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Abstract: The problem considered in this report is the *joint estimation of state and some parameters* for a class of truly nonlinear systems. The proposed method is *constructive* and guarantees *global convergence*. It has been inspired by the technique of high gain observer and by a recent result on linear adaptive observer. A numerical example is presented for illustration. Potential applications of the adaptive observer are adaptive control and fault detection and isolation.

Key-words: nonlinear system, adaptive observer, high gain observer, global state and parameter estimation.

(Résumé : tsvp)

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Observateur global pour une classe de systèmes non linéaires

Résumé : Pour l'estimation conjointe d'états et de paramètres d'une classe de systèmes non linéaires, nous proposons une méthode constructive permettant la conception d'observateurs adaptatifs avec convergence globale. Cette méthode a été inspirée par les techniques de l'observateur à grand gain et par un résultat récent sur l'observateur adaptatif linéaire. Un exemple numérique est présenté pour illustrer la performance de la méthode.

Mots-clé : systèmes non linéaire, observateur adaptatif, observateur à grand gain, estimation globale d'états et de paramètres.

1 Introduction

Joint estimation of states and parameters in linear and nonlinear systems with *adaptive observers* has been studied for more than two decades. Some early works on adaptive observers for linear systems can be found in (Lüders and Narendra, 1973; Kreisselmeier, 1977). Adaptive observers for nonlinear systems then have drawn more attentions of researchers. For instance, results on global state and parameter estimation have been reported in (Bastin and Gevers, 1988; Marino and Tomei, 1995b; Marino and Tomei, 1995a). However, these results are restricted to the nonlinear systems whose dynamics can be linearized by a coordinate change with output injection. Their applicability is limited by the restrictive linearization condition. More recently, some more general results on nonlinear systems have been published (Rajamani and Hedrick, 1995; Cho and Rajamani, 1997; Besançon, 2000). These methods do not require the considered nonlinear system to be linearizable, instead, they assume the existence of some Lyapounov function satisfying particular conditions. They are not constructive methods in the sense that there is no systematic way for the design of the required Lyapounov function, and it is not known how to systematically check their applicability to a given system.

Recently, a new approach to the design of adaptive observers for multi-input-multi-output (MIMO) linear time varying (LTV) systems has been proposed (Zhang and Delyon, 2001; Zhang, 2001). Due to the conceptual simplicity of this approach, it is possible to extend the result to some *truly nonlinear* systems[1] for the design of global adaptive observers. Since *high gain observers* (Gauthier et al., 1992; Gauthier and Kupka, 1994) ensure global state estimation for a class of truly nonlinear systems, it is a natural idea to combine the linear adaptive observer with high gain observers. Such an attempt has been reported in (Zhang and Xu, 2001). Up to our knowledge, it was the first *constructive* method for the design of global adaptive observers for a class of truly nonlinear systems. However, it is restricted to the estimation of a single parameter, requires a strong excitation condition, and is formulated in the form of differential-algebraic equations (DAE).

This paper improves the result of (Zhang and Xu, 2001) by introducing a new global adaptive observer enabling the estimation of more parameters with a more reasonable excitation condition. It is formulated in the form of ordinary differential equations (ODE), thus easier to be implemented. It is applicable to a class of single output nonlinear systems observable for all inputs with additional terms containing

[1] By "truly nonlinear system", we mean a nonlinear system that cannot be linearized by coordinate change and output injection.

unknown coefficients. Roughly speaking, if a high gain observer can be designed for a given nonlinear system, then it is possible to design an adaptive observer for the system obtained by adding into the state equation additional terms with unknown coefficients.

The paper is organized as follows. In Section 2 we formulate the considered problem. In Sections 3 and 4 we recall some results on high gain observer and on linear adaptive observer. In Section 5 we present an attempt to combine the high gain observer and the linear adaptive observer, that suggests the nonlinear adaptive observer presented in Section 6. A numerical example is presented in Section 7. Finally, some concluding remarks are drawn in Section 8.

