An introduction to Utility Maximization with Partial Observation

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Abstract: We give an overview of the theory and methods involved in portfolio optimization problems with partial observation. By "partial observation", we mean that the drift process and the driving Brownian motion appearing in the stochastic differential equation for the security prices are not directly observable for investors in the market. The history of security prices is assumed to constitute the *only* information available to investors and the investment processes are then required to be adapted to the natural filtration of the price processes. In the complete market case, we obtain explicit formulae for the optimal wealth process in a "Bayesian" context, when the drift vector is an unobserved random variable with known prior distribution; explicit representations for the optimal investment process are only derived when the stock drift is modelled as a Gaussian process. We also consider the case of incomplete market and characterize the optimal investment policies when price process of risky assets follows a stochastic volatility model.

Key-words: Portfolio optimization, filtering, partial observation, utility maximization, dynamic programming, Clark's formula, stochastic volatility.

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Maximisation d'utilité en observation partielle

Résumé : Nous donnons un aperçu de la théorie et des méthodes relatifs aux problèmes d'optimisation de portefeuilles en observation partielle. Par "observation partielle", nous entendons que le processus de drift et le mouvement Brownien apparaissant dans l'équation différentielle stochastique pour les prix des actifs risqués ne sont pas directement observables par les investisseurs dans le marché. L'histoire des prix des actifs risqués est supposée constituer la *seule* information disponible aux investisseurs et on exige des processus d'investissement qu'ils soient adaptés à la filtration engendrée par le processus des prix. Dans le cas des marchés complets, nous obtenons des formules explicites pour le processus de richesse optimale dans le contexte d'un investisseur Bayésien, lorsque le drift est une variable aléatoire de loi initiale connue; des formules explicites pour le processus d'investissement optimal ne sont présentées que lorsque le drift est modélisé par un processus Gaussien. Nous considérons aussi le cas des marchés incomplets et caractérisons les politiques d'investissement optimales lorsque le processus de prix des actifs risqués suit un modèle de volatilité stochastique.

Mots-clés : optimisation de portefeuilles, filtrage, observation partielle, programmation dynamique, maximisation d'utilité, formule de Clark, volatilité stochastique

1 Introduction

The problem of maximizing the expected utility from terminal wealth has been largely studied in the literature. For the case of complete markets, we refer to Karatzas and al. [11], Cox and Huang [3] or Ocone and Karatzas [23]. For the case of incomplete and/or constrained markets, we refer to Karatzas and al. [12], He and Pearson [9] and Cvitanić and Karatzas [4]. A salient feature in the above papers is the assumption of *complete observations* : investors have a complete knowledge of all the parameters involved in the stochastic differential equation for the asset prices. However, in practical situations one cannot observe these quantities directly : the quadratic variation and hence the diffusion coefficient of the stocks price process can be estimated precisely from the sample path, but neither the drift vector nor the underlying Brownian motion. Since prices are published and available to the public, it seems reasonable to consider a market model where economic agents can observe stock prices *only* : as the underlying price process evolves, investors observe the outcomes and thus obtain information about the true value of the stochastic parameters. In this framework, investment processes are required to be adapted to the filtration generated by the security prices, so that investor's portfolio choices should be affected by the information contained in the stock prices only. This situation is called the case of *partial observations* to distinguish it from the case of *complete observations* (where the variables which characterize the state of the financial market are all assumed directly observable).

The purpose of this work is to give an overview of the papers related with the standard portfolio problem in the case of partial observations. In section 2, we describe the market model and set the expected utility from terminal wealth maximization problem. Section 3 presents the solution when the market is complete. In subsection 3.1, the model with partial observations is transformed into one with complete observations, where all quantities involved are adapted to the natural filtration of the security prices. The well-known methods in the complete observations context can then be applied and for later comparison, we recall in subsection 3.2 the general results of Karatzas, Lehoczky and Shreve

[11]. In subsection 3.3, using the martingale approach of Lakner [18], we characterize the optimal wealth process and the optimal investment policies of the portfolio problem. This involves a process ξ which is the conditional expectation given the available information of the Radon-Nikodym derivative of the martingale measure with respect to the original probability measure.

In subsection 3.4, we show how to compute ξ in a "Bayesian" context, when the drift vector is an unobserved random variable with known prior distribution. Using the methodology developed in [18], we give several representations of the process ξ and derive very explicit formulae for the optimal wealth process. Let us mention that Karatzas and Zhao [16] also solve the portfolio optimization problem for a Bayesian investor but in a more specialized context, when the drift vector is additionaly assumed to be independent of the Brownian motion process which drives the assets prices. The special case of logarithmic utility function and normal prior distribution for the drift process was addressed by Browne and Whitt [2].

Subsection 3.5 is devoted to the computation of the process ξ in the general setting of our portfolio optimization problem. In [19], Lakner gives an explicit representation for ξ_t but his formula involves a process m which is the conditional expectation of the drift vector μ given the available information. In subsection 3.6, we specialize the dynamics of μ_t which allows us to compute m_t by means of the Kalman-Bucy filter. Using Clark's formula, we then obtain explicit representations for the optimal investment policies in terms of the previously described processes ξ and m and the deterministic conditional covariance function of μ_t. Particular details will be worked out for the logarithmic and power type utility functions. With the logarithmic utility function, the optimal portfolio can be formally derived from the complete observations case by replacing the unobservable drift process μ_t by its best estimate m_t : this property corresponds to the so-called separation principle. It is actually proved in Kuwana [17] that in the Merton type of optimal/consumption problem in a Black-Scholes market with unobservable drift vector, the separation principle holds if and only if the utility functions are logarithmic.

In section 4, we treat the case of incomplete markets. In this framework, Karatzas and Xue [15] study the problem of maximizing expected utility from consumption but with a restrictive assumption on the diffusion matrix which implies that the methodologies developed in section 3 for the complete market case are applicable. We then consider the portfolio optimization problem adressed by Pham and Quenez [26] where the stock prices follow a stochastic volatility model. In subsection 4.1, we use results from stochastic filtering theory to reduce the optimization problem with partial observations into one with complete observations. The well-developed martingale duality approach of [4] and [12] allows us to get explicit formulae providing the optimal investment policies; this task is carried out in subsection 4.2 where, in addition, the special case of logarithmic and power type utility functions is studied.

2 Financial markets

2.1 The model

Consider the following continuous time model for a financial market consisting of one risk free bond (or bank account) whose price process is assumed for simplicity to be equal 1 at each time t and n risky securities (stocks) with the price process $S = \{S_t = (S_1(t), \cdots, S_n(t))^*; t \in [0,T]\}$ governed by :

$$dS_i(t) = S_i(t)\Big[\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\Big] \tag{2.1}$$

$$S_i(0) = s_i \in \Re^+ \ . \tag{2.2}$$

Here $W = \{W_t = (W_1(t), \cdots, W_n(t))^* : t \in [0,T]\}$ is a d-dimensional Brownian motion (with possibly $d \geq n$) on a probability space $(\Omega, \mathcal{F}, P)$. We denote by $\{\mathcal{F}_t\}$ the P- augmentation of the filtration generated by W.

In equation (2.1), the drift (vector of the appreciation rates) $\mu = \{\mu_t = (\mu_1(t), \cdots, \mu_n(t); t \in [0,T]\}$ and the volatility matrix $\sigma = (\sigma_{ij}(t))_{1\leq i,j\leq n}$ are supposed to be measurable, adapted processes with respect to $\{\mathcal{F}_t\}$.

We introduce $\mathcal{F}^S = \{\mathcal{F}_t^S; t \in [0,T]\}$ the P- augmented filtration generated by the price process S; intuitively, $\{\mathcal{F}_t\}$ records the history of the market up to time t whereas $\{\mathcal{F}_t^S\}$ represents the information collected by observing asset prices up to time t.

From now on, we shall assume that only $\mathcal{F}^S$- adapted processes are observable, that is, we suppose that investors in this financial market model observe stock prices only. We call this situation the case of partial observation or partial information to distinguish it from the case of "full information" where economic agents have a continuous access to the entire history of the market represented by the filtration $\{\mathcal{F}_t\}$.

2.2 Portfolios and wealth processes

Let us consider an investor who can invest in the $(n+1)$ basic securities.

Definition 2.1 *A trading strategy for this agent is defined as a n-dimensional, measurable, $\mathcal{F}^S$-adapted process $\pi = \{\pi_t = (\pi_1(t), \cdots, \pi_n(t); t \in [0,T]\}$ such that:*

$$\int_0^T ||\sigma_t^* \pi_t||^2 \, dt < +\infty \; a.s.. \tag{2.3}$$

We regard $\pi_i(t)$ as the amount the agent invests in the i^{th} stock at date t.

Note that the fact that investors have only partial information is modelled by requiring that π is adapted to the filtration generated by the stock prices. In other words, in pursuing investment decisions, the agent has at his disposal information about the evolution of the asset prices but not about the entire history of the market.

We shall denote by $X_t^{\pi,x}$ the value of the portfolio corresponding to the investment process π and the initial investment x. It is given by :

$$X_t^{\pi,x} = x + \int_0^t \pi_u^* \mu_u \, du + \int_0^t \pi_u^* \sigma_u \, dW_u \; . \tag{2.4}$$

Definition 2.2 *An investment process* π *is called admissible (for the maximization of the expected utility from terminal wealth problem) if :*

$$X_t^{\pi,x} \geq 0, a.s., 0 \leq t \leq T \; .$$

We denote by $\mathcal{A}(x)$ *the class of admissible investment processes.*

2.3 The utility maximization problem

The agent has a utility function $U : [0, \infty) \rightarrow \Re \bigcup \{-\infty\}$ such that it is strictly increasing, strictly concave, continuously differentiable and satisfies :

$$\lim_{x \rightarrow +\infty} U'(x) = 0 \tag{2.5}$$

The pseudo-inverse function of the derivative of U will be denoted by I : $(0, \infty) \rightarrow \Re$ and is defined as follows :

$$I(y) = \inf\{x \geq 0 \, ; \, U'(x) \leq y\} \, . \tag{2.6}$$

Note that the function I actually becomes the inverse function of U' if $\lim_{x \rightarrow 0} U'(x) = \infty$ and by analogy with (2.5), it satisfies

$$\lim_{x \rightarrow \infty} I(y) = 0 \, . \tag{2.7}$$

Note that :

$$U(I(y)) \geq U(x) + y[I(y) - x] \; , \quad \forall x, y > 0 \, . \tag{2.8}$$

We address the following problem :

Problem 2.3 *For a given utility function* U *and a given initial capital* $x > 0$*, maximize the expected utility from terminal wealth,*

$$E[U(X_T^{\pi,x})] \tag{2.9}$$

over the class $\mathcal{A}'(x) = \{\pi \in \mathcal{A}(x); E[U^-(X_T^{\pi,x})] < +\infty\}$.

