Stability Analysis of Second Order Fluid Flow Models with State-Dependent Parameters

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Abstract: We study in this paper the stability of a fluid queue with an infinite capacity buffer. The input and service rates are governed by a general stochastic process and are allowed to depend on the fluid level in the buffer. The variability of the traffic is modeled by a Brownian motion and a local variance function which also depends on the fluid level in the buffer. The behavior of this second order fluid flow model is described by a reflected stochastic differential equation and, under stationarity and ergodicity assumptions, we obtain stability conditions for this general fluid queue.

Key-words: Fluid queues, Brownian motion, Reflected stochastic differential equations, Stability, Lindley's equation.

(Résumé : tsvp)

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Analyse de la stabilité de modèles fluides du second ordre avec paramètres dépendant de l'état

Résumé : Nous étudions dans cet article la stabilité d'une file d'attente fluide de capacité infinie. Les taux d'entrée et de service sont contrôlés par un processus stochastique général et peuvent dépendre du niveau de fluide dans la file. La variabilité du trafic est modélisée par un mouvement brownien et une fonction de variance locale qui dépend aussi du niveau de fluide dans la file. Le comportement de ce modèle fluide du second ordre est décrit par une équation différentielle stochastique réfléchie et, sous des hypothèses de stationnarité et d'ergodicité, nous obtenons des conditions de stabilité pour cette file d'attente générale.

Mots-clé : Files d'attente fluide, mouvement brownien, équations différentielles stochastiques réfléchies, stabilité, équation de Lindley.

1 Introduction

Fluid flow models are widely used in performance evaluation of high-speed communication network. Typically, the fluid represents information stored in a buffer and waiting for transmission in a network. The arrival and service processes are modulated by a random external environment and the quantity of interest is the behavior of the buffer level. A lot of papers, see among others [1], [10], [13], [7], [11], [12] and the references therein, considered the case where the random external environment is a continuous time Markov chain (CTMC) with a finite or infinite state space. In these papers, the input and output rates of the buffer are both piecewise constant, depending on the state of the Markov process. Extensions of this model can be found in [8] where the authors deal with the case where the input and output rates of the buffer may also depend on the buffer level. More recently, in [4] and [2], new models (called second order models) were introduced by adding a "white noise" factor which represents in practice the variability of the traffic during the transmission periods. The fluid level was thus described by a reflected Brownian motion modulated by a CTMC X. When the CTMC X is in state i, the fluid level is modeled by a reflected Brownian motion with drift b_i and variance parameter σ_i^2. In these two papers an expression for the tail distribution of the fluid level in the buffer is explicitly obtained. In the state-of-the-art [7], the author mentioned as further work the importance of the study of such models with arrivals and services depending on the fluid level.

In this paper, we consider a fluid flow model driven by a general stochastic process (not necessarily a Markov process) where the arrival and service rates are allowed to depend on the fluid level in the buffer. The variability of the traffic is modeled by a Brownian motion and a local variance function which also depends on the fluid level in the buffer. This generalization leads in particular to the use of the Itô's stochastic integral, and the well-known differential equations used in [11] become stochastic differential equations with reflection at 0. The fluid level process with these level-dependent drift and variance coefficients is, essentially, a reflected Brownian motion which has been altered, at each instant, by changing its drift and its variance. The main result of this paper is obtaining, under stationarity and ergodicity assumptions, stability conditions for such models, i.e. conditions for the existence of the limiting behavior for the fluid level in the buffer.

The remainder of the paper is organized as follows. The model and notation is introduced in the next section. In Section 3, we solve the Lindley's equation corresponding to that model. The main results and concluding remarks, concerning the stability of the fluid queue, are proved in Section 4.

2 Model and notation

Consider an infinite capacity buffer where fluid enters and exits according to the behavior of a general stochastic process denoted by $X = \{X(t),\ t \geq 0\}$. Let $Q(t)$ denote the amount of fluid in the buffer at time t. The input and service rates in the buffer are denoted respectively by $\lambda(X(t), Q(t))$ and $\mu(X(t), Q(t))$. They are both nonnegative and depend on the environment and the fluid level at time t. Their difference is called the local drift and is denoted by

$$b(X(t), Q(t)) = \lambda(X(t), Q(t)) - \mu(X(t), Q(t)).$$

The variability of the traffic at time t is represented by a Brownian motion $\{B_t\}$ and a local variance function $\sigma(X(t), Q(t))$. The behavior of the process $\{Q(t)\}$ is thus described by the following reflected stochastic differential equation (RSDE)

$$dQ(t) = b(X(t), Q(t))dt + \sigma(X(t), Q(t))dB_t + dL(t), \tag{1}$$

so-called because 0 is a reflecting barrier, where the stochastic process $L = \{L(t),\ t \geq 0\}$ is an increasing process introduced to prevent $\{Q(t)\}$ from being negative. Relation (1) is very compact and covers a broad range of situations.

Suppose, for instance, that we do not take into account the random part of relation (1) by taking $\sigma(X(t), Q(t)) = 0$. In that case, we obtain the following classical differential equation (see [11])

$$\frac{d}{dt}Q(t) = \begin{cases} b(X(t), Q(t)) & \text{if } Q(t) > 0 \\ b(X(t), Q(t))^+ & \text{if } Q(t) = 0, \end{cases}$$

by setting $dL(t) = b(X(t), Q(t))^- \mathbb{1}_{\{Q(t)=0\}}dt$, where $x^+ = \max(x, 0)$ and $x^- = (-x)^+$.

Let us now establish more rigorously the setting in which we will work from now on.

2.1 Mathematical framework

Let $(\Omega, \mathcal{F}, P)$ be a probabilistic space, endowed with a set of one-to-one measurable transformations $\{\theta_t : (\Omega, \mathcal{F}, P) \to (\Omega, \mathcal{F}, P)\}_{t\in\mathbb{R}}$ forming a flow, i.e. such that $\theta_t \circ \theta_s = \theta_{t+s}$ for all s and t. We assume that the probability P is θ_t-invariant, i.e. $P(\theta_t^{-1}A) = P(A)$ for all $t \in \mathbb{R}$ and $A \in \mathcal{F}$. We denote by $\{X(t),\ t \in \mathbb{R}\}$ a stochastic process with values in $\mathcal{X} \subset \mathbb{R}^d$, càdlàg, and compatible with the flow, i.e. such that $\forall s, t \in \mathbb{R}$ and $\omega \in \Omega$, $X(s, \theta_t\omega) = X(s + t, \omega)$. Note that the process X is deliberately dealt with as much generality as possible and is stationary because of the invariance of P under $\{\theta_t\}$. Let $\{B_t,\ t \in \mathbb{R}\}$ be a Brownian motion with real values, and compatible with the flow and let $(\mathcal{F}_t)$ be a right continuous filtration such that $\{B_t\}$ and $\{X(t)\}$ are $(\mathcal{F}_t)$-adapted. We assume that

