On the Number of Cylindrical Shells

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Abstract: Given a set P of n points in three dimensions, a cylindrical shell or zone cylinder is formed by two cylindrical cylinders with the same axis such that all points of P are between the two cylinders. We prove that the number of cylindrical shells enclosing P passing through combinatorially different subsets of P has size $\Omega(n^3)$ and $O(n^4)$ (previous known bound was $O(n^5)$).

Key-words: Computational geometry, metrology, convex hull

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Sur le dénombrement des paires de cylindres coaxiaux

Résumé : Étant donné un ensemble P de n points en dimension trois, on cherche à dénombré le nombre de paire de cylindres coaxiaux telle que tous les points de P soient compris entre les deux cylindres. Nous démontrons que le nombre de paires de cylindres coaxiaux contenant P est borné par $\Omega(n^3)$ et $O(n^4)$ (la meilleure borne connue était $O(n^5)$).

Mots-clés : Géométrie algorithmique, métrologie, enveloppe convexe

1 Introduction

Given a set P of n points in three dimensions. A cylindrical shell is formed by the space between two coaxial circular cylinders in 3D. The shell is said to enclose P, if the set P is between the two cylinders. The difference between the radii of the two cylinders is the width of the shell.

Finding the minimum width shell enclosing P is an important metrology problem, this width can be viewed as a measure of the quality of cylindrical object. Devillers and Preparata reduce the problem to linear programming in the special case of small width and find a provably good approximation [DP00]. Agarwal, Aronov and Sharir compute in quadratic time a constant factor approximation of the width [AAS00]. Har-Peled and Varadarajan propose to compute an $1+\epsilon$ approximation in time linear in n and exponential in $\frac{1}{\epsilon}$ [HPV01].

There is few works about exact algorithms to compute the width, In general position, 6 points defined a constant number of shells (less than 150) having these 6 points on their boundary [DMPT01], thus a naive algorithm will have $O(n^7)$ complexity. Agarwal, Aronov and Sharir [AAS00] have proposed a reduction to a convex hull in 10 dimensions which yields to a $O(n^5)$ complexity algorithm.

In this paper, we will prove upper and lower bounds for the number of combinatorially different enclosing cylindrical shells for a set of n points in three dimensional space. These bounds yields to a $O(n^4)$ complexity for the exact algorithm of Agarwal, Aronov and Sharir computing the smallest width enclosing shell.

2 Lower bound example

Theorem 1 *Given n points in 3 dimensions, the number of combinatorially different enclosing shells is $\Omega(n^3)$.*

Proof: We consider three sets of n points:

$$\begin{aligned} p_i &= (\frac{i}{n}, 0, 0),\ 1 \leq i \leq n \\ q_i &= (\frac{i}{n}, 0, \zeta),\ 1 \leq i \leq n \\ r_i &= (x_i, \frac{i}{n}, 0),\ 1 \leq i \leq n \end{aligned}$$

such that the r_i belongs to the circle $\mathcal{C}$ in plane $z = 0$ of center $c = (3, \frac{1}{2}, 0)$ and radius 6 and ζ is some positive parameter to be chosen later (see Figure 1).

Then for any triple (i, j, k) the aim is to define a shell such that the internal cylinder is tangent to line $y = z = 0$ at p_i and to line $y = \zeta - z = 0$ at q_j and such that the external cylinder is tangent (outside) to Circle $\mathcal{C}$ at r_k.

The two first tangencies are granted provided that the axis of the shell is directed by vector p_iq_j and passes through a point of line p_iy, that point is chosen at the intersection