2 Problem statement

In this paper, we consider the joint estimation of the state vector $x(t)$ and the parameter vector θ in the system

$$\dot{x}(t) = A_o x(t) + f(x(t)) + g(x(t))u(t) + \Psi(t)\theta \tag{1a}$$
$$y(t) = c_o x(t) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}$ are system state, input and output,

$$A_o = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad c_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2}$$

$f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^l$ are two nonlinear functions in the triangular form:

$$f(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix} \qquad g(x) = \begin{bmatrix} g_1(x_1) \\ g_2(x_1, x_2) \\ \vdots \\ g_n(x_1, \dots, x_n) \end{bmatrix} \tag{3}$$

$\Psi(t) \in \mathbb{R}^n \times \mathbb{R}^n$ is a *diagonal* matrix of known signals, possibly depending on u and y, $\theta \in \mathbb{R}^n$ is an unknown constant parameter. It is assumed that $u(t)$ and $\Psi(t)$ are both uniformly bounded.

We have assumed that the matrix $\Psi(t)$ is square diagonal, and therefore the number of unknown parameters is equal to the number of state variables. It is possible to generalize the result to the case with an arbitrary matrix $\Psi(t)$ and thus with a parameter vector θ of arbitrary dimension. Such results will be reported elsewhere.

We assume that system (1) cannot be linearized by coordinate change and output injection, otherwise classical methods could be applied to the linearized system.

3 A basic result on high gain observer

The high gain observer recalled in this section essentially follows (Gauthier et al., 1992). It is one of the two main bricks building up the nonlinear adaptive observer proposed in this paper. Consider the system

$$\dot{x}(t) = A_o x(t) + f(x(t)) + g(x(t))u(t) \tag{4a}$$
$$y(t) = c_o x(t) \tag{4b}$$

with the same definitions of $A_o, c_o, f(x), g(x)$ as in (2) and (3).

For any positive real number ρ, define

$$\Lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \rho^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \rho^{-(n-1)} \end{bmatrix} \tag{5}$$

Let S be the positive definite solution of the matrix equation

$$A_o^T S + S A_o + S = c_o^T c_o \tag{6}$$

and

$$k_o = \frac{1}{2} S^{-1} c_o^T \tag{7}$$

Assumption 1 *The functions $f(x)$ and $g(x)$ are globally Lipschitz.*

Assumption 2 *The input u stays in a bounded subset $\mathbb{U}$ of $\mathbb{R}^l$.*

Theorem 1 *Consider system* (4) *with* $A_o, c_o, f(x)$ *and* $g(x)$ *as defined in* (2) *and* (3). *Under Assumptions 1 and 2, for sufficiently large* ρ, *the ODE system*

$$\begin{aligned}\dot{\hat{x}}(t) = A_o\hat{x}(t) + f(\hat{x}(t)) + g(\hat{x}(t))u(t) \\ + \rho\Lambda^{-1}k_o(y(t) - c_o\hat{x}(t))\end{aligned} \tag{8}$$

with Λ *and* k_o *as defined in* (5) *and* (7) *is a global observer for the state of system* (4) *with exponential convergence, i.e., for all initial conditions* $x(t_0)$ *and* $\hat{x}(t_0)$, $\hat{x}(t) - x(t)$ *tends to zero exponentially fast when* $t \to \infty$.

See (Gauthier et al., 1992) for a proof of this theorem.

4 Adaptive observer for linear systems

This is the other main brick for our nonlinear adaptive observer.

Consider linear state space systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Psi(t)\theta \tag{9a}$$
$$y(t) = C(t)x(t) \tag{9b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$, $y(t) \in \mathbb{R}^m$ are respectively the state, input, output of the system, $A(t), B(t), C(t)$ are known time varying matrices of appropriate sizes, $\theta \in \mathbb{R}^p$ is an unknown constant parameter vector, $\Psi(t) \in \mathbb{R}^n \times \mathbb{R}^p$ is a matrix of known signals.

Under appropriate assumptions, the global exponential convergence of the following adaptive observer for system (9) has been established in (Zhang, 2001).

$$\dot{\Upsilon}(t) = [A(t) - K(t)C(t)]\Upsilon(t) + \Psi(t) \tag{10a}$$
$$\begin{aligned}\dot{\hat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t) + \Psi(t)\hat{\theta}(t) \\ &\quad + \left[K(t) + \Upsilon(t)\Gamma\Upsilon^T(t)C^T\Sigma(t)\right][y(t) - C(t)\hat{x}(t)]\end{aligned} \tag{10b}$$
$$\dot{\hat{\theta}}(t) = \Gamma\Upsilon^T(t)C^T(t)\Sigma(t)\,[y(t) - C(t)\hat{x}(t)] \tag{10c}$$

where $K(t), \Sigma(t), \Gamma$ are some design matrices.

