Practical Stabilization of Driftless Systems on Lie Groups

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Abstract: A general control design approach for the stabilization of controllable driftless nonlinear systems on finite dimensional Lie groups is presented. The approach is based on the concept of transverse functions, the existence of which is equivalent to the system's controllability. Its outcome is the *practical* stabilization of *any* trajectory —i.e. not necessarily a solution of the control system— in the state space. The possibility of applying the approach to an arbitrary controllable smooth driftless system follows in turn from the fact that any controllable homogeneous approximation of this system can be lifted (via a dynamic extension) to a system on a Lie group. Illustrative examples are given.

Key-words: practical stabilization, nonlinear system, Lie group, homogeneous vector field

Stabilisation pratique de systèmes sans dérive sur les groupes de Lie

Résumé : On présente une méthode générale de synthèse de lois de commande pour la stabilisation de systèmes de commande nonlinéaires, sans dérive, opérant sur des groupes de Lie. L'approche est basée sur l'utilisation de fonctions transverses, dont l'existence est équivalente à la commandabilité du système. A partir de cette méthode, il est possible de stabiliser pratiquement n'importe quelle trajectoire dans l'espace d'état, sans que celle-ci soit solution du système de commande. En outre, l'approche est aussi utilisable pour des systèmes sans dérive commandables généraux —i.e. pas nécessairement sur des groupes de Lie— en considérant une approximation homogène commandable et, éventuellement, une extension dynamique de celle-ci permettant de rendre les champs de vecteurs invariants par rapport à une opération de groupe. Des examples illustratifs sont donnés à la fin du rapport.

Mots-clés : stabilisation pratique, système nonlinéaire, groupe de Lie, champ de vecteurs homogène

Contents

1 Introduction

Let

$$\dot{g} = \sum_{i=1}^{m} u_i X_i(g) + P(g,t) \tag{1}$$

denote a control system on a Lie group G, with $X_1, \ldots, X_m$ left-invariant smooth vector fields (v.f.) which satisfy the Lie Algebra Rank Condition (LARC). The drift term $P(g,t)$ is assumed to belong to G_g —the tangent space of G at g—, and to depend continuously on t. The problem addressed in this paper is the *practical stabilization* of the identity element $e \in G$ via the asymptotic stabilization of a compact set contained in an arbitrary small neighborhood of e. Obviously, the main case of interest is when $\text{span}\{X_1(e), \ldots, X_m(e)\} \neq G_e$ —when $m < \dim(G)$, for instance.

Prior to commenting upon this particular control objective, it may be useful to explain why the Lie group framework is chosen here. A common motivation is that various physical systems are naturally modeled as systems on Lie groups. Rigid bodies in space and cart-like vehicles are well known examples. A second motivation, which we believe is less known although it is important with respect to the general relevance of this framework in Control Theory, is related to the possibility of locally approximating any smooth controllable driftless system by a controllable system on a Lie group. Let us be more specific about this point. A well known result, used in many control studies, is that any smooth controllable driftless systems on some arbitrary manifold M —not necessarily a Lie group—,

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x) \tag{2}$$

can be locally approximated, via the concept of homogeneous approximation, by a nilpotent system of the same dimension [14, 39]. What is less known, or —to our knowledge— has been less used in the Control literature devoted to feedback stabilization of nonlinear systems, is that this nilpotent system can in turn be embedded, via the concept of lifting of vector fields [35, 12], into a system on a Lie group —i.e. a system of type (1) with zero drift term P— whose dimension is thus given by the dimension of the Control Lie Algebra associated with the original nilpotent system. Note that this property is very much related to the important role played in formal Lie-algebraic techniques by Lie groups and left-invariant control systems on Lie groups —via the so called *universal control system* [41, 17]. A consequence of it is that the techniques developed in the present paper are not restricted to systems on Lie groups. They apply also to general systems like (2). Once the generality of systems on Lie group is understood, our next and last argument consists in showing that the Lie group framework is particularly well adapted to the present approach for both theoretical and control design reasons. The extensive use of the specific properties associated with systems on Lie groups in the proofs of the main results reported in the present paper is, by itself, a good illustration of this.

Let us now focus on control issues and put the practical stabilization objective considered here in perspective with the research effort devoted to driftless controllable systems like (2), mostly during the last decade. While controllability properties of these systems have been known for a long time —as a consequence of the classical Chow theorem [7]—, algorithms to compute open-loop controls in order to steer the system from one point to another have been proposed more recently. Most of these methods have been derived along one of the following three approaches. The first one is based on the concept of *highly oscillatory control inputs* used to generate motion in the direction of iterated Lie brackets of the control v.f. [21, 32, 40]. Another approach is based on properties associated with *nilpotent systems* which allow to calculate the solutions of the control system as explicit functions of the control inputs. The mapping so obtained can then be inverted to yield open-loop control expressions [18]. Finally, a third approach is based on the concept of *flatness* which allows to express both the solutions of the control system and the control inputs as functions of *flat outputs* —and their derivatives— [11, 25]. These flat outputs can then be used conveniently to generate trajectories of the control system. Since systems on a Lie group fit within the general formulation (2), the methods described above apply to these systems and can be particularized. This has been done in [19] for the approach based on highly oscillatory controls. Let us also mention [4] which aims at extending this kind of approach to mechanical systems with a drift term in their modeling equations.

The problem of asymptotic stabilization —via state feedback control— of an equilibrium point of (2) has also attracted much attention. From Brockett's theorem [3], it is well known that in general —e.g. if $m < \dim(M)$ and the control v.f. evaluated at the chosen equilibrium point are linearly independent— no smooth or even continuous pure state feedback can make the equilibrium point asymptotically stable. Different types of feedback laws have been considered to solve this problem —although not all of them guarantee Lyapounov stability. Discontinuous feedbacks [1, 6, 20], hybrid feedbacks [2, 28, 38] are two possibilities. Another one, more related to the present approach, consists in using continuous time-varying feedbacks. The relevance of this type of feedback for a three-dimensional system in the form (2) was first reported in [36] and a general existence result for an arbitrary controllable driftless system has been proved in [8]. Numerous design methods for time-varying stabilizers have also been developed, starting with Lipschitz continuous feedback laws [33, 42, 37] —that present the drawback of yielding slow (polynomial) convergence— and continuing with homogeneous feedbacks —that ensure exponential convergence, but are only continuous at the stabilized point [26, 34, 29, 27].

Despite these efforts, some important issues have not received a satisfactory solution. One of them concerns the compromise between speed of convergence and robustness of the stability property against modeling errors. For a few driftless systems, this type of robustness can be ensured by using a Lipschitz continuous time-varying feedback law [24], but this implies slow —not exponential— convergence to the origin for most of the system's trajectories. On the other hand, to our knowledge, no robust exponential stabilizer has been proposed until now. It is in fact possible that such a feedback does not exist for systems which do not satisfy Brockett's condition. A result in this direction has been proved in [23]. Efforts to circumvent the difficulty, by considering hybrid continuous/discrete time feedback laws [2, 28], have only brought partial results. For instance, such feedbacks can be made robust to unmodelled dynamics, but they are not robust against discretization uncertainties. Another issue is related to the trajectory stabilization problem. This problem is usually easier than point stabilization —in particular, the linearized approximation of the error system may be controllable— and various feedback solutions have been proposed for specific classes of nonlinear driftless systems, especially in the robotics literature [15, 37, 5]. Asymptotic stabilization is usually obtained under some conditions upon the reference trajectory. Typically it should not "converge to a fixed point". One could have hoped for the existence of a feedback law which would have uniformly guaranteed asymptotic stability independently of the considered trajectory —just as for linear controllable systems—, but negative results concerning this existence issue have been proven [22]. This has clear consequences in mobile robotics, when the control objective is the tracking of a reference vehicle whose trajectory is not known in advance. Indeed, which feedback law should be applied? How to reach the decision of switching from one control law to another? Is it possible to uniformly ensure stability by considering such a switching strategy? These are difficult questions which have seldom been studied so far.

The abovementionned difficulties suggest that, for nonlinear driftless systems, asymptotic stabilization will rarely be achieved in practice and that *practical* stabilization could be a more realistic control objective to pursue. The present paper goes into this direction by proposing a general approach for the practical stabilization of an arbitrary system (1). As already mentioned, the approach also applies to driftless systems with homogeneous v.f.. Also, by allowing an additive perturbation in the form of the drift term P in (1), and thanks to the Lie group structure, the approach extends directly to the trajectory (practical) stabilization problem. In particular, and in contrast with most of the published works on this subject, the reference trajectory is not required to be a solution of the unperturbed control system —i.e. a solution of (1) with $P = 0$. This is another argument in favor of the practical stabilization objective, since asymptotic stabilization of a trajectory basically requires that this trajectory be a solution of the control system. We believe that this type of robustness, seldom addressed explicitly in the past, could have interesting implications, and applications, in practice.

