# Lyapunov Exponents: When the Top joins the Bottom

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**Abstract:** In this paper, we consider Lyapunov exponents associated with iterates of random functions. When these functions are taken from an *irreducible* set, we show that the top and the bottom Lyapunov exponents are equal.

**Key-words:** Lyapunov exponents, stochastic recurrence. *AMS 1991 subject classifications* : Primary 15A52, 34D08; Secondary 68M20, 05C50.

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# Exposants de Lyapunov: Quand le Top rejoint le Bottom

**Résumé :** Dans cet article, on considère les exposants de Lyapunov associés aux itérées de fonctions aléatoires. Lorsque le support de ces fonctions est *irréductible*, nous montrons l'égalité des exposants de Lyapunov top et bottom.

**Mots-clés :** exposant de Lyapounov, récurrence stochastique. *classification AMS 1991* : Primaire 15A52, 34D08; Secondaire 68M20, 05C50.

# 1 Introduction

It is well known result that top and bottom Lyapunov exponents associated to the iterates of monotone homogeneous functions exist under a quite general setting (cf. [14]).
The focus of the present paper is to find general conditions under which the two exponents are equal. The only known conditions for this equality are roughly based or on boundedness condition of functions or on some fixed structure of the matrices when these functions admit matrix representation (cf. [1, 13]). The sufficient conditions we give in §3 extend those conditions based on the generalized notion of irreducibility defined in §2. Preliminary results that we need for the proof of Theorem 1 are presented in §4 and the proof is in §5.

# 2 Preliminaries

We say that $f : \mathbb{R}^d \to \mathbb{R}^k$ is *monotone* if for all $X, Y \in \mathbb{R}^d$, $X \leq Y \implies f(X) \leq f(Y)$, where $\leq$ denotes the usual product ordering of $\mathbb{R}^n$, for all $n$. We say that $f$ is *homogeneous* if for all $\lambda \in \mathbb{R}$ and $X \in \mathbb{R}^d$, $f(\lambda + X) = \lambda + f(X)$, where for all $X \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $\lambda + X$ denotes the vector with entries $\lambda + (X)_i$.

Let $\mathbb{T}_d(\mathbb{R})$ be the set of montone and homogeneous functions $\mathbb{R}^d \to \mathbb{R}^d$. Let $(\Omega, \rho)$ be a probability space and $\theta$ the shift operator on $\Omega$, namely a measurable map $\Omega \to \Omega$ preserving the measure $\rho$. Let $f : \Omega \to \mathbb{T}_d(\mathbb{R})$ be a measurable function. We denote $f(\theta^n\omega)$ by $f(\omega, n)$ or more simply $f_n$ for all $n \in \mathbb{Z}$. The support set of $f_n$ is denoted $\mathbb{S}$.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{R}^d$ defined by the stochastic recurrence:

$$X_{n+1} = f_n(X_n)$$

with an initial condition $X_0 \in \mathbb{R}^d$.

In the following we denote by $\mathbf{0}$ the vector $(0, \cdots, 0)^t \in \mathbb{R}^d$. We define $\mathsf{t}, \mathsf{b} : \mathbb{R}^d \to \mathbb{R}$ by

$$\forall X \in \mathbb{R}^d,\ \mathsf{t}(X) = \mathsf{t}X = \max_i (X)_i \quad \text{and} \quad \mathsf{b}(X) = \mathsf{b}X = \min_i (X)_i$$

If $\rho$ is stationary ergodic and if $f$ is integrable, that is $f(\cdot)(\mathbf{0}) \in L^1$, then it is proved in [14] that the *top–Lyapunov exponent*

$$\lim_{n\to\infty} \frac{\mathsf{t}X_n}{n} = \gamma_{\mathsf{t}} \tag{1}$$

and the *bottom–Lyapunov exponent*

$$\lim_{n\to\infty} \frac{\mathsf{b}X_n}{n} = \gamma_{\mathsf{b}}$$

exist a.s. and in $L^1$. This result is based on the Kingman's theorem for subadditive process [12, 8]. Its computation issue is an open and difficult problem that has been studied in many papers ([2], [3], [4], [6], [7], [11]). Besides, interesting results on the study of the cycle-time vector in deterministic case ($f_n$ constant) can be found in [10].