5 Combining the high gain observer and the linear adaptive observer

Now let us go back to system (1). Inspired by the results recalled in the two previous sections, we naturally try the adaptive observer of the following form:

$$\dot{\Upsilon}(t) = \rho(A_o - k_o c_o)\Upsilon(t) + \rho\Psi(t) \tag{11a}$$

$$\begin{aligned}\dot{\hat{x}}(t) &= A_o\hat{x}(t) + f(\hat{x}(t)) + g(\hat{x}(t))u + \Psi(t)\hat{\theta}(t) \\ &\quad + \Lambda^{-1}\left[\rho k_o + \Upsilon(t)\Gamma\Upsilon^T(t)c_o^T\right][y(t) - c_o\hat{x}(t)]\end{aligned} \tag{11b}$$

$$\dot{\hat{\theta}}(t) = \rho\Lambda^{-1}\Gamma\Upsilon^T(t)c_o^T\,[y(t) - c_o\hat{x}(t)] \tag{11c}$$

where k_o is as in (7), Λ as in (5), and $\Gamma \in \mathbb{R}^n \times \mathbb{R}^n$ is some positive definite matrix.

Following the convergence proof of the linear adaptive observer (10) as presented in (Zhang, 2001), let us try to analyze the convergence of the algorithm (11).

Combine (11b) and (11c) to obtain

$$\begin{aligned}\dot{\hat{x}} &= A_o\hat{x} + f(\hat{x}) + g(\hat{x})u + \Psi\hat{\theta} \\ &\quad + \rho\Lambda^{-1}k_o(y - c_o\hat{x}) + \Lambda^{-1}\Upsilon\rho^{-1}\Lambda\dot{\hat{\theta}}\end{aligned}$$

Let $\tilde{x} = \hat{x} - x$, $\tilde{\theta} = \hat{\theta} - \theta$ and notice that $\dot{\theta} = 0$, then

$$\begin{aligned}\dot{\tilde{x}} &= (A_o - \rho\Lambda^{-1}k_o c_o)\tilde{x} + f(\hat{x}) - f(x) \\ &\quad + g(\hat{x})u - g(x)u + \Psi\tilde{\theta} + \Lambda^{-1}\Upsilon\rho^{-1}\Lambda\dot{\tilde{\theta}}\end{aligned}$$

It is easy to check that $\Lambda A_o = \rho A_o \Lambda$ and $c_o\Lambda = c_o$.

Define $\tilde{z} = \Lambda\tilde{x}$ and $\tilde{\vartheta} = \rho^{-1}\Lambda\tilde{\theta}$, then

$$\dot{\tilde{z}} = \rho(A_o - k_o c_o)\tilde{z} + \xi + \rho\Psi\tilde{\vartheta} + \Upsilon\dot{\tilde{\vartheta}}$$

where

$$\begin{aligned}\xi &= \Lambda[f(\Lambda^{-1}\hat{z}) - f(\Lambda^{-1}z)] \\ &\quad + \Lambda[g(\Lambda^{-1}\hat{z}) - g(\Lambda^{-1}z)]u\end{aligned} \tag{12}$$

Note that we have used the fact that Ψ is diagonal and thus $\Lambda\Psi = \Psi\Lambda$.

Now define

$$\eta(t) = \tilde{z}(t) - \Upsilon(t)\tilde{\vartheta}(t)$$

then we get

$$\begin{aligned}\dot{\eta} &= \rho(A_o - k_o c_o)(\eta + \Upsilon\tilde{\vartheta}) + \xi + \rho\Psi\tilde{\vartheta} - \dot{\Upsilon}\tilde{\vartheta} \\ &= \rho(A_o - k_o c_o)\eta + \xi + [\rho(A_o - k_o c_o)\Upsilon + \rho\Psi - \dot{\Upsilon}]\tilde{\vartheta}\end{aligned}$$

Because Υ is generated by (11a), we have simply

$$\dot{\eta} = \rho(A_o - k_o c_o)\eta + \xi \tag{13}$$

In the linear case where $\xi = 0$, we could immediately conclude $\eta \to 0$ based on the stability of the matrix $\rho(A_o - k_o c_o)$. Here $\xi \neq 0$ and it depends on $\tilde{\vartheta}$. So the behavior of η has to be analyzed together with $\tilde{\vartheta}$. The definition of $\tilde{\vartheta}$ implies $\dot{\tilde{\vartheta}} = \rho^{-1}\Lambda\dot{\tilde{\theta}}$. Then, from (11c) and the fact that $\dot{\theta} = 0$, we obtain

$$\begin{aligned}\dot{\tilde{\vartheta}} &= \Gamma\Upsilon^T c_o^T [y - c_o\hat{x}] = -\Gamma\Upsilon^T c_o^T c_o\tilde{z} \\ &= -\Gamma\Upsilon^T c_o^T c_o(\eta + \Upsilon\tilde{\vartheta})\end{aligned}$$