- $(\Omega, \mathcal{F}, P, \{\theta_t\})$ is ergodic.
- The function $b : \mathcal{X} \times [0, +\infty) \longrightarrow \mathbb{R}$ is lower bounded.
- The function $\sigma : \mathcal{X} \times [0, +\infty) \longrightarrow \mathbb{R}^+$ is bounded.
- The functions b and σ are both Lipschitz in t with respect to second variable, i.e. $\exists C > 0$, $\forall t \in \mathbb{R}$ such that $\forall x, x' \geq 0$,

$$|b(X(t), x) - b(X(t), x')| + |\sigma(X(t), x) - \sigma(X(t), x')| \leq C|x - x'|.$$

- $\sup_{x\geq 0} b(X(0), x) \in L^1$. Note that this condition is always satisfied when the function b is bounded.

Under these conditions, for a fixed $u \in \mathbb{R}$ and for a fixed square integrable random variable $Y \geq 0$, the following RSDE

$$\begin{cases} dQ(t) &=& b(X(t), Q(t))dt + \sigma(X(t), Q(t))dB_t + dL(t) \quad \text{for } t \geq u \\ Q(u) &=& Y \\ Q(t) &\geq& 0 \quad \text{for } t \geq u \\ L(t) &=& \displaystyle\int_u^t \mathbb{1}_{\{Q(s)=0\}}dL(s) \quad \text{for } t \geq u \end{cases} \tag{2}$$

has a unique solution couple $(Q(t), L(t))_{t\geq u}$, $(\mathcal{F}_t)$-adapted, where $\{Q(t)\}$ and $\{L(t)\}$ are both continuous and nonnegative and $\{L(t)\}$ is increasing. The proof of this result can be found in [6]. We see that the process $\{L(t)\}$ is linked to the process $\{Q(t)\}$ interfering only when $Q(t) = 0$ and "prodding it upward" whenever $Q(t)$ has a tendency to go downwards while approaching zero, i.e. whenever intuitively,"$b(X(t), 0)dt + \sigma(X(t), 0)dB_t \leq 0$".

It is shown in [6, p. 118] that, for a fixed $u \in \mathbb{R}$, the couple $(Q(t), L(t))_{t\geq u}$ is the unique solution to the following system of equations in the unknown couple $(z(t), k(t))$

$$\begin{cases} z(t) &= \int_u^t [b(X(s), Q(s))ds + \sigma(X(s), Q(s))dB_s] + k(t) \quad \text{for } t \geq u \\ z(t) &\geq 0 \quad \text{for } t \geq u \\ k(t) &= \int_u^t \mathbb{1}_{\{z(s)=0\}} dk(s) \quad \text{for } t \geq u, \end{cases}$$

and, necessarily,

$$L(t) = \sup_{s\in[u,t]} \left(- \int_u^s [b(X(v), Q(v))dv + \sigma(X(v), Q(v))dB_v] \right)^+ . \tag{3}$$

Roughly speaking, since $L(t)$ is increasing only when $Q(t) = 0$, we formally have the (non rigorous) expression "$dL(t) = [b(X(t), Q(t))dt + \sigma(X(t), Q(t))dB_t]^- \mathbb{1}_{Q(t)=0}$".

We suppose that the buffer is initially empty, i.e. that $Q(0) = 0$, and we define the stability and the unstability of the fluid queue as follows.

Definition 2.1 *The queue is said to be stable if there exists an almost surely finite random variable W such that $Q(t)$ converges in law to W when t tends to infinity. The queue is said to be unstable if $\forall x \geq 0$, $\lim_{t\to\infty} P(Q(t) > x) = 1$.*

3 Lindley's stationarity equation

As usual, a key to state a stability criterion is to solve an equation of the Lindley-type (see [9]). The unique solution to the RSDE (2) is in fact a function of the real number u and the nonnegative random variable Y. We consider the solution when $Y = 0$ and we denote it by $(Q_u(t), L_u(t))_{t\geq u}$ to make visible the dependence on u. $Q_u(t)$ is thus the amount of fluid in the buffer at time t when the buffer is empty at time u. We set by convention $Q_u(t) = 0$ and $L_u(t) = 0$ for $t \leq u$. We then have the following results.

Proposition 3.1 *Let $(Y_t^1, k_t^1)_{t\geq u}$ and $(Y_t^2, k_t^2)_{t\geq u}$ be two solutions to the following RSDE*

$$\begin{cases} dY_t &= b(X(t), Y_t)dt + \sigma(X(t), Y_t)dB_t + dk_t \quad \text{for } t \geq u \\ Y_t &\geq 0 \quad \text{for } t \geq u \\ k_t &= \int_u^t \mathbb{1}_{\{Y_s=0\}} dk_s \quad \text{for } t \geq u \end{cases} \tag{4}$$

where $(k_t)_{t\geq u}$ is an increasing continuous process.
If $Y_u^1 \geq Y_u^2$ then we have, for all $t \geq u$,

1) $Y_t^1 \geq Y_t^2$,

2) $\forall h \geq 0$, $0 \leq k_{t+h}^1 - k_t^1 \leq k_{t+h}^2 - k_t^2$.

Proof. See Appendix A. ■

This proposition is used to prove the following lemma.

Lemma 3.2 *Let u, t, and t' be arbitrary real numbers with $u \le t \le t'$. For every $h > 0$, we have*

1) $0 \le Q_u(t) \le Q_{u-h}(t)$.

2) $0 \le L_{u-h}(t') - L_{u-h}(t) \le L_u(t') - L_u(t)$.

Proof. Consider the RSDE (4) with $Y_u = 0$. From equation (2), the unique solution to this equation is given by $Y_t^2 = Q_u(t)$ and $k_t^2 = L_u(t)$.

Let $h \ge 0$ and consider again the RSDE (4) with $Y_u = Q_{u-h}(u)$. From equation (2), the unique solution to this equation is given by $Y_t^1 = Q_{u-h}(t)$ and $k_t^1 = L_{u-h}(t) - L_{u-h}(u)$.