The value function of this problem is :

$$V(x) = \sup_{\pi \in \mathcal{A}'(x)} E[U(X_T^{\pi,x})] . \tag{2.10}$$

Definition 2.4 *A portfolio process $\hat{\pi} \in \mathcal{A}'(x)$ which attains the supremum in (2.10) is called optimal.*

3 The case of complete markets

Throughout this section, the number of risky assets is the same as the dimension of the driving Brownian motion W, that is, $n = d$ and the volatility coefficient σ_t is a constant invertible matrix σ.

3.1 Reduction to a completely observed model

Consider the positive local martingale $Z = \{Z_t; t \in [0,T]\}$ defined by the following equation :

$$dZ_t = -(\sigma^{-1}\mu_t)^* Z_t dW_t \ , \quad Z_0 = 1 \ . \tag{3.1}$$

We make the following assumption throughout this section :

Assumption 3.1 *The process Z is a $(P, \{\mathcal{F}_t\})$-martingale.*

We then define $\tilde{P}$ as the probability measure equivalent to P on $\mathcal{F}_T$ that admits the Radon-Nikodym derivative $Z(T)$:

$$\frac{d\tilde{P}}{dP} = Z(T) \ .$$

We also introduce the n-dimensional process
$\tilde{W} = \{\tilde{W}_t = (\tilde{W}_1(t), \cdots, \tilde{W}_n(t)); t \in [0,T]\}$ defined by :

$$\tilde{W}_t = W_t + \int_0^t \sigma^{-1}\mu_u \, du \ . \tag{3.2}$$

By Girsanov's theorem, $\tilde{W}$ is a Brownian motion under the new probability measure $\tilde{P}$ with respect to the filtration $\{\mathcal{F}_t\}$.

We now come to the following key-point which states that S and $\tilde{W}$ generate the same filtration. We define the n-dimensional return process $R = \{R_t = (R_1(t), \cdots, R_n(t)); t \in [0,T]\}$ by

$$dS_i(t) = S_i(t) dR_i(t)\ , \quad i = 1, \cdots, n\,. \tag{3.3}$$

We have the following decompositions for the return process :

$$dR_t = \mu_t dt + \sigma dW_t \tag{3.4}$$

and

$$dR_t = \sigma d\tilde{W}_t\,. \tag{3.5}$$

Relations (3.3) and (3.5) imply then that S,R and $\tilde{W}$ generate the same filtration so that $\mathcal{F}^S$ is continuous. Moreover, note that we can use the return process R of (3.3) as the observation process.

The wealth equation (2.4) can be rewritten in terms of the Brownian motion $\tilde{W}$ as

$$X_t^{\pi,x} = x + \int_0^t \pi_u^* \sigma\, d\tilde{W}_u\ . \tag{3.6}$$

Problem 2.3 is thus equivalent to :
Maximize

$$E[U(X_T^{\pi,x})]$$

subject to

$$dX_t^{\pi,x} = \pi_u^* \sigma\, d\tilde{W}_u\,, \quad X_0^{\pi,x} = x\ ,$$

over all $\mathcal{F}_t^S$-adapted policies $\pi_t \in \mathcal{A}'(x)$.

This approach will be convenient (see section 3.3) for solving our partially observed optimization problem.

However, there is a more systematic way to reduce the optimization Problem 2.3 with partial information to the case of a model where all coefficients are adapted to the observation process. This methodology is useful to provide more explicit formulae for the optimal investment process and the optimal wealth. To this purpose, consider the process $\bar{W} = \{\bar{W}_t = (\bar{W}_1(t), \cdots, \bar{W}_n(t)); t \in [0,T]\}$ defined by :

$$\bar{W}_t = \tilde{W}_t - \int_0^t \sigma^{-1} E[\mu_u | \mathcal{F}_u^S] \, du \,, \tag{3.7}$$

where $\bar{W}$ is the "innovation process" in the filtering theory. An intuitive interpretation of the terminology "innovation process" is the following : By means of the relation,

$$d\bar{W}_t = \bar{W}_{t+dt} - \bar{W}_t = \tilde{W}_{t+dt} - \tilde{W}_t - \sigma^{-1} E[\mu_t | \mathcal{F}_t^S] \, dt$$

we note that $d\bar{W}_t$ is the "innovation" part of the new observation obtained between t and $t + dt$, because it is the difference between the new observation and what we expect to observe given the past observations.

By classical results in filtering theory (see for example Bensoussan [1], Kallianpur [10] or Pardoux [24]), $\bar{W}$ is a $\mathcal{F}^S$-brownian motion so that we can now describe the dynamics of the partially observed model within a framework of a complete observation model, where all quantities involved are adapted to the observation process R :

$$dS_t = S_t [E[\mu_t | \mathcal{F}_t^S] dt + \sigma \, d\bar{W}_t] \,. \tag{3.8}$$

However, the price we pay for this is that the constant μ in (2.1) is replaced by the more complicated process $E[\mu_t | \mathcal{F}_t^S]$.

In the next section, we briefly recall the solution for Problem 2.3 in the context of full information, both for easy reference and later comparison in the treatment of the partial information case.

3.2 Utility maximization problem in the case of complete information

We thus assume that the drift process μ and the driving Brownian Motion W appearing in the stochastic differential equation (2.1) for the security prices are observable for investors in the market and that the trading strategies are adapted to the filtration $\{\mathcal{F}_t\}$.

We recall the solution of Problem 2.3 in the case of full information (see Karatzas, Lehoczky and Shreve [11]).

Theorem 3.2 *Suppose that for every constant $y \in (0, \infty)$,*

$$E[Z_T I(yZ_T)] < \infty \tag{3.9}$$

Then the optimal level of terminal wealth is

$$X_T^{\hat{\pi},x} = I(\hat{y} Z_T) \tag{3.10}$$

where the constant $\hat{y}$ is uniquely determined by

$$E[Z_T I(\hat{y} Z_T)] = x \; . \tag{3.11}$$

The optimal wealth process $X^{\hat{\pi},x}$ and the optimal investment process $\hat{\pi}$ are implicitly determined by :

$$X_t^{\hat{\pi},x} = \frac{1}{Z_t} E[Z_T I(yZ_T) | \mathcal{F}_t] = x + \int_0^t \hat{\pi}_u^* \sigma \, d\tilde{W}_u \tag{3.12}$$

3.3 Maximization of expected utility from terminal wealth

In section 3.1, we have transformed our market model into one with complete observations. This way we can apply the methodology for the case when full information is available to economic agents to our case of restricted information. Let us notice that in the formulae (3.10)–(3.12) providing the optimal level of terminal wealth and the optimal investment process in the case of full information, the process Z enters in a prominent fashion. In the case of restricted information, it seems natural to project the $(P, \{\mathcal{F}_t\})$-martingale Z to the available information at time t and hence, to consider the process $\xi = \{\xi_t; t \in [0,T]\}$ defined by :

$$\xi(t) = E[Z_t|\mathcal{F}_t^S] = E[E[Z_T|\mathcal{F}_t]|\mathcal{F}_t^S] = E[Z_T|\mathcal{F}_t^S]\,. \tag{3.13}$$

Note that ξ is a $(P, \mathcal{F}^S)$-martingale with $E[\xi_T] = 1$.

The following relations from stochastic filtering theory will be useful in the sequel of this paper :

For every $\mathcal{F}_t^S$-measurable random variable V, $\mathcal{F}_u$-measurable random variable Y, and $\mathcal{F}_u^S$-measurable random variable W with $0 \le t \le u \le T$, we have :

$$\tilde{E}[V] = E[\xi_t V]\,, \tag{3.14}$$

$$\tilde{E}[Y|\mathcal{F}_t^S] = \frac{1}{\xi_t} E[Z_u Y|\mathcal{F}_t^S]\,, \tag{3.15}$$

and

$$\tilde{E}[W|\mathcal{F}_t^S] = \frac{1}{\xi_t} E[\xi_u W|\mathcal{F}_t^S]\,. \tag{3.16}$$

By taking $W = \frac{1}{\xi_u}$, the last identity implies that $\frac{1}{\xi}$ is a $(\tilde{P}, \mathcal{F}^S)$-martingale. Since $\mathcal{F}^S$ is generated by $\tilde{W}$, $\frac{1}{\xi}$, and also ξ, must have a continuous version.