Putting together the equations of η and $\tilde{\vartheta}$ yields

$$\dot{\eta} = \rho(A_o - k_o c_o)\eta + \xi \tag{14a}$$

$$\dot{\tilde{\vartheta}} = -\Gamma\Upsilon^T c_o^T c_o(\eta + \Upsilon\tilde{\vartheta}) \tag{14b}$$

Define the Lyapounov function candidate

$$V(t) = \eta^T S\eta + \tilde{\vartheta}^T\Gamma^{-1}\tilde{\vartheta}$$

with S being the solution of (6). Then,

$$\begin{aligned}\frac{dV(t)}{dt} &= -\rho\eta^T S\eta + 2\eta^T S\xi - 2\tilde{\vartheta}^T\Upsilon^T c_o^T c_o(\eta + \Upsilon\tilde{\vartheta}) \\ &= 2\eta^T S\xi - [\eta^T\ \tilde{\vartheta}^T]M[\eta^T\ \tilde{\vartheta}^T]^T \qquad (15)\end{aligned}$$

with

$$M = \begin{bmatrix}\rho S & c_o^T c_o\Upsilon \\ \Upsilon^T c_o^T c_o & 2\Upsilon^T c_o^T c_o\Upsilon\end{bmatrix}$$

The matrix S has been designed to be positive definite. If the matrix $2\Upsilon^T c_o^T c_o\Upsilon$ was positive definite and uniformly bounded from below, then M could be made positive definite by choosing a sufficiently large ρ, and in (15) the second term would be able to dominate the first term. In that case, it would be possible to conclude the negative definiteness of $dV(t)/dt$. Unfortunately, the matrix $2\Upsilon^T c_o^T c_o \Upsilon$ cannot be positive definite in general. As a matter of fact, c_o is a row vector, and thus the rank of $2\Upsilon^T c_o^T c_o \Upsilon$ cannot be larger than 1.

At this point, we see that the main difficulty is how to make the matrix $2\Upsilon^T c_o^T c_o \Upsilon$ positive definite. It would be possible if the number of outputs was larger than n. This gives an important hint for solving the problem: artificially building up a system with many outputs equivalent to system (1). This idea is fully developed in the following section.

6 Adaptive observer based on extended system

As mentioned in the previous section, it will be possible to design a globally convergent adaptive observer for system (1) if we can build an equivalent system with many states and outputs. One possible solution is to extend the system by delaying in time the state, input and output, as explained in the following.

For any integer k and some real constant $\Delta > 0$, define

$$
\begin{aligned}
x^k(t) &= x(t - k\Delta) \\
u^k(t) &= u(t - k\Delta) \\
y^k(t) &= y(t - k\Delta) \\
\Psi^k(t) &= \Psi(t - k\Delta)
\end{aligned}
$$

Then these variables satisfy

$$
\dot{x}^k(t) = A_o x^k(t) + f(x^k(t)) + g(x^k(t))u^k(t) + \Psi^k(t)\theta \qquad (16a)
$$

$$
y^k(t) = c_o x^k(t) \qquad (16b)
$$

Consider the extended system

$$
\begin{bmatrix} \dot{x}^0(t) \\ \dot{x}^1(t) \\ \vdots \\ \dot{x}^{m-1}(t) \end{bmatrix} = \begin{bmatrix} A_o & 0 & \cdots & 0 \\ 0 & A_o & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_o \end{bmatrix} \begin{bmatrix} x^0(t) \\ x^1(t) \\ \vdots \\ x^{m-1}(t) \end{bmatrix}
+ \begin{bmatrix} f(x^0(t)) \\ f(x^1(t)) \\ \vdots \\ f(x^{m-1}(t)) \end{bmatrix} + \begin{bmatrix} g(x^0(t))u^0(t) \\ g(x^1(t))u^1(t) \\ \vdots \\ g(x^{m-1}(t))u^{m-1}(t) \end{bmatrix} + \begin{bmatrix} \Psi^0(t) \\ \Psi^1(t) \\ \vdots \\ \Psi^{m-1}(t) \end{bmatrix} \theta \tag{17a}
$$