The approach here considered to achieve practical stabilization is based on the existence of bounded functions which are *transverse* to a set of v.f. [31]. Intuitively, one can make a comparison between this approach and the general open-loop control design algorithm developed in [21, 40]. In those papers, the idea was to add additional virtual control inputs involving sinusoids with fixed and high enough frequencies in order to ensure uniform boundedness of tracking errors by a pre-specified threshold. Here, the threshold is directly related to the size of periodic transverse functions whose associated frequencies are the new inputs. An interesting feature associated with this type of frequency adaptation is that the control frequencies may —depending essentially on the reference trajectory— tend to zero so that oscillations are not systematic. Note that the same idea can be traced back, in the context of mobile robots, to [10], a work itself adapted from control techniques used for induction motors [9].

The paper is organized as follows. In Section 2, the main technical result, later used for control design, about the existence and construction of transverse functions is stated. This result may be viewed as a generalization —and a simplification— of the transverse function theorem reported in [31]. It clarifies the connections between transverse functions and Lie groups, which had not been identified in [31]. In Section 3, the concept of transverse functions is used to solve the practical stabilization problem presented at the beginning of this introduction. The key point is that transverse functions allow to reformulate the practical stabilization problem as a trivial asymptotic stabilization of the identity element of the Lie group associated with the original system. In section 4, we show how the approach applies to systems with homogeneous v.f., via a dynamic extension given by the *lifting theorem* [35, 12]. Finally, the approach is illustrated by several examples in Section 5.

2 Main result

Let us first introduce some notation.

- $|.|$ is the Euclidean norm in $\mathbb{R}^n$.
- Let v denote a function of the variable θ, which depends smoothly on some parameter ε —possibly vector-valued—, we write $v = \mathrm{o}(|\varepsilon|^k)$ (resp. $v = \mathrm{O}(|\varepsilon|^k)$) if $\frac{v(\theta)}{|\varepsilon|^k} \longrightarrow 0$ as $|\varepsilon| \longrightarrow 0$ (resp. if $\frac{v(\theta)}{|\varepsilon|^k} \leq K$ in some neighborhood of $\varepsilon = 0$) uniformly with respect to θ.
- The tangent space of a manifold M at a point p is denoted as M_p.
- The differential of a smooth mapping f between manifolds, at a point p, is denoted as $df(p)$.
- $\mathbb{T}^k$ is the torus of dimension k, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

We shall also use standard notation relative to Lie groups —see e.g [13, 43] for more details on this topic.

- G denotes a finite dimensional Lie group, with Lie algebra —of left-invariant v.f.— $\mathfrak{g}$.
- The identity element of G is denoted by e.
- Left and right translations are denoted by l and r respectively, i.e. $l_\sigma(\tau) = r_\tau(\sigma) = \sigma\tau$.
- As usual, if $X \in \mathfrak{g}$ and $p \in G$, $\exp tX$ is the solution at time t of $\dot{g} = X(g)$ with initial condition $g(0) = e$.
- The adjoint representation of G is Ad, i.e. for $\sigma \in G$, $\mathrm{Ad}(\sigma) = dI_\sigma(e)$ with $I_\sigma : G \longrightarrow G$ defined by $I_\sigma(g) = \sigma g \sigma^{-1}$.
- The differential of Ad is ad, defined by $(\mathrm{ad}X, Y) = [X, Y]$.

The following definition of a *graded basis* of $\mathfrak{g}$ will be used also.

Definition 1 *Let* $X_1, \ldots, X_m \in \mathfrak{g}$ *denote independent v.f. such that* $\mathrm{Lie}(X_1, \ldots, X_m) = \mathfrak{g}$, *with* $\dim(\mathfrak{g}) = n$. *Define*

$$\mathfrak{g}^k \triangleq \mathrm{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{j-1}}, X_{i_j}] \ldots]]] : i_1, \ldots i_j \leq m \text{ and } j \leq k\}$$

and let $K = \min\{k : \mathfrak{g}^k = \mathfrak{g}\}$. *A* graded basis *of* $\mathfrak{g}$ *associated with* $X_1, \ldots, X_m$ *is an ordered basis* $\{X_1, \ldots, X_n\}$ *of* $\mathfrak{g}$ *associated with two mappings* $\lambda, \rho : \{m+1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ *such that:*

1. *For any* $k = 1, \ldots, K$, $\mathrm{span}\{X_1, X_2, \ldots, X_{m_k}\} = \mathfrak{g}^k$ *for some integer* m_k.
2. *For* $m_1 \leq m_{k-1} < i \leq m_k$, $X_i = [X_{\lambda(i)}, X_{\rho(i)}]$ *with* $X_{\lambda(i)} \in \mathfrak{g}^a$, $X_{\rho(i)} \in \mathfrak{g}^b$, *and* $a + b = k$.

Note that $m_1 = m$ and $m_K = n$. With any graded basis of $\mathfrak{g}$, one can associate a *weight vector* $(r_1, \ldots, r_n)$ defined by

$$r_i = k \Longleftrightarrow X_i \in \mathfrak{g}^k \setminus \mathfrak{g}^{k-1} \Longleftrightarrow m_{k-1} + 1 \leq i \leq m_k$$

Note that, $1 = r_1 \leq r_2 \leq \cdots \leq r_n = K$ and, from Definition 1,

$$\forall i > m_1 = m, \qquad r_i = r_{\lambda(i)} + r_{\rho(i)}$$

The main result, on which the design of a practical stabilizer will rely, is the following.

Theorem 1 *Let* G *denote a Lie group of dimension* n *with Lie algebra* $\mathfrak{g}$. *Let* $X_1, \ldots, X_m \in \mathfrak{g}$ *denote independent v.f. Then the following properties are equivalent:*

1. $\mathrm{Lie}\{X_1, \ldots, X_m\} = \mathfrak{g}$
2. *For any neighborhood* $\mathcal{U}$ *of* e *in* G, *there exists a function* $f \in \mathcal{C}^\infty(\mathbb{T}^{n-m}; \mathcal{U})$ *such that*

$$\forall \theta \in \mathbb{T}^{n-m}, \quad G_{f(\theta)} = \mathrm{span}\{X_1(f(\theta)), \ldots, X_m(f(\theta))\} \oplus df(\theta)(\mathbb{T}_\theta^{n-m}) \tag{3}$$

Furthermore, with $\{X_1, \ldots, X_n\}$ *denoting a graded basis of* $\mathfrak{g}$, *a possible choice for* f *is given by*

$$\forall \theta = (\theta_{m+1}, \ldots, \theta_n) \in \mathbb{T}^{n-m}, \qquad f(\theta) = f_n(\theta_n) f_{n-1}(\theta_{n-1}) \cdots f_{m+1}(\theta_{m+1}) \tag{4}$$

with $f_i : \mathbb{T} \longrightarrow G$ *defined by*

$$f_i(\theta_i) = \left(\exp \varepsilon_i^{r_i} \frac{\sin 2\theta_i}{4} X_i\right) \left(\exp \varepsilon_i^{r_{\rho(i)}} \cos \theta_i X_{\rho(i)}\right) \left(\exp \varepsilon_i^{r_{\lambda(i)}} \sin \theta_i X_{\lambda(i)}\right) \tag{5}$$

for adequately chosen real numbers $\varepsilon_i > 0$ $(m+1 \leq i \leq n)$.

As in [31], and for the sake of conciseness, functions which satisfy (3) will be called "transverse" to the v.f. $X_1, \ldots, X_n$, or just *transverse functions* (for this set of v.f.) when no ambiguity is possible. The choice of the parameters $\varepsilon_{m+1}, \ldots, \varepsilon_n$ is further specified in Lemma 3 used in the proof of the theorem.

Proof: To prove that Property 2 implies Property 1, we refer to [31] —the proof is a direct consequence of the Frobenius theorem. To show that Property 1 implies Property 2, we need to show that (3) is satisfied with the function f defined by (4), for some values of the parameters ε_i $(i = m+1, \ldots, n)$ in (5). We indicate below the main steps of the proof in the form of three lemmas.