# 3 Main result

Let $E = \{1, \ldots, d\}$. Let $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. We denote by $\mathbb{S}^+$ the semi-group generated by $\mathbb{S}$. We denote by $e_i$ the vector of dimension $d$ with all components equal to $-\infty$ except for the $i$-th component equal to 0.

**Extension on $\hat{\mathbb{R}}$** Let $g \in \mathbb{T}_d(\mathbb{R})$. $g$ is extended on $\hat{\mathbb{R}}$ as follow: if $X$ is a vector of $\hat{\mathbb{R}}^d$, we define $X^N$ as a vector obtained from $X$ replacing all its components equal to $-\infty$ by $-N$ and we defined $g(X)$ as the limit $\lim_{N\to\infty} g(X^N)$. Integrability and monotonicity of $g$ implies that this limit is in $\hat{\mathbb{R}}^d$.

**Definition 1 (path).** *We say that $g \in \mathbb{S}^+$ realizes a* path *from $i \in E$ to $j \in E$ if $g(e_i)_j > -\infty$ and we denote it by $i \xrightarrow{g} j$. The length of the path is defined by $|g| = \min\{l \geq 1 \mid g \in \mathbb{S}^l\}$.*

**Definition 2 (irreducibility).** *$\mathbb{S}$ is said to be* irreducible *if $\forall i, j \in E$, $\exists g \in \mathbb{S}^+$ such that $i \xrightarrow{g} j$,*

**Definition 3 (row-allowability).** *$\mathbb{S}$ is said to be* row-allowable *if $\forall g \in \mathbb{S}$, $\forall i \in E$, $\exists j \in E$ such that $j \xrightarrow{g} i$.*

Here is the main result:

**Theorem 1.** *Assume that the sequence $\{f_n\}$ is i.i.d. If $\mathbb{S}$ is irreducible and row-allowable then $\gamma_{\mathsf{t}} = \gamma_{\mathsf{b}} = \gamma$ and therefore for all $i$,*

$$\lim_{n\to\infty} \frac{(X_n)_i}{n} = \gamma \text{ a.s. and in } L_1.$$

**Remark 1.** The result of Theorem 1 can be easily generalized to the sequence of functions with Markov dependence. For stationary ergodic case, the irreducibility does no more guarantee the equality $\gamma_{\mathsf{t}} = \gamma_{\mathsf{b}}$.

**Example 1.** Here are two main examples of monotone homogeneous map. The first example is

$$g : \mathbb{R}^d \to \mathbb{R}^d, \qquad g(X) = \log(M \exp(X)) \ , \tag{2}$$

where $\exp(X) = (\exp(X_1), \ldots, \exp(X_d))^t$, $\log(X) = (\log(X_1), \ldots, \log(X_d))^t$, and $M$ is a $d \times d$ nonnegative matrix with at least one strictly positive entry per row (the later condition is row-allowability and this ensures that $f(\mathbb{R}^d) \subset \mathbb{R}^d$). If each map $f_k$ is of the form $f_k(X) = \log(M_k \exp(X))$, for some $M_k$, then the Lyapunov exponent (1) coincides with the classical top Lyapunov exponent [5] of the random product of nonnegative matrices $M_n \ldots M_1$, which is defined by:

$$\gamma = \text{a. s.} \lim_{n\to\infty} \frac{1}{n} \log \|M_n \cdots M_1\| \ ,$$

for any norm $\|\cdot\|$. In this case, if we denote $\hat{\mathbb{S}}$ the support set of $M_k$, $\mathbb{S}$ is irreducible if $\forall i, j \in E$, $\exists M \in \hat{\mathbb{S}}^+$ such that $M_{ij} > 0$.