$$
\begin{bmatrix} y^0(t) \\ y^1(t) \\ \vdots \\ y^{m-1}(t) \end{bmatrix} = \begin{bmatrix} c_o & 0 & \cdots & 0 \\ 0 & c_o & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_o \end{bmatrix} \begin{bmatrix} x^0(t) \\ x^1(t) \\ \vdots \\ x^{m-1}(t) \end{bmatrix} \tag{17b}
$$

where m is a chosen number of delays. Note that the extended system has mn states and m outputs, but the number of unknown parameters remains to be n. As discussed in the previous section, when $m \geq n$, applying algorithm (11) to the extended system may lead to an adaptive observer with global convergence.

In order to achieve an efficient implementation of the proposed adaptive observer by avoiding large sparse matrices, we will formulate it as the collection of several parallel coupled observers, each corresponding to one delayed time in the extended system.

Assumption 3 *Let $\Upsilon(t) \in \mathbb{R}^n \times \mathbb{R}^n$ be generated through the ODE*

$$\dot{\Upsilon}(t) = \rho(A_o - k_o c_o)\Upsilon(t) + \rho\Psi(t) \tag{18}$$

with some initial condition and some $\rho > 0$. Denote

$$\Upsilon^k(t) = \Upsilon(t - k\Delta) \tag{19}$$

For a given integer $m \geq n$, assume that there exits a constant $\alpha > 0$ such that, for all t, the following inequality holds:

$$\sum_{k=0}^{m-1} \Upsilon^{k^T}(t) c_o^T c_o \Upsilon^k(t) \geq \alpha I \tag{20}$$

Remark 1 Because the matrix $\rho(A_o - k_o c_o)$ is asymptotically stable, $\Upsilon(t)$ is bounded for bounded $\Psi(t)$. Moreover, as the solution of the ODE (18) at the time $\rho^{-1}t$ is

$$\Upsilon(\rho^{-1}t) = e^{(A_o - k_o c_o)(t-t_0)}\Upsilon(\rho^{-1}t_0) + \int_{t_0}^{t} e^{(A_o - k_o c_o)(t-s)}\Psi(\rho^{-1}s)ds$$

the upper bound of $\Upsilon(t)$ is independent of ρ for a given upper bound of $\Psi(t)$. $\square$

Now we are ready to introduce the adaptive observer for the extended system through the following theorem.

Theorem 2 *Let $\Gamma \in \mathbb{R}^n \times \mathbb{R}^n$ be any symmetric positive definite matrix. Under Assumptions 1, 2 and 3, for sufficiently large $\rho > 0$, the ODE system for $k = 0, \ldots, m-1$*

$$\begin{aligned}
\dot{\hat{x}}^k(t) &= A_o\hat{x}^k(t) + f(\hat{x}^k(t)) + g(\hat{x}^k(t))u^k(t) \\
&\quad + \Psi^k(t)\hat{\theta}(t) + \rho\Lambda^{-1}k_o[y^k(t) - c_o\hat{x}^k(t)] \\
&\quad + \Lambda^{-1}\Upsilon^k(t)\Gamma\sum_{i=0}^{m-1}\Upsilon^{i^T}(t)c_o^T[y^i(t) - c_o\hat{x}^i(t)] \qquad (21a)
\end{aligned}$$

$$\dot{\hat{\theta}}(t) = \rho\Lambda^{-1}\Gamma\sum_{i=0}^{m-1}\Upsilon^{i^T}(t)c_o^T[y^i(t) - c_o\hat{x}^i(t)] \qquad (21b)$$

is a global adaptive observer for the extended system (17), i.e., for any initial conditions $x(t_0), \hat{x}^k(t_0)$, $\hat{\theta}(t_0)$ and for all $\theta \in \mathbb{R}^n$, the errors $\hat{x}^k(t) - x^k(t)$ and $\hat{\theta}(t) - \theta$ tend to zero when $t \to \infty$.

Note that there are m state estimation equations with $k = 0, 1, \ldots, m-1$, each being of dimension n. These ODEs are all coupled through the last term.