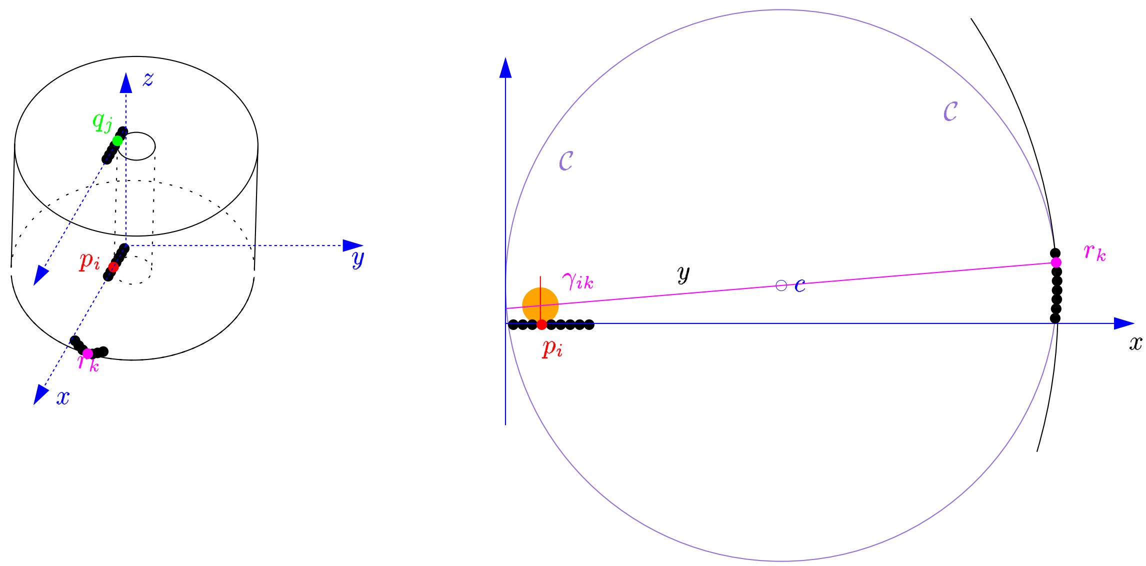

Figure 1: $\Omega(n^3)$ lower bound example.

point γ_{ik} of cr_k with $p_i y$. Then the cylinder with that axis passing through p_i also passes through q_j and is tangent to plane Oxz so that points $p_l, l \neq i$ and $q_l, l \neq j$ are outside that cylinder.

We have now to consider now the cylinder with the same axis passing through r_k. First we can remark that the circle of center γ_{ik} passing through r_k is tangent externally to $\mathcal{C}$ since its radius is bigger than the radius of $\mathcal{C}$. Since the cylinder is not vertical, its intersection with the horizontal plane is not a circle but an ellipse, although, by choosing ζ large enough it is possible to make this ellipse close to a circle and to preserve the property that $\mathcal{C}$ is tangent inside to the cylinder. ■

3 Projection theorem

We will prove in this section, that the convex hull of certain special configuration of points in d dimensions cannot reach the worst case complexity of $O(n^{\lfloor \frac{d}{2} \rfloor})$. More precisely if the set of $2n$ points is obtained by taking n points in an hyperplane and n other points which are the projection of the first n points in another hyperplane which do not intersect the convex hull of the n points, then the convex hull in d dimension is the extrusion of the convex hull in the hyperplanes and has complexity $O(n^{\lfloor \frac{d-1}{2} \rfloor})$.

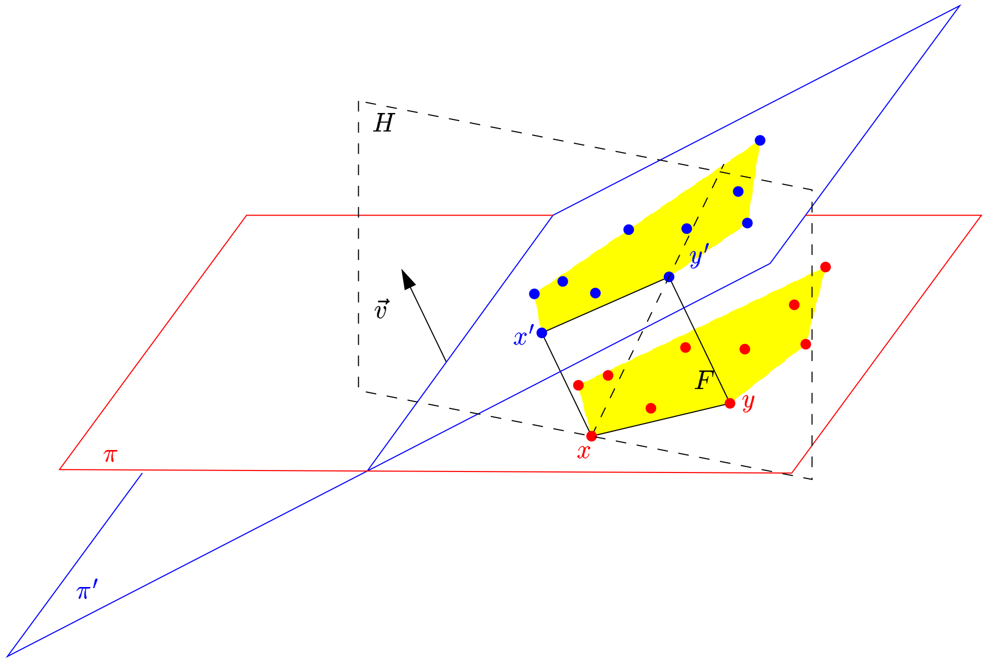

Figure 2: For the proof of Theorem 2.