We have $Y_u^2 = 0 \le Q_{u-h}(u) = Y_u^1$, so we can apply proposition 3.1 which leads to

$$Q_{u-h}(t) = Y_t^1 \ge Y_t^2 = Q_u(t),$$

and, since $t' \ge t$,

$$L_{u-h}(t') - L_{u-h}(t) = k_{t'}^1 - k_t^1 \le k_{t'}^2 - k_t^2 = L_u(t') - L_u(t),$$

which completes the proof. ■

The stationary equation of the Lindley-type is given in the following proposition.

Proposition 3.3 *There exists a nonnegative random variable W which is either finite a.s. or infinite a.s., and a process $\{L(t,v)\}_{t \ge v}$, increasing in t and decreasing in v, satisfying, whenever W is finite, $\forall t \ge v$,*

$$W \circ \theta_t = W \circ \theta_v + \int_v^t b(X(s), W \circ \theta_s)ds + \int_v^t \sigma(X(s), W \circ \theta_s)dB_s + L(t,v) \quad P \text{ a.s.} \tag{5}$$

Proof. Consider again the solution $(Q_u(s), L_u(s))_{s \ge u}$ to equation (2) with $Y = 0$ and $Q_u(s) = 0$ and $L_u(s) = 0$ for $s \le u$. Since $(Q_u(s), L_u(s))$ is the solution to (2) then, for every $t \in \mathbb{R}$, $(Q_u(s) \circ \theta_t, L_u(s) \circ \theta_t)_{s \ge u}$ is the solution to

$$\left\{ \begin{array}{lcl} dQ(s) & = & b(X(s) \circ \theta_t, Q(s))ds + \sigma(X(s) \circ \theta_t, Q(s))d(B_s \circ \theta_t) + dL(s) \\ Q(u) & = & 0 \\ Q(s) & \ge & 0 \\ L(s) & = & \displaystyle\int_u^s \mathbb{1}_{\{Q(x)=0\}} dL(x) \end{array} \right. \tag{6}$$

The processes $\{X(t)\}$ and $\{B_t\}$ being compatible with the flow $\{\theta_t\}$, we have that, for $\omega \in \Omega$, $X(s, \theta_t\omega) = X(s+t, \omega)$ and $B_s \circ \theta_t(\omega) = B_{s+t}(\omega)$. It thus follows that, system (6) is equal to

$$\left\{ \begin{array}{lcl} dQ(s) & = & b(X(s+t), Q(s))ds + \sigma(X(s+t), Q(s))dB_{s+t} + dL(s) \\ Q(u) & = & 0 \\ Q(s) & \ge & 0 \\ L(s) & = & \displaystyle\int_u^s \mathbb{1}_{\{Q(x)=0\}} dL(x) \end{array} \right.$$

By the uniqueness of their solution, we have

$$\forall t, s \in \mathbb{R} \quad (Q_u(s) \circ \theta_t, L_u(s) \circ \theta_t) = (Q_{u+t}(s+t), L_{u+t}(s+t)) \quad \text{P a.s.} \tag{7}$$

Let u, t and t' be real numbers such that $u \leq t \leq t'$. From Lemma 3.2, $Q_u(0)$ increases when u decreases and the difference $L_u(t') - L_u(t)$ decreases when u decreases. We thus define the following limits

$$W = \lim_{u \searrow -\infty} \nearrow Q_u(0) \text{ and } L(t', t) = \lim_{u \searrow -\infty} \searrow (L_u(t') - L_u(t)).$$

Intuitively, W is the level of fluid in the buffer at time $t = 0$ when the buffer is empty at time $-\infty$.

The rest of the proof is divided in two steps.

Step 1: We show that the event $\{W = +\infty\}$ is θ_t-invariant for every $t \in \mathbb{R}$. We detail the proof only for nonnegative values of t, the opposite case being similar.

Let u, v, and t be real numbers such that $u \leq v \leq t$ with $u \leq 0$ and $t \geq 0$. We have

$$\begin{aligned} Q_u(0) \circ \theta_t &= Q_{u+t}(t) \\ &= Q_{u+t}(v) + \int_v^t b(X(s), Q_{u+t}(s))ds + \int_v^t \sigma(X(s), Q_{u+t}(s))dB_s \\ &\quad + L_{u+t}(t) - L_{u+t}(v) \\ &= Q_{u+t-v}(0) \circ \theta_v + \int_v^t b(X(s), Q_{u+t-s}(0) \circ \theta_s)ds + \int_v^t \sigma(X(s), Q_{u+t-s}(0) \circ \theta_s)dB_s \\ &\quad + L_{u+t}(t) - L_{u+t}(v), \end{aligned} \tag{8}$$

where the first and third equalities are due to relation (7), and the second one is obtained by using relation (2).

If $v = 0$, we obtain $u \leq 0 \leq t$ and

$$\begin{aligned} Q_u(0) \circ \theta_t &= Q_{u+t}(0) + \int_0^t b(X(s), Q_{u+t-s}(0) \circ \theta_s)ds + \int_0^t \sigma(X(s), Q_{u+t-s}(0) \circ \theta_s)dB_s \\ &\quad + L_{u+t}(t) - L_{u+t}(0). \end{aligned} \tag{9}$$

Consider separately the three terms of the right hand side of (9).

$\int_0^t b(X(s), Q_{u+t-s}(0) \circ \theta_s)ds$ is uniformly lower bounded with respect to u because b is lower bounded.

From Lemma 3.2, we have for $u + t < 0$,

$$L_{u+t}(t) - L_{u+t}(0) \leq L_0(t) - L_0(0) = L_0(t) < \infty \quad \text{a.s.}$$

Thus, $L_{u+t}(t) - L_{u+t}(0)$ is uniformly bounded with respect to u a.s., for $u + t < 0$.

The function σ is bounded, so there exists a sequence of random variables $\{u_k\}_{k \in \mathbb{N}}$ tending to $-\infty$ as $k \to +\infty$, and an $(\mathcal{F}_t)$-adapted process $\{l(s)\}_{s \leq t}$, such that

$$\sigma(X(s), Q_{u_k+t-s}(0) \circ \theta_s) \underset{k \to \infty}{\longrightarrow} l(s) \quad \text{a.s.}$$

By using the dominated convergence theorem, we get

$$\int_0^t \sigma(X(s), Q_{u_k+t-s}(0) \circ \theta_s)dB_s \underset{k \to \infty}{\longrightarrow} \int_0^t l(s)dB_s \quad \text{in } L^2$$

where L^2 denotes the set of square integrable processes on $(\Omega, \mathcal{F}, P)$. We may suppose that this convergence occurs almost surely, by extracting a subsequence from the sequence $\{u_k\}_{k\in\mathbb{N}}$. Hence, we may suppose that the sequence $\left\{\int_0^t \sigma(X(s), Q_{u_k+t-s}(0)\circ\theta_s)dB_s\right\}_{k\in\mathbb{N}}$ is bounded almost surely.