By standard calculations we first prove that:

Lemma 1 *There exist analytic functions* $v_{i,j}$ $(i \in \{m+1, \ldots, n\},\ j \in \{1, \ldots, n\})$ *such that*

$$\frac{\partial f_i}{\partial \theta_i}(\theta_i) = \sum_{j=1}^{n} v_{i,j}(\theta_i)\, X_j(f_i(\theta_i)) \tag{6}$$

with

$$v_{i,j} = \begin{cases} \mathrm{O}(|\varepsilon_i|^{r_j}) & \text{if } j < i \text{ and } r_j < r_i \\ \mathrm{o}(|\varepsilon_i|^{r_j}) & \text{if } j < i \text{ and } r_j = r_i \\ \dfrac{\varepsilon_i^{r_i}}{2} + \mathrm{o}(|\varepsilon_i|^{r_i}) & \text{if } j = i \\ \mathrm{O}(|\varepsilon_i|^{r_j}) & \text{if } j > i \end{cases} \tag{7}$$

From this lemma we then prove that:

Lemma 2 *There exist analytic functions* $a_{j,i}$ $(j \in \{m+1, \ldots, n\},\ i \in \{1, \ldots, n\})$ *such that*

$$\frac{\partial f}{\partial \theta_i}(\theta) = \sum_{j=1}^{n} a_{j,i}(\theta)\, X_j(f(\theta)) \tag{8}$$

with

$$a_{j,i} = \begin{cases} \mathrm{O}(|\varepsilon_{m+1}, \ldots, \varepsilon_i|^{r_j}) & \text{if } j < i \text{ and } r_j < r_i \\ \displaystyle\sum_{k<r_j} \varepsilon_i^k \mathrm{O}(|\varepsilon_{m+1}, \ldots, \varepsilon_{i-1}|^{r_j - k}) + \mathrm{o}(|\varepsilon_{m+1}, \ldots, \varepsilon_i|^{r_j}) & \text{if } j < i \text{ and } r_j = r_i \\ \dfrac{\varepsilon_i^{r_i}}{2} + \displaystyle\sum_{k<r_i} \varepsilon_i^k \mathrm{O}(|\varepsilon_{m+1}, \ldots, \varepsilon_{i-1}|^{r_i - k}) + \mathrm{o}(|\varepsilon_{m+1}, \ldots, \varepsilon_i|^{r_i}) & \text{for } j = i \\ \mathrm{O}(|\varepsilon_{m+1}, \ldots, \varepsilon_i|^{r_j}) & \text{for } j > i \end{cases} \tag{9}$$

Note that this lemma would be a direct consequence of Lemma 1 if the partial derivative of f with respect to θ_i were equal to the partial derivative of f_i with respect to θ_i. This would happen, for example, if the group operation were the vector addition in a vector space. Note also that, if all O and o terms in the above expressions were equal to zero, then Theorem 1 would follow directly from (8-9) and from the fact that $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$. Although this is not the case, it is not very difficult to show that these terms can be neglected provided that the ε_i's are adequately chosen, as stated in the following lemma.

Lemma 3 *There exist $n-m$ numbers $\beta_{m+1}, \ldots, \beta_n$, and $\varepsilon_0 > 0$, such that choosing*

$$(\varepsilon_{m+1}, \ldots, \varepsilon_n) \;=\; \varepsilon\,(\beta_{m+1}, \ldots, \beta_n)\;,$$

with $0 < \varepsilon < \varepsilon_0$, yields

$$\forall \theta \in \mathbb{T}^{n-m}, \qquad \mathrm{D}_n(\theta) \triangleq \mathrm{Det}(a_{j,i}(\theta))_{j,i=m+1,\ldots,n} \neq 0\;. \tag{10}$$

In view of (8), (10) is equivalent to (3), so that the proof of the theorem follows. The proofs of the above three lemmas are reported in the appendix. ∎

3 Application to practical stabilization

3.1 Point stabilization

Consider system (1) and assume —without loss of generality— that $X_1, \ldots, X_m$ are independent. We show in the following proposition how the concept of transverse functions can be used to design control laws for system (1) which make $g = e$ *practically stable*. Let us remark that no assumption is made on P. For instance, when the projection of $P(e,t)$ onto $(\mathrm{span}\{X_1(e), \ldots, X_m(e)\})^{\perp}$ does not tend to zero as t tends to infinity, no control law can make $g = e$ an equilibrium of the system, and asymptotic stabilization of this element is pointless.

Proposition 1 *Let $\mathcal{U}$ denote a neighborhood of e, and $f : \mathbb{T}^{n-m} \longrightarrow \mathcal{U}$ a transverse function. Then along any solution $g(.)$ of (1), and along any trajectory $\theta(.)$ in $\mathbb{T}^{n-m}$,*

$$\frac{d}{dt}(f(\theta)g^{-1}) = -dr_{g^{-1}}(f(\theta))\left(\sum_{i=1}^{m} u_i X_i(f(\theta)) - \sum_{i=m+1}^{n} \dot{\theta}_i \frac{\partial f}{\partial \theta_{m+1}}(\theta) + dl_{f(\theta)g^{-1}}(g)P(g,t)\right) \tag{11}$$

Let $Z \in \mathfrak{g}$ denote a v.f. which leaves an open domain $\mathcal{D} \subseteq G$ invariant and for which e is an asymptotically stable equilibrium point (globally in $\mathcal{D}$). Assume that $\mathcal{U} \subset \mathcal{D}$. Then the dynamic feedback law $(u, \dot{\theta})(\theta, g)$ defined by

$$\sum_{i=1}^{m} u_i(\theta,g) X_i(f(\theta)) - \sum_{i=m+1}^{n} \dot{\theta}_i(\theta,g) \frac{\partial f}{\partial \theta_{m+1}}(\theta) = -dl_{f(\theta)g^{-1}}(g)P(g,t) - dr_g(f(\theta)g^{-1})Z(f(\theta)g^{-1}) \tag{12}$$

ensures practical stabilization of $g = e$ in the sense that the set $f(\mathbb{T}^{n-m}) \subset \mathcal{U}$ is asymptotically stable for the closed-loop system and that every solution initialized at $t = 0$ at a point $g(0)$ such that $f(\theta(0))g^{-1}(0) \in \mathcal{D}$ converges to this set.

Proof: The proof of (11) is easily obtained by differentiating the equality $fg^{-1}g = f$ and using the fact that

$$\left(dr_g(fg^{-1})\right)^{-1} = dr_{g^{-1}}(f)$$

By applying the feedback law (12) to (1), one deduces from (11) that the solutions of the closed-loop system are such that

$$\frac{d}{dt}(f(\theta)g^{-1}) = Z(f(\theta)g^{-1})$$

Since, by assumption, e in an asymptotically stable equilibrium point of the v.f. Z, it follows that along any trajectory of the controlled system, $f(\theta(t))g^{-1}(t)$ tends to e as t tends to infinity, and thus that the distance of $g(t)$ to the set $f(\mathbb{T}^{n-m})$ tends to zero. Asymptotic stability of this set follows in turn from the fact that $f(\theta)$ takes values in a compact set. ∎

3.2 Trajectory stabilization

We show how Proposition (1) directly applies to the problem of practical stabilization of a trajectory on a Lie group. Let $g_r(.)$ denote an arbitrary smooth trajectory in G. For any basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, there exist smooth functions $v_{r,i}$ $(i = 1, \ldots, n)$ such that

$$\dot{g}_r = \sum_{i=1}^{n} v_{r,i} X_i(g_r)$$

Furthermore, if $Y_1, \ldots, Y_n, Z_1, \ldots, Z_n$ are left-invariant v.f. in $\mathfrak{g}$ and σ, τ are solutions to the differential equations

$$\dot{\sigma} = \sum_{i=1}^{n} v_i Y_i(\sigma), \qquad \dot{\tau} = \sum_{i=1}^{n} u_i Z_i(\tau)$$

then one verifies that the following identity holds

$$\frac{d}{dt}(\sigma^{-1}\tau) = \sum_{i=1}^{n} u_i Z_i(\sigma^{-1}\tau) - \mathrm{Ad}(\tau^{-1}\sigma) Y_i(\sigma^{-1}\tau)$$

One deduces the following *error system* associated with the trajectory stabilization problem

$$\begin{aligned} \frac{d}{dt}(g_r^{-1}g) &= \sum_{i=1}^{m} u_i X_i(g_r^{-1}g) - \sum_{i=1}^{n} v_{r,i}(t) \mathrm{Ad}(g^{-1}g_r) X_i(g_r^{-1}g) \\ &= \sum_{i=1}^{m} u_i X_i(g_r^{-1}g) + P(g_r^{-1}g, t) \end{aligned} \tag{13}$$

with

$$P(\tilde{g},t) \triangleq -\sum_{i=1}^{n} v_{r,i}(t)\mathrm{Ad}(\tilde{g}^{-1})X_i(\tilde{g})$$

Since this system has the same form as (1), Proposition 1 applies to it directly and provides practical stabilizers of the trajectory $g_r(.)$.

4 The case of homogeneous systems

In this section, we show how the results of the previous sections apply to driftless controllable systems with homogeneous vector fields. The study of such systems is motivated by several reasons. One of them is that any controllable smooth driftless control system on $\mathbb{R}^n$ can be approximated by a controllable system with homogeneous v.f. [39, 14]. While this approximation is local in general, there are also physical systems which admit an homogeneous representation in a large domain. The modeling by chained systems of the kinematic equations of several non-holonomic wheeled mobile robots is a well known example.

The main tool used to apply the results of the previous sections to homogeneous systems is the so-called lifting theorem [35] which specifies how homogeneous systems can be viewed as systems on Lie groups. This explains in part the importance given to Lie groups in the formal Lie-algebraic literature [17, 41]. In this literature, free Lie algebras and free systems [16] are usually considered. They correspond to the framework for the original lifting theorem where nilpotent v.f. are lifted to a free Lie group. While this is well justified from a theoretical standpoint and the sake of generality, it can be interesting, for practical purposes and computational efficiency, to lift the v.f. associated with a specific control system under consideration to the smallest possible Lie group —i.e. the embedding Lie group with the smallest dimension. This possibility, investigated in [12], will be used here.