Lemma 3.3 *For every* $\pi \in \mathcal{A}'(x)$,

$$\tilde{E}[X_T^{\pi,x}] \le x\,. \tag{3.17}$$

Proof :

The substitution of the expression (3.2) verified by $\tilde{W}$ in (2.4) leads to :

$$X_t^{\pi,x} = x + \int_0^t \pi_u^* \sigma \, d\tilde{W}_u\,. \tag{3.18}$$

This shows, in particular, that the process $X^{\pi,x}$ is a nonnegative $(\tilde{P},\mathcal{F}^S)$ local martingale, hence by Fatou's Lemma, a supermartingale. It satisfies then :

$$\tilde{E}[X_T^{\pi,x}] \leq \tilde{E}[X_0^{\pi,x}] = x \; . \tag{3.19}$$

□

We introduce the function $\Xi : (0,\infty) \to (0,\infty]$ defined by

$$\Xi(y) = E[\xi_T I(y\xi_T)] = \tilde{E}[I(y\xi_T)] \, . \tag{3.20}$$

Lemma 3.4 *If*

$$\Xi(y) < \infty \quad \textit{for every } y \in (0,\infty) \tag{3.21}$$

then there exists a unique constant $\hat{y} \in (0,\infty)$ such that :

$$\Xi(\hat{y}) = x \, . \tag{3.22}$$

Proof : Under the assumption (3.21), the function Ξ inherits from I the property of being a continuous, strictly decreasing mapping of $(0,\infty)$ into $(0,\infty)$. Consequently, there exists a unique constant $\hat{y} \in (0,\infty)$ such that $\Xi(\hat{y}) = x$. □

We define :

$$\zeta^x = I(\hat{y}\xi_T) \; . \tag{3.23}$$

Lemma 3.5 *The random variable ζ^x satisfies*

$$E[\xi_T \zeta^x] = \tilde{E}[\zeta^x] = x \, , \tag{3.24}$$

$$E[(U(\zeta^x))^-] < \infty \; , \tag{3.25}$$

and for every portfolio $\pi \in \mathcal{A}'(x)$, we have :

$$E[U(X_T^{\pi,x})] \leq E[U(\zeta^x)] \; . \tag{3.26}$$

Proof : From the definitions of ζ^x and Ξ, it follows that :

$$\tilde{E}[\zeta^x] = E[\xi_T I(\hat{y}\xi_T)] = \Xi(\hat{y}) = x \,, \tag{3.27}$$

so we have proved the statement (3.24) of the lemma.

Next, inequality (2.8) implies that

$$\begin{aligned} U(\zeta^x) &\geq U(1) + \hat{y}\xi_T(\zeta^x - 1) \\ &\geq -|U(1)| - \hat{y}\xi_T \end{aligned} \tag{3.28}$$

since

$$U(1) + \hat{y}\xi_T\zeta^x \geq -|U(1)| \;.$$

Hence, we obtain

$$E[(U(\zeta^x))^-] \leq |U(1)| + \hat{y}E[\xi_T] \tag{3.29}$$

and (3.25) follows by the martingale property of the process ξ.

Finally, let $\pi \in \mathcal{A}'(x)$ be a portfolio satisfying $E[(U(X_T^{\pi,x}))^-] < \infty$. Using (2.8),(3.24) and the supermartingale property of the process $\xi X^{\pi,x}$, we get

$$\begin{aligned} E[U(\zeta^x)] &\geq EU(X_T^{\pi,x}) + E[\hat{y}\xi_T(\zeta^x - X_T^{\pi,x})] \\ &\geq E(U(X_T^{\pi,x})) \,, \end{aligned} \tag{3.30}$$

Hence, (3.26) holds and our proof is complete.

□

Remark 3.6 *From Lemma 3.5, we infer that if there exists a portfolio* $\hat{\pi} \in \mathcal{A}'(x)$ *such that* $\zeta^x = X_T^{\hat{\pi},x}$*, then* $\hat{\pi}$ *is optimal. The existence of such an investment process* $\hat{\pi}$ *is adressed in the following proposition.*

Proposition 3.7 *Suppose that* $\Xi(y) < \infty$ *for every* $y \in (0, \infty)$.

Then there exists a unique (up to equivalence) investment process $\hat{\pi} \in \mathcal{A}'(x)$ *such that :*

$$X_T^{\hat{\pi},x} = \zeta^x \,. \tag{3.31}$$

The corresponding wealth process is

$$X_t^{\pi,x} = \tilde{E}[\zeta^x|\mathcal{F}_t^S] \,. \tag{3.32}$$

Proof : Consider the positive $(\tilde{P}, \mathcal{F}_t^S)$-martingale

$$M_t = \tilde{E}[\zeta^x|\mathcal{F}_t^S] \,. \tag{3.33}$$

By the representation theorem for Brownian Motion ([14], p.184), M admits the stochastic integral representation :

$$M_t = M_0 + \int_0^t \psi_u^* \, d\tilde{W}_u \tag{3.34}$$

for some $\mathcal{F}^S$-adapted process ψ satisfying :

$$\int_0^T ||\psi_s||^2 \, ds < +\infty \,, \quad a.s. \tag{3.35}$$

The process $\hat{\pi}_t \equiv (\sigma^{-1})^* \psi_t$ is a n-dimensional, measurable, $\mathcal{F}_t^S$-adapted process such that :

$$\int_0^T ||\sigma^* \hat{\pi}_t||^2 \, dt = \int_0^T ||\psi_t||^2 \, dt < +\infty \,, \quad a.s.$$

Hence, the process $\hat{\pi}$ is a investment process according to definition 2.1 and :

$$M_t = M_0 + \int_0^t \hat{\pi}_u^* \sigma \, d\tilde{W}_u \,. \tag{3.36}$$

By virtue of the identity $M_0 = \tilde{E}(\zeta^x)$ and (3.24), it follows that $X_t^{\hat{\pi},x} = x + M_t - M_0 = M_t$, hence $\hat{\pi} \in \mathcal{A}'(x)$ and satisfies (3.31)-(3.32).

We then have to show the uniqueness (up to equivalence) of $\hat{\pi}$. Assume by contradiction, that $\hat{\pi}_i$ are investment processes satisfying (3.31)-(3.32) and such that $\hat{\pi}_i \in \mathcal{A}'(x)$ for $i = 1, 2$. Let N_i be the $(\tilde{P}, \mathcal{F}^S)$-local martingales defined by

$$N_i(t) = \int_0^t \hat{\pi}_i^*(u)\sigma \, d\tilde{W}_u = X_t^{\hat{\pi}_i, x} - x \ . \tag{3.37}$$

It appears that N_i is bounded from below by $-x$ thus by Fatou's lemma, N_i is a $(\tilde{P}, \mathcal{F}^S)$-supermartingale for $i = 1, 2$.

Equations (3.37) and (3.24) imply that $\tilde{E}[N_i(T)] = \tilde{E}[X_T^{\hat{\pi}_i, x}] - x = \tilde{E}[\zeta^x] - x = 0$. Therefore N_i is a $(\tilde{P}, \mathcal{F}^S)$-martingale for $i = 1, 2$. Using (3.31), we infer that $X_T^{\hat{\pi}_1, x} - X_T^{\hat{\pi}_2, x} = N_1(T) - N_2(T) = 0$, thus the martingale $N_1 - N_2$ is zero and its quadratic variation is also zero. Therefore,

$$\int_0^T ||(\hat{\pi}_1(u) - \hat{\pi}_2(u))^* \sigma||^2 \, du = 0$$

and the equivalence of $\hat{\pi}_1$ and $\hat{\pi}_2$ follows. □

In conclusion, we have proved the following result :

Theorem 3.8 *Suppose that $\Xi(y) < \infty$ for every $y \in (0, \infty)$. Then the optimal terminal wealth is given by :*

$$\zeta^x = I(\hat{y}\xi_T), \tag{3.38}$$

where $\hat{y}$ is the constant from (3.22). The unique optimal investment process $\hat{\pi}$ and the corresponding wealth process $X^{\hat{\pi}, x}$ satisfy the equation

$$\tilde{E}\left[X_T^{\hat{\pi}, x} \middle| \mathcal{F}_t^S\right] = X_t^{\hat{\pi}, x} = x + \int_0^t \hat{\pi}_u^* \sigma \, d\tilde{W}_u \ . \tag{3.39}$$

In the previous theorem, a formula is derived for the optimal level of terminal wealth but the optimal investment process $\hat{\pi}$ is only implicitely determined by equation (3.39). Therefore, the main objective now on will be to work out an explicit representation for the optimal investment process. Since the process ξ plays an important role in the equations (3.38) and (3.39), it is necessary to find a way to compute it. This was done by Lakner [19] who derived an explicit formula for the process ξ involving the conditional expectation of the drift μ_t given $\mathcal{F}_t^S$. However, for the Bayesian case, we don't need such developments and expose in the next section a direct approach for computing the process ξ : this allows us to present an explicit formula for the optimal level of terminal wealth but we can only give an explicit representation for the optimal investment process when the utility function is logarithmic.

3.4 A Bayesian problem : explicit computation of ξ.

In this section, we suppose that $\mu = (\mu_1, \cdots, \mu_n)^*$ is a n-dimensional, $\mathcal{F}_0$-measurable random variable with a known distribution ν. This situation is usually refered as the "Bayesian" case. This is the model studied by Lakner [18] and we will follow his methodology to compute the process ξ.

Remark 3.9 *For the sake of completeness, we mention that Karatzas and Zhao [16] solved Problem 2.3 for a Bayesian investor but only in the special case where μ is independent of the process W under P and σ is the identity matrix.*

Our objective is to give an explicit representation for the process ξ using the special structure of the drift μ. We need the conditional distribution of μ given $\mathcal{F}_t^S$, $\nu(.|\mathcal{F}_t^S)$, under the probability measure P. It is given by (see Appendix, Lemma A.1 of [18]) :

$$\nu(B|\mathcal{F}_t^S) = \frac{\int_B H(t, \tilde{W}_t, b)\, \nu(db)}{\int_{\Re^n} H(t, \tilde{W}_t, b) \nu(db)}, \ B \in \mathcal{B}(\Re^n), \tag{3.40}$$

where the mapping $H : [0,T] \times \Re^n \times \Re^n \to (0, \infty)$ is defined by :

$$H(t, x, b) = \exp\Big((\sigma^{-1}b)^* x - \frac{1}{2}||\sigma^{-1}b||^2 t\Big), \tag{3.41}$$

and is such that :

$$Z_t = \Big(H(t, \tilde{W}_t, \mu)\Big)^{-1} . \tag{3.42}$$

We thus have :

$$E\left[\mu_i|\mathcal{F}_t^S\right] = \frac{\int_{\Re^n} H(t,\tilde{W}_t,b)b_i\,\nu(db)}{\int_{\Re^n} H(t,\tilde{W}_t,b)\,\nu(db)},\ i=1,\cdots,n. \tag{3.43}$$

whenever $E[|\mu_i|]<\infty$.