Suppose that for a fixed ω we have $W(\omega) = +\infty$. We thus have, by definition of W,

$$\lim_{k\longrightarrow\infty} Q_{u_k(\omega)+t}(0,\omega) = W(\omega) = +\infty.$$

By writing the relation (9) with u_k instead of u and by taking the limit when $k\to\infty$, we get

$$\lim_{k\longrightarrow\infty} Q_{u_k(\omega)}(0,\theta_t\omega) = +\infty.$$

By definition of W, we obtain that $W(\theta_t\omega) = +\infty$. This means that the event $\{W = +\infty\}$ is θ_t-invariant and thus, by ergodicity, we have

$$P(W = +\infty) = 0 \quad \text{or} \quad P(W = +\infty) = 1.$$

Step 2: Suppose that W is finite a.s. and consider the relation (8). As the function b is continuous with respect to the second variable, and since $Q_{u+t-s}(0)\circ\theta_s$ converges increasingly to $W\circ\theta_s$ when u tends to $-\infty$, we get

$$\int_v^t b(X(s), Q_{u+t-s}(0)\circ\theta_s)ds \underset{u\searrow-\infty}{\longrightarrow} \int_v^t b(X(s), W\circ\theta_s)ds.$$

As the function σ is continuous with respect to the second variable and bounded, we get

$$\int_v^t \sigma(X(s), Q_{u+t-s}(0)\circ\theta_s)dB_s \underset{u\searrow-\infty}{\longrightarrow} \int_v^t \sigma(X(s), W\circ\theta_s)dB_s \quad \text{in } L^2.$$

As seen previously, we may consider, by taking a subsequence, that this convergence holds almost surely.

Finally, by definition, we have $L_{u+t}(t) - L_{u+t}(v) \longrightarrow L(t,v)$ a.s. when $u\to-\infty$ and so, by taking the limit when $u\longrightarrow-\infty$ in relation (8), we obtain the desired relation (5). ∎

Proposition 3.4 *The amount of fluid $\{Q_0(t)\}_{t\geq 0}$ converges in law to W, that is*

$$Q_0(t) \underset{t\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} W.$$

Proof. Since the process X is stationary, and as $Q_0(0) = 0$, we have that

$$\forall u \leq 0,\ Q_u(0) \overset{\mathcal{L}}{=} Q_0(-u).$$

Then by letting $u\to-\infty$, we get $Q_0(t) \underset{t\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} W$. ∎

Note that in the case where $W = +\infty$ a.s., we have

$$\forall x \geq 0, \quad P(Q_0(t) > x) = P(Q_{-t}(0) > x) \underset{t\to\infty}{\longrightarrow} 1.$$

4 Stability of the queue

In order to establish the main result of this paper, we need the two following technical lemmas.

Lemma 4.1 *Let V be a nonnegative, a.s. finite random variable such that, for every $t \in \mathbb{R}$, $V \circ \theta_t - V \in L^1$. Then $E(V \circ \theta_t - V) = 0$ for every $t \in \mathbb{R}$.*

Proof. The proof is fairly standard (see for instance [3, p. 77]). For all $M > 0$, the random variables $(V \circ \theta_t) \wedge M$ and $V \wedge M$ are both in L^1 and, since the probability P is θ_t-invariant, we have

$$E((V \circ \theta_t) \wedge M - V \wedge M) = 0.$$

Moreover,

$$|(V \circ \theta_t) \wedge M - V \wedge M| \leq |V \circ \theta_t - V| \in L^1$$

and $(V \circ \theta_t) \wedge M - V \wedge M$ converges to $V \circ \theta_t - V$ as M tends to infinity. By the dominated convergence theorem, we obtain that

$$E(V \circ \theta_t \wedge M - V \wedge M) \underset{M \to \infty}{\longrightarrow} E(V \circ \theta_t - V),$$

and so we conclude that $E(V \circ \theta_t - V) = 0$. ■

Lemma 4.2 *For all $t, u \in \mathbb{R}$ such that $u \leq t$, we have*

1) $\sup_{x \geq 0} |b(X(0), x)| \in L^1$.

2) $\int_u^t \sup_{x \geq 0} |b(X(v), x)| dv \in L^1$.

3) $L_u(t) \in L^1$.

Proof. Let m be the lower bound of function b. For the first point, we have

$$\begin{aligned} E\left(\sup_{x \geq 0} |b(X(0), x)|\right) &\leq E\left(\sup_{x \geq 0} |b(X(0), x) - m|\right) + |m| \\ &= E\left(\sup_{x \geq 0} b(X(0), x)\right) - m + |m| \\ &< +\infty \quad \text{by hypothesis.} \end{aligned}$$

For the second point, by Fubini's theorem and by the stationarity of $\{X(t)\}$, we get

$$\begin{aligned} E\left(\int_u^t \sup_{x \geq 0} |b(X(v), x)| dv\right) &= \int_u^t E\left(\sup_{x \geq 0} |b(X(v), x)|\right) dv \\ &= (t - u) E\left(\sup_{x \geq 0} |b(X(0), x)|\right) \\ &< +\infty \quad \text{from point 1).} \end{aligned}$$

For the third point, recall that in view of (3),

$$L_u(t) = \sup_{s\in[u,t]} \left(- \int_u^s [b(X(v), Q_u(v))dv + \sigma(X(v), Q_u(v))dB_v] \right)^+ .$$

Hence, as $L_u(t) \geq 0$, we have

$$\begin{aligned} 0 \leq E(L_u(t)) &= E\left(\sup_{s\in[u,t]} \left(- \int_u^s [b(X(v), Q_u(v))dv + \sigma(X(v), Q_u(v))dB_v] \right)^+ \right) \\ &\leq E\left(\sup_{s\in[u,t]} \left| \int_u^s [b(X(v), Q_u(v))dv + \sigma(X(v), Q_u(v))dB_v] \right| \right) \\ &\leq E\left(\sup_{s\in[u,t]} \left| \int_u^s b(X(v), Q_u(v))dv \right| \right) + E\left(\sup_{s\in[u,t]} \left| \int_u^s \sigma(X(v), Q_u(v))dB_v \right| \right). \end{aligned}$$