The second example is

$$f : \mathbb{R}^d \to \mathbb{R}^d, \qquad f_i(x) = \max_{1\leq j\leq d} (M_{ij} + x_j) \ , \tag{3}$$

where $M$ is a $d \times d$ matrix with entries in $\mathbb{R} \cup \{-\infty\}$, such that each row contains at least one finite entry. If each map $f_k$ is of the form (3) for some matrix $M_k$, the Lyapunov exponent (1) coincides with the Lyapunov exponent of the random product of matrices $M_n \ldots M_1$ in the max-plus semiring [1, 6]. Here $\mathbb{S}$ is irreducible if $\forall i, j \in E$, $\exists M \in \mathbb{S}^+$ such that $M_{ij} > -\infty$.

# 4 Backward coupling class

In this section we will assume that $\mathbb{S}$ is irreducible for an arbitraly given path relation and that this relation is transitive, namely if $i \xrightarrow{g} j$ and $j \xrightarrow{g'} k$ then $i \xrightarrow{g' \circ g} k$. In the next section, we show that Definition 1 defines a transitive relation.

**Notations** For $J \subset E$, we introduce the following notations:

1. *Forward and backward path connections by* $g \in \mathbb{S}^+$:

$$I_f(J, g) = \{i \in E \mid \exists j \in J \ \ s.t. \ \ i \xrightarrow{g} j\}, \qquad I_b(J, g) = \{i \in E \mid \exists j \in J \ \ s.t. \ \ j \xrightarrow{g} i\}.$$

2. *Random path connections*: $\forall n \in \mathbb{Z}$,

$$\begin{aligned} I(\omega, J, n) &= I_b(J, f(\omega, -1) \cdots f(\omega, n)) \ \text{ for } n \leq -1, \\ &= I_f(J, f(\omega, n) \cdots f(\omega, 0)) \ \text{ for } n \geq 0. \end{aligned}$$

3. *Recurrent states of* $I(\omega, J, n)$ *in backward*:

$$S_b(\omega, J) = \{J' \subset E \mid \text{Card}[n \leq -1 \mid I(\omega, J, n) = J'] = \infty\}.$$

4. *Maximum size of path connections*:

$$N_f = \max\{\text{Card}[I_f(j, g)] \mid j \in E, \ g \in \mathbb{S}^+\}, \quad N_b = \max\{\text{Card}[I_b(j, g)] \mid j \in E, \ g \in \mathbb{S}^+\}.$$

5. *Maximum size of random path connections in backward*:

$$N_b(\omega, j) = \max_{n \leq -1} \text{Card}[I(\omega, j, n)].$$

The following lemma will be crucial to characterize the backward coupling class.

**Lemma 1.** *Let*

$$H(\omega, j, i, B) = \big\{n \leq -1 \mid i \in I(\omega, j, n) \ \textit{and} \ f_{n-1} \circ \cdots \circ f_{n-|B|} = B\big\}.$$

$\forall i, j \in E$, $\forall B \in \mathbb{S}^+$, $\text{Card}[H(\omega, j, i, B)] = \infty$ *a.s.*

*Proof.* This is an immediate consequence of the irreducibility of $\mathbb{S}$ and of the fact that conditionnally to the event $i \in I(\omega, j, n)$, we have $\mathbb{P}(f_{n-1} \circ \cdots \circ f_{n-|B|} = B) > 0$. □

**Lemma 2.** *Let* $i_0 \in E$ *and* $M \in \mathbb{S}^+$ *s.t.* $\text{Card}[I_b(i_0, M)] = N_b$. *The following almost sure equalities hold :*

1. $\forall j \in E$, $N_b(\omega, j) = N_b$,
2. $\forall j \in E$, $S_b(\omega, j) = S_b = \cup_{B \in \mathbb{S}^+} I_b(i_0, M \circ B)$, *where* $\text{Card}[I_b(i_0, M)] = N_b$.

*Proof.* We first prove the first a.s. equality.