Proof of Theorem 2 Combine (21a) and (21b):

$$\begin{aligned}
\dot{\hat{x}}^k(t) &= A_o\hat{x}^k(t) + f(\hat{x}^k(t)) + g(\hat{x}^k(t))u^k + \Psi^k(t)\hat{\theta}(t) \\
&\quad + \rho\Lambda^{-1}k_o[y^k(t) - c_o\hat{x}^k(t)] + \Lambda^{-1}\Upsilon^k(t)\rho^{-1}\Lambda\dot{\hat{\theta}}(t)
\end{aligned}$$

Let $\tilde{x}^k(t) = \hat{x}^k(t) - x^k(t)$, $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$ and notice that $\dot{\theta} = 0$, then

$$\begin{aligned}\dot{\tilde{x}}^k(t) &= (A_o - \rho\Lambda^{-1}k_oc_o)\tilde{x}^k(t) + f(\hat{x}^k(t)) - f(x^k(t)) \\ &+ g(\hat{x}^k(t))u^k - g(x^k(t))u^k + \Psi^k(t)\tilde{\theta}(t) \\ &+ \Lambda^{-1}\Upsilon^k(t)\rho^{-1}\Lambda\dot{\tilde{\theta}}(t)\end{aligned}$$

Define

$$\begin{aligned}z^k(t) &= \Lambda x^k(t) \\ \hat{z}^k(t) &= \Lambda\hat{x}^k(t) \\ \tilde{z}^k(t) &= \Lambda\tilde{x}^k(t) \\ \tilde{\vartheta}(t) &= \rho^{-1}\Lambda\tilde{\theta}(t)\end{aligned}$$

Then,

$$\dot{\tilde{z}}^k(t) = \rho(A_o - k_oc_o)\tilde{z}^k(t) + \xi^k(t) + \rho\Psi^k(t)\tilde{\vartheta}(t) + \Upsilon^k(t)\dot{\tilde{\vartheta}}(t)$$

with

$$\begin{aligned}\xi^k(t) &= \Lambda[f(\Lambda^{-1}\hat{z}^k(t)) - f(\Lambda^{-1}z^k(t))] \\ &+ \Lambda[g(\Lambda^{-1}\hat{z}^k(t)) - g(\Lambda^{-1}z^k(t))]u^k(t)\end{aligned} \tag{22}$$

Note that we have used the fact that Ψ is diagonal and thus $\Lambda\Psi = \Psi\Lambda$.

Now define

$$\eta^k(t) = \tilde{z}^k(t) - \Upsilon^k(t)\tilde{\vartheta}(t)$$

then we get

$$\begin{aligned}\dot{\eta}^k(t) &= \rho(A_o - k_oc_o)[\eta^k(t) + \Upsilon^k(t)\tilde{\vartheta}(t)] + \xi^k(t) \\ &+ \rho\Psi^k(t)\tilde{\vartheta}(t) - \dot{\Upsilon}^k(t)\tilde{\vartheta}(t) \\ &= \rho(A_o - k_oc_o)\eta^k(t) + \xi^k(t) \\ &+ [\rho(A_o - k_oc_o)\Upsilon^k(t) + \rho\Psi^k(t) - \dot{\Upsilon}^k(t)]\tilde{\vartheta}(t)\end{aligned}$$

Recall that $\Upsilon(t)$ is generated by (18) and $\Upsilon^k(t)$ is defined by (19), thus $\Upsilon^k(t)$ satisfies

$$\dot{\Upsilon}^k(t) = \rho(A_o - k_oc_o)\Upsilon^k(t) + \rho\Psi^k(t)$$

Therefore, we obtain

$$\dot{\eta}^k(t) = \rho(A_o - k_o c_o)\eta^k(t) + \xi^k(t)$$

The definition of $\tilde{\vartheta}(t)$ implies $\dot{\tilde{\vartheta}} = \rho^{-1}\Lambda\dot{\tilde{\theta}}(t)$. Then, from (21b) and the fact that $\dot{\theta} = 0$, we obtain

$$\begin{aligned}
\dot{\tilde{\vartheta}}(t) &= \Gamma \sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T [y^i(t) - c_o \hat{x}^i(t)] \\
&= -\Gamma \sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T c_o \tilde{x}^i(t) \\
&= -\Gamma \sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T c_o \tilde{z}^i(t) \\
&= -\Gamma \sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T c_o [\eta^i(t) + \Upsilon^i(t)\tilde{\vartheta}(t)]
\end{aligned}$$

Putting together the equations of η and $\tilde{\vartheta}$ yields

$$\dot{\eta}^k(t) = \rho(A_o - k_o c_o)\eta^k(t) + \xi^k(t) \tag{23a}$$

$$\dot{\tilde{\vartheta}}(t) = -\Gamma \sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T c_o [\eta^i(t) + \Upsilon^i(t)\tilde{\vartheta}(t)] \tag{23b}$$

