{u} &=& \mathbf{K} * \omega,\ \text{where } \mathbf{K}(\mathbf{y}) = \dfrac{1}{2\pi \mid \mathbf{y}|^2}(-y_2, y_1), \\ \omega(0,x,\omega_0) &=& \omega_0(x). \end{cases} \tag{6}$$

If the law $\mu$ of the initial condition of (6) is concentrated on $L^1\left(\mathbb{R}^2\right) \cap L^\infty_c\left(\mathbb{R}^2\right)$ (i.e. the subspace of $L^1\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right)$ of functions with compact support), Constantin and Wu [10] have shown that Equation (6) has a unique statistical solution with initial condition $\mu$. Moreover, if $\mu$ is concentrated on a closed ball of $L^1\left(\mathbb{R}^2\right) \cap L^\infty_c\left(\mathbb{R}^2\right)$, the spatial correlations of the velocity $\mathbf{u}$ are related to moments (see Vaillant [27]):

$$\begin{aligned} \int \mathbf{u}_i(t,x,\omega_0)\mu(d\omega_0) &= \langle\ M_1(t), \mathbf{K}_i(x-.)\ \rangle_{L^2(\mathbb{R}^2)},\ \ i=1,2 && \text{(mean velocity)}, \\ \frac{1}{2}\int \mathbf{u}^2(t,x,\omega_0)\mu(d\omega_0) &= \tfrac{1}{2}\langle\ M_2(t), \mathbf{K}(x-.).\mathbf{K}(x-.)\ \rangle_{L^2(\mathbb{R}^2)\otimes L^2(\mathbb{R}^2)} && \text{(mean kinetic energy)}. \end{aligned}$$

This representation mainly explains our interest in simulating moments.

We now turn to the statistical study of the model equation (2). We recall that the uncertainty on initial conditions of (2) is supposed to come from a random parameter $\theta$ with law $\nu$. Admittedly, this assumption is quite restrictive from a physical point of view. Nevertheless, it is very useful for numerical reasons: generally, the law of the random initial condition of (2) is a measure on an infinite dimensional functional space; the parametrization allows us to reduce it to the law of a finite dimensional random variable.
From now on, we denote by $\Phi$ the one to one application assigning to any parameter $a \in [-1,1]$ an initial condition $\Phi(a) := p_0(.,a)$.
We now prove that the model problem (2) has a low Reynolds number in the sense of Definition 2.1. This is a straightforward consequence of the following proposition:

**Proposition 2.3.** *Suppose that*

**(H1)** $\exists\ \varepsilon \in ]0,1[$ *such that*

$$b \in C_b^{2+\varepsilon}\left(\mathbb{R}^2\right),\ \sigma \in C_b^{2+\varepsilon}\left(\mathbb{R}^2\right) \text{ and, for any } (x,y) \in \mathbb{R}^2,\ \sigma(x,y) \geq \sigma_* > 0, \tag{7}$$

*where, for $k \in \mathbb{N}$, $C_b^{k+\varepsilon}(\mathbb{R}^n)$ denotes Hölder spaces of functions (see, e.g., [20]).*
*In particular there exists a strictly positive constant $L$ such that,*

$$\forall (x,y,z,u) \in \mathbb{R}^4, \;\; |b(x,y)-b(z,u)| + |\sigma(x,y)-\sigma(z,u)| \leq L\left(|x-z|+|y-u|\right). \tag{8}$$

*Suppose also that the function $\Phi$ satisfies*

**(H2)** *$\Phi([-1\,,\,1]) \subset C_b^{2+\varepsilon}(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ and $\Phi([-1\,,\,1])$ is a set of probability density functions,*

**(H3)** *$\Phi$ is Lipschitz continuous for the norm in $L^1(\mathbb{R})$.*

**(H4)** *$\Phi$ is such that*

$$\sup_{a\in[-1,1]} \|p_0(.,a)\|_{W^{2,1}(\mathbb{R})} < +\infty.$$

*Then, for any $a \in [-1\,,\,1]$, the equation*

$$\begin{cases} \frac{\partial p(t,x,a)}{\partial t} & = -\frac{\partial}{\partial x}\left(u_b(t,x,a)p(t,x,a)\right) + \dfrac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\left(u_\sigma(t,x,a)\right)^2 p(t,x,a)\Big), \\ p(0,x,a) & = p_0(x,a), \\ u_b(t,x,a) & := \int_{\mathbb{R}} b(x,y)p(t,y,a)dy, \\ u_\sigma(t,x,a) & := \int_{\mathbb{R}} \sigma(x,y)p(t,y,a)dy, \end{cases} \tag{9}$$

*has a unique solution in the set of probability densities, and this solution satisfies*

$$p(.,.,a) =: (S\circ\Phi)(a) \in C_b^{1,2+\varepsilon}([0,T]\times\mathbb{R}) \cap C\left([0,T], L^2(\mathbb{R})\right),$$

*where $C_b^{1,2+\varepsilon}([0,T]\times\mathbb{R})$ is the set of functions from $[0,T]\times\mathbb{R}$ to $\mathbb{R}$ of class $C^1$ in time and $C_b^{2+\varepsilon}$ in space.*

*Proof.* As functions $b$ and $\sigma$ are Lipschitz continuous and bounded, there is a unique solution $\mathbb{P}$ to the nonlinear martingale problem with coefficients $\int b(x,y)\mathbb{P}_t(dy)$ and $\int \sigma(x,y)\mathbb{P}_t(dy)$ and initial condition $p_0(x,a)dx$ (see Sznitman [25]). Moreover, any density $p(t,.,a)$ solution to (9) is the density of $Y_t(a)$, where the process $(Y_t(a))_{t\in[0,T]}$ satisfies the following stochastic differential equation :

$$\begin{cases} dY_t(a) & = \left(\int_{\mathbb{R}} b(x,y)\,p(t,y,a)dy\right)|_{x=Y_t(a)}\,dt + \left(\int_{\mathbb{R}} \sigma(x,y)\,p(t,y,a)dy\right)|_{x=Y_t(a)}\,dW_t, \\ Y_0(a) & \text{with law } \left[\Phi(a)\right](x)dx. \end{cases}$$

Thus $p(t,.,a)$ is also the density of $X_t(a)$, where the process $(X_t(a))_{t\in[0,T]}$ satisfies the following nonlinear SDE:

$$\begin{cases} dX_t(a) & = \mathbb{E}\left(b(x,X_t(a))\right)|_{x=X_t(a)}\,dt + \mathbb{E}\left(\sigma(x,X_t(a))\right)|_{x=X_t(a)}\,dW_t, \\ X_0(a) & \text{with law } \left[\Phi(a)\right](x)dx. \end{cases} \tag{10}$$

Then the uniqueness of $\mathbb{P}$ implies the uniqueness of the density solution of (9). Moreover, from Equation (10), the functions $u_b$ and $u_\sigma$ satisfy:

$$u_b(t,x,a) = \mathbb{E}b\left(x, X_t(a)\right), \quad u_\sigma(t,x,a) = \mathbb{E}\sigma\left(x, X_t(a)\right). \tag{11}$$

Using Equalities (11) and Hypothesis (7), one easily checks that the functions $u_b$ and $u_\sigma$ are of class $C_b^2$ in $x$. They are also Hölder continuous of order $1/2$ in $t$. Indeed, for any $(s,t,x,a) \in [0,T]^2 \times \mathbb{R} \times [-1,1]$,

$$\begin{aligned}
\mathbb{E}\left|X_t(a) - X_s(a)\right|^2 &\leq 2\mathbb{E}\left\{\int_s^t \mathbb{E}b\left(x, X_\tau(a)\right) d\tau\right\}^2 \\
&\quad +2\mathbb{E}\left\{\int_s^t \mathbb{E}\sigma\left(x, X_\tau(a)\right) dW\tau\right\}^2, \\
&\leq 2\left\|b\right\|^2_{L^\infty(\mathbb{R}^2)} \left|t-s\right|^2 + 2\left\|\sigma\right\|^2_{L^\infty(\mathbb{R}^2)} \left|t-s\right|.
\end{aligned}$$

Hence, as $b$ and $\sigma$ are Lipschitz functions,

$$\begin{aligned}
\left|u_b\left(t,x,a\right) - u_b\left(s,x,a\right)\right| + \left|u_\sigma\left(t,x,a\right) - u_\sigma\left(s,x,a\right)\right| &= \left|\mathbb{E}b\left(x, X_t(a)\right) - \mathbb{E}b\left(x, X_s(a)\right)\right| \\
&\quad + \left|\mathbb{E}\sigma\left(x, X_t(a)\right) - \mathbb{E}\sigma\left(x, X_s(a)\right)\right|, \\
&\leq L\mathbb{E}\left|X_t(a) - X_s(a)\right| \\
&\leq L\sqrt{2T\left\|b\right\|^2_{L^\infty(\mathbb{R}^2)} + 2\left\|\sigma\right\|^2_{L^\infty(\mathbb{R}^2)}}\sqrt{|t-s|}.
\end{aligned}$$