Consider these two terms separately. We have

$$\begin{aligned} E\left(\sup_{s\in[u,t]} \left| \int_u^s b(X(v), Q_u(v))dv \right| \right) &\leq E\left(\sup_{s\in[u,t]} \int_u^s |b(X(v), Q_u(v))| \, dv \right) \\ &= E\left(\int_u^t |b(X(v), Q_u(v))| \, dv \right) \\ &\leq E\left(\int_u^t \sup_{x\geq 0} |b(X(v), x)| dv \right) \\ &< +\infty \quad \text{from point 2)}. \end{aligned}$$

Let M denote the bound of the function σ. By the Burkholder-Davis-Gundy inequality [5, p. 166], there exists a constant C such that

$$\begin{aligned} E\left(\sup_{s\in[u,t]} \left| \int_u^s \sigma(X(v), Q_u(v))dB_v \right| \right) &\leq CE\left(\left(\int_u^t [\sigma(X(v), Q_u(v))]^2 dv \right)^{1/2} \right) \\ &\leq C(t-u)^{1/2} M < +\infty, \end{aligned}$$

which completes the proof. ■

The main contribution of this paper is the following theorem. It establishes that the buffer does not fill indefinitely as long as, on average, the drift is negative when the level of fluid in the buffer is high. The second assertion means that, since the drift is in average positive, the fluid builds up in the buffer. It can also be noted that the random component in the equation does not affect the stability condition.

Theorem 4.3

1) *If* $E(\limsup_{x\to\infty} b(X(0), x)) < 0$ *then the queue is stable.*

2) *If* $E(\inf_{x\geq 0} b(X(0), x)) > 0$ *then the queue is unstable.*

Proof. The assumption $\sup_{x\geq 0} b(X(0),x) \in L^1$ implies that $\limsup_{x\to\infty} b(X(0),x)) \in L^1$. To prove the first assertion it suffices to prove that if $W = +\infty$ a.s. then $E(\limsup_{x\to\infty} b(X(0),x)) \geq 0$.

So, suppose that $W = +\infty$ a.s. and let $u < 0$ and $t \geq 0$. Since P is θ_t-invariant we have

$$E(Q_u(0)\circ\theta_t) = E(Q_u(0)).$$

From Lemma 3.2, $Q_u(0)$ is decreasing in u, thus we have $E(Q_u(0) - Q_{u+t}(0)) \geq 0$. Combining these two relations, we obtain that

$$E(Q_u(0)\circ\theta_t - Q_{u+t}(0)) = E(Q_u(0) - Q_{u+t}(0)) \geq 0.$$

Replacing the value of $Q_u(0)\circ\theta_t - Q_{u+t}(0)$ by its expression in relation (9), we get, since the martingale part has a null expectation,

$$0 \leq E\left(\int_0^t b(X(s), Q_{u+t-s}(0)\circ\theta_s)ds\right) + E(L_{u+t}(t) - L_{u+t}(0)). \tag{10}$$

Observing that $Q_{v+t-s}(0)\circ\theta_s$ increases from $Q_{u+t-s}(0)\circ\theta_s$ to $+\infty$ when v decreases from u to $-\infty$, we obtain that

$$\begin{aligned} b(X(s), Q_{u+t-s}(0)\circ\theta_s) &\leq \sup_{v\leq u} b(X(s), Q_{v+t-s}(0)\circ\theta_s) \\ &= \sup_{x\geq Q_{u+t-s}(0)\circ\theta_s} b(X(s),x) \\ &\leq \sup_{x\geq 0} b(X(s),x), \end{aligned}$$

and $\int_0^t \sup_{x\geq 0} b(X(s),x)ds \in L^1$ from the point 2) of Lemma 4.2. By definition of W, we have $\lim_{u\searrow -\infty} \nearrow Q_u(0) = W = +\infty$, thus using the dominated convergence theorem, we get

$$E\left(\int_0^t \sup_{x\geq Q_{u+t-s}(0)\circ\theta_s} b(X(s),x)ds\right) \underset{u\searrow-\infty}{\longrightarrow} E\left(\int_0^t \limsup_{x\to+\infty} b(X(s),x)ds\right).$$

On the other hand, we have using the second assertion of Lemma 3.2, that $dL_{u+t}(s) \leq dL_0(s)$, for $u \leq -t$. Thus

$$0 \leq L_{u+t}(t) - L_{u+t}(0) = \int_0^t \mathbb{1}_{\{Q_{u+t}(s)=0\}} dL_{u+t}(s) \leq \int_0^t \mathbb{1}_{\{Q_{u+t}(s)=0\}} dL_0(s). \tag{11}$$

We have seen in the previous section that the difference $L_{u+t}(t) - L_{u+t}(0)$ converges decreasingly to a limit $L(t,0)$ when $u \searrow -\infty$. Since $W = +\infty$, we obtain by using the dominated convergence theorem that

$$L_{u+t}(t) - L_{u+t}(0) \underset{u\searrow-\infty}{\longrightarrow} L(t,0) = 0.$$

Thanks to Lemma 4.2, $L_{u+t}(t) - L_{u+t}(0) \in L^1$, and thus

$$E(L_{u+t}(t) - L_{u+t}(0)) \underset{u\searrow-\infty}{\longrightarrow} E(L(t,0)) = 0.$$

Using these results, the relation (10), at point $t=1$, leads to

$$0 \leq E\left(\int_0^1 \limsup_{x\to+\infty} b(X(s),x)ds\right) = E\left(\limsup_{x\to+\infty} b(X(0),x)\right),$$

by Fubini's theorem and the stationarity of the process $\{X(t)\}$. This completes the proof of the first assertion.

To prove the second assertion, let us suppose that $W<+\infty$ a.s., and let $t\geq 0$.