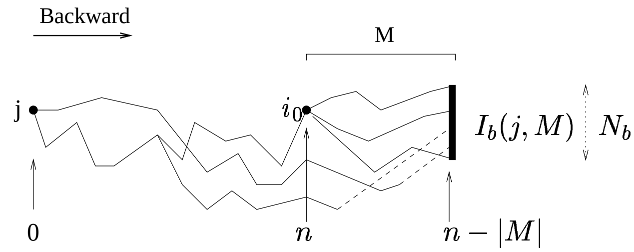


Figure 1: $n \in H(\omega, j, i_0, M)$

We have (see Fig.1): $\forall n \in H(\omega, j, i_0, M)$, $I_b(i_0, M) \subset I(\omega, j, n - |M|)$. Therefore $I_b(i_0, M) = I(\omega, j, n - |M|)$. Since $H(\omega, j, i_0, M)$ is a.s. non empty, this implies that $N_b(\omega, j) = N_b$ a.s.

We next show the second equality. We have $\forall n \leq 0$, $S_b(\omega, j) \subset \cup_{k<n} I(\omega, j, k)$. If $n \in H(\omega, j, i_0, M)$, $\forall k \leq n$, $\exists B \in \mathbb{S}^+$ s.t. $I(\omega, j, k) = I_b(i_0, M \circ B)$. Hence $S_b(\omega, j) \subset \cup_{B \in \mathbb{S}^+} I(i_0, M \circ B)$. Now $\forall B \in \mathbb{S}^+$, $\mathrm{Card}[H(\omega, j, i_0, M \circ B)] = \infty$ *a.s.* Hence $I(i_0, M \circ B)$ are recurrent states. □

**Definition 4 (BC-class).** *$i, j \in E$ are in the same bacward coupling class (BC-class) if the property: $\exists N(\omega) \in ]-\infty, 0[$ such that $I(\omega, i, N) = I(\omega, j, N)$ holds a.s. and we will denote this property by $i \sim j$.*

**Lemma 3.** *We have the following properties:*

1. *$\sim$ is an equivalence relation.*
2. *If $\mathbb{P}(\exists n \leq -1, I(\omega, i, n) \cap I(\omega, j, n) \neq \emptyset) = 1$, then $i \sim j$.*
3. *If $\exists B \in \mathbb{S}^+$, $\exists i_0 \in E$ s.t. $\{i, j\} \subset I_b(i_0, B)$, then $i \sim j$.*

*Proof.* 1) It is immediate that $i \sim i$ and that $i \sim j$ implies $j \sim i$. If $i \sim j$ and $j \sim k$, then for almost all $\omega$, $\exists N, N' < 0$ s.t. $I(\omega, i, N) = I(\omega, j, N)$ and $I(\omega, j, N') = I(\omega, k, N')$. Then, we have $I(\omega, i, \min(N, N')) = I(\omega, k, \min(N, N'))$.

2) If $\mathbb{P}(\exists n \leq -1, I(\omega, i, n) \cap I(\omega, j, n) \neq \emptyset) = 1$, then for almost all $\omega$, $\exists n \leq -1$ and $x \in E$ s.t. $x \in I(\omega, i, n) \cap I(\omega, j, n)$.

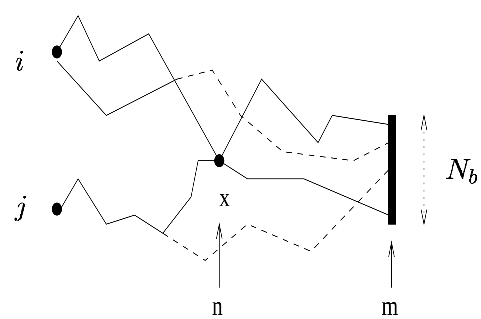


Figure 2: $x \in I(\omega, i, n) \cap I(\omega, j, n)$.

If $m < n$ is the time at which $\mathrm{Card}[I(\theta^n \omega, x, m - n)] = N_b$ ($m$ is a.s. finite by Lemma 2, cf. Fig.2), then $I(\omega, j, m) = I(\theta^n \omega, x, m - n) = I(\omega, i, m)$. Therefore $i \sim j$.