Then, as $\Phi(a) \in C_b^{2+\varepsilon}(\mathbb{R})$, Theorem 5.1.9 in Lunardi [20] shows that Equation (9) has a unique solution in $C_b^{1,2+\varepsilon}([0,T]\times\mathbb{R})$. We can then define the operator

$$\begin{array}{rcl}
S : \Phi\left([-1,1]\right) & \rightarrow & C_b^{1,2+\varepsilon}\left([0,T]\times\mathbb{R}\right) \\
p_0 = \Phi(a) & \mapsto & Sp_0 := p\left(.,.,a\right).
\end{array} \tag{12}$$

Moreover, observe that $\Phi\left([-1,1]\right) \in C_b(\mathbb{R}) \cap W^{2,1}(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$. Then, owing to Assumption (7), the Fokker-Planck equation (9) has a unique solution in $C\left([0,T];L^2(\mathbb{R})\right)$ (see e.g. Cessenat et al [7], p.89), and

$$S \circ \Phi\left([-1\ ,\ 1]\right) \subset C_b^{1,2+\varepsilon}\left([0,T]\times\mathbb{R}\right) \cap C\left([0,T];L^2(\mathbb{R})\right).$$

$\square$

Before turning our attention to the question of moments of the statistical solution, we state and prove a technical regularity result which will be needed in the proof of Lemma 5.1.

**Proposition 2.4.** *Under the hypotheses of Proposition 2.3, the mapping*

$$a \in [-1\ ,\ 1] \mapsto \left(So\Phi\right)(a) \in L^\infty\left([0,T];L^1(\mathbb{R})\right)$$

*is Lipschitz continuous.*

*Proof.* We sketch the proof in Vaillant [27]. Take $a^1,\ a^2 \in [-1\ ,\ 1]$ and, for $i = 1$ or $2$, set

$$\left\{\begin{array}{l} p_t^i = p\left(T-t, ., a^i\right), \\ u_b^i(t) = \int_{\mathbb{R}} b(.,y) p_t^i(y) dy, \quad v_\sigma^i(t) = \frac{1}{2}\left(\int_{\mathbb{R}} \sigma(x,y) p_t^i(y) dy\right)^2, \\ L_t^i = v_\sigma^i(t) \partial_{xx}^2 + \left(2\partial_x v_\sigma^i(t) - u_b^i(t)\right)\partial_x, \\ k_t^i = \partial_x u_b^i(t) - \partial_{xx}^2 v_\sigma^i(t), \\ F_t = \left(v_\sigma^1(t) - v_\sigma^2(t)\right)\partial_{xx}^2 p_t^1 + \left\{\left(u_b^2(t) - u_b^1(t)\right) + 2\left(\partial_x v_\sigma^1(t) - \partial_x v_\sigma^2(t)\right)\right\}\partial_x p_t^1 \\ \qquad + \left\{\partial_x u_b^2 - \partial_x u_b^1 + \partial_{xx}^2 v_\sigma^1(t) - \partial_{xx}^2 v_\sigma^2(t)\right\} p_t^1. \end{array}\right. \tag{13}$$

Apply the Feynman-Kac formula to Equation (9): for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$p_t^i(x) = \mathbb{E}\left\{p_0^i\left(X_T^{i,t,x}\right)\exp\left(-\int_t^T k_\theta^i\left(X_\theta^{i,t,x}\right)d\theta\right)\right\}, \tag{14}$$

$$\begin{aligned} q_t(x) &:= p_t^2(x) - p_t^1(x) \\ &= \mathbb{E}\left\{\left(p_0^2\left(X_T^{2,t,x}\right) - p_0^1\left(X_T^{2,t,x}\right)\right)\exp\left(-\int_t^T k_\theta^2\left(X_\theta^{2,t,x}\right)d\theta\right)\right\} \\ &\quad -\mathbb{E}\left\{\int_t^T F_s\left(X_s^{2,t,x}\right)\exp\left(-\int_t^s k_\theta^2\left(X_\theta^{2,t,x}\right)d\theta\right)ds\right\}, \end{aligned} \tag{15}$$

where $X_\cdot^{i,t,x}$ is the Markov process whose infinitesimal generator is $L_t^i$ and such that $X_t^{i,t,x} = x$ *a.s.* Moreover, as $b \in C_b^2\left(\mathbb{R}^2\right)$, for $\alpha \in \{0,1,2\}$ and $i = 1$ or $2$,

$$\begin{aligned} \|u_b^i(t)\|_{L^\infty(\mathbb{R})} &= \|\partial_x^\alpha \int_{\mathbb{R}} \sigma(.,y) p_t^i(y) dy\|_{L^\infty(\mathbb{R})}, \\ &\leq \|\partial_x^\alpha \sigma\|_{L^\infty(\mathbb{R}^2)}. \end{aligned}$$

Similarly,

$$\left\{\begin{array}{l} \sup\limits_{t\in[0,T]} \|\partial_x^\alpha v_t^i\|_{L^\infty(\mathbb{R})} < +\infty, \quad \sup\limits_{t\in[0,T]} \|\partial_x^\alpha u_t^i\|_{L^\infty(\mathbb{R})} < +\infty, \\ \|\partial_x^\alpha v_\sigma^1(t) - \partial_x^\alpha v_\sigma^2(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x^\alpha u_b^1(t) - \partial_x^\alpha u_b^2(t)\|_{L^\infty(\mathbb{R})} \leq C\|q_t\|_{L^1(\mathbb{R})}, \\ |F_t(x)| \leq C\|q_t\|_{L^1(\mathbb{R})}\left(\left|p_t^1(x)\right| + \left|\partial_x p_t^1(x)\right| + \left|\partial_{xx}^2 p_t^1(x)\right|\right). \end{array}\right. \tag{16}$$

In view of these bounds and of Equality (15), we deduce

$$\begin{aligned}\int_{\mathbb{R}} \mid q_t(x) \mid dx &\leq C(T) \int_{\mathbb{R}} \left\{ \mathbb{E} \left| p_0^2 \left( X_T^{2,t,x} \right) - p_0^1 \left( X_T^{2,t,x} \right) \right| + \int_t^T \mathbb{E} \left| F_s \left( X_s^{2,t,x} \right) \right| ds \right\} dx, \\ &=: C(T) \left( A(t) + B(t) \right). \end{aligned} \tag{17}$$

Moreover, owing to Hypothesis (7), we have already observed that the functions $u_b$ and $u_\sigma$ are of class $C_b^2$ in $x$ and Hölder continuous of order $1/2$ in $t$. Hence the density $\Gamma(s,y,t,x)$ of the law of $X_s^{t,x}$ is exponentially bounded (see Friedman [15] p. 139-150):

$$\forall\, \overline{\sigma} > \|\sigma\|_{L^\infty(\mathbb{R}^2)}, \ \exists C_0 > 0, \ \ \Gamma(s,y,t,x) \leq \frac{C_0}{\sqrt{s-t}} \exp\left( -\frac{(x-y)^2}{2\overline{\sigma}(s-t)} \right). \tag{18}$$

In view of Inequalities (16), the constant $C_0$ can be chosen uniform in $a \in [-1,1]$ . Thus we have

$$\begin{aligned}\int_{\mathbb{R}} \mathbb{E} \left| p_0^2 \left( X_T^{2,t,x} \right) - p_0^1 \left( X_T^{2,t,x} \right) \right| dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} |q_T(y)| \, \Gamma(T,y,t,x) dy dx, \\ &\leq \int_{\mathbb{R}} \left\{ |q_T(y)| \int_{\mathbb{R}} \frac{C_0}{\sqrt{T-t}} exp \left( -\frac{(x-y)^2}{2\overline{\sigma}(T-t)} \right) dx \right\} dy, \\ &\leq C \int_{\mathbb{R}} |q_T(y)| \, dy, \end{aligned}$$

from which

$$A(t) \ \leq \ C \|p_0^2 - p_0^1\|_{L^1(\mathbb{R})}. \tag{19}$$

Before estimating $B(t)$, we observe that the hypotheses **(H1)** and **(H2)** ensure that, for any $a \in [-1,1]$, the solution $p(.,.,a)$ to Equation (9) belongs to $C\left([0,T], W^{2,1}(\mathbb{R})\right)$ (see Cannarsa-Vespri [5]). Moreover, from the Feynman-Kac formula (14) and Inequality (18), one easily shows that

$$\sup_{a\in[-1,1]} \left( \sup_{t\in[0,T]} \|p(t,.,a)\|_{L^1(\mathbb{R})} \right) < +\infty.$$