In that case, $\int_0^t \sigma(X(s),W\circ\theta_s)dB_s$ is integrable since σ is bounded. Concerning function b, we have, from point 2) of Lemma 4.2

$$\begin{aligned} E\left(\left|\int_0^t b(X(s),W\circ\theta_s)ds\right|\right) &\leq E\left(\int_0^t |b(X(s),W\circ\theta_s)|\,ds\right) \\ &\leq E\left(\int_0^t \sup_{x\geq 0}|b(X(s),x)|\,ds\right) < +\infty. \end{aligned}$$

Moreover, $L(t,0)$ is also integrable since

$$0\leq L(t,0)=\lim_{u\searrow-\infty,\ u\leq 0}\searrow [L_u(t)-L_u(0)]\leq L_0(t)-L_0(0)=L_0(t),$$

and from point 3) of Lemma 4.2, $L_0(t)\in L^1$. It thus follows from relation (5), for $v=0$, that $W\circ\theta_t - W$ is integrable and, from Lemma 4.1, we obtain $E(W\circ\theta_t-W)=0$. This leads, by using again Fubini's Theorem and the stationarity of the process $\{X(t)\}$, to

$$\begin{aligned} 0=E(W\circ\theta_t-W) &= E\left(\int_0^t b(X(s),W\circ\theta_s)ds\right)+E(L(t,0)) \\ &= \int_0^t E\left(b(X(s),W\circ\theta_s)\right)ds + E(L(t,0)) \\ &= tE\left(b(X(0),W)\right)+E(L(t,0)). \end{aligned} \tag{12}$$

Now, since, $L(t,0)\geq 0$, we obtain at point $t=1$,

$$0\geq E\left(b(X(0),W)\right)\geq E\left(\inf_{x\geq 0} b(X(0),x)\right),$$

which completes the proof. ■

Corollary 4.4 *If* $E(\limsup_{x\to\infty} b(X(0),x))<0$ *then for every* $t\geq 0$*, we have*

$$E(L(t,0)) = -tE(b(X(0),W)).$$

Proof. If $E(\limsup_{x\to\infty} b(X(0),x))<0$, we have from Theorem 4.3 that the queue is stable, which means that W is a.s. finite. We thus get immediately the result from relation (12). ■

This corollary leads to the following interesting remarks in the case where the condition $E(\limsup_{x\to\infty} b(X(0),x))<0$ is satisfied, which implies in particular, from Theorem 4.3, that the queue is stable.

1) $E(b(X(0), W)) \leq 0$, since $L(t, 0)$ is nonnegative.

2) If $E(b(X(0), W)) < 0$ then $L(t, 0) > 0$ on a non negligible set. This means that we are sure that the buffer empties in stationary regime, i.e. $P(W = 0) > 0$. Indeed, if $W > 0$ a.s. then from relation (11) with $u \searrow -\infty$, we obtain $L(t, 0) = 0$.

3) If $E(b(X(0), W)) = 0$ then $L(t, 0) = 0$, since $L(t, 0)$ is nonnegative. This means that the buffer may or may not be emptied in stationary regime, i.e. we can have either $P(W = 0) = 0$ or $P(W = 0) > 0$. To illustrate these situations, let us consider the following deterministic cases:

   - $\sigma(X(t), x) = 0$ and $b(X(t), x) = 0$. In that case, we have $dQ(t) = 0$ which means that $Q(t) = 0$ and so $W = 0$. We thus get $E(b(X(0), W)) = 0$ and the buffer is always empty.
   - $\sigma(X(t), x) = 0$ and $b(X(t), x) = \cos(x)$. In that case, we have $dQ(t) = \cos(Q(t))dt$ which means that $Q(t) = 2\arctan(e^t) - \pi/2$ and so $W = \pi/2$. We thus have $E(b(X(0), W)) = \cos(\pi/2) = 0$, and the buffer is never empty in stationary regime.

   Besides, we may note, from relation (5) that, since $L(t, 0) = 0$, $\{W \circ \theta_s\}_s$ satisfies a conventional (non reflected) SDE [5].

Appendix A. Proof of Proposition 3.1

The proof of the first point of Proposition 3.1 can be found in [6]. Roughly speaking, the second point states that, since Y^1 is above Y^2, then Y^1 hits 0 less often than Y^2.

Formally, let $t \geq u$, $h > 0$. The process $\{k_t^1\}$ being increasing, we have $k_{t+h}^1 - k_t^1 \geq 0$. It remains to prove that $(k_{t+h}^1 - k_t^1) - (k_{t+h}^2 - k_t^2) \leq 0$.

By definition of k_t, we have

$$
\begin{aligned}
(k_{t+h}^1 - k_t^1) - (k_{t+h}^2 - k_t^2) &= \int_t^{t+h} \mathbb{1}_{\{Y_s^1=0\}} dk_s^1 - \int_t^{t+h} \mathbb{1}_{\{Y_s^2=0\}} dk_s^2 \\
&= \int_t^{t+h} \mathbb{1}_{\{Y_s^1=0\}} d(k^1 - k^2)_s + \int_t^{t+h} \left(\mathbb{1}_{\{Y_s^1=0\}} - \mathbb{1}_{\{Y_s^2=0\}}\right) dk_s^2 \\
&\leq \int_t^{t+h} \mathbb{1}_{\{Y_s^1=0\}} d(k^1 - k^2)_s,
\end{aligned}
$$

where we have used the result of the first point, that is $Y_s^1 \geq Y_s^2$, to get $\mathbb{1}_{\{Y_s^1=0\}} \leq \mathbb{1}_{\{Y_s^2=0\}}$.

In order to complete the proof, it suffices to show that $\int_t^{t+h} \mathbb{1}_{\{Y_s^1=0\}} d(k^1 - k^2)_s = 0$.

Let us define

$$\tau = \inf\{v \geq t | Y_v^1 = 0\}.$$

τ is a stopping time taking its values in $[t, +\infty]$. Since $Y_s^1 > 0$ for $s \in [t, \tau)$, we have

$$\int_t^{\tau \wedge (t+h)} \mathbb{1}_{\{Y_s^1=0\}} d(k^1 - k^2)_s = 0,$$

and thus,

$$\int_t^{t+h} \mathbb{1}_{\{Y_s^1=0\}} d(k^1 - k^2)_s = \int_{\tau \wedge (t+h)}^{t+h} \mathbb{1}_{\{Y_s^1=0\}} d(k^1 - k^2)_s.$$

Finally, we have to show that

$$\int_{\tau\wedge(t+h)}^{t+h} \mathbb{1}_{\{Y_s^1=0\}} d(k^1-k^2)_s = 0. \tag{13}$$

By definition of τ, we have $Y_\tau^1 = 0$. Since, from the first point of the proposition, we have $0 \le Y_s^2 \le Y_s^1$, it follows that $Y_\tau^2 = Y_\tau^1 = 0$. So the idea is to show that since at time τ both processes are equal, they are also equal for $s > \tau$, when τ is finite, that is $Y_s^2 = Y_s^1$ for $s \ge \tau$. We will then obtain $dk_s^1 = dk_s^2$ for $s \ge \tau$, which leads to relation (13).