In what follows, we give several representations of the process ξ.

Proposition 3.10 *The process ξ is given by the following explicit representation :*

$$\xi_t = \left(\int_{\Re^n} H(t,\tilde{W}_t,b)\,\nu(db)\right)^{-1}, \tag{3.44}$$

where

$$\tilde{W}_j(t) = \sum_{i=1}^n \int_0^t (\sigma^{-1})_{ij} S_j^{-1}(u)\,dS_j(u),\ j=1,\cdots,n.$$

Proof : We first determine a regular conditional distribution for $(\tilde{W}_t,\mu)$ given $\mathcal{F}_t^S$: it is easily feasible since for every $A,B\in\mathcal{B}(\Re^n)$,

$$P(\tilde{W}_t\in A,\mu\in B) = 1_{\{\tilde{W}_t\in A\}}\,\nu(B|\mathcal{F}_t^S)\ . \tag{3.45}$$

It remains to extend the right-hand side of (3.45) from the class of measurable rectangles of $\mathcal{B}(\Re^n\times\Re^n)$ to the entire $\mathcal{B}(\Re^n\times\Re^n)$.

By (3.42), we can compute the conditional expectation of Z_t given $\mathcal{F}_t^S$ by integrating $\Big(H(t,.,.)\Big)^{-1}$ with respect to the conditional distribution for $(\tilde{W}_t,\mu)$ given $\mathcal{F}_t^S$. This gives the formula (3.44) for the process ξ, using (3.45) and (3.40).

□

Proposition 3.11 *Suppose that for every constant $K_1>0$,*

$$E[||\mu||^2\exp\{K_1||\mu||\}] < +\infty. \tag{3.46}$$

Then we have the following representation :

$$\xi_t^{-1} = 1 + \int_0^t \int_{\Re^n} H(u, \tilde{W}_u, b)(\sigma^{-1}b)^* \nu(db) d\tilde{W}_u \tag{3.47}$$

Furthermore, ξ^{-1} satisfies the following linear stochastic differential equation :

$$\xi_t^{-1} = 1 + \int_0^t \xi_u^{-1} (\sigma^{-1} E[\mu|\mathcal{F}_u^S])^* \, d\tilde{W}_u \tag{3.48}$$

and, in addition to (3.47), ξ also has the following representation :

$$\xi_t = \exp\Big(- \int_0^t (\sigma^{-1} E[\mu|\mathcal{F}_u^S])^* \, d\tilde{W}_u + \frac{1}{2} \int_0^t ||\sigma^{-1} E[\mu|\mathcal{F}_u^S]||^2 \, du \Big). \tag{3.49}$$

Proof : (3.48) follows from (3.47) using (3.43) and (3.44), and (3.49) is an straightforward consequence of (3.48) by Ito's rule; thus all we have to show is statement (3.47).

Define the mapping $J : [0, T] \times \Re^n \to \Re$ by :

$$J(t, x) = E[H(t, x, \mu)], \tag{3.50}$$

which is finite since

$$\begin{aligned} E[H(t, x, \mu)] &\leq E[\exp\{(\sigma^- 1\mu)^* x\}] && (3.51) \\ &\leq \exp(||\sigma^{-1}|| \, ||x||) + E[||\mu||^2 \exp\{||\sigma^- 1|| \, ||\mu|| \, ||x||\}] \\ &< +\infty \, , \end{aligned}$$

where we have used assumption (3.46) with $K_1 = ||\sigma^{-1}|| \, ||\mu||$. We refer the reader to the Appendix, Lemma A.2. of [18] for the proof that J is in $\mathcal{C}^{1,2}$ and that the expectation and differentiation are exchangeable, i.e.,

$$\frac{\partial}{\partial x_i} J(t, x) = E\Big[\frac{\partial}{\partial x_i} H(t, x, \mu)\Big] \, , \quad i = 1, \cdots, n \, , \tag{3.52}$$

$$\frac{\partial^2}{\partial^2 x_i x_j} J(t,x) = E\Big[\frac{\partial^2}{\partial^2 x_i x_j} H(t,x,\mu)\Big] \ , \quad i,j = 1,\cdots,n \ , \tag{3.53}$$

and

$$\frac{\partial}{\partial t} J(t,x) = E\Big[\frac{\partial}{\partial t} H(t,x,\mu)\Big] \ , \quad t \in [0,T] \ . \tag{3.54}$$

By (3.44), it is easy to see that for every $t \in [0,T]$,

$$\xi_t^{-1} = J(t,\tilde{W}_t) \ . \tag{3.55}$$

thus Ito's rule applied to $J(t,\tilde{W}_t)$ leads to :

$$d\xi_t^{-1} = \frac{\partial}{\partial t} J(t,\tilde{W}_t)dt + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} J(t,\tilde{W}_t) d\tilde{W}_i(t) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} J(t,\tilde{W}_t)dt \ . \tag{3.56}$$

By virtue of (3.52),(3.53) and (3.54), the right-hand side of (3.56) can be written under the form :

$$\sum_{i=1}^{n} E[H(t,\tilde{W}_t,\mu)(\sigma^{-1}\mu)_i] d\tilde{W}_i(t) \ , \tag{3.57}$$

where we have used the identity

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_i^2} = 0 \ .$$

Equation (3.47) follows since $J(0,\tilde{W}(0)) = 1$. □

Remark 3.12 *Note that assumption (3.46) is satisfied when, for example, μ follows a n-dimensional normal distribution.*

The previous results allow explicit computation of the optimal terminal wealth. Plugging relation (3.44) for ξ_t to (3.38), we obtain the following expression under the assumption that $\Xi(y) < \infty$ for $y \in (0, \infty)$:

$$\zeta^x = I\left(\hat{y}\left(\int_{\Re^n} H(T, \tilde{W}_T, b)\nu(db)\right)^{-1}\right) \tag{3.58}$$

where $\hat{y}$ is the constant given by (3.22).

Example 3.13 *: Logarithmic utility function $u(x) = \log x$.*

In this case, $I(y) = \frac{1}{y}$ and the function Ξ introduced in (3.20) is given by $\Xi(y) = \frac{1}{y}$. Thus $\Xi(y) < \infty$ for every $y \in (0, \infty)$ and $\hat{y} = \frac{1}{x}, x > 0$. The optimal terminal wealth of (3.38) takes the simple form

$$\zeta^x = x\xi_T^{-1} , \tag{3.59}$$

and from (3.39), we get

$$X_t^{\hat{\pi},x} = x\xi_t^{-1} = x + \int_0^t \hat{\pi}_u^* \sigma \, d\tilde{W}_u , \tag{3.60}$$

using the $(\tilde{P}, \mathcal{F}^S)$-martingale property of the process ξ^{-1}. By using representation (3.48) of ξ^{-1}, we provide another expression for the wealth process $X^{\hat{\pi},x}$:

$$X_t^{\hat{\pi},x} = x + x\int_0^t \xi_u^{-1}(\sigma^{-1}E[\mu|\mathcal{F}_u^S])^* \, d\tilde{W}_u , \tag{3.61}$$

under the additional assumption (3.46).

Identifying (3.60) and (3.61) gives an explicit formula for the optimal investment process

$$\hat{\pi}_i(t) = X_t^{\hat{\pi},x} S_i^{-1}(t) \sum_{j=1}^{n} \left((\sigma\sigma^*)^{-1}\right)_{ij} E[\mu_j|\mathcal{F}_t^S] , \tag{3.62}$$

which can be rewritten, using (3.43), in the following form :

$$\hat{\pi}_i(t) = xS_i^{-1}(t)\sum_{j=1}^{n}\Big((\sigma\sigma^*)^{-1}\Big)_{ij}\int_{\Re^n} H(t,\tilde{W}_t,b)b_j\nu(db)\ . \tag{3.63}$$

Remark 3.14 *It appears that in (3.62), μ enters only through its conditional expectation given $\mathcal{F}_t^S$. Therefore, the optimal portfolio process can be formally derived by writing the corresponding formulae with deterministic μ, i.e. in the complete observation case, and then replace μ by its best estimate $E[\mu|\mathcal{F}_t^S]$. This property corresponds to the so-called separation principle.*

By the separation principle, we mean the following : Let $S(\theta)$ denote the optimal control in the complete observation case with known parameter θ. Then $S(\hat{\theta})$ is the optimal control in the partial observation case, where $\hat{\theta} = E[\theta|\mathcal{F}_t^S]$ is the estimate of θ based on the observation $\mathcal{F}_t^S$. It is known that the separation principle holds in the quadratic linear partial observation control problems. On the other hand, it is actually proved in Kuwana [17] that in the Merton type of optimal/consumption problem in a Black-Scholes market with unobservable drift vector, the separation principle holds if and only if the utility functions are logarithmic.