Then observe that $\partial_x p(t,.,a)$ and $\partial^2_{xx} p(t,.,a)$ satisfy PDEs similar to (9) so that, repeating the same kind of arguments and using Hypothesis **(H4)**, one can show that

$$\sup_{a\in[-1,1]} \left( \sup_{t\in[0,T]} \|p(t,.,a)\|_{W^{2,1}(\mathbb{R})} \right) < +\infty. \tag{20}$$

We can now estimate $B(t)$ in the same way than $A(t)$. Indeed, using (18) and the third line of (16):

$$\begin{aligned}
&\int_{\mathbb{R}} \left( \int_t^T \mathbb{E} \left| F_s \left( X_s^{t,x} \right) \right| ds \right) dx \\
&\qquad \leq C \int_{\mathbb{R}} \int_t^T \|q_s\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} \sum_{i=0}^{2} \left| \partial_y^i p_t^1(y) \right| \frac{1}{\sqrt{s-t}} e^{-\frac{(x-y)^2}{\overline{\sigma}(s-t)}} dx dy \\
&\qquad = C \int_t^T \|q_s\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} \sum_{i=0}^{2} \left| \partial_y^i p_t^1(y) \right| \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{s-t}} e^{-\frac{(x-y)^2}{\overline{\sigma}(s-t)}} dx \right\} dy ds \\
&\qquad \leq C \int_t^T \|q_s\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} \sum_{i=0}^{2} \left| \partial_y^i p_t^1(y) \right| dy ds \\
&\qquad \leq C \sup_{t\in[0,T]} \|p_t^1\|_{W^{2,1}(\mathbb{R})} \int_t^T \|q_s\|_{L^1(\mathbb{R})} ds, \\
&\qquad \leq C \sup_{a\in[-1,1]} \left( \sup_{t\in[0,T]} \|p(t,.,a)\|_{W^{2,1}(\mathbb{R})} \right) \int_t^T \|q_s\|_{L^1(\mathbb{R})} ds.
\end{aligned}$$

We now use Property (20) and get that, for any $t \in [0,T]$,

$$\left\| p(t,.,a^1) - p(t,.,a^2) \right\|_{L^1(\mathbb{R})} \leq C \left( \|p_0^2 - p_0^1\|_{L^1(\mathbb{R})} + \int_0^t \left\| p(s,.,a^1) - p(s,.,a^2) \right\|_{L^1(\mathbb{R})} ds \right).$$

Finally, we deduce from the previous inequality that

$$\begin{aligned}
\left\| S\circ\Phi(a^1) - S\circ\Phi(a^2) \right\|_{L^\infty([0,T],L^1(\mathbb{R}))} &= \sup_{t\in[0,T]} \left\| p(t,.,a^1) - p(t,.,a^2) \right\|_{L^1(\mathbb{R})} \\
&\leq C\|\Phi(a^2) - \Phi(a^1)\|_{L^1(\mathbb{R})} \\
&\quad + C \int_0^T \left\| S\circ\Phi(a^1) - S\circ\Phi(a^2) \right\|_{L^\infty([0,t],L^1(\mathbb{R}))} dt.
\end{aligned}$$

The Lipschitz continuity of $S \circ \Phi$ then follows from the Gronwall lemma and the Lipschitz continuity of the function $\Phi$. □

**Remark 2.5.** *The restrictive assumption (20) can be relaxed if the kernel $\sigma$ is constant, as for the Burgers or the vorticity equations. Indeed, in that case, we can write (9) in an integral form:*

$$p(t,x,a) = G_t * p_0(x,a) - \int_0^t G_{t-s} * \partial_x \left( u_b(s,x,a) p(s,x,a) \right) ds,$$

*where* $G_t(x) = (2\pi\sigma^2 t)^{-1} \exp\left(-\frac{x^2}{2\sigma^2 t}\right)$ *is the heat kernel. By classical properties of* $G_t$ *and Inequalities (16), it is then easy to deduce that*

$$\left\| p(t,.,a^1) - p(t,.,a^2) \right\|_{L^1(\mathbb{R}^2)} \leq C(T) \left\| p_0(.,a^1) - p_0(.,a^2) \right\|_{L^1(\mathbb{R}^2)}.$$

In view of Definition 2.1 and Proposition 2.3, the unique statistical solution of (2) with initial condition $\mu = \nu o \Phi^{-1}$ is

$$m := \mu \circ S^{-1} \;=\; \nu \circ (S \circ \Phi)^{-1}. \tag{21}$$

The following proposition gives conditions under which the time marginals of the measure $m$ has moments up to order $k$.

**Proposition 2.6.** *Assume that the hypotheses of Proposition 2.3 hold. Assume also that the measure* $\nu$ *and the function* $\Phi$ *satisfy*

$$\exists k \in \mathbb{N}, \qquad \int_{-1}^{1} \|\Phi(a)\|^k_{L^2(\mathbb{R})}\, \nu(da) \;<+\infty. \tag{22}$$

*Then, for any* $t \in [0,T]$*, the measure* $m_t := \mu \circ S_t$ *has moments up to order* $k$.

*Proof.* We need to prove that the measure $m_t$ satisfies Condition (4), that is,

$$\int_{L^2(\mathbb{R})} \|p\|^k_{L^2(\mathbb{R})}\, m_t\,(dp) \;\; <+\infty.$$

By definition of $m_t$ one has

$$\begin{aligned}\int_{L^2(\mathbb{R})} \|p\|^k_{L^2(\mathbb{R})}\, m_t\,(dp) &= \int_{L^2(\mathbb{R})} \|p\|^k_{L^2(\mathbb{R})} \left(\nu \circ (S_t \circ \Phi)^{-1}\right)(dp), \\ &= \int_{-1}^{1} \|(S_t \circ \Phi)\,(a)\|^k_{L^2(\mathbb{R})}\, \nu(da).\end{aligned}$$

As in the proof of Lemma 2.4, the solution $(S_t \circ \Phi)\,(a)$ to Equation (9) satisfies a Feynman-Kac formula, from which Inequalities (16) and (18) easily lead to

$$\|(S_t \circ \Phi)\,(a)\|_{L^2(\mathbb{R})} \leq C\, \|\Phi(a)\|_{L^2(\mathbb{R})}.$$

The result follows from assumption (22). □

In view of (5), the first moment of the measure $m_t$ is then defined by

$$\begin{aligned}\forall f \in L^2(\mathbb{R}),\;\; \langle M_1(t), f\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \langle S_t p_0, f\rangle_{L^2(\mathbb{R})}\, \mu\,(dp_0), \\ &= \int_{-1}^{1} \langle S_t \circ \Phi(a), f\rangle_{L^2(\mathbb{R})}\, \nu(da).\end{aligned} \tag{23}$$

In order to develop an approximation method of the moments, we now prove a probabilistic representation of these moments.

# 3 A probabilistic representation of the moments.

Our objective is now to give a probabilistic representation of the moments of the statistical solution of (2), in order to be able to approximate these moments by a stochastic particle method. For the sake of simplicity, we limit ourselves to considering the first moment. The extension to moments of higher order is straightforward (see Corollary 3.4).
We recall that, for any $a \in [-1,1]$, the solution $(S_t \circ \Phi)(a)$ of Equation (9) is the density of the random variable $X_t(a)$, where $X_.(a)$ is the weak solution of the stochastic differential equation (10). Hence, for any bounded and measurable function $f$,

$$\langle (S_t \circ \Phi)(a) \ , \ f \rangle_{L^2(\mathbb{R})} \quad = \quad \mathbb{E} f\left(X_t(a)\right),$$

and we have

$$\begin{aligned} \langle M_1(t), f\rangle_{L^2(\mathbb{R})} &= \int_{-1}^{1} \langle (S_t \circ \Phi)(a) \ , \ f \rangle_{L^2(\mathbb{R})} \, \nu(da), \\ &= \int_{-1}^{1} \mathbb{E} f\left(X_t(a)\right) \nu(da). \end{aligned}$$

This representation suggests the following naïve method of simulation:

1. Consider $N_1$ realizations $a_l$, $1 \le l \le N_1$, of a random variable $\theta$ with law $\nu$.
2. For each initial parameter $a_l$, consider $N_2$ i.i.d. random variables $X_0^i(a_l)$, with common law $[\Phi(a_l)](x)dx$, and define the particle system:

$$1 \le i \le N_2, \begin{cases} dX_t^{i,N_2}(a_l) &= \frac{1}{N_2} \sum_{j=1}^{N_2} b\left(X_t^{i,N_2}(a_l), X_t^{j,N_2}(a_l)\right) dt \\ & \quad + \frac{1}{N_2} \sum_{j=1}^{N_2} \sigma\left(X_t^{i,N_2}(a_l), X_t^{j,N_2}(a_l)\right) dW_t^i, \\ X_0^{i,N_2}(a_l) &= X_0^i(a_l). \end{cases} \tag{24}$$

   Then,

$$\mathbb{E} f\left(X_t(a_l)\right) \simeq \frac{1}{N_2} \sum_{i=1}^{N_2} f\left(X_t^{i,N_2}(a_l)\right). \tag{25}$$

   This latter approximation is understood in the following sense: the measure-valued process $\frac{1}{N_2}\sum_{i=1}^{N_2} \delta_{X_.^{i,N_2}}$ converges in law, as $N_2$ tends to infinity, to $\delta_{\mathbb{P}_{X_.(a_l)}}$. This type of convergence is the so called *propagation of chaos*. We refer to Méléard [21] or Sznitman [25] for further details.