Consider the Lyapounov function candidate

$$V(t) = \sum_{k=0}^{m-1} \eta^{k^T}(t) S \eta^k(t) + \tilde{\vartheta}^T(t)\Gamma^{-1}\tilde{\vartheta}(t)$$

$$\begin{aligned}
\frac{dV(t)}{dt} &= -\rho \sum_{k=0}^{m-1} \eta^{k^T}(t) S \eta^k(t) + 2 \sum_{k=0}^{m-1} \eta^{k^T}(t) S \xi^k(t) \\
&\quad - 2\tilde{\vartheta}^T(t) \sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T c_o [\eta^i(t) + \Upsilon^i(t)\tilde{\vartheta}(t)] \\
&= 2 \sum_{k=0}^{m-1} \eta^{k^T}(t) S \xi^k(t) - \sum_{k=0}^{m-1} \left(\begin{bmatrix} \eta^k(t) \\ \tilde{\vartheta}(t) \end{bmatrix}^T M_k(\rho) \begin{bmatrix} \eta^k(t) \\ \tilde{\vartheta}(t) \end{bmatrix} \right)
\end{aligned} \tag{24}$$

with

$$M_k(\rho) = \begin{bmatrix} \rho S & c_o^T c_o \Upsilon^k(t) \\ \Upsilon^{k^T}(t) c_o^T c_o & \dfrac{2}{m} \displaystyle\sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T c_o \Upsilon^i(t) \end{bmatrix}$$

The matrix S has been designed to be positive definite. By Assumption 3, the matrix

$$\frac{2}{m} \sum_{i=0}^{m-1} \Upsilon^{i^T}(t) c_o^T c_o \Upsilon^i(t)$$

is positive definite and uniformly bounded from below. Moreover, $\Upsilon^k(t)$ is uniformly bounded from above due to the asymptotic stability of the matrix $A_o - k_o c_o$ and the boundedness of $\Psi^k(t)$. It follows that, for sufficiently large ρ, the matrix $M_k(\rho)$ is positive definite.

Because f and g are globally Lipschitz and have the triangular form, it can be shown that ξ^k as defined in (22) satisfies the inequality

$$\|\xi^k(t)\| \leq \kappa(\rho^{-1}) \|\tilde{z}^k(t)\| = \kappa(\rho^{-1}) \|\eta^k(t) + \Upsilon^k(t)\tilde{\vartheta}(t)\|$$

with $\kappa(\rho^{-1}) > 0$ a polynomial in ρ^{-1} depending on the Lipschitz constants of the functions f and g and on the bound of u. Therefore, there exist polynomials $\mu_1(\rho^{-1})$ and $\mu_2(\rho^{-1})$ such that

$$\eta^{k^T}(t) S \xi^k(t) \leq \mu_1(\rho^{-1}) \|\eta^k(t)\|^2 + \mu_2(\rho^{-1}) \|\eta^k(t)\| \cdot \|\tilde{\vartheta}(t)\|$$

It follows that

$$\begin{aligned} \frac{dV(t)}{dt} \leq\ & 2 \sum_{k=0}^{m-1} \Big(\mu_1(\rho^{-1}) \|\eta^k(t)\|^2 + \mu_2(\rho^{-1}) \|\eta^k(t)\| \cdot \|\tilde{\vartheta}(t)\| \Big) \\ & - \sum_{k=0}^{m-1} \left(\begin{bmatrix} \eta^k(t) \\ \tilde{\vartheta}(t) \end{bmatrix}^T M_k(\rho) \begin{bmatrix} \eta^k(t) \\ \tilde{\vartheta}(t) \end{bmatrix} \right) \end{aligned}$$

The first sum at the right hand side of the inequality does not increase with ρ. For sufficiently large ρ, each term in the second sum is positive definite and increases linearly with ρ. Therefore, for sufficiently large ρ, $dV(t)/dt$ is negative definite. We can then conclude that $\eta^k(t) \to 0$ and $\tilde{\vartheta}(t) \to 0$, and thus $\tilde{z}^k(t) = \eta^k(t) + \Upsilon^k(t)\tilde{\vartheta}(t) \to 0$. It follows that $\tilde{x}^k(t) \to 0$ and $\tilde{\theta}(t) \to 0$. □

7 Numerical example

In this section we show a simulation example of the proposed adaptive observer. The simulated system is

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + \psi_1(t)\theta_1 \\ \dot{x}_2(t) &= 10\sin x_2(t) + \psi_2(t)\theta_2 \\ y(t) &= x_1(t)\end{aligned}$$

with the parameters $\theta_1 = 1$, $\theta_2 = 1.5$. The excitation signals are $\psi_1(t) = \sin 2t + \cos 20t$ and $\psi_2(t) = \sin 3t + \cos 27t$. Note that, in principle, adding some nonlinear functions of x_1 into the state equations would not increase any difficulty, since x_1 is directly measured by y.