Theorem 2 *Let π, π' be two hyperplanes in $\mathbb{R}^d$ and $\vec{v}$ be a vector of $\mathbb{R}^d$ which does not belong to the directions of π or π'. Let π^+ be one of the half hyperplane of π delimited by π'. Let $\mathcal{U} \subset \pi^+$ be a finite set of n points and $\mathcal{U}' \subset \pi'$ be the projection of $\mathcal{U}$ on π' along direction $\vec{v}$. Let $\mathcal{V} = \mathcal{U} \cup \mathcal{U}'$. The convex hull $CH(\mathcal{V})$ of $\mathcal{V}$ has the same asymptotic complexity as $CH(\mathcal{U})$, that is $O(n^{\lfloor \frac{d-1}{2} \rfloor})$.*

Proof: The space is divided in four quadrants by the hyperplanes π and π' (see Figure 2). Three of these quadrants are obviously separated from $\mathcal{V}$ by π or π' and thus cannot intersect $CH(\mathcal{V})$. Consider a face F of $CH(\mathcal{V})$ in the fourth quadrant, we first claim that $\vec{v}$ belongs to the direction of F. Indeed, if we denote H an hyperplane supporting F and n_H the normal to H, x a vertex of F in π and x' its projection on π', y' a vertex of F in π' and y its projection on π. Since H is a supporting hyperplane we must have x' and y on the same side of H that is $\text{sign}(\vec{xx'} \cdot n_H) = \text{sign}(\vec{y'y} \cdot n_H)$, but since $\vec{xx'}$ and $\vec{y'y}$ are collinear to $\vec{v}$ and in opposite directions, we get that $\vec{v}$ belongs to H.

In fact we have that the faces of $CH(\mathcal{V})$ are either faces of $CH(\mathcal{U})$, faces of $CH(\mathcal{U}')$ or faces linking a face of $CH(\mathcal{U})$ to its projection in $CH(\mathcal{U}')$. Thus the total number of faces of $CH(\mathcal{V})$ is three times the number of faces of $CH(\mathcal{U})$. ■

Unfortunately, we have not found any suitable generalization of Theorem 2 where π and π' are k-flats with $k \leq d-2$. For example if $\mathcal{U} = \{p_i = (i, i^2, 0, 0); 1 \leq i \leq n\}$ and $\mathcal{U}' = \{p'_i = (0, 0, i, i^2); 1 \leq i \leq n\}$ then the $O(n^2)$ simplices $(p_i, p_{i+1}, p_j, p_{j+1})$ are convex hull faces.

4 Minimum width cylindrical shell algorithm

Given a set $\mathcal{S}$ of n 3D points Agarwal, Aronov and Sharir [AAS00, page 511 col.2] exhibit a transformation mapping a 3D point $p = (x, y, z)$ in two half spaces $P^\star$ and $P^\dagger$ in 10 dimensions :

$$\begin{aligned} P^\star \ : \quad \varphi_9 \leq \quad & (x^2+y^2) + 2x\varphi_1 + 2y\varphi_2 + 2z\varphi_3 - 2xy\varphi_4 - 2xz\varphi_5 - 2yz\varphi_6 \\ & +(y^2+z^2)\varphi_7 + (x^2+z^2)\varphi_8 \\ P^\dagger \ : \quad \varphi_{10} \geq \quad & (x^2+y^2) + 2x\varphi_1 + 2y\varphi_2 + 2z\varphi_3 - 2xy\varphi_4 - 2xz\varphi_5 - 2yz\varphi_6 \\ & +(y^2+z^2)\varphi_7 + (x^2+z^2)\varphi_8 \end{aligned}$$

where $\varphi_1, \ldots \varphi_{10}$ are the coordinates in 10 dimensions.