We finally get:

$$\langle M_1(t), f\rangle_{L^2(\mathbb{R})} \simeq \frac{1}{N_1}\sum_{l=1}^{N_1} \frac{1}{N_2}\sum_{i=1}^{N_2} f\left(X_t^{i,N_2}(a_l)\right). \tag{26}$$

This approximation is numerically very expensive, as it requires $N_1 \times N_2$ simulations of paths of stochastic processes. The method with a much lower cost which is developed in Section 4 relies on the construction of a stochastic process whose marginal laws are equal to $[(S_t \circ \Phi)(a)](x)dx \otimes \nu(da)$. In view of Equation (10) and Equations (11), a natural candidate is the weak solution (if it exists) of the following stochastic differential equation:

$$\begin{cases} dX_t \;=\; u_b(t, X_t, \theta)\, dt + u_\sigma(t, X_t, \theta)\, dW_t, \\ \forall A \in \mathcal{B}(\mathbb{R} \times [-1,1]), \;\; \mathbb{P}((X_0, \theta) \in A) = \int_A [\Phi(a)](x) dx \;\nu(da), \\ \theta \text{ random variable independent of } W_\cdot, \end{cases} \tag{27}$$

where the functions $u_b$ and $u_\sigma$ are the coefficients of Equation (9). However, we cannot deduce a numerical method from (27). Indeed, we cannot simulate the solution of (27) since we do not know the coefficients $u_b(t,x,a)$ and $u_\sigma(t,x,a)$ without solving Equation (9). To overcome this problem, we will observe that, for any function $g \in C_b(C([0,T], \mathbb{R}), \mathbb{R})$,

$$\mathbb{E}^\nu g(X_\cdot) = \int_{-1}^{1} \mathbb{E} g(X_\cdot(a))\, \nu(da), \tag{28}$$

where $X_\cdot(a)$ is the solution of (10) (see Theorem 3.1 below). From this property, we will prove that $X_\cdot$ is the unique solution of the nonlinear SDE

$$\begin{cases} dX_t & = \;\; \mathbb{E}[b(x, X_t) \mid \theta]\,|_{x=X_t}\, dt + \mathbb{E}[\sigma(x, X_t) \mid \theta]\,|_{x=X_t}\, dW_t,\; t \le T \\ (X_0, \theta) & \quad\;\; \text{with law } [\Phi(a)](x)dx \;\nu(da), \\ \theta & \quad\;\; \text{random variable independent of } W_\cdot, \end{cases} \tag{29}$$

(see Theorem 3.2 below). We now prove our conjectures.

**Theorem 3.1.** *Suppose that the hypotheses of Proposition 2.3 hold. Then*

**(i)** *The stochastic differential equation (27) has a unique weak solution.*

**(ii)** *For any function $g \in C_b(C([0,T], \mathbb{R}), \mathbb{R})$,*

$$\mathbb{E}^\nu g(X_\cdot) = \int_{-1}^{1} \mathbb{E} g(X_\cdot(a))\, \nu(da), \tag{30}$$

*where $X_\cdot(a)$ is the solution of (10) and, if $\mathbb{P}_W$ stands for the law of the Brownian motion $W_\cdot$, $\mathbb{E}^\nu$ is the expectation operator under the product measure $\mathbb{P}_W \otimes \nu$.*

*Proof.* We first need to check that the mapping $a \in [-1,1] \mapsto \mathbb{E} g(X_\cdot(a))$ is Lebesgue measurable. It is actually continuous, as proven in Lemma 5.1 whose proof is postponed to the Appendix.

*Proof of* **(i)**. Observe that Equation (27) can be rewritten as

$$\begin{cases} dX_t \;=\; u_b\left(t, X_t, \theta_t\right) dt + u_\sigma\left(t, X_t, \theta_t\right) dW_t, \\ d\theta_t \;=\; 0, \\ \forall A \in \mathcal{B}\left(\mathbb{R} \times [-1,1]\right), \quad \mathbb{P}\left((X_0, \theta_0) \in A\right) = \int_A \left[\Phi(a)\right](x) dx \nu(da). \end{cases} \tag{31}$$

In the proof of Proposition 2.3, we have shown that Hypothesis (7) ensures that the functions $u_b$ and $u_\sigma$ are Hölder continuous in $t$. Similarly, one easily check that Hypothesis (8) implies boundedness and Lipschitz continuity in $x$ of $u_b(t,x,a)$ and $u_\sigma(t,x,a)$. Finally, Lemma 2.4 ensures that $u_b$ and $u_\sigma$ also are Lipschitz continuous in $a$. Thus Equation (31), and of course Equation (27), has a unique solution in law. □

*Proof of* **(ii)**. We first fix some notation.

- $E = C\left([0,T], \mathbb{R} \times [-1,1]\right)$ and $(y_t)$ is the canonical process on $E$.
- For each $a \in [-1,1]$, $c_.(a)$ denotes the constant application $t \in [0,T] \mapsto a$.
- For each $a \in [-1,1]$ and any function $\phi \in C^2\left(\mathbb{R}\right)$, we set

$$L_t^a \phi(x) \;=\; u_b\left(t,x,a\right) \phi'(x) + \frac{1}{2} u_\sigma^2\left(t,x,a\right) \phi''(x), \tag{32}$$

and, for each function $\psi \in C^2\left(\mathbb{R} \times [-1,1]\right)$,

$$A_t \psi(x,a) \;=\; \left(L_t^a \psi(.,a)\right)(x). \tag{33}$$

Observe that $A_t$ is the infinitesimal generator of the Markov process $(X_., \theta_.)$, unique solution in law of Equation (31). In other words, the law of $(X_., \theta_.)$ is the unique solution of the martingale problem associated with operator $A_t$, that is the only probability measure $\mathbb{P}^\nu$ on $E$ satisfying the following properties:

**(a)** $\mathbb{P}^\nu o y(0)^{-1} = \left[\Phi(a)\right](x) dx \otimes \nu(da)$,

**(b)** for any function $\psi \in C_b^2\left(\mathbb{R} \times [-1,1]\right)$, the process $M_.\left(\psi, A\right)$, defined by

$$M_t\left(\psi, A\right) \;=\; \psi\left(y(t)\right) - \psi\left(y(0)\right) - \int_0^t A_s \psi\left(y(s)\right) ds \tag{34}$$

is a $\mathbb{P}^\nu$-martingale.

By uniqueness, Equality (30) will thus be proved if we show that the probability measure $\mathcal{Q}$, defined by

$$\forall g \in C_b\left(C\left([0,T], \mathbb{R} \times [-1,1]\right), \mathbb{R}\right), \quad \langle \mathcal{Q}, g \rangle := \int_{-1}^1 \mathbb{E} g\left(X_.(a), c_.(a)\right) \nu(da), \tag{35}$$

also satisfies Properties **(a)** and **(b)**.
By definition of the process $X_\cdot(a)$, solution of Equation (10), $\mathcal{Q}$ obviously satisfies Property **(a)**. Now let $p \in \mathbb{N}$, $h \in C_b\left((\mathbb{R} \times [-1,1])^p\right)$, $\psi \in C^2\left(\mathbb{R} \times [-1,1]\right)$ and $(t_1, \ldots, t_p, s, t) \in [0,T]^p$, such that

$$0 \leq t_1 \leq \ldots, \leq t_p \leq s \leq t.$$

In order to show that $\mathcal{Q}$ satisfies Property **(b)**, it is sufficient to check that

$$\mathbb{E}^{\mathcal{Q}}\left[h\left(y(t_1), \ldots, y(t_p)\right)\left(M_t(\psi, A) - M_s(\psi, A)\right)\right] = 0.$$

Observe that $L^a_\cdot$ is the infinitesimal generator of the process $X_\cdot(a)$. Thus, applying Itô's formula to the function $\psi$ and using definitions (33) and (35), we get

$$\begin{aligned} &\mathbb{E}^{\mathcal{Q}}\left[h\left(y(t_1), \ldots, y(t_p)\right)\left(M_t(\psi, A) - M_s(\psi, A)\right)\right] \\ &\quad = \int_{-1}^{1} \mathbb{E}\left[h\left(\left(X_{t_1}(a), a\right), \ldots, \left(X_{t_p}(a), a\right)\right) \int_s^t u_\sigma\left(\tau, X_\tau(a), a\right) \partial_x \psi\left(\tau, X_\tau(a), a\right) dW_\tau\right] \nu(da) \\ &\quad = 0 \end{aligned}$$

since $u_\sigma$ is a bounded function. □

We now identify the solution of Equation (27) as the unique solution of a nonlinear SDE, which will allow us to develop our stochastic particle method. This is the main result of this part I.