3) Let Card$[I_b(i,M)] = N_b$. If $n \in H(\omega,i,i,M)$, $I(\omega,i,n-|M|) = I_b(i,M)$.

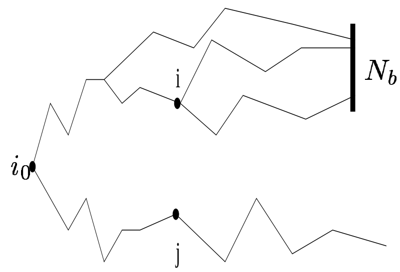


Figure 3: $\{i,j\} \subset I_b(i_0,B)$.

But we also have $I(\omega,j,n-|M|) \subset I_b(i,M)$. Otherwise $\exists x \in I(\omega,j,n-|M|)$ s.t. $x \notin I_b(i,M)$. Since $\{i,j\} \subset I_b(i_0,B)$, that would imply that Card$[I_b(i_0, B\circ f_{-1}\circ .. \circ f_{-n} \circ M)] > N_b$ which is impossible. Therefore $\cup_{n<0}(I(\omega,j,n) \cap I(\omega,i,n))$ is a.s. non-empty and from the point 2, $i \sim j$.

□

**Lemma 4.** *$E$ admits a unique partition in $K$ BC-classes $(C_1,..,C_K)$ which only depend on $\mathbb{S}$ and which has the following properties:*

1. *$\forall i \in \{1,\cdots,K\}$, $C_i$ can be constructed recursively as follow: $C_1 = BC(1)$ (the BC-class which contains $1$), $C_2 = BC(\min\{i \in E-C_1\})$,.., $C_K = BC(\min\{i \in E-C_1..-C_{K-1}\})$.*
2. *$\forall j \in \{1,..,K\}$, $\forall B \in \mathbb{S}^+$, if $C_i \cap I_f(C_j,B) \neq \emptyset$, then $C_i \cap I_f(E-\cup_{k\neq j}C_k, B) = \emptyset$.*
3. *$i \sim j$ if and only if $\exists B \in \mathbb{S}^+$ s.t. $I_b(i,B) \cap I_b(j,B) \neq \emptyset$.*
4. *If $i \neq j$, then $\forall n \in \mathbb{Z}$, $\forall \omega \in \Omega$, $I(\omega,C_i,n) \cap I(\omega,C_j,n) = \emptyset$.*
5. *$\forall n \in \mathbb{Z}$, $\forall \omega \in \Omega$, $\forall i$, $\exists j = \sigma_n(i)$, $I(\omega,C_i,n) \subset C_j$ and $I(\omega,C_i,n)$ is not empty. If $n \geq 0$, $I(\omega,C_i,n) = C_{\sigma_n(i)}$.*
6. *$\{I(\omega,i,n)\}_{n\leq -1}$ is a positive recurrent Markov chain. Its positive recurrent states are $S_b$. $\{I(\omega,C_i,n)\}_{n\geq 0}$ is a positive recurrent Markov chain. Its positive recurrent states are the $K$ BC-classes with stationary probability $\pi$ which is independent of $i$.*

*Proof.* 1) The BG-class containing $i$, $BC(i)$, has the following characterization. Due to the point 3 of Lemma 3, we know that all elements of $I(\omega,i,n)$ are in the same class for all $n<0$. Therefore,

$$BC(i) = \cup\{j \mid \{i,j\} \subset J \in S_b\},$$

which is uniquely defined depending only on $\mathbb{S}$.

2) Let $x \in I_f(j,B) \cap C_i$, $j \in C_l$ and $y \in I_f(k,B) \cap C_i$, $k \in C_m$, $l \neq m$. Assume that with probability $p>0$, $\cup_{n<0}\{I(\omega,j,n) \cap I(\omega,k,n)\} = \emptyset$. Then with probability, $\mathbb{P}(f_{-1}\circ\cdots\circ f_{-|B|} = B) \times p > 0$, $I(\omega,x,n) \neq I(\omega,y,n)$, for all $n<0$. This would imply that $x \not\sim y$.