For that purpose, let us introduce a sequence $(\varphi_n)_n$ of functions of $\mathcal{C}^2(\mathbb{R},\mathbb{R})$ such that $\varphi_n'(0) = \varphi_n''(0) = 0$ and

$$\begin{array}{rcl} 0 & \le & \varphi_n(x) \underset{n\to+\infty}{\nearrow} -(x^-)^2 \\ 0 & \ge & \varphi_n'(x) \underset{n\to+\infty}{\nearrow} 2x^- \\ 0 & \ge & \varphi_n''(x) \underset{n\to+\infty}{\searrow} -2\mathbb{1}_{\{x<0\}}. \end{array}$$

This can be done by taking for instance

$$\varphi_n(x) = \begin{cases} 0 & \text{if} \quad x \ge 0 \\ \dfrac{nx^3}{3} & \text{if} \quad -\dfrac{1}{n} < x < 0 \\ -x^2 - \dfrac{x}{n} - \dfrac{1}{3n^2} & \text{if} \quad x \le -\dfrac{1}{n}. \end{cases}$$

Let us define $Z_s = Y_s^2 - Y_s^1$ and $\overline{Z}_s = Z_{s\vee\tau} = Y_{s\vee\tau}^2 - Y_{s\vee\tau}^1$. We set for convenience $Z_{+\infty} = 0$, so that $\overline{Z}_\tau = 0$. Applying Itô's formula to $\varphi_n(Z_s)$, we get

$$d\varphi_n(Z_s) = \varphi_n'(Z_s)dZ_s + \frac{1}{2}\varphi_n''(Z_s)d <Z>_s,$$

and if τ_1 and τ_2 are uniformly bounded stopping times such that $\tau_1 \le \tau_2$, we obtain

$$\varphi_n(Z_{\tau_2}) - \varphi_n(Z_{\tau_1}) = \int_{\tau_1}^{\tau_2} \varphi_n'(Z_s)dZ_s + \frac{1}{2}\int_{\tau_1}^{\tau_2} \varphi_n''(Z_s)d <Z>_s . \tag{14}$$

Let $T \in [t, t+h]$ and consider the stopping times $\tau_1 = (t\vee\tau)\wedge(t+h)$ and $\tau_2 = (T\vee\tau)\wedge(t+h)$. Note that, in fact $\tau_1 = \tau\wedge(t+h)$, since by definition of τ, we have $t \le \tau$. We have $\tau_1 \le \tau_2 \le t+h$ and $\varphi_n(Z_{\tau_2}) - \varphi_n(Z_{\tau_1}) = \varphi_n(\overline{Z}_T) - \varphi_n(\overline{Z}_t)$. Indeed,

- if $\tau \ge t+h$, $\tau_2 = \tau_1 = t+h$ and $\overline{Z}_T = Z_{T\vee\tau} = Z_\tau = Z_{t\vee\tau} = \overline{Z}_t$
- if $\tau < t+h$, $\tau_2 = T\vee\tau$, which gives $Z_{\tau_2} = Z_{T\vee\tau} = \overline{Z}_T$ and $\tau_1 = \tau$, which gives $Z_{\tau_1} = Z_\tau = \overline{Z}_t$.

Relation (14) can thus be written as

$$\varphi_n(\overline{Z}_T) - \varphi_n(\overline{Z}_t) = \int_{\tau\wedge(t+h)}^{(T\vee\tau)\wedge(t+h)} \varphi_n'(Z_s)dZ_s + \frac{1}{2}\int_{\tau\wedge(t+h)}^{(T\vee\tau)\wedge(t+h)} \varphi_n''(Z_s)d <Z>_s .$$

By distinguishing again the cases $\tau \ge t+h$ and $\tau < t+h$, we have

$$\mathbb{1}_{\{s<\tau\}}\mathbb{1}_{\{s\in[\tau\wedge(t+h),(T\vee\tau)\wedge(t+h))\}} = 0 \text{ and } \mathbb{1}_{\{s\ge\tau\}}\mathbb{1}_{\{s\in[\tau\wedge(t+h),(T\vee\tau)\wedge(t+h))\}} = \mathbb{1}_{\{s\ge\tau\}}\mathbb{1}_{\{s\in[t,T)\}},$$

thus we get

$$\varphi_n(\overline{Z}_T) - \varphi_n(\overline{Z}_t) = \int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s)dZ_s + \frac{1}{2}\int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n''(Z_s)d <Z>_s .$$

Recalling that $Z_s = Y_s^2 - Y_s^1$ and using relation (4), we obtain

$$\begin{aligned}\varphi_n(\overline{Z}_T) - \varphi_n(\overline{Z}_t) &= \int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s)\Big[b(X(s),Y_s^2) - b(X(s),Y_s^1)\Big]ds \\ &+ \int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s)\Big[\sigma(X(s),Y_s^2) - \sigma(X(s),Y_s^1)\Big]dB_s \\ &+ \int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s)d(k^2-k^1)_s \\ &+ \frac{1}{2}\int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n''(Z_s)\Big[\sigma(X(s),Y_s^2) - \sigma(X(s),Y_s^1)\Big]^2 ds.\end{aligned}$$

Now, since the conditions $s \geq \tau$ and $s \in [t,T)$ imply that $(s\vee\tau)\wedge(t+h) = s$, we have

$$\begin{aligned}\varphi_n(\overline{Z}_T) - \varphi_n(\overline{Z}_t) &= \int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s)\Big[b(X(s),Y^2_{(s\vee\tau)\wedge(t+h)}) - b(X(s),Y^1_{(s\vee\tau)\wedge(t+h)})\Big]ds \\ &+ \int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s)\Big[\sigma(X(s),Y^2_{(s\vee\tau)\wedge(t+h)}) - \sigma(X(s),Y^1_{(s\vee\tau)\wedge(t+h)})\Big]dB_s \\ &+ \int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s)d(k^2-k^1)_s \\ &+ \frac{1}{2}\int_t^T \mathbb{1}_{\{s\geq\tau\}}\varphi_n''(Z_s)\Big[\sigma(X(s),Y^2_{(s\vee\tau)\wedge(t+h)}) - \sigma(X(s),Y^1_{(s\vee\tau)\wedge(t+h)})\Big]^2 ds.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\varphi_n'(\overline{Z}_s) &= \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(\overline{Z}_s) + \mathbb{1}_{\{s<\tau\}}\varphi_n'(\overline{Z}_s) \\ &= \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s) + \mathbb{1}_{\{s<\tau\}}\varphi_n'(0) \\ &= \mathbb{1}_{\{s\geq\tau\}}\varphi_n'(Z_s), \text{ since } \varphi_n'(0) = 0,\end{aligned}$$

and the same result holds for φ_n'', so we get

$$\begin{aligned}\varphi_n(\overline{Z}_T) - \varphi_n(\overline{Z}_t) &= \int_t^T \varphi_n'(\overline{Z}_s)\Big[b(X(s),Y^2_{(s\vee\tau)\wedge(t+h)}) - b(X(s),Y^1_{(s\vee\tau)\wedge(t+h)})\Big]ds \\ &+ \int_t^T \varphi_n'(\overline{Z}_s)\Big[\sigma(s,Y^2_{(s\vee\tau)\wedge(t+h)}) - \sigma(s,Y^1_{(s\vee\tau)\wedge(t+h)})\Big]dB_s \\ &+ \int_t^T \varphi_n'(\overline{Z}_s)d(k^2-k^1)_s \\ &+ \frac{1}{2}\int_t^T \varphi_n''(Z_s)\Big[\sigma(X(s),Y^2_{(s\vee\tau)\wedge(t+h)}) - \sigma(X(s),Y^1_{(s\vee\tau)\wedge(t+h)})\Big]^2 ds.\end{aligned}$$