**Theorem 3.2.** *Suppose that the hypotheses of Proposition 2.3 hold. Then there exists a unique weak solution to the SDE (29), where* $(t,x) \mapsto \mathbb{E}\left[b\left(x, X_t\right) \mid \theta\right]$ *and* $(t,x) \mapsto \mathbb{E}\left[\sigma\left(x, X_t\right) \mid \theta\right]$ *stand for continuous modifications of the conditional expectation processes. Moreover, the law of the solution is* $\mathbb{P}_{X_\cdot(a)} \otimes \nu(da)$, *where* $X_\cdot(a)$ *is the unique weak solution of Equation (10).*

*Proof.* The existence of a solution results from Theorem 3.1, since Equalities (11) and (28) imply that

$$u_b(t,x,a) = \mathbb{E}\left[b\left(x, X_t\right) \mid \theta = a\right], \quad u_\sigma(t,x,a) = \mathbb{E}\left[\sigma\left(x, X_t\right) \mid \theta = a\right], \tag{36}$$

and that continuous modifications of $\left(\mathbb{E}\left[b\left(x, X_t\right) \mid \theta\right]\right)$ and $\left(\mathbb{E}\left[\sigma\left(x, X_t\right) \mid \theta\right]\right)$ exist in view of the Kolmogorov-Centsov criterion.
To get the uniqueness, one can easily show that two solutions $Y^1_\cdot = \left(X^1_\cdot, \theta\right)$ and $Y^2_\cdot = \left(X^2_\cdot, \theta\right)$ of the SDE (29), when constructed on the same probability space with the same Brownian motion, satisfy

$$\mathbb{E}^\nu\left(\sup_{t \leq T}\left|X^1_t - X^2_t\right|^2\right) \leq C(T) \int_0^T \mathbb{E}^\nu\left(\sup_{s \leq t}\left|X^1_s - X^2_s\right|^2\right) dt.$$

We refer to Vaillant [27] for the standard calculation. □

**Remark 3.3.** *Equation (29) reduces to the classical nonlinear stochastic differential equation (10) when $\nu$ is a Dirac measure.*

Finally, in view of Definition (5), a straightforward consequence of Theorems 3.1 and 3.2 is that moments of a statistical solution of the McKean-Vlasov equation (2) have the following probabilistic representation:

**Corollary 3.4.** *Suppose that the hypotheses of Proposition 2.6 hold. Let $\left(X^i_. \ ,1 \leq i \leq k\right)$ be independent copies of the processus $X_.$ solution of (29). Then, for any continuous and bounded fonction*
$f \in L^2(k) = \overset{k}{\otimes} L^2(\mathbb{R})$,

$$\langle M_k(t) \ , \ f \rangle_{L^2(k)} = \mathbb{E}^\nu f\left(X^1_t, \ldots, X^k_t\right). \tag{37}$$

# 4 Approximation of the moments. The stochastic particle method.

We now use the nonlinear SDE (29) to construct a particle system $\left(X^{i,N}_. \ 1 \leq i \leq N\right)$ which coincides with the system (24) when $\nu$ is a Dirac measure, and such that

$$\mathbb{E}^\nu \left| \langle M_1(t) \ , \ f \rangle_{L^2(\mathbb{R})} - \frac{1}{N}\sum_{i=1}^N f\left(X^{i,N}_t\right) \right| \underset{N \to +\infty}{\longrightarrow} 0. \tag{38}$$

First, consider the case where $\nu$ is a Dirac measure in a point $a \in [-1,1]$. The SDE (29) reduces to (10), which is the limit equation for the particle system (24). Formally, this system can be constructed from the SDE (10) as follows: consider $N$ independent copies $\left(X^i_.(a) \ ,1 \leq i \leq N\right)$ of the solution of (10) and replace the coefficients of this equation by estimates got by a Monte Carlo method:

$$\begin{aligned}
\mathbb{E}\left[b\left(x, X_t(a)\right)\right] &\simeq \frac{1}{N}\sum_{i=1}^N b\left(x, X^i_t(a)\right), \\
\mathbb{E}\left[\sigma\left(x, X_t(a)\right)\right] &\simeq \frac{1}{N}\sum_{i=1}^N \sigma\left(x, X^i_t(a)\right).
\end{aligned}$$

In order to simulate the SDE (29), we generalize this method: we replace the coefficients $\mathbb{E}\left[b\left(x, X_t\right) \mid \theta = a\right]$ and $\mathbb{E}\left[\sigma\left(x, X_t\right) \mid \theta = a\right]$ by their estimates constructed with $N$ independent copies $\left(X^i_t \ , \ \theta^i\right)$, $1 \leq i \leq N$ of the pair $\left(X_t \ , \ \theta\right)$. There is a wide litterature about such estimators of regression functions: we refer, e.g., to Hardle [16], Hardle *et al.* [17], Chu and Marron [9] or Juditsky *et al* [19]. We consider in this article the two following *kernel* estimators:

- if the measure $\nu$ is discrete, concentrated on a finite number of points $a_l$, $1 \leq l \leq M$, we choose the so-called *regressogram* estimator (see, e.g., Bouleau-Lépingle [4]):

$$r_1(a_l) := \begin{cases} \dfrac{\sum_{j=0}^N b\left(x, X_t^j\right) \mathbb{I}(\theta^j = a_l)}{\sum_{j=0}^N \mathbb{I}(\theta^j = a_l)} & \text{if } \sum_{j=0}^N \mathbb{I}(\theta^j = a_l) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

- if the measure $\nu$ has a density, we choose the so-called *Nadaraya-Watson* estimator (see Hardle [16], Chu-Marron [9]):

$$r_2(a) := \begin{cases} \dfrac{\sum_{j=0}^N b\left(x, X_t^j\right) G\left((\theta^j - a)/h_N\right)}{\sum_{j=0}^N G\left((\theta^j - a)/h_N\right)} & \text{if } \sum_{j=0}^N G\left((\theta^j - a)/h_N\right) \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{40}$$

where $h_N$ is a strictly positive real number and $G$ is a positive function such that $G(0) \neq 0$, for example a Gaussian density.

**Remark 4.1.** *The above choices of the estimators are mainly motivated by the simplicity of formulae (39) and (40), which allows us to estimate the convergence rate of the particle method that we define below. However, we think that our results should easily be extended to more accurate estimators, for example those developed in the references mentioned above.*

Replacing, in the SDE (29), the exact coefficients by one of the formulae (39) or (40), we get the particle system:

$$1 \leq i \leq N \begin{cases} dX_t^{i,N} &= \sum_{j=1}^N \alpha_{ij} b(X_t^{i,N}, X_t^{j,N})dt + \sum_{j=1}^N \alpha_{ij}\sigma(X_t^{i,N}, X_t^{j,N})dW_t^i, \\ X_t^{i,N}\,|_{t=0} &= X_0^i, \end{cases} \tag{41}$$

where the $\left(X_0^i, \theta^i\right)$ are independent copies with common law $[\Phi(a)](x)dx\ \nu(da)$ and

$$\text{if } \nu \text{ is discrete,} \qquad \alpha_{ij} = \frac{\mathbb{I}(\theta^i = \theta^j)}{\sum_{k=1}^N \mathbb{I}(\theta^i = \theta^k)}, \tag{42}$$

$$\text{if } \nu \text{ has a density,} \qquad \alpha_{ij} = \frac{G\left((\theta^i - \theta^j)/h_N\right)}{\sum_{k=1}^N G\left((\theta^i - \theta^k)/h_N\right)}. \tag{43}$$

Observe that the system (41) generalizes the system (24). Indeed, if $\nu$ is a Dirac measure, the weights $\alpha_{ij}$ reduce to the usual value $1/N$.
By Corollary 3.4, we thus define an approximation formula for the first moment:

$$\langle M_1(T), f\rangle_{L^2(\mathbb{R})} = \mathbb{E}^\nu f\left(X_T\right) \simeq \frac{1}{N}\sum_{i=1}^N f\left(\overline{X}_T^{i,N}\right), \tag{44}$$

where $\overline{X}^{i,N}_{\cdot}$ , $1 \leq i \leq N$ is defined by discretizing the SDE (41) by the Euler scheme with constant step $\Delta t = T/K$ ($t_k = k\Delta t,\ 0 \leq k \leq K$):

$$\begin{cases} \overline{X}^{i,N}_{t_{k+1}} &= \overline{X}^{i,N}_{t_k} + \sum_{j=1}^{N} \alpha_{ij} b(\overline{X}^{i,N}_{t_k}, \overline{X}^{j,N}_{t_k})\Delta t + \sum_{j=1}^{N} \alpha_{ij} \sigma(\overline{X}^{i,N}_{t_k}, \overline{X}^{j,N}_{t_k}) \left( W^i_{t_{k+1}} - W^i_{t_k} \right), \\ \overline{X}^{i,N}_0 &= X^i_0, \end{cases} \tag{45}$$

In the second part of this work [26], we get estimates on the accuracy of the approximation method (44). We motivate our results by the following numerical experiments.