4.1 Lifting of an homogeneous system to a system on a Lie group

Prior to stating, in the form of a proposition, a version of the lifting theorem adapted to our present objectives, let us recall a few basic definitions and properties about homogeneity (for more details, we refer the reader to [14]).

Given $\lambda > 0$ and a *weight vector* $r = (r_1, \ldots, r_n)$ $(r_i > 0\ \forall i)$, a *dilation* Δ_λ^r on $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad \Delta_\lambda^r x \triangleq (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)$$

A function $f \in \mathcal{C}^0(\mathbb{R}^n;\mathbb{R})$ is *homogeneous of degree d with respect to the family of dilations* $(\Delta_\lambda^r)_{\lambda>0}$, or more concisely Δ^r*-homogeneous of degree* d, if

$$\forall \lambda > 0, \quad f(\Delta_\lambda^r x) = \lambda^d f(x)$$

A smooth v.f. X on $\mathbb{R}^n$ is Δ^r*-homogeneous of degree* τ if, for any $i = 1, \ldots, n$, the function $x \longmapsto X_i(x)$ is Δ^r-homogeneous of degree $\tau + r_i$.

Finally, amidst the many properties of homogeneous v.f. let us just recall that smooth homogeneous v.f. are polynomial, and that the Lie bracket of two homogeneous v.f of degree τ_1 and τ_2 is a homogeneous v.f of degree $\tau_1 + \tau_2$.

The possibility of lifting a set of homogeneous v.f. to a Lie group, with dimension equal to the dimension of the Lie algebra generated by the v.f., is summarized in the following proposition.

Proposition 2 *(Rothschild-Stein [35], Folland [12]) Let $X_1, \dots, X_m$ denote smooth v.f. on $\mathbb{R}^n$, $\mathbb{R}$-independent*[1]*, Δ^r-homogeneous of degree $-\tau_1, \dots, -\tau_m < 0$ respectively, and which satisfy the LARC at the origin. Let $\bar{n}$ denote the dimension, over $\mathbb{R}$, of* $\mathrm{Lie}(X_1, \dots, X_m)$*. Then, there exist*

i) a lifting $\bar{X}_i$ of X_i on $\mathbb{R}^{\bar{n}}$ $(1 \le i \le m)$ of the following form:

$$\forall \bar{x} = (x, y) \in \mathbb{R}^{\bar{n}}\,, \quad \bar{X}_i(\bar{x}) = \begin{pmatrix} X_i(x) \\ Y_i(x, y) \end{pmatrix}$$

ii) a smooth mapping $\varphi : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \longrightarrow \mathbb{R}^{\bar{n}}$

such that

1. *$\bar{X}_1, \dots, \bar{X}_m$ satisfy the LARC at the origin.*
2. *$\bar{X}_1, \dots, \bar{X}_m$ are $\Delta^{\bar{r}}$-homogeneous of degree $-\tau_1, \dots, -\tau_m$ respectively, for some $\bar{r} \in \mathbb{R}^{\bar{n}}$.*
3. *$\mathbb{R}^{\bar{n}}$, endowed with the composition law $(\bar{x}, \bar{y}) \longmapsto \bar{x} \star \bar{y} \triangleq \varphi(\bar{x}, \bar{y})$ is a Lie group, and $\bar{X}_1, \dots, \bar{X}_m$ are left-invariant v.f. for this composition law.*

We sketch below a constructive proof of the proposition.

Sketch of proof: Let $X_1, \dots, X_{\bar{n}}$ denote Δ^r-homogeneous v.f. of degree $-\tau_1, \dots, -\tau_{\bar{n}}$ respectively that form a basis of $\mathrm{Lie}(X_1, \dots, X_m)$. Denote by $\psi : \mathbb{R}^{\bar{n}} \longrightarrow \mathbb{R}^n$ the mapping defined by

$$\psi(\alpha) = \exp(\alpha X) \triangleq e^{\alpha X}(\mathrm{id})(0) \tag{14}$$

with

$$\alpha X \triangleq \sum_{i=1}^{\bar{n}} \alpha_i X_i\,, \qquad e^Z \triangleq \sum_{k=0}^{\infty} \frac{Z^k}{k!}$$

and id the identity function on $\mathbb{R}^n$. Let us recall that each Z^k in the definition of e^Z is the differential operator defined inductively on smooth functions f by $Z(f) = L_Z f$ —the Lie derivative of f along Z— and $Z^k(f) = Z(Z^{k-1}(f))$. Note also that the sum in the definition

[1] i.e. $\sum_{i=1}^m \alpha_i X_i = 0 \Longrightarrow \alpha_1 = \cdots = \alpha_m = 0$

of e^Z is finite for any Δ^r-homogeneous vector field of negative degree. Finally, let us recall that $\psi(\alpha)$ is just the solution at time $T=1$ of the differential equation

$$\dot{x} = \sum_{i=1}^{\bar{n}} \alpha_i X_i(x)\,, \qquad x(0)=0$$

Using the fact that $X_1,\dots,X_{\bar{n}}$ are Δ^r-homogeneous of degree $-\tau_1,\dots,-\tau_{\bar{n}}$ respectively, one easily verifies that

$$\forall \lambda > 0\,,\ \forall \alpha \in \mathbb{R}^{\bar{n}}\,, \quad \psi(\Delta_\lambda^\tau \alpha) = \Delta_\lambda^r \psi(\alpha) \tag{15}$$

By the Campbell-Baker-Hausdorff formula, there exists a mapping $C:\mathbb{R}^{\bar{n}}\times\mathbb{R}^{\bar{n}} \longrightarrow \mathbb{R}^{\bar{n}}$ such that

$$\forall(\alpha,\beta)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}^{\bar{n}}\,, \quad e^{\alpha X}e^{\beta X} = e^{C(\alpha,\beta)X} \tag{16}$$

This mapping defines a group operation on $\mathbb{R}^{\bar{n}}$. Furthermore, for any $\lambda \neq 0$, the linear mapping on Lie$(X_1,\dots,X_m)$ which maps a Δ^r-homogeneous vector field X of degree $-\tau$ to $\lambda^\tau X$ is an automorphism. It follows that

$$\forall\lambda>0, \forall(\alpha,\beta)\in\mathbb{R}^{\bar{n}}\times\mathbb{R}^{\bar{n}}\,, \quad C(\Delta_\lambda^\tau\alpha, \Delta_\lambda^\tau\beta) = \Delta_\lambda^\tau C(\alpha,\beta) \tag{17}$$

Let us distinguish two cases.

Case 1: $\mathbf{n=\bar{n}}$. No lifting of the v.f. is then necessary. It is not difficult to prove that ψ is a global diffeomorphism on $\mathbb{R}^n$, and $C(\alpha,\beta)$ is the expression of the composition law in the coordinates α. The function φ in Proposition 2 is subsequently given by

$$\varphi(x,y) = \psi(C(\psi^{-1}(x),\psi^{-1}(y)))$$

Case 2: $\mathbf{n<\bar{n}}$. Let $Y_1,\dots,Y_{\bar{n}}$ denote the v.f. on $\mathbb{R}^{\bar{n}}$ defined by

$$Y_i(0)=e_i\,, \quad Y_i(\alpha) = \frac{\partial C}{\partial\beta}(\alpha,0)Y_i(0)$$

with $e_i=(0,\dots,0,1,0,\dots,0)^T$ the i-th canonical vector in $\mathbb{R}^{\bar{n}}$. By construction, the Y_i's are left-invariant v.f. on $\mathbb{R}^{\bar{n}}$ w.r.t. the product $(\alpha,\beta)\longmapsto C(\alpha,\beta)$. It also follows from (17) that these v.f. are Δ^τ-homogeneous.