3) Assume that $i \not\sim j$ and $I_b(i,B) \cap I_b(j,B) \neq \emptyset$. Due to the point 2, $\exists k,k'$ s.t. $I_b(BC(i),B) \cup I_b(BC(j),B) \subset C_k$ and $I_b(E,B) \cap C_{k'} = \emptyset$. Then a.s. Card$[I(\omega,E,n)] \searrow 0$

when $n \to -\infty$. But Card$[I(\omega, E, n)] \geq 1$. The converse implication is obvious. This result shows that the BC-classes are deterministic notions that only depend on $\mathbb{S}$.

4) If $i \not\sim j$, $\forall B \in \mathbb{S}^+, \forall k \in E$, $\{i,j\} \not\subset I_b(k,B)$ (Lemma 3, point 3) and $I_b(i,B) \cap I_b(j,B) = \emptyset$ (point 3).

5) For $n \leq -1$: $\forall k \in E$, $I(k, C_i, n)$ is never empty because every coordinate has at least one antecedent on which it depends. The inclusion is a consequence of the point 2.

For $n \geq 0$: the non emptiness and the inclusion are a consequence of the point 3. The fact that every coordinate has at least one antecedent implies the equality.

6) It directly follows from the previous results. □

# 5 Proof of Theorem 1

**Lemma 5.** *The path defined by Definition 1 is transitive.*

*Proof.* If $i \xrightarrow{g} j$ and $j \xrightarrow{g'} k$, then by monotonicity and homogeneity $[g' \circ g(e_i)]_k \geq g'(e_j)_k + g(e_i)_j > -\infty$. □

Let $C_i(n) = C_{\sigma_n(i)}$, $\forall n \in \mathbb{Z}$. For all $J \subset E$, we denote by $[J]$ the vector with 0 on $k \in J$ and $-\infty$ on $k \notin J$. For all $n > 0$ and $m > n$, we define $X^i_n = (f_{n-1} \circ .. \circ f_0 \circ [C_i])_{j \in C_i(n)}$ and $X^i_{mn} = (f_m \circ .. \circ f_n \circ [C_i(n)])_{j \in C_i(m+1)}$.

**Proof of Theorem 1.** It is clear that $\mathsf{t}X^i_{mn}$ is subadditive. If $X^s$ is a stationary version of $X^i$ (i.e. $C_i$ chosen under stationary probability $\pi$), then $\mathsf{t}X^s_{mn}$ is a stationary ergodic subadditive process and from Kingman's theorem:

$$\lim_{n\to\infty} \frac{\mathsf{t}X^s_n}{n} \text{ exists a.s. and in } L_1.$$

But from irreducibility $\forall i \in E$, $\lim_{n\to\infty} \frac{\mathsf{t}X^s_n}{n} = \lim_{n\to\infty} \frac{\mathsf{t}X^i_n}{n} = \gamma_{\mathsf{t}}$. The same result holds for $\gamma_{\mathsf{b}}$.

Choose $j$ s.t. Card$[n : \mathsf{t}X^i_n = (X^i_n)_j] = +\infty$ a.s. Such a $j$ exists, since $E$ is finite. Let $M \in \mathbb{S}^l$ such that $N_f = \text{Card}[I_f(j, M)]$. Let $H = \{n : \mathsf{t}X^i_{n-l} = (X_{n-l})_j \text{ and } C_i(n) = I_f(j, M)\}$. It is clear that Card$[H] = \infty$ a.s. Hence one can construct an increasing subsequence $T_n \in H$ for which we have:

$$0 \leq \mathsf{t}X^i_{T_n} - \mathsf{b}X^i_{T_n} \leq \mathsf{t}M(\mathbf{0}) - \mathsf{b}M(e_j) \leq \mathsf{t}M(\mathbf{0}) - \inf_{i\in E} \mathsf{b}M(e_i) < \infty$$

Therefore $\lim_{n\to\infty} \frac{\mathsf{t}X^i_{T_n} - \mathsf{b}X^i_{T_n}}{T_n} = 0$ a.s.

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