## 5 Numerical tests

**Burgers equation.** In this section, we study the numerical behavior of the particle method by computing moments of statistical solutions of the Burgers and the Navier-Stokes equations.

Fix $a \in \mathbb{R}$ and consider the Burgers equation on $[0,T] \times \mathbb{R}$:

$$\begin{cases} \partial_t V(t,x,a) &= -V(t,x,a)\partial_x V(t,x,a) + \frac{1}{2}\sigma^2 \partial^2_{xx} V(t,x,a), \\ V(0,x,a) &= 1 - H(x-a), \end{cases} \tag{46}$$

where $H = \mathbb{I}_{\mathbb{R}^+}$ is the Heaviside function. It is well known (see Bossy-Talay [3]) that the solution $V(t,x,a)$ of (46) is given by

$$V(t,x,a) = 1 - \mathbb{E} H(x - X_t(a)), \tag{47}$$

where the process $X_{\cdot}(a)$ is the unique weak solution of the nonlinear SDE

$$\begin{cases} dX_t(a) &= [1 - \mathbb{E} H(x - X_t(a))]\,|_{x=X_t(a)}\, dt + \sigma dW_t, \\ X_0(a) &= a \ a.s. \end{cases} \tag{48}$$

Consider a real valued random variable $\theta$ with law $\nu$. Theorem 3.2 does not hold in that case since the function $H$ is discontinuous at 0. Nevertheless, we think that a combination of our Section 3 with the technique of [26] leads to

$$\mathbb{E}^{\nu} V(t,x,\theta) = 1 - \mathbb{E}^{\nu} H(x - X_t), \tag{49}$$

where the process $X_{\cdot}$ is a weak solution of

$$\begin{cases} dX_t &= \{1 - \mathbb{E}[H(x - X_t) \mid \theta]\}\,|_{x=X_t}\, dt + \sigma dW_t, \\ X_0 &= \theta. \end{cases} \tag{50}$$

As explained in Section 4, we approximate the law of the process $X_.$ by the empirical measure of the particles $\left(X_.^{i,N},\ 1 \le i \le N\right)$, satisfying

$$\begin{cases} dX_t^{i,N} & = \sum_{j=1}^{N} \alpha_{ij} \left(1 - H\left(X_t^{i,N} - X_t^{j,N}\right)\right) dt + \sigma dW_t^i, \\ X_0^{i,N} & = \theta^i, \end{cases} \tag{51}$$

where the $\theta^i, 1 \le i \le N$, are independent copies of $\theta$, and the weights $\alpha_{ij}$ are defined by (42) or (43). Finally, denoting by $\left(\overline{X}_.^{i,N}, 1 \le i \le N\right)$ the Euler scheme with discretization step $\Delta t = T/K$ applied to the SDE (51), we define an approximation formula for the mean solution of the Burgers equation (46) :

$$\mathbb{E}^{\nu} V\left(T, x, \theta\right) \simeq \overline{V}(T, x) := 1 - \frac{1}{N} \sum_{i=1}^{N} H\left(x - \overline{X}_T^{i,N}\right). \tag{52}$$

Following Bossy and Talay [3], we now study the convergence error of this algorithm for the norm in $L^1(\mathbb{R})$:

$$\mathcal{E}\left(N,\ \Delta t,\ \nu\right) = \mathbb{E}\left\|\mathbb{E}^{\nu} V\left(T, ., \theta\right) - \overline{V}(T, .)\right\|_{L^1(\mathbb{R})}. \tag{53}$$

Given a seed $\rho$ of the random number generator, we approximate $\left\|\mathbb{E}^{\nu} V\left(T, ., \theta\right) - \overline{V}(T, .)\right\|_{L^1(\mathbb{R})}$ by

$$\begin{aligned} \left\|\mathbb{E}^{\nu} V\left(T, ., \theta\right) - \overline{V}(T, .)\right\|_{L^1(\mathbb{R})} & \simeq \sum_{i=1}^{N-1} \left(\overline{X}_T^{i+1,N} - \overline{X}_T^{i,N}\right) \left|\left[\mathbb{E}^{\nu} V\left(T, x, \theta\right)\right]\big|_{x=\overline{X}_T^{i,N}} - \overline{V}(T, \overline{X}_T^{i,N})\right| \\ & := \mathcal{E}_{\rho}\left(N,\ \Delta t,\ \nu\right). \end{aligned}$$

We produce independent particle systems by choosing different values $\rho_k$; we have

$$\mathcal{E}\left(N,\ \Delta t,\ \nu\right) \simeq \frac{1}{20} \sum_{k=1}^{20} \mathcal{E}_{\rho_k}\left(N,\ \Delta t,\ \nu\right).$$

For a fixed $a \in \mathbb{R}$, we compute the solution $V(T, x, a)$ of the Burgers equation (46) by using the Cole-Hopf formula:

$$V(T, x, a) = \frac{\int_{\mathbb{R}} V_0(y, a) \exp\left(-\frac{1}{\sigma^2}\left[\frac{(x-y)^2}{2T} + \int_0^y V_0(z, a) dz\right]\right) dy}{\int_{\mathbb{R}} \exp\left(-\frac{1}{\sigma^2}\left[\frac{(x-y)^2}{2T} + \int_0^y V_0(z, a) dz\right]\right) dy}.$$

As $V_0(\cdot, a) = 1 - H(\cdot - a)$, this formula reduces to the more convenient form :

$$V(T, x, a) = 1 - \frac{\operatorname{erfc}\left(\frac{-(x-a)}{\sqrt{2\sigma^2 T}}\right)}{\operatorname{erfc}\left(\frac{-(x-a)}{\sqrt{2\sigma^2 T}}\right) + \exp\left(\frac{T - 2(x-a)}{2\sigma^2}\right)\left(2 - \operatorname{erfc}\left(\frac{T - (x-a)}{\sqrt{2\sigma^2 T}}\right)\right)}, \tag{54}$$

where erfc$(x)$ is defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp\left(-u^2\right) du.$$

This function is preprocessed in the standard C library "math.h". If the measure $\nu$ is discrete, we compute $\mathbb{E}^\nu V(T, x, \theta)$ by computing the right hand side of (54) exactly. If the measure $\nu$ has a density, we approximate the right hand side of (54) by means of an adaptative Simpson's rule improved by a Richardson-Romberg extrapolation.
Finally, observe that, as the interaction kernel for the Burgers equation is the Heaviside function, a quick way to compute the drift term $\sum_{j=1}^N \alpha_{ij} H\left(\overline{X}_{t_k}^{i,N} - \overline{X}_{t_k}^{j,N}\right)$ is to sort the particles at each time step: indeed, if the family $\left(\overline{X}_{t_k}^{i,N},\ 1 \le i \le N\right)$ is increasing,

$$\sum_{j=1}^N \alpha_{ij} H\left(\overline{X}_{t_k}^{i,N} - \overline{X}_{t_k}^{j,N}\right) = \sum_{j=1}^i \alpha_{ij} \quad \text{and} \quad \overline{V}\left(t_k, \overline{X}_{t_k}^{i,N}\right) = \frac{i}{N}.$$

For all the numerical tests described below, we set $\sigma^2 = 0.1$. This value of the diffusion coefficient is small enough to ensure an accurate simulation of the particles and large enough for an accurate computation of the Cole-Hopf formula (54).
First suppose that the measure $\nu$ is discrete, concentrated in $M$ points, and define the weights $\alpha_{ij}$ as in (42).

| | Number of particles | | | | |
|---|---|---|---|---|---|
| $M$ | 250 | 1000 | 4000 | 16000 | 64000 |
| 2 | 6.381 | 3.237 | 1.795 | 0.760 | 0.425 |
| 5 | 8.322 | 4.142 | 2.366 | 0.989 | 0.518 |

$T = 0.5$,
$\sigma^2 = 0.1$,
$\Delta t = 0.01$.