Using the definitions of ψ and the Y_i's, one shows that

$$\forall i=1,\dots,\bar{n}\,, \quad d\psi(\alpha)Y_i(\alpha) = X_i(\psi(\alpha)) \tag{18}$$

From (18) and the fact that the vectors $X_i(0)$ $(i=1,\dots,\bar{n})$ span $\mathbb{R}^n$, the linear mapping $d\psi(\alpha)$ is onto, for any α in a neighborhood of the origin. Therefore, there exists a projection matrix $P : \mathbb{R}^{\bar{n}} \longrightarrow \mathbb{R}^{\bar{n}-n}$ —i.e., $P\alpha = (\alpha_{i_1}, \ldots, \alpha_{i_{\bar{n}-m}})^T$ for some integers $i_1, \ldots, i_{\bar{n}-m}$ — such that

$$\alpha \longmapsto \bar{\psi}(\alpha) \triangleq \begin{pmatrix} \psi(\alpha) \\ P\alpha \end{pmatrix}$$

is a local change of coordinates. From (15) and the special form of P,

$$\bar{\psi}(\Delta_\lambda^\tau \alpha) = \Delta_\lambda^{\bar{\tau}} \bar{\psi}(\alpha) \quad \text{with} \quad \bar{r} = (r, \tau_{i_1}, \ldots, \tau_{i_{\bar{n}-m}})$$

This homogeneity property implies that $\bar{\psi}$ is a global change of coordinates. The dynamic extension of the v.f. X_i is defined by $\bar{X}_i \triangleq \bar{\psi}_* Y_i$ —i.e. $\bar{X}_i(\psi(\alpha)) = d\bar{\psi}(\alpha) Y_i(\alpha)$— , and the group operation by

$$\varphi(\bar{x}, \bar{y}) = \bar{\psi}(C(\bar{\psi}^{-1}(\bar{x}), \bar{\psi}^{-1}(\bar{y})))$$

Then, one shows that all properties of Proposition 2 are satisfied with this choice. ∎

Remark : The combination of Theorem 1 and Proposition 2 implies the results stated in Theorem 1 of [31] and, in particular, the existence of transverse functions for any smooth controllable driftless system. In fact, this combination is more general because it points out complementary results. It is also more directly applicable to feedback control design, as illustrated by Proposition 1 and a forthcoming corollary. Besides the clarification brought by using the framework of Lie groups, one of the complementary results is the specification of a smaller value for the number of variables on which transverse functions depend. More precisely, this number is equal to the difference between the dimension of the Lie algebra generated by the v.f. of a controllable homogeneous approximation of the system and the number of control v.f. Another interesting complementary result is the expression (4)–(5) of such a transverse function. This expression is more concise and explicit than the construction proposed in the proof of Theorem 1 in [31].

4.2 Application to practical stabilization

Theorem 1 and Propositions 1 and 2 yield the following corollary.

Corollary 1 *(Practical feedback linearization) Let $X_1, \ldots, X_m$ denote smooth v.f. on $\mathbb{R}^n$ satisfying the assumptions of Proposition 2. Let $\bar{X}_1, \ldots, \bar{X}_m$ and φ be the v.f. and the mapping evoked in this proposition. Consider the control system*

$$\dot{\bar{x}} = \sum_{i=1}^{m} u_i \bar{X}_i(x) + P(x,t) \tag{19}$$

with $P(x,t) \in \mathbb{R}^{\bar{n}}$. Then, for any transverse function $\bar{f} : \mathbb{T}^{\bar{n}-m} \longrightarrow \mathbb{R}^{\bar{n}}$ associated with the v.f. $\bar{X}_1, \ldots, \bar{X}_m$, the following feedback control

$$(u, -\dot{\theta})^T = -H^{-1}(\theta) \left(\frac{\partial \varphi}{\partial \bar{x}}(\varphi(\bar{f}(\theta), \bar{x}^{-1}), \bar{x}) \bar{U} + \frac{\partial \varphi}{\partial \bar{y}}(\varphi(\bar{f}(\theta), \bar{x}^{-1}), \bar{x}) P(x,t) \right) \tag{20}$$

with $H(\theta) \in \mathbb{R}^{\bar{n}\times\bar{n}}$ *the invertible matrix defined by*

$$H(\theta) = \left(\bar{X}_1(\bar{f}(\theta)) \;\cdots\; \bar{X}_1(\bar{f}(\theta)) \;\frac{\partial \bar{f}}{\partial \theta_{m+1}}(\theta) \;\cdots\; \frac{\partial \bar{f}}{\partial \theta_{\bar{n}}}(\theta)\right)$$

linearizes the "practical error system" with state vector $z \triangleq \varphi(\bar{f}(\theta), \bar{x}^{-1})$ *to the form*

$$\dot{z} \;=\; \bar{U}\ . \tag{21}$$

Proof: The existence of the v.f. $\bar{X}_i$ and the mapping φ are guaranteed by Proposition 2. Existence of transverse functions is guaranteed by Theorem 1. The proof then follows directly from Proposition 1, with the linearizing feedback (20) obtained by particularizing (12) to the homogeneous case. ∎

The above corollary indicates how any feedback which globally stabilizes the origin of the linear system (21) induces a feedback law for the system (19) which globally stabilizes the set $\bar{f}(\mathbb{T}^{\bar{n}-m})$. For instance, exponential stability is obtained by choosing $\bar{U}(z) = Kz$, with K any Hurwitz stable matrix. Note that uniform asymptotic boundedness of all control variables is also ensured in this case, and that these variables converge to zero when the perturbation $P(x,t)$ is identically equal to zero, or when it does not depend upon x and $P(t)$ tends to zero. Moreover, if the convergence of $P(t)$ to zero is exponential, then $\theta(t)$ itself converges to a point in $\mathbb{T}^{\bar{n}-m}$.

5 Examples

5.1 Chained systems

The (perturbed) single-chain system of dimension n with two control inputs is defined by

$$\dot{x} = u_1 X_1(x) + u_2 X_2 + P(x,t) \tag{22}$$

with

$$X_1(x) = (1, 0, x_2, \ldots, x_{n-1})^T \quad \text{and} \quad X_2 = (0, 1, 0, \ldots, 0)^T$$

and P some additive drift term. The unperturbed chained system —corresponding to the case where $P \equiv 0$— is a particular homogeneous system, and it is well known that $\text{Lie}(X_1, X_2)$ is of dimension n, and is spanned by the v.f. $X_1, \ldots, X_n$ with

$$X_i \triangleq (\text{ad}^{i-2} X_1, X_2) = (-1)^i\, e_i \quad (i = 3, \ldots, n) \tag{23}$$

and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the i-th canonical vector on $\mathbb{R}^n$. According to Proposition 2, X_1 and X_2 are left-invariant w.r.t. a group operation on $\mathbb{R}^n$. One easily verifies that this group operation is given by

$$x \star y = \varphi(x, y) = e^{Ay_1} x + y \tag{24}$$

with A the $n \times n$ matrix whose only non-zero entries are $a_{i+1,i} = 1$, for $i = 2, \dots, n-1$. More explicitly, the components of $x \star y$ are defined by

$$(x \star y)_i = \begin{cases} x_i + y_i & \text{if } i = 1,2 \\ x_i + y_i + \displaystyle\sum_{j=2}^{i-1} \frac{y_1^{i-j}}{(i-j)!} x_j & \text{otherwise} \end{cases} \tag{25}$$

Let us now proceed with the computation of transverse functions for this system. The n v.f. defined by (23) form a graded basis of Lie(X_1, X_2) in the sense of Definition 1, and the associated weight-vector is given by

$$r = (1, 1, 2, \dots, n-1) \tag{26}$$

From the definition of X_1 and X_2, and (23) for the subsequent v.f. X_i, one easily calculates

$$\exp \alpha_i X_i = \begin{cases} \alpha_i \, e_i & \text{for } i = 1,2 \\ (-1)^i \alpha_i \, e_i & \text{otherwise} \end{cases}$$

Using (26), this implies that each function f_i in (5) is given by

$$f_i(\theta_i) = (-1)^i \varepsilon_i^{i-1} \frac{\sin 2\theta_i}{4} e_i \star (-1)^{i-1} \varepsilon_i^{i-2} \cos \theta_i e_{i-1} \star \varepsilon_i \sin \theta_i e_1$$

The expression of $f = f_n \cdots f_3$ can then be calculated from (25). Then, for any transverse function, and any Hurwitz stable matrix K, it follows from Corollary 1 and (24), that the feedback

$$(u, -\dot{\theta})^T = H^{-1}(\theta) \left(e^{Ax_1} K e^{-Ax_1} (x - f(\theta)) + A(x - f(\theta)) P_1(x,t) - P(x,t) \right)$$

with

$$H(\theta) = \left(X_1(f(\theta)) \; X_2(f(\theta)) \; \frac{\partial f}{\partial \theta_{m+1}}(\theta) \; \cdots \; \frac{\partial f}{\partial \theta_n}(\theta) \right)$$

makes the set $f(\mathbb{T}^{n-m})$ exponentially stable. This was the main result of [30].