Remark 3.15 *Browne and Whitt [2] solved the problem of maximizing a logarithmic utility function for a Bayesian investor where μ follows a one-dimensional normal distribution with mean f and variance l^2. In this special case, our formula for the optimal investment process takes the simpler form*

$$\hat{\pi}(t) = S_t^{-1}X_t^{\hat{\pi},x}\frac{l^2\tilde{W}_t + f\sigma}{\sigma(l^2t+\sigma^2)}, \tag{3.64}$$

where $X_t^{\hat{\pi},x}$ is determined by (3.60) with

$$\xi_t^{-1} = \frac{|\sigma|}{\sqrt{l^2t+\sigma^2}}\exp\Big(-\frac{f^2}{2l^2}+\frac{(l^2\tilde{W}_t+f\sigma)^2}{2l^2(l^2t+\sigma^2)}\Big)\ . \tag{3.65}$$

Example 3.16 *: Power utility function $u(x) = \frac{x^\alpha}{\alpha}$.*

We assume that (3.46) holds and that μ is bounded, i.e., there exists a constant $K_2 > 0$ such that :

$$||\mu||^2 \leq K_2 \text{ , } a.s. \tag{3.66}$$

We have $u'(x) = x^{\alpha-1}$ and $I(y) = y^{-\beta}$ with $\beta = \frac{1}{1-\alpha}$. The function defined in (3.20) takes the form $\Xi(y) = \hat{y}^{-\beta}\tilde{E}[(\xi_T)^{-\beta}]$.

First, we show that $\Xi(y) < \infty$, for every $y \in (0, \infty)$. From (3.41), (3.44) and using Jensen's inequality, we get :

$$\begin{aligned}
\Xi(y) &\leq \hat{y}^{-\beta}\tilde{E}\left[\int_{\Re^n} H(T, \tilde{W}_T, b)^{\beta} \, \nu(db)\right] \\
&\leq \hat{y}^{-\beta}\tilde{E}\left[\int_{\Re^n} \exp\left(\beta(\sigma^{-1}b)^* \tilde{W}_T - \frac{\beta}{2}||\sigma^{-1}b||^2 T\right) \nu(db)\right] .
\end{aligned}$$

The process $\left(\exp\{\beta(\sigma^{-1}b)^*\tilde{W}_t - \frac{\beta^2}{2}||\sigma^{-1}b||^2 t\}\right)_{t\geq 0}$ is a $(\tilde{P}, \{\mathcal{F}_t\})$-martingale with expectation equal to one so that :

$$\begin{aligned}
\Xi(y) &\leq \hat{y}^{-\beta}\int_{\Re^n} \exp\left[\frac{\alpha}{2(1-\alpha)^2}||\sigma^{-1}b||^2 T\right] \nu(db) \\
&\leq \hat{y}^{-\beta}\exp\left[\frac{\alpha}{2(1-\alpha)^2}||\sigma^{-1}||^2 T K_2\right] ,
\end{aligned}$$

hence $\Xi(y) < \infty$.

Theorem 3.8 gives then the optimal wealth process for problem 2.3 :

$$\zeta^x = (\hat{y}\xi_T)^{-\beta} \text{ ,} \tag{3.67}$$

where $\hat{y}$ is determined in (3.22).

In the next section, we provide a representation for the process ξ defined in (3.13) for the model described in (2.1)–(2.2).

3.5 A general representation for the process ξ.

We introduce the conditional mean vector $m = \{m_t = (m_1(t), \cdots, m_n(t); t \in [0,T]\}$ and the covariance matrix $\gamma = \{\gamma_t = (\gamma_1(t), \cdots, \gamma_n(t)); t \in [0,T]\}$ of μ_t :

$$m_t = E[\mu_t | \mathcal{F}_t^S] \tag{3.68}$$

and

$$\gamma_t = E[(\mu_t - m_t)(\mu_t - m_t)^* | \mathcal{F}_t^S] \; . \tag{3.69}$$

Using Lakner's result (see [19], Theorem 3.1), we give now a representation for ξ involving the process m, and a characterization of ξ^{-1} as a unique solution of a stochastic differential equation.

Theorem 3.17 *Suppose that*

$$E[||\mu_t||] < +\infty \; , \quad t \in [0,T] \, , \tag{3.70}$$

and that the n-dimensional process m is continuous. Then the process $\frac{1}{\xi}$ satisfies the stochastic differential equation

$$d\Big(\frac{1}{\xi_t}\Big) = \frac{1}{\xi_t}(\sigma^{-1} m_t)^* d\tilde{W}_t \tag{3.71}$$

and we have the representation :

$$\xi_t = exp\Big(- \int_0^t (\sigma^{-1} m_u)^* \, d\tilde{W}_u + \frac{1}{2} \int_0^t ||\sigma^{-1} m_u||^2 \, du \Big) \; . \tag{3.72}$$

Proof : From the definition of Z in (3.1) and Ito's rule, it appears that $\frac{1}{Z}$ is the solution of the stochastic differential equation

$$d\Big(\frac{1}{Z_t}\Big) = \frac{1}{Z_t}(\sigma^{-1} \mu_t)^* \, d\tilde{W}_t \; . \tag{3.73}$$

We obtain the following representation for $\frac{1}{Z}$:

$$\frac{1}{Z_t} = 1 + \int_0^t (\sigma^{-1}\mu_u)^* \frac{1}{Z_u}\, d\tilde{W}_u \ . \tag{3.74}$$

Taking the conditional expectation with respect to $\mathcal{F}_t^S$ in (3.74) and using Theorem 5.14 of [21], we obtain :

$$\tilde{E}\Big[\frac{1}{Z_t}\Big|\mathcal{F}_t^S\Big] = 1 + \int_0^t \tilde{E}\Big[(\sigma^{-1}\mu_u)^* \frac{1}{Z_u}\Big|\mathcal{F}_u^S\Big]\, d\tilde{W}_u \ , \tag{3.75}$$

provided that the following two conditions hold

$$\tilde{E}\Big[\Big|\sum_{i=1}^n \mu_i(u) s_{ij} \frac{1}{Z_u}\Big|\Big] < +\infty \ , \quad j = 1, \cdots, n, \quad u \in [0, T] \tag{3.76}$$

and

$$\int_0^T \Big(\tilde{E}\Big[\frac{1}{Z_u}\sum_{i=1}^n \mu_i(u) s_{ij}\Big|\mathcal{F}_u^S\Big]\Big)^2 du < +\infty \ , \quad a.s. \tag{3.77}$$

where s_{ij} is the (i,j)-entry of the matrix $\left(\sigma^{-1}\right)^*$.
Moreover, using (3.70), one can easily check that :

$$\begin{aligned} \tilde{E}\Big[\Big|\sum_{i=1}^n \mu_i(u) s_{ij} \frac{1}{Z_u}\Big|\Big] &= E\Big[\Big|\sum_{i=1}^n \mu_i(u) s_{ij}\Big|\Big] \\ &\leq \sum_{i=1}^n |s_{ij}| E[|\mu_i(u)|] \\ &\leq \sum_{i=1}^n |s_{ij}| E[||\mu_u||] \\ &< +\infty \ , \end{aligned} \tag{3.78}$$

and, by the Kallianpur-Striebel formula (3.15), the left-hand side of (3.77) can be writen as

$$\int_0^T \frac{1}{{\xi_u}^2}\Big(E\Big[\sum_{i=1}^n \mu_i(u)s_{ij}\Big|\mathcal{F}_u^S\Big]\Big)^2 du = \int_0^T \frac{1}{{\xi_u}^2}\Big(\sum_{i=1}^n m_i(u)s_{ij}\Big)^2 du \;. \tag{3.79}$$

This last integral is finite because of the continuity of m and ξ. Now, we deduce from (3.15) that the left-hand side of (3.75) is equal to $\frac{1}{\xi_t}$. By (3.15), we obtain

$$1+\int_0^t \tilde{E}\Big[(\sigma^{-1}\mu_u)^*\frac{1}{Z_u}\Big|\mathcal{F}_u^S\Big]\,d\tilde{W}_u = 1+\int_0^t \frac{1}{\xi_u}(\sigma^{-1}m_u)^*\,d\tilde{W}_u \;, \tag{3.80}$$

which implies (3.71) and thus (3.72).

□

In equation (3.72), the process m enters in a prominent fashion but at this level of generality, we still do not have a computable representation for it. In order to give an explicit formula for ξ, we then have to specialize the dynamics of the drift μ; this program is carried out in the next section.

3.6 A Gaussian setting : explicit representation for the process ξ

We suppose now, as in Lakner [19], that the n-dimensional drift process μ is governed by the stochastic differential equation

$$d\mu_t = \alpha(\delta-\mu_t)dt+\beta d\hat{W}_t \;, \tag{3.81}$$

where μ_0 is an $\mathcal{F}_0$-measurable random variable with gaussian probability law of mean m_0 and covariance matrix γ_0. Here, the process $\hat{W}=\{\hat{W}_t=(\hat{W}_1(t),\cdots,\hat{W}_n(t))^*; t\in[0,T]\}$ is an n-dimensional $(P,\{\mathcal{F}_t\})$-Brownian motion, independant of W under P. We assume that α, δ and β are known deterministic constants, valued respectively in $\Re^{n\times n}$, $\Re^n$ and $\Re^{n\times n}$ and that β is invertible. Moreover, we suppose :

$$tr(\gamma_0) + T||\beta||^2 < K_1 \ , \tag{3.82}$$

where

$$K_1 = \frac{1}{360T||\sigma^{-1}||^2 K} \ , \tag{3.83}$$

and

$$K = \max_{0\leq t\leq T} ||e^{-\alpha t}||^2 \ . \tag{3.84}$$

Then, by Lakner's result ([19], Lemma 4.1), Assumption 3.1 is satisfied and the process Z is a $(P, \{\mathcal{F}_t\})$-martingale; furthermore

$$E\left[{Z_T}^5 + {Z_T}^{-4}\right] + \tilde{E}\left[{\xi_T}^4 + {\xi_T}^{-5}\right] < \infty \ . \tag{3.85}$$

Recall that the return process R of (3.3) corresponds to the observation process and satisfies

$$dR_t = \mu_t dt + \sigma\, dW_t \ . \tag{3.86}$$

We are thus in the framework of the Kalman-Bucy filter and we deduce the following equations for m_t and γ_t

Proposition 3.18 *m is the unique $\mathcal{F}^S$-measurable solution of the stochastic differential equation (3.87) and γ is the unique solution of the deterministic Ricatti equation (3.88)*

$$dm_t = \left[-\alpha - \gamma_t(\sigma\sigma^*)^{-1}\right] m_t dt + \gamma_t(\sigma\sigma^*)^{-1} dR_t + \alpha\delta dt \tag{3.87}$$

$$\frac{d\gamma}{dt}(t) = -\gamma_t(\sigma\sigma^*)^{-1}\gamma_t - \alpha\gamma_t - \gamma_t\alpha^* + \beta\beta^* \tag{3.88}$$

with the initial conditions m_0 and γ_0.