Tab. 1 - $\mathcal{E}(N,\ \Delta t,\ \nu).10^{-2}$. *Influence of N.*

Table 1 presents the results of the numerical computation of the error $\mathcal{E}(N,\ \Delta t,\ \nu)$ for a fixed time step $\Delta t = 0.01$ and an increasing number of simulated particles. From a column to the next, the number of particles is multiplied by 4, and one observes that the error is roughly divided by 2. Thus the convergence rate of the particle method seems to be $1/\sqrt{N}$.

We now investigate the influence of the number of points on which the measure $\nu$ is concentrated, $M$. For that purpose, we test the algorithm, with $N = 16000$ particles, for a sequence $(\nu_M)_{M\in\mathbb{N}}$ of initial measures defined by

$$\nu_M := \frac{1}{2^M+1} \sum_{p=0}^{2^M+1} \delta\left(\frac{2p}{2^M+1}\right).$$

The results are gathered in Table 2. Observe that, when $M$ increases, the error first decreases then increases. This behaviour could be explained by a convergence rate estimate of the form

$$\mathcal{E}\left(N,\ \Delta t,\ \nu_M\right) = \bigcirc\left(\frac{M^a}{N} + \frac{1}{M^b} + \Delta t^c\right), \quad a, b, c \in ]0, +\infty[. \tag{55}$$

The second term in the right hand side of (55) should be an estimate on the convergence rate of the measure $\nu_M$ to the uniform measure on $[0, 2]$. Consequently, this would suggest that the particle method with weights (42) could be applied even if the measure $\nu$ has a density. In the second part of this work [26], we will prove this conjecture in the case of a McKean-Vlasov equation with smooth interaction kernels.

| M | $\mathcal{E}\left(N,\ \Delta t,\ \nu_M\right).10^{-2}$ |
|---|---|
| 2 | 0.76 |
| 3 | 0.693 |
| 5 | 0.655 |
| 9 | 0.608 |
| 17 | 0.587 |
| 33 | 0.572 |
| 65 | 0.554 |
| 129 | 0.585 |
| 257 | 6.439 |
| 513 | 8,952 |

$N = 16000$,
$\Delta t = 0.01$,
$T = 0.5$,
$\sigma^2 = 0.1$.

Tab. 2 - *Convergence error. Influence of M.*

Finally, we test the influence of the time discretization step $\Delta t$ on the convergence error of the particle method. We fix $T = 1$ and the number of particles is chosen large enough, namely $N = 100000$, so that the effect of the statistical error is negligible compared to the error induced by the time discretization. The results are gathered in Table 3.

| $\Delta t$ | $\mathcal{E}\left(N, \Delta t,\ \nu\right).10^{-2}$ |
|---|---|
| 1/2 | 6.866 |
| 1/4 | 3.293 |
| 1/8 | 1.636 |
| 1/16 | 0.902 |
| 1/32 | 0.775 |
| 1/64 | 0.485 |
| 1/128 | 0.424 |
| 1/256 | 0.41 |

$N = 100000$,
$M = 5$,
$T = 1$,
$\sigma^2 = 0.1$.

Tab. 3 - *Convergence error. Influence of $\Delta t$.*

Observe that, when the time step is divided by 2, the error is also roughly divided by 2 until the time step becomes so small that the influence of $N$ must be taken into account. This suggests a convergence rate of order $\Delta t$. In the case of the Burgers equation with deterministic initial condition, Bossy [1] has recently proved this optimal estimate on the convergence rate of the particle method with constant weights $1/N$. For numerical tests, see Bossy, Fezoui & Piperno [2].

We now assume that the probability measure $\nu$ is uniform on $[-1,1]$. We study the numerical behaviour of the particle method whose weights are defined by (43). We fix $T = 0.5$, $\Delta t = 0.01$ and we suppose that the bandwidth $h_N$ is of the form $h_N = (1/N)^{1/a}$. Table 4 gathers the results of numerical tests for different values of $N$ and $a$. The higher convergence rate depending on $N$ is observed for $a = 5$, where it is close to $1/\sqrt{N}$. Then this convergence rate decreases when $a$ increases. This behaviour is characteristic of the Nadaraya-Watson estimator (40) for a regression function (see e.g. Theorem 1 in [19]): if the regression function is Hölder continuous of order $s$, the convergence rate of the Nadaraya-Watson estimator for the mean square norm is of order $h_N^{2s} + \frac{1}{Nh_N}$. Consequently, if $h_N = (1/N)^{1/a}$, the leading term of this convergence rate estimate is $\left(\frac{1}{N}\right)^{\frac{2s}{a}}$, which increases with $a$. Furthermore, there exists an optimal choice of $h_N$, for which the error is of order $(1/N)^{2s/(2s+1)}$. In particular, if the regression function is of class $C^\infty$, the error is of order $1/N$. Observe that we are in this situation, since the considered regression function is the smooth solution of the Burgers equation (46) given by the Cole-Hopf formula (54).

| | Number of particles | | | | | | |
|---|---|---|---|---|---|---|---|
| $a$ | 250 | 500 | 1000 | 2000 | 4000 | 8000 | 16000 |
| 2 | 4.051 | 3.158 | 2.309 | 1.668 | 1.288 | 1.065 | 0.687 |
| 5 | 4.06 | 3.21 | 2.289 | 1.669 | 1.212 | 1.061 | 0.684 |
| 7 | 4.08 | 3.24 | 2.34 | 1.723 | 1.377 | 1.147 | 0.757 |
| 10 | 4.143 | 3.301 | 2.357 | 1.831 | 1.471 | 1.255 | 0.876 |
| 15 | 4.150 | 3.352 | 2.411 | 1.904 | 1.575 | 1.392 | 1.007 |

$\Delta t = 0.01$,
$T = 0.5$,
$\sigma^2 = 0.1$.

Tab. 4 - $\mathcal{E}(N, \Delta t, \nu).10^{-2}$. *Influence of $N$.*

**Navier-Stokes equation.** We now test our method by computing moments of the statistical solution of the 2D incompressible Navier-Stokes equation:

$$(t,x) \in [0,T] \times \mathbb{R}, \begin{cases} \partial_t \omega(t,x,a) &= -\nabla.\left(\mathbf{u}(t,x,a)\omega(t,x,a)\right) + \dfrac{\sigma^2}{2}\Delta\omega(t,x,a), \\ \mathbf{u} &= \mathbf{K} * \omega, \text{ where } \mathbf{K}(\mathbf{y}) = \dfrac{1}{2\pi \mid \mathbf{y}\mid^2}(-y_2, y_1), \\ \omega(0,x,a) &= \omega_0(x,a). \end{cases} \tag{56}$$

Assuming that $\omega_0$ is a probability density function, one can show (see, e.g., Roberts [24]), that the mapping

$$t \mapsto L(t,\ a) := \int_{\mathbb{R}^2} \|x\|^2 \omega(t,x,a)dx$$

grows linearly :

$$L(t,\ a) = L(0,\ a) + 2\sigma^2 t.$$

Consequently,

$$\mathbb{E}^\nu L(t,\ \theta) = \mathbb{E}^\nu L(0,\ \theta) + 2\sigma^2 t. \tag{57}$$

We verify that our particle method captures this linear behaviour. Consider the 2D particle system

$$\begin{cases} \overline{X}^{i,N}_{(k+1)\Delta t} = \overline{X}^{i,N}_{k\Delta t} + \Delta t \sum_{j=1}^N \mathbf{K}_\varepsilon \left(\overline{X}^{i,N}_{k\Delta t},\ \overline{X}^{j,N}_{k\Delta t}\right) + \sigma\left(W^i_{(k+1)\Delta t} - W^i_{k\Delta t}\right), \\ \left(\overline{X}^{i,N}_0,\ \theta^i\right) \text{ with law } \omega_0(x,a)dx\ \nu(da), \end{cases} \tag{58}$$

where the weights $\alpha_{ij}$ are defined by (43), and

$$\mathbf{K}_\varepsilon(x) = \frac{4\varepsilon^4 + \left(\|x\|^2 + 3\varepsilon^2\right)\|x\|^2}{2\pi\left(\|x\|^2+\varepsilon^2\right)^3}\left(\ -x_2, x_1\ \right),\quad \varepsilon > 0,$$

is a smooth approximation of the Biot-Savart kernel $\mathbf{K}$ (see, e.g., Bossy and Talay[3]). We test the approximation formula

$$\forall k \le K,\quad \mathbb{E}^\nu L(k\Delta t,\ \theta) \simeq A_N(k\Delta t) = \frac{1}{N}\sum_{i=1}^N \left\|\overline{X}^{i,N}_{k\Delta t}\right\|^2. \tag{59}$$

Suppose that $\theta$ is uniformly distributed on $[0,1]$ and that $\omega_0(.,\theta)$ is a Gaussian density with mean zero and covariance matrix $\theta^2 Id_{\mathbb{R}^2}$. Fix $\varepsilon = 10^{-5}$, $\Delta t = 0.01$ and $\sigma^2 = 0.1$, so that the slope of $\mathbb{E}^\nu L(t,\theta)$ is 0.2. We test the particle method by computing the empirical mean slope $\overline{s}$ of $A_N$ and its empirical variance $Var(s)$ over the time interval $[0,\ 1]$. The numerical results for different numbers $N$ of particles are presented in Table 5.

| N | $(0.2-\overline{s}).10^{-2}$ | $Var(s).10^{-3}$ |
|---|---|---|
| 250 | 1.0 | 7.8 |
| 500 | 0.7 | 4.1 |
| 1000 | 0.8 | 2.7 |
| 2000 | 0.5 | 0.7 |
| 4000 | 0.3 | 0.3 |
| 8000 | 0.2 | 0.2 |
| 16000 | 0.03 | 0.18 |

$\Delta t = 0.01$,
$T = 0.5$,
$\sigma^2 = 0.1$,
$\varepsilon = 10^{-5}$.