5.2 Systems on SO(3) with two control inputs

Consider the underactuated system on the matrix Lie group $\mathbf{SO(3)}$:

$$\dot{R} = R \left(u_1 S(e_1) + u_2 S(e_2) \right) \tag{27}$$

where e_i denotes, as in the above example, the i-th canonical vector in $\mathbb{R}^3$, and S is the operator associated with the vector product, i.e. $S(e)x = e \times x$. Since $S(e_1)$, $S(e_2)$, and

$$S(e_3) = [S(e_1), S(e_2)] = S(e_1)S(e_2) - S(e_2)S(e_1)$$

form a basis of $\mathfrak{so}(3)$, the Lie algebra generated by $RS(e_1)$ and $RS(e_2)$ is isomorphic to $\mathfrak{so}(3)$. This basis is a graded basis in the sense of Definition 1, with

$$r = (1,1,2)$$

By application of Theorem 1, a transverse function is obtained by setting

$$F(\theta) = F_3(\theta)\, F_2(\theta)\, F_1(\theta) \tag{28}$$

with

$$F_1(\theta) \triangleq \exp \varepsilon \sin\theta S(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon \sin\theta) & -\sin(\varepsilon\sin\theta) \\ 0 & \sin(\varepsilon\sin\theta) & \cos(\varepsilon\sin\theta) \end{pmatrix} \tag{29}$$

$$F_2(\theta) \triangleq \exp \varepsilon \cos\theta S(e_2) = \begin{pmatrix} \cos(\varepsilon \cos\theta) & 0 & \sin(\varepsilon\cos\theta) \\ 0 & 1 & 0 \\ -\sin(\varepsilon\cos\theta) & 0 & \cos(\varepsilon\cos\theta) \end{pmatrix} \tag{30}$$

$$F_3(\theta) \triangleq \exp \varepsilon^2 \frac{\sin 2\theta}{4} S(e_3) \begin{pmatrix} \cos(\varepsilon^2 \frac{\sin 2\theta}{4}) & -\sin(\varepsilon^2 \frac{\sin 2\theta}{4}) & 0 \\ \sin(\varepsilon^2 \frac{\sin 2\theta}{4}) & \cos(\varepsilon^2 \frac{\sin 2\theta}{4}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{31}$$

Lemma 4 *The function F defined by (28) is a transverse function for any $\varepsilon \in (0, \frac{\pi}{2})$.*

Proof: From the definition of F,

$$\frac{\partial F}{\partial \theta} = F_3 \frac{\partial \alpha_3}{\partial \theta} S(e_3) F_2 F_1 + F_3 F_2 \frac{\partial \alpha_2}{\partial \theta} S(e_2) F_1 + F_3 F_2 F_1 \frac{\partial \alpha_1}{\partial \theta} S(e_1) \tag{32}$$

with

$$\alpha_3(\theta) \triangleq \varepsilon^2 \frac{\sin 2\theta}{4}\,, \quad \alpha_2(\theta) \triangleq \varepsilon \cos\theta\,, \quad \alpha_1(\theta) \triangleq \varepsilon\sin\theta$$

Using the fact that for any rotation matrix R and any vector $\omega \in \mathbb{R}^3$, $R^T S(\omega) R = S(R^T \omega)$, we deduce from (32) that

$$\frac{\partial F}{\partial \theta} = F S(\omega)$$

with

$$\omega = \frac{\partial \alpha_1}{\partial \theta} e_1 + \frac{\partial \alpha_2}{\partial \theta} F_1^T e_2 + \frac{\partial \alpha_3}{\partial \theta} (F_2 F_1)^T e_3$$

The transversality condition is equivalent to the fact that the third component of the vector ω above is different from zero, for any value of θ. Using (29) and (30), a straightforward computation yields

$$\begin{aligned} \omega_3(\theta) &= \varepsilon \sin\theta \sin(\varepsilon\sin\theta) + \varepsilon^2 \frac{\cos 2\theta}{2} \cos(\varepsilon\sin\theta)\cos(\varepsilon\cos\theta) \\ &= \varepsilon^2 \left(\sin^2\theta \frac{\sin(\varepsilon\sin\theta)}{\varepsilon\sin\theta} + \frac{\cos^2\theta - \sin^2\theta}{2} \cos(\varepsilon\sin\theta)\cos(\varepsilon\cos\theta) \right) \end{aligned}$$

Since

$$\forall \varepsilon \in (0, \frac{\pi}{2}), \forall \theta \qquad \frac{\sin(\varepsilon \sin\theta)}{\varepsilon \sin\theta} > \frac{1}{2} \quad \text{and} \quad 0 < \cos(\varepsilon\sin\theta)\cos(\varepsilon\cos\theta) \leq 1$$

it is simple to verify that $\omega_3(\theta) > 0$ when $\varepsilon \in (0, \frac{\pi}{2})$.

∎

5.3 Unicycle

Kinematic equations of the unicycle are

$$\begin{cases} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \end{cases}$$

For stabilization purposes, this system is often transformed, via a diffeormorphic change of coordinates and new equivalent control variables, to the three dimensional single-chain system with two inputs. The solution described in Section 5.1 then applies directly. However, we would like to show now that the preliminary transformation into a chained system is not necessary, and that the control design can be performed easily by using natural system coordinates. The reason why this is possible is that, as this is well known, the above equations define a left-invariant control system on the Lie group $\mathbb{R}^2 \times S^1$. The group operation can be obtained from the formalism of homogeneous matrices, i.e.

$$\begin{pmatrix} R(\theta_1) & X_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R(\theta_2) & X_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R(\theta_1+\theta_2) & X_1 + R(\theta_1)X_2 \\ 0 & 1 \end{pmatrix} \tag{33}$$

with $X_i = (x_i, y_i) \in \mathbb{R}^2$ and $R(\theta_i)$ the rotation matrix of angle θ_i.

The v.f.

$$b_1(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, [b_1, b_2](\theta) = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} \tag{34}$$

form a graded basis of Lie(b_1, b_2), with $r = (1, 1, 2)$. A transverse function f is obtained by using (4). This yields

$$f(\alpha) = \left(\exp \varepsilon^2 \frac{\sin 2\alpha}{4}[b_1, b_2]\right)\left(\exp \varepsilon \cos\alpha\, b_2\right)\left(\exp \varepsilon \sin\alpha\, b_1\right)$$

After simple calculations, using (33) and (34), one finally obtains the following expression for f:

$$f(\alpha) = \begin{pmatrix} \varepsilon \cos(\varepsilon \cos\alpha) \sin\alpha \\ \varepsilon \sin(\varepsilon\cos\alpha)\sin\alpha - \varepsilon^2 \dfrac{\sin 2\alpha}{4} \\ \varepsilon \cos\alpha \end{pmatrix} \tag{35}$$

The transversality condition is equivalent to the fact that the matrix

$$H(\alpha) \triangleq \left(b_1(f(\alpha)) \quad b_2(f(\alpha)) \quad \frac{\partial f}{\partial \alpha}(\alpha) \right)$$

is invertible for any α. Using (34) and (35), one shows that

$$\begin{aligned} \det H(\alpha) &= \varepsilon^2 \sin^2 \alpha + \varepsilon^2 \frac{\cos 2\alpha}{2} \cos(\varepsilon \cos \alpha) \\ &= \frac{\varepsilon^2}{2} + \varepsilon^2 \frac{\cos 2\alpha}{2} (\cos(\varepsilon \cos \alpha) - 1) \end{aligned}$$

and it follows from this equation that the transversality condition is satisfied provided that $\varepsilon \in (0, \frac{\pi}{2})$. ∎

Appendix: Proof of Lemmas 1, 2, and 3

The following technical claims are used at various places in the proofs of Lemmas 1 and 2.

Claim 1 *Let Y and Z denote two time-dependent left-invariant v.f. on G, and σ, τ solutions of $\dot{\sigma} = Y(\sigma, t)$ and $\dot{\tau} = Z(\tau, t)$ respectively. Then $\nu \triangleq \sigma\tau$ is a solution of $\dot{\nu} = \mathrm{Ad}(\tau^{-1})\, Y(\nu, t) + Z(\nu, t)$.*

This is well known. No proof needed.

Claim 2 *Let $X_1, \ldots, X_n$ denote a graded basis of the Lie algebra $\mathfrak{g}$ of a Lie group G. Let $p, q \in \{1, \ldots, n\}$, $\alpha_p \in \mathbb{R}$, and $s \in \mathbb{N}$. Then, there exist analytic functions $g_1, \ldots, g_n$ such that*

$$\sum_{j=s}^{\infty} \frac{\alpha_p^j}{j!}(\mathrm{ad}^j X_p, X_q) = \sum_{k=1}^{n} g_k(\alpha_p) X_k$$

Furthermore, if $\alpha_p : (\varepsilon, \theta) \in \mathbb{R} \times \mathbb{T} \longmapsto \alpha_p(\varepsilon, \theta)$ denotes an analytic function such that $\alpha_p = \mathrm{O}(|\varepsilon|^{r_p})$, then $g_k(\alpha_p)$ is an analytic function such that $g_k(\alpha_p) = \mathrm{O}(|\varepsilon|^{\max\{s\, r_p, r_k - r_q\}})$

Proof: Let $c_{i,j}^k$ denote the structural constants associated with $X_1, \ldots, X_n$, i.e.

$$[X_i, X_j] = \sum_k c_{i,j}^k X_k$$

From Definition 1, $r_k > r_i + r_j \Longrightarrow c_{i,j}^k = 0$. From this fact, one easily obtains by induction on j:

$$\begin{aligned}
(\mathrm{ad}^j X_p, X_q) &= \sum_{k_j : r_{k_j} \le j r_p + r_q} \; \sum_{k_1, \ldots, k_{j-1} = 1}^{n} c_{p,q}^{k_1} c_{p,k_1}^{k_2} \cdots c_{p,k_{j-1}}^{k_j} \; X_{k_j} \\
&= \sum_{k_j : r_{k_j} \le j r_p + r_q} a_{j,k_j} X_{k_j}
\end{aligned}$$

with

$$|a_{j,k_j}| \le (Mn)^j \quad \text{and} \quad M \triangleq \max c_{i,j}^k \tag{36}$$