It follows that the conditional covariance matrix $\gamma_t = \gamma(t)$ is deterministic. When $n > 1$, the Ricatti equation (3.88) has no explicit solution. However, (3.88) can be solved in the following way (see [30] for the general methodology to solve Ricatti equations) : let $\Phi : [0,T] \to \Re^{n\times n}$ be the fundamental solution of the deterministic equation

$$\frac{d\Phi}{dt}(t) = \left[-\alpha - \gamma(t)(\sigma\sigma^*)^{-1}\right]\Phi(t) \ , \tag{3.89}$$

$$\Phi(0) = I_n \tag{3.90}$$

where I_n is the identity matrix.

Then, m_t is determined in terms of γ and Φ by :

$$m_t = \Phi(t)\left[m_0 + \int_0^t \Phi^{-1}(s)\gamma(s)(\sigma\sigma^*)^{-1}dR_s + \left(\int_0^t \Phi^{-1}(s)ds\right)\alpha\delta\right] \ . \tag{3.91}$$

We can now state the following result due to Lakner (1998), which yields a representation for the optimal investment process, $\hat{\pi}$, corresponding to Problem 2.3.

Theorem 3.19 *Suppose that U is twice continuously differentiable on $(0,\infty)$ and that*

$$I(y) < K_2(1 + y^{-5}) \ , \tag{3.92}$$

$$-I'(y) < K_2(1 + y^{-2}) \ , \tag{3.93}$$

for some $K_2 > 0$. Then the optimal investment process is

$$\hat{\pi}_t = \hat{y}(\sigma\sigma^*)^{-1}\frac{1}{\xi_t}E\left[I'(\hat{y}\xi_T){\xi_T}^2\left(-\gamma(t)(\Phi^*(t))^{-1}\int_t^T \Phi^*(u)(\sigma^*)^{-1}\,d\bar{W}_u - m_t\right)\middle|\mathcal{F}_t^S\right] \ , \tag{3.94}$$

ξ is given by (3.72), m is given by (3.91), and the constant $\hat{y}$ is uniquely determined by (3.22).

Proof : The main technique of the proof is the use of the Malliavin derivative D acting on the subset of the class of functionnals of $\tilde{W}$ called $D_{1,1}$ (for the definition of D and $D_{1,1}$, see Ocone and Karatzas [23]). For details, the reader is referred to Lakner ([19], subsection 5) for the (lengthy, and rather demanding) proof of this theorem. Note that the optimal investment strategy under complete information is derived using the same technique (see [23]). □

Example 3.20 *Logarithmic utility function* $u(x) = \log(x)$

In this case, $I(y) = \frac{1}{y}$*, thus* $-I'(y) = \frac{1}{y^2}$ *and (3.92)–(3.93) are satisfied. Moreover* $\Xi(y) = \tilde{E}[\frac{1}{y\xi_T}] = \frac{1}{y}$ *,therefore,* $\Xi(y) < \infty$ *for every* $y \in (0,\infty)$ *and* $\hat{y} = \frac{1}{x}$*. Formula (3.94) provides the optimal investment process* $\hat{\pi}$

$$\hat{\pi}_t = x(\sigma\sigma^*)^{-1}\frac{1}{\xi_t}m_t \ , \tag{3.95}$$

and Equation (3.38) yields the optimal terminal wealth :

$$X_T^{\hat{\pi},x} = \frac{1}{\hat{y}\xi_T} = \frac{x}{\xi_T} \ . \tag{3.96}$$

From the $(\tilde{P},\mathcal{F}^S)$*-martingale property of the process* $\frac{1}{\xi}$ *and* $X^{\hat{\pi},x}$*, Equ. (3.96) imply :*

$$X_t^{\hat{\pi},x} = \frac{x}{\xi_t} \ . \tag{3.97}$$

We thus obtain the feedback form for the optimal portfolio process

$$\hat{\pi}_t = (\sigma\sigma^*)^{-1}m_t X_t^{\hat{\pi},x} \ . \tag{3.98}$$

Recall that in the full information case, the optimal investment process is given by the feedback form ([23]) :

$$(\sigma\sigma^*)^{-1}\mu_t X^{\hat{\pi},x}(t) \ , \tag{3.99}$$

Therefore, in the case of partial observation, the optimal investment process of (3.98) can be derived by substituting the drift process μ_t *by its best estimate*

$E[\mu_t|\mathcal{F}_t^S]$ in equation (3.99). This property corresponds to the so-called separation principle as already stated in Remark 3.14.

Example 3.21 *Power type utility function $u(x) = \frac{x^\alpha}{\alpha}$ whith $\alpha < 0$.*

In this case, $I(y) = y^{\frac{1}{\alpha-1}}$ and $-I'(y) = \frac{1}{1-\alpha} y^{\frac{1}{\alpha-1}}$, so that (3.92) and (3.93) hold. Moreover, using Jensen's inequality and the $(\tilde{P}, \mathcal{F}^S)$-martingale property of the process ξ, we get

$$\begin{aligned} \tilde{E}[I(\hat{y}\xi_T)] &= \hat{y}^{\frac{\alpha}{\alpha-1}} E[\xi_T{}^{\frac{1}{\alpha-1}}] \\ &\leq \hat{y}^{\frac{\alpha}{\alpha-1}} E[\xi_T]^{\frac{1}{\alpha-1}} \\ &\leq \hat{y}^{\frac{1}{\alpha-1}} < \infty , \end{aligned} \tag{3.100}$$

and (3.21) is satisfied.

From (3.94), we have

$$\hat{\pi}_t = \frac{1}{1-\alpha} (\sigma\sigma^*)^{-1} m_t X_t^{\hat{\pi},x} + G_t , \tag{3.101}$$

where

$$G_t = y^{\frac{1}{\alpha-1}} \frac{1}{1-\alpha} \frac{1}{\xi_t} (\sigma\sigma^*)^{-1} \gamma(t) (\Phi(t))^{-1} E\left[\xi_T^{\frac{\alpha}{\alpha-1}} \int_t^T \Phi^*(u)(\sigma^*)^{-1} \, d\bar{W}_u \middle| \mathcal{F}_t^S \right] . \tag{3.102}$$

The optimal investment process for this utility function under complete information is (see [23])

$$\frac{1}{1-\alpha} (\sigma\sigma^*)^{-1} \mu_t X_t^{\hat{\pi},x} . \tag{3.103}$$

Therefore, the separation principle does not hold for power utility functions : formal substitution of $E[\mu|\mathcal{F}_t^S]$ in the expression (3.103) does not yield the correct expression for the optimal investment process in the partial observation case because of the additional non-zero term G_t in equation (3.101).

4 The case of incomplete markets

A first step towards an analysis of incomplete markets with partial information was taken by Karatzas and Xue [15] : they adressed the maximization of expected total utility from consumption problem, in the case where investors can observe security prices only and when the dimension of the driving Brownian motion is strictly greater than the number of stocks. Of course, the utility maximization from terminal wealth problem with partial information may be treated by superposition using the same methodology. The general idea is also to reduce the model with partial observations to a completely observable one, where the martingale approach of [12] can be applied.

However, Karatzas and Xue impose a restrictive condition on the "diffusion" matrix $a(t) = \sigma(t)\sigma^*(t)$, which is assumed to be nondegenerate in the following sense : for some $\epsilon > 0$,

$$\zeta^* a(t,\omega)\zeta \geq \epsilon ||\zeta||^2 \,, \quad \text{for every } \zeta \in \Re^d \,, (t,\omega) \in [0,T] \times \Omega. \tag{4.1}$$

As already known, this condition guarantees, roughly speaking, that "there are exactly as many stocks as there are sources of uncertainty in the market model". Therefore, in such a context with assumption (4.1), the methodology developed in section 3 for the complete market case can be applied to solve the utility maximization problem adressed by Karatzas and Xue.

To the best of our knowledge, the only contribution to the theory of utility maximization with partial observation in the setting of incomplete markets is the paper of Pham and Quenez [26]. In this paper, the authors solve Problem 2.3, when the risky securities follow a stochastic volatility model. More precisely, the system of stochastic differential equations (2.1)–(2.2) verified by the price process S is replaced by

$$dS_t = \mu_t dt + \sigma(t, S_t, Y_t) dW_t \,, \tag{4.2}$$

$$dY_t = \eta_t dt + \rho(t, S_t, Y_t) dW_t + \gamma(t, S_t, Y_t) dB_t \,, \tag{4.3}$$

where it is assumed that the volatility is influenced by some latent process Y.

W and B are independent Brownian motions defined on $(\Omega, \mathcal{F}, P)$ and valued respectively in $\Re^n$ and $\Re^d$. We denote by $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_t; 0 \le t \le T\}$ the natural filtration of $(\tilde{W}, \tilde{B})$. $\mu = \{\mu_t; t \in [0,T]\}$ (resp. $\eta = \{\eta_t; t \in [0,T]\}$) is a $\Re^n$-valued (resp. $\Re^d$-valued) adapted process. The known functions $\sigma(t,s,y)$, $\rho(t,s,y)$ and $\gamma(t,s,y)$ are measurable mappings from $[0,T] \times \Re^n \times \Re^d$ into $\Re^{n\times n}$,$\Re^{d\times n}$ and $\Re^{d\times d}$ and satisfy :

Assumption 4.1 (i) *The functions $\sigma(t,.,.)$, $\rho(t,.,.)$ and $\gamma(t,.,.)$ are Lipschitz-continuous in $(s,y) \in \Re^n \times \Re^d$, uniformly in $t \in [0,T]$.*

(ii) *For all (t,s,y), the $n \times n$ and $d \times d$ matrix $\sigma(t,s,y)$ and $\gamma(t,s,y)$ are nonsingular.*

(iii) *The function $\sigma\sigma^*$ is continuous in $(t,s,y) \in [0,T] \times \Re^n \times \Re^d$ and for all (t,s), the function $\sigma\sigma^*(t,s,.)$ is one-to-one from $\Re^d$ into a subset Σ of the set of $n \times n$ positive definite matrices, and its inverse function, denoted $\varsigma(t,s,.)$, is continuous with respect to $(t,s,z) \in [0,T] \times \Re^n \times \Sigma$*

Recall that partial observation means here that investors can observe neither the Brownian motions W and B nor the drift μ and η, but only the stock price process S.