Tab. 5 - *Convergence error. Influence of* $N$.

These results show that the particle method actually converges, but it is very difficult to conjecture estimates on the rate of convergence in terms of $N$, $h_N$, $\varepsilon$.

# Appendix

**Lemma 5.1.** *Suppose that the hypotheses of Proposition 2.3 hold. Then, for any function* $g \in C_b\left(C\left([0,T],\mathbb{R}\right),\mathbb{R}\right)$*, the mapping*

$$a \in [-1,1] \mapsto \mathbb{E} g\left(X_.(a)\right)$$

*is continuous.*

*Proof.* Let $g \in C_b\left(C\left([0,T],\mathbb{R}\right)\right)$. For any $a \in [-1,1]$, let $\mathbb{P}_{X_.(a)}$ denote the law of the process $X_.(a)$. Let a sequence $(a_n) \subset [-1,1]$ converging to $a$. We have to verify that the sequence $\left(\mathbb{P}_{X_.(a_n)}\right)$ weakly converges to $\mathbb{P}_{X_.(a)}$. Owing to the boundedness of the functions $b$ and $\sigma$, it is clear that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left\{\sup_{t\in[0,T]} \left|X_t(a_n) - X_s(a_n)\right|^4\right\} \leq C(T)\,(t-s)^2 .$$

The sequence $\left(\mathbb{P}_{X_.(a_n)}\right)$ is thus tight, so there exists a subsequence of $\left(\mathbb{P}_{X_.(a_n)}\right)$, that we abusively denote by $\left(\mathbb{P}_{X_.(a_n)}\right)$, which weakly converges to a probability measure $\mathbb{P}^\infty$ on $C\left([0,T],\mathbb{R}\right)$. It remains to prove that $\mathbb{P}^\infty$ is equal to $\mathbb{P}_{X_.(a)}$. As Equation (10) has a unique solution in law, this is equivalent to show that $\mathbb{P}^\infty$ is the unique solution of the martingale problem associated with the operator $L_t^a$ defined in (32).
So let $\psi \in C_b^2\left(\mathbb{R}\right)$, $p \in \mathbb{N}$, $h \in C_b\left(\mathbb{R}^p\right)$, and $(t_1,\ldots,t_p,s,t) \in [0,T]^p$, such that

$$0 \leq t_1 \leq \ldots, \leq t_p \leq s \leq t.$$

For any $\alpha \in [-1,1]$, we set

$$M_t\left(\psi, L^\alpha\right) \;=\; \psi\left(x(t)\right) - \psi\left(x(0)\right) - \int_0^t L_\tau^\alpha \psi\left(x(\tau)\right) d\tau, \tag{60}$$

where $x(.)$ is the canonical process on $C\left([0,T],\mathbb{R}\right)$. For any probability measure $m$ on $C\left([0,T],\mathbb{R}\right)$, we set

$$\Delta(m,\alpha) := \mathbb{E}^m\left[h\left(x\left(t_1\right),\ldots,x\left(t_p\right)\right)\left(M_t\left(\psi,L^\alpha\right) - M_s\left(\psi,L^\alpha\right)\right)\right]. \tag{61}$$

We have to prove

**(a)** $\forall\tilde{\psi} \in C_b\left(\mathbb{R}\right),\;\; \mathbb{E}^{\mathbb{P}^\infty}\left[\tilde{\psi}\left(x(0)\right)\right] - \int_{\mathbb{R}} \tilde{\psi}(x)\left[\Phi(a)\right](x)dx = 0,$

**(b)** $\Delta\left(\mathbb{P}^\infty, a\right) = 0.$

As $\left(\mathbb{P}_{X_\cdot(a_n)}\right)$ weakly converges to $\mathbb{P}^\infty$, for any function $\tilde{\psi} \in C_b(\mathbb{R})$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^\infty}\left[\tilde{\psi}(x_0)\right] - \int_{\mathbb{R}} \tilde{\psi}(x)\left[\Phi(a)\right](x)dx &= \lim_{n\to+\infty}\left\{\mathbb{E}^{\mathbb{P}_{X_\cdot(a_n)}}\left[\tilde{\psi}(x_0)\right] - \int_{\mathbb{R}} \tilde{\psi}(x)\left[\Phi(a)\right](x)dx\right\} \\
&= \lim_{n\to+\infty}\int_{\mathbb{R}} \tilde{\psi}(x)\left\{\left[\Phi(a_n)\right](x) - \left[\Phi(a)\right](x)\right\}dx \\
&\leq \left\|\tilde{\psi}\right\|_{L^\infty(\mathbb{R})} \lim_{n\to+\infty} \left\|\left[\Phi(a_n)\right] - \left[\Phi(a)\right]\right\|_{L^1(\mathbb{R})} \\
&= 0.
\end{aligned}$$

It now remains to prove Property **(b)**. One has

$$\begin{aligned}
\Delta\left(\mathbb{P}^\infty, a\right) &= \Delta\left(\mathbb{P}^\infty, a\right) - \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a\right) \\
&\quad + \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a\right) - \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a_n\right) \\
&\quad + \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a_n\right).
\end{aligned}$$

As $\left(\mathbb{P}_{X_\cdot(a_n)}\right)$ weakly converges to $\mathbb{P}^\infty$,

$$\lim_{n\to+\infty}\left\{\Delta\left(\mathbb{P}^\infty, a\right) - \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a\right)\right\} = 0. \tag{62}$$

Moreover, the measure $\mathbb{P}_{X_\cdot(a_n)}$ is solution of the martingale problem associated with operator $L_t^{a_n}$, so

$$\Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a_n\right) = 0, \quad \forall n \in \mathbb{N}. \tag{63}$$

Finally, one has

$$\begin{aligned}
&\Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a\right) - \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a_n\right) \\
&\quad = \mathbb{E}\left[h\left(X_{t_1}(a_n), \ldots, X_{t_p}(a_n)\right)\int_s^t \left(L_\tau^a - L_\tau^{a_n}\right)\psi\left(X_\tau(a_n)\right)d\tau\right], \\
&\quad = \mathbb{E}\left[h\left(X_{t_1}(a_n), \ldots, X_{t_p}(a_n)\right)\int_s^t \left(u_b(\tau, X_\tau(a_n), a) - u_b(\tau, X_\tau(a_n), a_n)\right)\psi'\left(X_\tau(a_n)\right)d\tau\right] \\
&\qquad + \mathbb{E}\left[h\left(X_{t_1}(a_n), \ldots, X_{t_p}(a_n)\right)\int_s^t \frac{1}{2}\left(u_\sigma^2(\tau, X_\tau(a_n), a) - u_\sigma^2(\tau, X_\tau(a_n), a_n)\right)\psi''\left(X_\tau(a_n)\right)d\tau\right].
\end{aligned}$$

By boundedness of the functions $\psi$, $\psi'$, $\psi''$, and $h$ and Properties (7) and (8) of $b$ and $\sigma$, it is easy to check that

$$\Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a\right) - \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a_n\right) \leq C \sup_{\tau\in[0,T]} \left\|\left(S_\tau \circ \Phi\right)(a_n) - \left(S_\tau \circ \Phi\right)(a)\right\|_{L^1(\mathbb{R})}, \tag{64}$$

(7) where the operator $S \circ \Phi$ has been defined in Proposition 2.3. Then, owing to Proposition 2.4,

$$\lim_{n\to+\infty}\left\{\Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a\right) - \Delta\left(\mathbb{P}_{X_\cdot(a_n)}, a_n\right)\right\} = 0. \tag{65}$$

Property **(b)** results from (62), (63) and (65). □

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