Therefore,

$$\begin{aligned}
\sum_{j=s}^{\infty} \frac{\alpha_p^j}{j!}(\mathrm{ad}^j X_p, X_q) &= \sum_{j=s}^{\infty} \frac{\alpha_p^j}{j!} \sum_{k_j : r_{k_j} \le j r_p + r_q} a_{j,k_j} X_{k_j} \\
&= \sum_{k=1}^{n} \; \sum_{r_k \le j r_p + r_q,\, s \le j} \frac{\alpha_p^j}{j!} a_{j,k_j} \; X_k \\
&= \sum_{k=1}^{n} g_k(\alpha_p) X_k
\end{aligned}$$

with

$$g_k(\alpha_p) = \sum_{r_k \le j r_p + r_q,\, s \le j} \frac{\alpha_p^j}{j!} a_{j,k_j} \tag{37}$$

It follows from (36) that each g_k is an analytic function of α_p. Furthermore, if $\alpha_p = \mathrm{O}(|\varepsilon|^{r_p})$ is an analytic function of ε and θ, then $g_k(\alpha_p)$ is analytic and it follows, by considering the term of lowest order in (37), that

$$g_k(\alpha_p) = \mathrm{O}(|\varepsilon|^{\max\{s\, r_p, r_k - r_q\}})$$

Note that the equality is uniform w.r.t. θ because $g_k(\alpha_p)$ is periodic w.r.t θ. ∎

Proof of Lemma 1: In order to simplify the notation, we denote by f_i' the derivative of f_i w.r.t θ_i. We also let $X_\lambda = X_{\lambda(i)}$, $X_\rho = X_{\rho(i)}$, $X = X_i$, and denote by α_λ, α_ρ, and α the functions defined by

$$\alpha_\lambda(\varepsilon_i, \theta_i) = \varepsilon_i^{r_{\lambda(i)}} \sin\theta_i\,,\ \alpha_\rho(\varepsilon_i, \theta_i) = \varepsilon_i^{r_{\rho(i)}} \cos\theta_i\,,\ \alpha(\varepsilon_i, \theta_i) = \varepsilon_i^{r_i} \frac{\sin 2\theta_i}{4} \tag{38}$$

With this notation, $f_i = (\exp \alpha X)(\exp \alpha_\rho X_\rho)(\exp \alpha_\lambda X_\lambda)$. Using Claim 1 and the fact that $\mathrm{Ad}(\exp Y)Z = (\exp \mathrm{ad} Y, Z)$,

$$\begin{aligned}
f_i' &= \alpha_\lambda' X_\lambda(f_i) + \alpha_\rho' \mathrm{Ad}(\exp -\alpha_\lambda X_\lambda) X_\rho(f_i) + \alpha' \mathrm{Ad}((\exp -\alpha_\lambda X_\lambda)(\exp -\alpha_\rho X_\rho)) X(f_i) \\
&= \alpha_\lambda' X_\lambda(f_i) + \alpha_\rho' \sum_{j=0}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} (\mathrm{ad}^j X_\lambda, X_\rho)(f_i) \\
&\quad + \alpha' \sum_{j,k=0}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} \frac{(-\alpha_\rho)^k}{k!} (\mathrm{ad}^j X_\lambda, (\mathrm{ad}^k X_\rho, X))(f_i) \\
&= \alpha_\lambda' X_\lambda(f_i) + \alpha_\rho' X_\rho(f_i) - \alpha_\rho' \alpha_\lambda [X_\lambda, X_\rho](f_i) + \alpha' X(f_i) \\
&\quad + \alpha_\rho' \sum_{j=2}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} (\mathrm{ad}^j X_\lambda, X_\rho)(f_i) + \alpha' \sum_{j+k=1}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} \frac{(-\alpha_\rho)^k}{k!} (\mathrm{ad}^j X_\lambda, (\mathrm{ad}^k X_\rho, X))(f_i)
\end{aligned}$$

From (38) and the fact that $X = [X_\lambda, X_\rho]$ —by definition 1—, the above equality yields

$$\begin{aligned}
f_i' =& \alpha_\lambda' X_\lambda(f_i) + \alpha_\rho' X_\rho(f_i) + \frac{\varepsilon_i^{r_i}}{2} X(f_i) \\
&+ \alpha_\rho' \sum_{j=2}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} (\mathrm{ad}^j X_\lambda, X_\rho)(f_i) + \alpha' \sum_{j+k=1}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} \frac{(-\alpha_\rho)^k}{k!} (\mathrm{ad}^j X_\lambda, (\mathrm{ad}^k X_\rho, X))(f_i)
\end{aligned} \tag{39}$$

By application of Claim 2,

$$\alpha_\rho' \sum_{j=2}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} (\mathrm{ad}^j X_\lambda, X_\rho) = \sum_{k=1}^{n} \alpha_\rho'\, g_k(\alpha_\lambda) X_k \tag{40}$$

for some analytic functions $g_1, \ldots, g_n$, and using (38),

$$\alpha'_\rho\, g_k(\alpha_\lambda) = \mathrm{O}(|\varepsilon|^{\max\{r_{\rho(i)}+2r_{\lambda(i)}, r_k\}}) = \mathrm{O}(|\varepsilon|^{\max\{r_i+r_{\lambda(i)}, r_k\}}) \tag{41}$$

Similarly, by applying Claim 2 again, one shows the existence of analytic functions $h_1, \ldots, h_n$ such that

$$\alpha' \sum_{j+k=1}^{\infty} \frac{(-\alpha_\lambda)^j}{j!} \frac{(-\alpha_\rho)^k}{k!} (\mathrm{ad}^j X_\lambda, (\mathrm{ad}^k X_\rho, X)) = \sum_{k=1}^{n} \alpha'\, h_k(\alpha_\lambda, \alpha_\rho) X_k \tag{42}$$

with

$$\alpha'\, h_k(\alpha_\lambda, \alpha_\rho) = \mathrm{O}(|\varepsilon|^{\max\{r_i+1, r_k\}}) \tag{43}$$

From (39), (40), and (42), we finally get

$$\begin{aligned} f'_i = &\left(\alpha'_\lambda + \alpha'_\rho\, g_\lambda(\alpha_\lambda) + \alpha'\, h_\lambda(\alpha_\lambda, \alpha_\rho)\right) X_\lambda(f_i) \\ &+ \left(\alpha'_\rho + \alpha'_\rho\, g_\rho(\alpha_\lambda) + \alpha'\, h_\rho(\alpha_\lambda, \alpha_\rho)\right) X_\rho(f_i) \\ &+ \left(\varepsilon_i^{r_i}/2 + \alpha'_\rho\, g_i(\alpha_\lambda) + \alpha'\, h_i(\alpha_\lambda, \alpha_\rho)\right) X(f_i) \\ &+ \sum_{k \notin \{\lambda,\rho,i\}} \left(\alpha'_\rho\, g_k(\alpha_\lambda) + \alpha'\, h_k(\alpha_\lambda, \alpha_\rho)\right) X_k(f_i) \end{aligned}$$

From here, Lemma 1 easily follows, by using (38), (41), and (43).

Proof of Lemma 2: From Claim 1, and relations (4) and (6), one deduces that

$$\frac{\partial f}{\partial \theta_i} = \sum_{k=1}^{n} v_{i,k} \mathrm{Ad}(f_{m+1}^{-1} \cdots f_{i-1}^{-1}) X_k(f) \tag{44}$$

From the fact that $\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1)\mathrm{Ad}(g_2)$ and (5),

$$\begin{aligned} \mathrm{Ad}(f_{m+1}^{-1} \cdots f_{i-1}^{-1}) &= \prod_{j=m+1}^{i-1} \mathrm{Ad}(f_j^{-1}) \\ &= \prod_{j=m+1}^{i-1} \mathrm{Ad}(\exp -\alpha_{\lambda(j)} X_{\lambda(j)}) \mathrm{Ad}(\exp -\alpha_{\rho(j)} X_{\rho(j)}) \mathrm{Ad}(\exp -\alpha_j X_j) \end{aligned} \tag{45}$$

where $\alpha_{\lambda(j)}$, $\alpha_{\rho(j)}$, and α_j are defined according to (38). By application of Claim 2, for any $p, q = 1, \ldots, n$

$$\mathrm{Ad}(\exp -\alpha_p X_p) X_q = \sum_{j=1}^{n} h_{p,q}^j X_j$$

for some analytic functions $h_{p,q}^j$. Moreover, if $\alpha_p = \mathrm{O}(|\varepsilon|^{r_p})$ is an analytic function then, $h_{p,q}^j(\alpha_p) = \mathrm{O}(|\varepsilon|^{r_j - r_q})$. By applying this property recursively, one deduces from (45) that

$$\mathrm{Ad}(f_{m+1}^{-1} \cdots f_{i-1}^{-1}) X_k = \sum_{j=1}^{n} g_{i,k}^j X_j \tag{46}$$

for some analytic functions $g^j_{i,k}$ which depend on $\varepsilon_{m+1} \dots, \varepsilon_{i-1}, \theta_{m+1}, \dots, \theta_{i-1}$, and are such that

$$g^j_{i,k} = \mathrm{O}(|\varepsilon_{m+1}, \dots, \varepsilon_{i-1}|^{r_j - r_k}) \tag{47}$$