4.1 Reduction to a completely observed model

The first steps for solving Problem 2.3 are similar to the ones in the complete market case.

First, we construct a new probability measure under which the stock prices become local martingales. Let us then define the risk-premia processes :

$$\lambda_t = \sigma(t, S_t, Y_t)^{-1}\mu_t \tag{4.4}$$

$$\alpha_t = \gamma(t, S_t, Y_t)^{-1}(\eta_t - \rho(t, S_t, Y_t)\lambda_t) \tag{4.5}$$

and we assume that they satisfy the integrability condition :

$$\int_0^T ||\lambda_t||^2 + ||\alpha_t||^2 \, dt < \infty \ , \quad a.s. \tag{4.6}$$

Consider also the positive local martingale Z defined by :

$$dZ_t = -Z_t(\lambda_t^* dW_t + \alpha_t^* dB_t) \ , \quad Z_0 = 1 \ . \tag{4.7}$$

We make the usual assumption in filtering theory :

Assumption 4.2 *The process Z is a $(P, \tilde{\mathcal{F}})$-martingale.*

We can now carry out the absolutely continuous change of probability measure, suggested by the Girsanov theorem.

We introduce the auxiliary probability measure $\tilde{P}$ defined by :

$$\frac{d\tilde{P}}{dP} = Z(T) \ , \tag{4.8}$$

where Z is defined in (4.7). Then, the n-dimensional process :

$$\tilde{W}_t = W(t) + \int_0^t \lambda_u \, du \ , \tag{4.9}$$

and the d-dimensional process :

$$\tilde{B}_t = B_t + \int_0^t \alpha_u \, du \tag{4.10}$$

are independant $(\tilde{P}, \tilde{\mathcal{F}})$-Brownian motions.

Plugging (4.9)–(4.10) into (4.2)–(4.3), we obtain the dynamics of (S, Y) under $\tilde{P}$

$$dS_t = \sigma(t, S_t, Y_t) d\tilde{W}_t \ , \tag{4.11}$$

$$dY_t = \rho(t, S_t, Y_t) d\tilde{W}_t + \gamma(t, S_t, Y_t) d\tilde{B}_t \ . \tag{4.12}$$

In order to transform the partially observed model (4.11)–(4.12) into one with complete observations, we need the following lemma :

Lemma 4.3 *Under Assumption 4.1 and Assumption 4.2, the filtration $\mathcal{F}^S$ is the augmented filtration of $(\tilde{W}, \tilde{B})$.*

Proof : Let $\mathcal{F}^{S,Y}$ be the augmented filtration of (S, Y). First, we shall show that $\mathcal{F}^{S,Y} = \mathcal{F}^S$. Obviously, $\mathcal{F}^S \subset \mathcal{F}^{S,Y}$. In order to show the reverse inclusion, we consider the quadratic variation process of S, $< S >_t$, which is given by :

$$< S >_t = \int_0^t \sigma\sigma^*(u, S_u, Y_u)\, du \ , \quad 0 \le t \le T \ . \tag{4.13}$$

From the continuity of the function $\sigma\sigma^*$ and of the processes S and Y, it follows that the process $(\sigma\sigma^*(t, S_t, Y_t), 0 \le t \le T)$ is $\mathcal{F}^S$-adapted. Moreover, by Assumption 4.1(iii), the process Y may be rewritten in term of the continuous ς, as follows :

$$Y_t = \varsigma(t, S_t, \sigma\sigma^*(t, S_t, Y_t)) \ , \quad 0 \le t \le T \ . \tag{4.14}$$

Therefore, Y is also $\mathcal{F}^S$-adapted and $\mathcal{F}^{S,Y} = \mathcal{F}^S$.

From (4.11)–(4.12), one can easily derive the identities :

$$\tilde{W}_t = \int_0^t \sigma^{-1}(u, S_u, Y_u) dS_u \ , \tag{4.15}$$

$$\tilde{B}_t = \int_0^t \gamma^{-1}(u, S_u, Y_u)[dY_u - \rho(u, S_u, Y_u)\sigma^{-1}(u, S_u, Y_u) dS_u] \ , \tag{4.16}$$

for all $t \in [0, T]$, which imply that $\tilde{\mathcal{F}} \subset \mathcal{F}^{S,Y} = \mathcal{F}^S$. Conversely, under Assumption 4.1 (i), by Protter([27], Thm V.3.7), (4.11)–(4.12) has a unique solution (S, Y), which must be $\tilde{\mathcal{F}}$-adapted, thus $\mathcal{F}^S = \mathcal{F}^{S,Y} \subset \tilde{\mathcal{F}}$ and finally $\mathcal{F}^S = \tilde{\mathcal{F}}$.

□

In the case of partial observation, we have to project the process Z of (4.7) to the available information to the agents given by $\mathcal{F}^S$. Hence, we define

$$\xi_t = E[Z(t)|\mathcal{F}_t^S] = E[E[Z(T)|\tilde{\mathcal{F}}_t]|\mathcal{F}_t^S] = E[Z(T)|\mathcal{F}_t^S] \ . \tag{4.17}$$

It is clear that the process ξ is a strictly positive $(P, \mathcal{F}^S)$-martingale with $E[\xi_T] = 1$. Moreover, using identity (3.16), we note that $\frac{1}{\xi}$ is a $(\tilde{P}, \mathcal{F}^S)$-martingale. From Lemma 4.3 which states that $\mathcal{F}^S$ is generated by $(\tilde{W}, \tilde{B})$, we deduce then that $\frac{1}{\xi}$ and also ξ must be continuous processes.

We now make the standing assumption on the risk premia processes (λ, α) of the stochastic volatility model.

Assumption 4.4 *For all* $t \in [0, T]$, $\quad E||\lambda_t|| + E||\alpha_t|| < \infty$.

We introduce the processes (N, M) defined by

$$N_t = \tilde{W}_t - \int_0^t \tilde{\lambda}_u \, du \ , \tag{4.18}$$

$$M_t = \tilde{B}_t - \int_0^t \tilde{\alpha}_u \, du \ , \tag{4.19}$$

where $\tilde{\lambda} = (\tilde{\lambda}_t, t \in [0, T])$ and $\tilde{\alpha} = (\tilde{\alpha}_t, t \in [0, T])$ are given by :

$$\tilde{\lambda}_t = E[\lambda_t|\mathcal{F}_t^S] \ , \tag{4.20}$$

$$\tilde{\alpha}_t = E[\alpha_t|\mathcal{F}_t^S] \ . \tag{4.21}$$

N and M correspond to the so-called "innovation processes" and by classical results in stochastic filtering theory, they are $(P, \mathcal{F}^S)$-independant Brownian motions.

By using similar arguments as in Theorem 3.3 or Lakner ([19], Thm 3.1), we provide an explicit form of ξ in terms of the innovation processes (N, M):

Proposition 4.5 *Under Assumption 4.1, 4.2 and 4.4, we have :*

$$\xi_t = \exp\Big(- \int_0^t \tilde{\lambda}_u^* \, dN_u - \int_0^t \tilde{\alpha}_u^* dM_u - \frac{1}{2} \int_0^t ||\tilde{\lambda}_u||^2 + ||\tilde{\alpha}_u||^2 \, du \Big) \ . \tag{4.22}$$

The dynamics of (S, Y) under $(P, \mathcal{F}^S)$ are given by

$$dS_t = \tilde{\mu}_t dt + \sigma(t, S_t, Y_t) dN_t \ , \tag{4.23}$$

$$dY_t = \tilde{\eta}_t dt + \rho(t, S_t, Y_t) dN_t + \gamma(t, S_t, Y_t) dM_t \ , \tag{4.24}$$

where $\tilde{\mu}$ and $\tilde{\eta}$ are $\mathcal{F}^S$-adapted processes defined by :

$$\tilde{\mu}_t = \sigma(t, S_t, Y_t) \tilde{\lambda}_t \ , \tag{4.25}$$

$$\tilde{\eta}_t = \rho(t, S_t, Y_t) \tilde{\lambda}_t + \gamma(t, S_t, Y_t) \tilde{\alpha}_t \ . \tag{4.26}$$

Hence, we have transformed the original partial observation model (4.2)–(4.3) into one with complete observations; all the processes involved in (4.23)–(4.24) are adapted to the same filtration $\mathcal{F}^S$, the one on which investors will have to base investment decisions.

4.2 Martingale dual approach under partial observation

We can now extend the known results for full information case to our case with partial information. We present results similar to the ones in [4], [7] and [12].

The next lemma develops a martingale representation theorem for $(P, \mathcal{F}^S)$-local martingales with respect to N and M.