Taking the expectation, and since the martingale part has a null expectation, we obtain

$$\begin{aligned}E(\varphi_n(\overline{Z}_T)) - E(\varphi_n(\overline{Z}_t)) = &\int_t^T E\Big(\varphi_n'(\overline{Z}_s)\Big[b(X(s),Y^2_{(s\vee\tau)\wedge(t+h)}) - b(X(s),Y^1_{(s\vee\tau)\wedge(t+h)})\Big]\Big)ds \\ &+ E\left(\int_t^T \varphi_n'(\overline{Z}_s)d(k^2-k^1)_s\right) \\ &+ \int_t^T E\left(\frac{1}{2}\varphi_n''(\overline{Z}_s)\Big[\sigma(X(s),Y^2_{(s\vee\tau)\wedge(t+h)}) - \sigma(X(s),Y^1_{(s\vee\tau)\wedge(t+h)})\Big]^2\right) ds.\end{aligned}$$

Taking now the limit when $n \to +\infty$, we get

$$
\begin{aligned}
E((\overline{Z}_T^-)^2) - E((\overline{Z}_t^-)^2) = & \int_t^T E\Big(-2\overline{Z}_s^- \left[b(X(s), Y^2_{(s\vee\tau)\wedge(t+h)}) - b(X(s), Y^1_{(s\vee\tau)\wedge(t+h)}) \right] \Big) ds \\
& + E\left(\int_t^T -2\overline{Z}_s^- d(k^2 - k^1)_s \right) \\
& + \int_t^T E\left(\mathbb{1}_{\{\overline{Z}_s < 0\}} \left[\sigma(X(s), Y^2_{(s\vee\tau)\wedge(t+h)}) - \sigma(X(s), Y^1_{(s\vee\tau)\wedge(t+h)}) \right]^2 \right) ds.
\end{aligned}
$$

Note that since $t \leq \tau$, we have $\overline{Z}_t = Z_\tau = 0$, so $E((\overline{Z}_t^-)^2) = 0$. We treat the first and the third term of this last relation together and the second one separately.

- Consider the first and the third term. Let $s \in [t, T)$. If $s \geq \tau$, we have $Z_{(s\vee\tau)\wedge(t+h)} = Z_s = \overline{Z}_s$ and if $s < \tau$, we have $\overline{Z}_s = Z_\tau = 0$. So, since the functions b and σ are Lipschitz we get

$$
\begin{aligned}
2\overline{Z}_s^- \left| b(X(s), Y^2_{(s\vee\tau)\wedge(t+h)}) - b(X(s), Y^1_{(s\vee\tau)\wedge(t+h)}) \right| & \leq 2C\overline{Z}_s^- |\overline{Z}_s| \\
& = 2C(\overline{Z}_s^-)^2,
\end{aligned}
$$

  and

$$
\begin{aligned}
\mathbb{1}_{\{\overline{Z}_s < 0\}} \left[\sigma(X(s), Y^2_{(s\vee\tau)\wedge(t+h)}) - \sigma(X(s), Y^1_{(s\vee\tau)\wedge(t+h)}) \right]^2 & \leq C^2 \mathbb{1}_{\{\overline{Z}_s < 0\}} |\overline{Z}_s|^2 \\
& = C^2 (\overline{Z}_s^-)^2.
\end{aligned}
$$

- Consider now the second term. We have $\overline{Z}_s^- = \overline{Z}_s^- \mathbb{1}_{\{\overline{Z}_s < 0\}}$, and

$$
0 \leq \mathbb{1}_{\{\overline{Z}_s < 0\}} dk_s^1 \leq \mathbb{1}_{\{Y_s^2 - Y_s^1 < 0\}} dk_s^1 \leq \mathbb{1}_{\{Y_s^1 > 0\}} dk_s^1 = 0,
$$

  by definition of the process (k_s^1). It thus follows that

$$
\int_t^T \overline{Z}_s^- dk_s^1 = 0,
$$

  which leads to

$$
\int_t^T -2\overline{Z}_s^- d(k^2 - k^1)_s = -2 \int_t^T \overline{Z}_s^- dk_s^2 \leq 0.
$$

  In particular, we get

$$
E\left(\int_t^T -2\overline{Z}_s^- d(k^2 - k^1)_s \right) \leq 0.
$$

Putting together these results, we obtain

$$
E((\overline{Z}_T^-)^2) \leq C(C+2) \int_t^T E((\overline{Z}_s^-)^2) ds.
$$

According to Gronwall's lemma, we obtain that $E((\overline{Z}_T^-)^2) = 0$, for every $T \in [t, t+h]$, which means that $\overline{Z}_T \geq 0$, that is $Y^2_{T\vee\tau} \geq Y^1_{T\vee\tau}$, for every $T \in [t, t+h]$, whenever τ is finite. Now, since the first point of the lemma states that $Y_s^1 \geq Y_s^2$ for every $s \geq u$, we conclude that

$$
Y^1_{T\vee\tau} = Y^2_{T\vee\tau}, \text{ for every } T \in [t, t+h].
$$

So, for $\tau \leq s \leq t+h$, we have $Y_s^1 = Y_s^2$, and thus

$$\begin{aligned} dk_s^1 &= dY_s^1 - b(X(s), Y_s^1)ds - \sigma(X(s), Y_s^1)dB_s \\ &= dY_s^2 - b(X(s), Y_s^2)ds - \sigma(X(s), Y_s^2)dB_s \\ &= dk_s^2, \end{aligned}$$

which means that

$$\int_{\tau \wedge (t+h)}^{t+h} \mathbb{1}_{\{Y_s^1 = 0\}} d(k^1 - k^2)_s = 0,$$

and this completes the proof.

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