Agarwal, Aronov and Sharir proves that the complexity of the intersection of the $P^\star$ and $P^\dagger$ bounds the complexity of the enclosing shells, and since we have an intersection of $2n$ half spaces in 10 dimensions get an $O(n^5)$ bound.

An intersection of half spaces problem can be transformed in a convex hull of points problem through duality if we know a point inside the intersection. Here we can remark that for some α large enough (larger than the square distance between the origin and its farthest neighbor in $\mathcal{S}$), point $(0,0,0,0,0,0,0,0,0,\alpha)$ is certainly inside the half spaces. By a translation of that vector, the hyperplane

$$0 = \psi_0 + \sum_{i=1}^{10} \psi_i \varphi_i$$

is first transform in the hyperplane

$$\begin{aligned} 0 &= \psi_0 + \sum_{i=1}^{9} \psi_i \varphi_i + \psi_{10}(\varphi_{10} - \alpha + \alpha) \\ 0 &= (\psi_0 + \psi_{10}\alpha) + \sum_{i=1}^{10} \psi_i \phi_i \end{aligned}$$

where ϕ_i are the coordinates in the new frame. This equation is normalized by dividing it by ψ_8 and by duality we get the point

$$\left(\frac{\psi_0 + \psi_{10}\alpha}{\psi_8}, \frac{\psi_1}{\psi_8}, \frac{\psi_2}{\psi_8}, \frac{\psi_3}{\psi_8}, \frac{\psi_4}{\psi_8}, \frac{\psi_5}{\psi_8}, \frac{\psi_6}{\psi_8}, \frac{\psi_7}{\psi_8}, \frac{\psi_9}{\psi_8}, \frac{\psi_{10}}{\psi_8}\right)$$

Through this process, we get to points $p^\star$ and $p^\dagger$

$$\begin{aligned}
p^\star &= \left(\frac{x^2+y^2}{x^2+z^2}, \frac{2x}{x^2+z^2}, \frac{2y}{x^2+z^2}, \frac{2z}{x^2+z^2}, \right.\\
&\qquad \left. \frac{-2xy}{x^2+z^2}, \frac{-2xz}{x^2+z^2}, \frac{-2yz}{x^2+z^2}, \frac{y^2+z^2}{x^2+z^2}, \frac{1}{x^2+z^2}, 0\right)\\
p^\dagger &= \left(\frac{x^2+y^2}{x^2+z^2} + \frac{\alpha}{x^2+z^2}, \frac{2x}{x^2+z^2}, \frac{2y}{x^2+z^2}, \frac{2z}{x^2+z^2}, \right.\\
&\qquad \left. \frac{-2xy}{x^2+z^2}, \frac{-2xz}{x^2+z^2}, \frac{-2yz}{x^2+z^2}, \frac{y^2+z^2}{x^2+z^2}, 0, \frac{1}{x^2+z^2}\right)\\
&= p^\star + \frac{1}{x^2+z^2}(\alpha, 0, 0, 0, 0, 0, 0, 0, -1, 1)
\end{aligned}$$

We remark that $p^\star$ belongs to the half-hyperplane $\pi^+ : \phi_{10} = 0, \phi_9 \geq 0$ $p^\dagger$ is the projection of $p^\star$ on $\pi' : \phi_9 = 0$ along direction $\vec{v} = (\alpha, 0, 0, 0, 0, 0, 0, 0, -1, 1)$. The hypotheses for Theorem 2 are verified and we conclude that the complexity of the intersections of the $P^\star$ and $P^\dagger$ ($p \in \mathcal{S}$) can be reduced to $O(n^4)$. We can state the following theorem, improving the previous result by a factor of n :

Theorem 3 *Given a set $\mathcal{S}$ of n points in $\mathbb{R}^3$, a minimum-width cylindrical shell containing $\mathcal{S}$ can be computed in $O(n^4)$ time.*

5 Conclusion and open problems

We have proven that the number of combinatorially different cylindrical shells enclosing a set of n points is $\Omega(n^3)$. This result apply to the number of all shells. Of course in general position, the minimum width shell is unique, we may ask what is the maximal number of minimum width shells in any position and what is the maximal number of shells whose width is locally minimal in general position.

We have also proven that the number of combinatorially different shells is $O(n^4)$ which reduces the complexity of Agarwal, Aronov and Sharir algorithm to that complexity. A clear open problem is to reduce the gap between these lower and upper bounds.

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