Lemma 4.6 *Let m be any $(P, \mathcal{F}^S)$-local martingale with $m_0 = 0$. Then, there exists an $\Re^n$-valued process ϕ and an $\Re^d$-valued process ψ which are $\mathcal{F}^S$-adapted processes, P-a.s. square-integrable and such that :*

$$m_t = \int_0^t \phi_u^* \, dN_u + \int_0^t \psi_u^* \, dM_u \ , \quad 0 \le t \le T. \tag{4.27}$$

Proof : From Bayes rule, it is easily checked that the process $\tilde{m} = (\tilde{m}_t; 0 \leq t \leq T)$ given by :

$$\tilde{m}_t = m_t \xi_t^{-1} , \quad 0 \leq t \leq T , \tag{4.28}$$

is a $(\tilde{P}, \mathcal{F}^S)$-local martingale. We can then apply the usual martingale representation theorem since $\mathcal{F}^S$ is generated by the Brownian motions $(\tilde{W}, \tilde{B})$. By Theorem ([14], p.184) in Karatzas and Shreve, there exists an $\Re^n$-valued process $\tilde{\phi}$ and an $\Re^d$-valued process $\tilde{\psi}$ which are $\mathcal{F}^S$-adapted processes, P-a.s. square integrable such that :

$$\tilde{m}_t = \int_0^t \tilde{\phi}_u^* \, d\tilde{W}_u + \int_0^t \tilde{\psi}_u^* \, d\tilde{B}_u , \quad 0 \leq t \leq T. \tag{4.29}$$

From the definition of N and M, the left hand side of (4.29) can be rewritten in the following form :

$$\int_0^t [\tilde{\phi}_u^* \tilde{\lambda}_u + \tilde{\psi}_u^* \tilde{\alpha}_u] \, dt + \int_0^t \tilde{\phi}_u^* \, dN_u + \int_0^t \tilde{\psi}_u^* \, dM_u , \quad 0 \leq t \leq T. \tag{4.30}$$

Finally, Ito's rule applied to $m_t = \tilde{m}_t \xi_t$, (4.22) and (4.30) lead to

$$m_t = \int_0^t \phi_u^* \, dN_u + \int_0^t \psi_u^* \, dM_u , \quad 0 \leq t \leq T , \tag{4.31}$$

where $\phi = (\phi_t; 0 \leq t \leq T)$ and $\psi = (\psi_t; 0 \leq t \leq T)$ are both $\mathcal{F}^S$-adapted processes, P-a.s. square integrable, valued respectively in $\Re^n$ and $\Re^d$, such that $\phi_t = \xi_t(\tilde{\phi}_t - \tilde{m}_t \tilde{\lambda}_t)$ and $\psi_t = \xi_t(\tilde{\psi}_t - \tilde{m}_t \tilde{\alpha}_t)$.
□

Remark 4.7 *Note that Lemma 4.6 couldn't be derived directly from usual martingale representation theorem since $\mathcal{F}^S$ is, in general, strictly larger than the augmented filtration generated by the $(P, \mathcal{F}^S)$-Brownian motions N and M.*

We introduce now the standart tools involved in the martingale duality formulation of Problem 2.3.

Let us define the $(P, \mathcal{F}^S)$-exponential local martingale

$$Z_t^\nu = \exp\Big(- \int_0^t \tilde{\lambda}_u^* \, dN_u - \int_0^t \nu_u^* \, dM_u - \frac{1}{2} \int_0^t ||\tilde{\lambda}_u||^2 + ||\nu_u||^2 \, du \Big) \ , \qquad (4.32)$$

for any $\mathcal{F}^S$-adapted, $\Re^d$-valued process $\nu = (\nu_t; 0 \leq t \leq T)$, which satisfies $\int_0^T ||\nu_u||^2 \, dt < \infty$.

Remark 4.8 *By Lemma 4.6, note that the previous family of local martingales Z^ν correspond to the so-called* equivalent local martingales measures*(for more details, see [7]).*

We denote by $\mathcal{H}$ the Hilbert space of $\mathcal{F}^S$-adapted, $\Re^d$-valued process ν such that $E\Big[\int_0^T ||\nu_u||^2 \, dt \Big] < \infty$.

We need to define the superhedging price of any contingent claim :

Definition 4.9 *Let H be a contingent claim, i.e. a nonnegative $\mathcal{F}_T^S$-measurable random variable. The superhedging price of H is defined by :*

$$u_0 = \inf\{ x \geq 0 \, ; \exists \pi \in \mathcal{A}'(x) \ , X^{\pi,x}(T) \geq H \ a.s. \} \ . \qquad (4.33)$$

The next theorem shows that the dual formulation of the superhedging price developed by El Karoui and Quenez [7] in the case of complete observations still holds in the case of partial observations.

Theorem 4.10 *Let H be a contingent claim. Then,*

$$u_0 = J_0 = \sup_{\nu \in \mathcal{H}} E[Z_T^\nu H], \qquad (4.34)$$

and when $J_0 < \infty$, there exists $\hat{\pi} \in \mathcal{A}'(J_0)$ such that $X_T^{\hat{\pi}, J_0} \geq H$. Moreover, in this case, for any $\nu^* \in \mathcal{H}$, the following conditions are equivalent :

(i) *ν^* achieves the supremum in (4.34),*

(ii) *H is attainable, i.e. there exists $\hat{\pi} \in \mathcal{A}'(J_0)$ such that $X_T^{J_0,\hat{\pi}} = H$, and the process $Z^{\nu^*} X^{\hat{\pi},J_0}$ is a $(P, \mathcal{F}^S)$-martingale.*

Proof : The technics are similar to the ones used in [7] for the Brownian case, since here the filtration $\mathcal{F}^S$ is a Brownian filtration. The basic difference is that we have to use Lemma 4.6 instead of the martingale representation theorem. □

Let us introduce $\tilde{U}$, the polar function of U defined by :

$$\tilde{U}(y) = \max_{x>0}[U(x) - xy] = U(I(y)) - yI(y)\ , \quad 0 < y < \infty\ , \tag{4.35}$$

where the function I is defined in (2.6). Hereafter, the following hypothesis are supposed to be satisfied as in Karatzas and al. [12] :

Assumption 4.11 (i) *$c \to cU'(c)$ is nondecreasing on $(0, \infty)$,*

(ii) *There exists $\alpha \in (0, 1)$ and $\gamma \in (1, \infty)$, such that $\alpha U'(x) \geq U'(\gamma x), \forall x \in (0, \infty)$,*

(iii) *For all $y \in (0, \infty)$, there exists $\nu \in \mathcal{H}$ such that $E[\tilde{U}(yZ_T^\nu)] < \infty$.*

We consider now the dual optimization problem of (2.9) :

$$\tilde{V}(z) = \inf_{\nu \in \mathcal{H}} E[\tilde{U}(zZ_T^\nu)]\ , \quad z > 0\ . \tag{4.36}$$

The question of existence in the dual problem (4.36) is adressed in the next proposition by the use of similar methods as in ([12], Thm 12.1).

Proposition 4.12 *Under Assumption 4.1, for all $z > 0$, the dual problem (4.36) admits a solution, $\nu^*(z) \in \mathcal{H}$.*

The solution of our portfolio optimization problem (2.9) is then related to the solution of the dual problem (4.36) as follows.

Theorem 4.13 *Assume that Assumption 4.1 holds. Then, for all $x > 0$, there exists an optimal investment process $\hat{\pi}$ for Problem 2.3 and the associated optimal wealth process $X^{\hat{\pi},x}$ is given by :*

$$X_t^{\hat{\pi},x} = E\Big[\frac{Z_T^{\nu^*(z_x)}}{Z_t^{\nu^*(z_x)}} I(z_x Z_T^{\nu^*(z_x)}) | \mathcal{F}_t^S\Big] , \tag{4.37}$$

where $z_x > 0$ is such that
$E\Big[Z_T^{\nu^*(z_x)} I(z_x Z_T^{\nu^*(z_x)})\Big] = x$,
or
$z_x \in Argmin_{z>0}(\tilde{V}(z) + xz)$.

Proof : The proof is similar as in Theorem 11.6 of Karatzas and al. ([12]). □

Example 4.14 *Logarithmic utility function $u(x) = \log(x)$.*

In this case, $I(y) = \frac{1}{y}$ and $\tilde{U}(y) = -(1 + \log(y))$. For all $z > 0$, the dual problem (4.36) admits the solution $\nu^*(z) = 0$ and $z_x = \frac{1}{x}$. Theorem 4.13 gives the optimal wealth

$$X_t^{\hat{\pi},x} = E\Big[\frac{Z_T^0}{Z_t^0} \frac{1}{z_x Z_t^0} | \mathcal{F}_t^S\Big] = \frac{x}{Z_t^0} , \tag{4.38}$$

and the optimal investment process is in the feedback form

$$\hat{\pi}_t = \sigma(t, S_t, Y_t)^{-1^*} \tilde{\lambda}_t X_t^{\hat{\pi},x} . \tag{4.39}$$

The optimal investment process for this utility function under full information is :

$$\sigma(t, S_t, Y_t)^{-1^*} \lambda_t X_t^{\hat{\pi},x} , \tag{4.40}$$

(Ocone and Karatzas, [23]). Therefore, we can formally derive Equation (4.39) if we substitute λ_t by its conditional mean $\tilde{\lambda}_t$ in (4.40). This property corresponds to the so-called separation principle as already stated in Remark 3.14.

Example 4.15 *Power type utility function* $u(x) = \frac{x^\alpha}{\alpha}$ *with* $0 < \alpha < 1$.

In this case, $I(y) = y^{\frac{1}{\alpha-1}}$ *and* $\tilde{U}(z) = \frac{z^{-p}}{p}$ *with* $p = \frac{p}{1-p}$. *Let us note from Theorem 4.13 that the optimal wealth process is strictly positive, i.e.*

$$P(X_t^{\hat{\pi},x}, \forall t \in [0,T]) = 1\,.$$

Hence, the value function $V(x)$ *of (2.10) is identical to the one of Problem 2.3 but with a strictly positive constraint on the wealth. By making an appropriate change of variable in (2.10), Pham and Quenez [26] show that the solution of the utility maximization Problem 1 can be directly derived without using the duality relation of Theorem 4.13. As it will be quite long to expose here their arguments, the reader is refered to ([26], subsection 4.2) for more details.*

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