By using the fact that $\mathrm{Ad}(e)X_k = X_k$, one has also

$$\varepsilon_{m+1} = \cdots = \varepsilon_{i-1} = 0 \Longrightarrow g^j_{i,k} = \delta_{j,k} \, , \; \forall i \tag{48}$$

From (46),

$$\sum_{k=1}^{n} v_{i,k} \mathrm{Ad}(f^{-1}_{m+1} \cdots f^{-1}_{i-1}) X_k(f) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} v_{i,k} g^j_{i,k} \right) X_j(f)$$

so that, by (44), (8) is satisfied with

$$a_{j,i} \triangleq \sum_{k=1}^{n} v_{i,k} \, g^j_{i,k} = A + B + C \tag{49}$$

with

$$A = \sum_{k<j} v_{i,k} \, g^j_{i,k} \, , \quad B = v_{i,j} \, g^j_{i,j} \, , \quad C = \sum_{k>j} v_{i,k} \, g^j_{i,k} \tag{50}$$

Lemma 2 follows from this decomposition. First, we remark that, from Lemma 1 and (47), A, B, and C in (50) are $\mathrm{O}(|\varepsilon_{m+1}, \dots, \varepsilon_i|^{r_j})$. This proves (9) for $j < i$ and $r_j < r_i$, and for $j > i$.

For $j < i$ and $r_j = r_i$, A vanishes at $\varepsilon_{m+1} = \cdots = \varepsilon_{i-1} = 0$ because of (48). It accounts for the sum in (9) (up to higher-order terms). $B = \mathrm{o}(|\varepsilon_{m+1}, \dots, \varepsilon_i|^{r_j})$ because of (7). Finally, using Lemma 1 and (48) again, it follows that $C = \mathrm{o}(|\varepsilon_{m+1}, \dots, \varepsilon_i|^{r_j})$ too.

For $j = i$, A accounts for the sum in (9). From Lemma 1 and (47), B accounts for the term $\varepsilon_i^{r_i}/2$ and, as in the previous case, $C = \mathrm{o}(|\varepsilon_{m+1}, \dots, \varepsilon_i|^{r_i})$.

∎

Proof of Lemma 3: The lemma is a direct consequence of the following property which will be proved by induction.

$$\begin{aligned} &\forall k = m+1, \dots, n, \; \exists \eta_k \in \mathbb{R}^{k-m}, \; \exists \alpha_k \neq 0 : \\ &\qquad (\varepsilon_{m+1}, \dots, \varepsilon_k) = \varepsilon_k \eta_k \quad \Longrightarrow \quad \mathrm{D}_k = \alpha_k \varepsilon_k^{r_{m+1} + \cdots + r_k} + \mathrm{o}(|\varepsilon_k|^{r_{m+1} + \cdots + r_k}) \end{aligned} \tag{51}$$

with D_k the function defined by

$$\mathrm{D}_k(\theta) \triangleq \mathrm{Det}(a_{j,i}(\theta))_{j,i=m+1,\dots,k}$$

Let us first prove (51) for $k = m+1$. From Lemma 2,

$$a_{m+1,m+1} = \frac{1}{2} \varepsilon_{m+1}^{r_{m+1}} + \mathrm{o}(|\varepsilon_{m+1}|^{r_{m+1}})$$

Since $\mathrm{D}_{m+1} = a_{m+1,m+1}$, (51) follows with $\eta_{m+1} = 1$ and $\alpha_{m+1} = 1/2$.

Let us now prove that if (51) is satisfied for k (with $m+1 \leq k < n$), then it is also satisfied for $k+1$. Let

$$(\varepsilon_{m+1}, \ldots, \varepsilon_{k+1}) = \varepsilon_{k+1}\eta_{k+1} \quad \text{with} \quad \eta_{k+1} = (\gamma_{k+1}\eta_k, 1) \tag{52}$$

In these equalities, η_k denotes the vector involved in (51), whereas γ_{k+1} is a design parameter whose value will be specified below. By (52), $(\varepsilon_{m+1}, \ldots, \varepsilon_k) = (\varepsilon_{k+1}\gamma_{k+1})\eta_k$. Therefore, one easily obtains from Lemma 2 the following equalities:

$$\forall q < k+1, \quad a_{j,q} = \mathrm{O}(|\varepsilon_{k+1}\gamma_{k+1}|^{r_j}) \tag{53}$$

$$a_{j,k+1} = \begin{cases} \mathrm{O}(|\varepsilon_{k+1}|^{r_j}) & \text{for } j < k+1 \text{ and } r_j < r_{k+1} \\ \mathrm{O}(|\varepsilon_{k+1}|^{r_j})\,\mathrm{O}(|\gamma_{k+1}|) + \mathrm{o}(|\varepsilon_{k+1}|^{r_j}) & \text{for } j < k+1 \text{ and } r_j = r_{k+1} \\ \varepsilon_{k+1}^{r_{k+1}}\left(\frac{1}{2} + \mathrm{O}(|\gamma_{k+1}|)\right) + \mathrm{o}(\varepsilon_{k+1}^{r_{k+1}}) & \text{for } j = k+1 \end{cases} \tag{54}$$

By definition,

$$\mathrm{D}_{k+1} = a_{k+1,k+1}\mathrm{D}_k + \sum_{j=1}^{k} a_{j,k+1}\mathrm{Cof}(a_{j,k+1}) \tag{55}$$

where $\mathrm{Cof}(a_{j,k+1})$ denotes the cofactor of $a_{j,k+1}$. From (53),

$$\mathrm{Cof}(a_{j,k+1}) = \mathrm{O}(|\varepsilon_{k+1}\gamma_{k+1}|^{r_{m+1}+\cdots+r_{j-1}+r_{j+1}+\cdots+r_{k+1}}) \tag{56}$$

From the induction hypothesis (51), and the fact that, by a recursive application of (52), $\varepsilon_k = \varepsilon_{k+1}\gamma_{k+1}$,

$$\mathrm{D}_k = \alpha_k(\varepsilon_{k+1}\gamma_{k+1})^{r_{m+1}+\cdots+r_k} + \mathrm{o}(|\varepsilon_{k+1}\gamma_{k+1}|^{r_{m+1}+\cdots+r_k}) \tag{57}$$

From (54) and (57),

$$a_{k+1,k+1}D_k = \alpha_k\gamma_{k+1}^{r_{m+1}+\cdots+r_k}\left(\frac{1}{2} + \mathrm{O}_1(|\gamma_{k+1}|)\right)\varepsilon_{k+1}^{r_{m+1}+\cdots+r_{k+1}} + \mathrm{o}(|\varepsilon_{k+1}|^{r_{m+1}+\cdots+r_{k+1}}) \tag{58}$$

From (54) and (56), for any $j = 1, \ldots, k$,

$$a_{j,k+1}\mathrm{Cof}(a_{j,k+1}) = \varepsilon_{k+1}^{r_{m+1}+\cdots+r_{k+1}}\mathrm{O}_2(|\gamma_{k+1}|^{r_{m+1}+\cdots+r_k+1}) + \mathrm{o}(|\varepsilon_{k+1}|^{r_{m+1}+\cdots+r_{k+1}}) \tag{59}$$

Therefore, from (55), (58), and (59),

$$\mathrm{D}_{k+1} = \alpha_k\gamma_{k+1}^{r_{m+1}+\cdots+r_k}\left(\frac{1}{2} + \mathrm{O}_3(|\gamma_{k+1}|)\right)\varepsilon_{k+1}^{r_{m+1}+\cdots+r_{k+1}} + \mathrm{o}(|\varepsilon_{k+1}|^{r_{m+1}+\cdots+r_{k+1}})$$

Then, (51) follows by choosing $\gamma_{k+1} > 0$ small enough so as to ensure that

$$\alpha_{k+1} \triangleq \alpha_k\gamma_{k+1}^{r_{m+1}+\cdots+r_k}\left(\frac{1}{2} + \mathrm{O}_3(|\gamma_{k+1}|)\right)$$

is different from zero.

We thus have proved by induction the existence of $n-m$ numbers $\gamma_{m+1},\ldots,\gamma_n$ (with $\gamma_{m+1}=1$) such that choosing —see (52)—

$$(\varepsilon_{m+1},\ldots,\varepsilon_n)=\varepsilon_n\left(\prod_{i=m+2}^{n}\gamma_i,\ldots,\gamma_n,1\right)$$

yields $\forall\theta, \mathrm{D}_n(\theta)\neq 0$, provided that ε_n is itself chosen small enough. This is precisely the result of Lemma 3. ∎

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