The parameters of the adaptive observer are: $m = 5$, $\Delta = 0.1$, $\rho = 8$, $\Gamma = \mathrm{diag}([3,1])$. The initial values are $\hat{x}^k(0) = [0,0]^T$, $\hat{\theta}(0) = [0.5, 0.5]^T$.

The state estimation errors are plotted in Figure 1, and the parameter estimates in Figure 2. Note that the state estimates seem to converge earlier than the parameter estimates. In fact, it is $\hat{x}^0(t) - x^0(t)$ that is plotted. The delayed state estimation error $\hat{x}^{m-1}(t) - x^{m-1}(t)$ converges $m\Delta = 0.5$ second later, approximatively at the same time as the parameter estimates.

8 Conclusion

We have considered in this paper the joint estimation of state and some parameters for a class of nonlinear systems. The proposed method for the design of adaptive observers is *constructive* and guarantees *global convergence* under appropriate assumptions.

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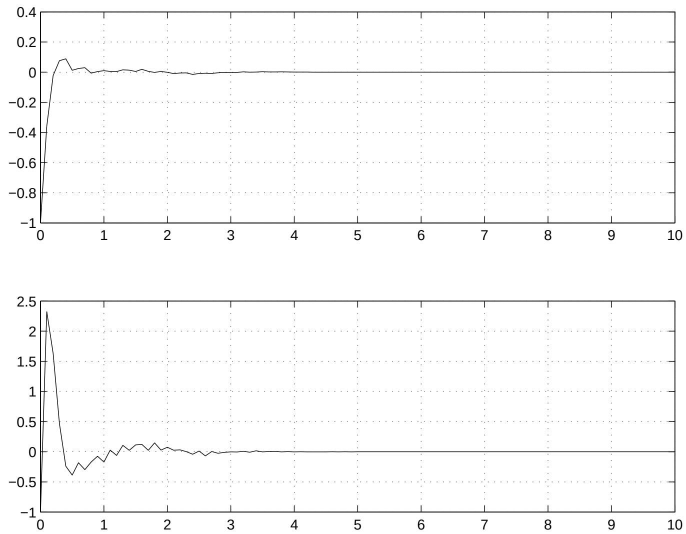

Figure 1: Simulation example: state estimation errors $\tilde{x}_1(t)$ (upper) and $\tilde{x}_2(t)$ (lower).

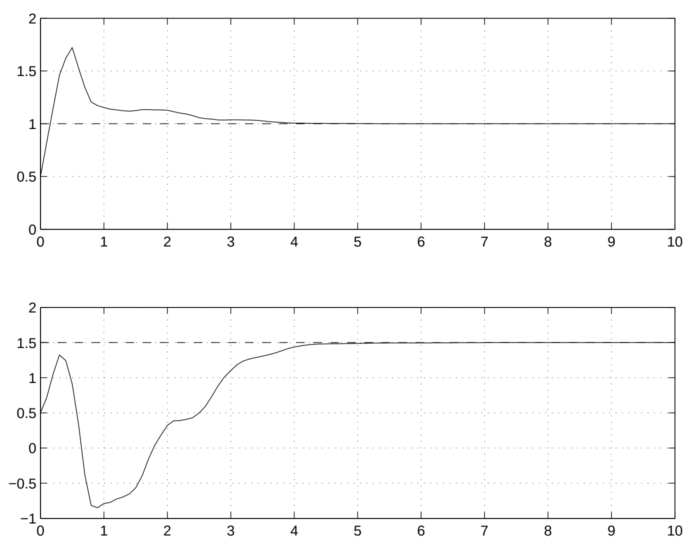

Figure 2: Simulation example: parameter estimates $\hat{\theta}_1(t)$ (upper) and $\hat{\theta}_2(t)$ (lower). The true parameter values are shown by the dashed lines.

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