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A Geometric Method and Dimension Split Algorithm for The 3-D Compressible Navier-Stokes Equations in Turbomachine*

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Abstract: In this paper,using classical tensor calculus, we derive the compressible Navier-Stokes Equation on a two dimensional manifold as a stream surface , define stream fuction and derive the equation satisfied by stream function. Based on this a new algorithm is proposaled which is called dimension splite algorithm. This new method is difference from domain decomposition method. In domain decompositiom method we have to solve 3-D problem in each subdomain but we solve two dimensional problem in each subdomain only in our new method.A numerical experiment for turbomachinery is devoted.

Key-words: Stream Layer, Stream surface, Dimensional splitting method, Navier-Stokes equations, Turbomachinery Flow.

Subject Classification(AMS): 65N30, 76U05, 76M05

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Une méthode géométrique et un algorithme de décomposition dimensionnele pour les équations de Navier-Stokes 3-D compressible dans une turbine

Résumé : En utilisant le calcul tensoriel, nous dérivons les équations de Navier-Stokes sur une variété bidimiensionnelle qui repésente la surface de courant. Nous définissons ensuite la fonction de courant et nous dérivons les équations qu'elle satisfait. Un nouvel algorithme est ensuite proposé basée sur la décomposition dimensionnelle. Ce nouvel algorithme est différent de la méthode de décomposition de domaine. Dans une méthode de décomposition de domaine il faut résoudre des problèmes tridimensionnels dans chaque sous-domaine, alors qu'avec notre méthode nous avons seulement des problèmes bidimensionnels à résoudre dans chaque sous-domaine. Une expérience numérique concernant l'écoulement dans une turbine est présentée.

Mots-clés : couche de courant, surface de courant, méthode de décomposition directionnelle, équations de Navier-Stokes, écoulement dans une turbine.

1 Introduction

In [1], the authors studied two dimensional flow on the stream surface, derived a nonlinear boundary value problem satisfied by stream function defined on the stream surface,and studied its finite element approximation. In this paper we used classical tensor calculation to derive Navier-Stokes Equations on the stream surface which are difference from that in [4]. Based this author proposal a new method called "dimension split method".Main idea is to split the 3-D flow into a series of 2-D problems on the stream surface and an one dimensional problem. Essential idea is how to generate the Riemann Structures of the stream surface by using an order differential equation with initial data. Indeed this method is a new kind of domain decomposition method. In our method,the three dimensional domain occupied by the fluid is decomposed into several stream layers, therefore a parallel algorithm can be applied. But the method is different from the classical domain decomposition method because we only solve a two dimensional problem in each subdomain(stream surface layer),instead of solving a 3-D problem.

As well known that 0n 2-D plane flow one can introduce a stream function ψ:

$$\frac{\partial\psi}{\partial x} = -u_x, \quad \frac{\partial\psi}{\partial y} = u_y,$$

and convert the Navier-Stokes equations in velocity-pressure form

$$-\lambda\Delta u + (u\cdot\nabla)u + \nabla p = f,$$

in the stream function form

$$\lambda\Delta^2\psi + \frac{\partial(\psi,\Delta\psi)}{\partial(x,y)} = \mathrm{rot} f, \tag{1.1}$$

where $u = (u_x, u_y)$ is the velocity field. Here the flow is assumed to be flat even the trajectory of the fluid particle is plane curve. The plane flows means the flow on any plane perpendicular to z-axes have the same dynamical behaviors.

By similar manner a two dimensional surface flow,as a flow on the two dimensional Riemann manifold S with metric tensor $a_{\alpha\beta}$,if only if

(a) the velocity vector $u = (u^1, u^2)$ should be on its tangent space TS;

(b) the flows on any geodesic superparallel surface have the same dynamical behavior.

We can derive the Navier-Stokes equations described two dimensional viscous flow on the 2-D maniflod

$$\partial_t u + \overset{*}{\nabla}_u u - \nu \overset{*}{\Delta} u + Ku + \overset{*}{\nabla} p = f, \mathrm{div} u = o. \tag{1.2}$$

which are different from [4],where K is Gaussian curvature of S and

$$\overset{*}{\Delta}\psi = a^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta \bullet$$

is a Laplace-Betrami operator on S.

If a flow is three dimensionl and we can define stream surface which is a two dimensionl manifold S such that the velocity of the fluid at a point on it is on tangent space TS at this point we derive Navier-Stokes equations on it under a semigeodesic coodinate system:

$$-\lambda(\overset{*}{\Delta} u^\alpha + Ku^\alpha) + u^\beta\overset{*}{\nabla}_\beta u^\alpha + a^{\alpha\beta}\overset{*}{\nabla}_\beta p = f^\alpha + \lambda(-\frac{\partial^2 u^\alpha}{\partial\xi^2} + 2(H\delta^\alpha_\lambda + b^\alpha_\lambda)\frac{\partial u^\lambda}{\partial\xi}), \tag{1.3}$$

$$-\lambda(\frac{\partial^2 u^3}{\partial\xi^2} + 2H\frac{\partial u^3}{\partial\xi}) + \frac{\partial}{\partial\xi}p = 2\lambda(u^\alpha\overset{*}{\nabla}_\beta H + b^\alpha_\beta\overset{*}{\nabla}_\alpha u^\beta) - b_{\alpha\beta}u^\alpha u^\beta + f^3, \tag{1.4}$$

$$\overset{*}{\mathrm{div}}\, u + \frac{\partial u^3}{\partial\xi} = 0.$$

It is shows that a flow is 2-D flow on a 2-D manifold if only if

$$-\frac{\partial^2 u^\alpha}{\partial \xi^2} + 2(H\delta^\alpha_\lambda + b^\alpha_\lambda)\frac{\partial u^\lambda}{\partial \xi} = 0.$$

and third component of exterior fore f satisfies

$$-2\lambda(u^\alpha \overset{*}{\nabla}_\beta H + b^\alpha_\beta \overset{*}{\nabla}_\alpha u^\beta) - b_{\alpha\beta}u^\alpha u^\beta = f^3.$$

It is obvious that $f^3 = 0$ for the plane flow owing to $b_{\alpha\beta} = 0$ and $H = 0$.

We define a stream function ψ on stream surface S:

$$\varepsilon u^\alpha = \varepsilon^{\alpha\beta} \overset{*}{\nabla}_\beta \psi, \quad \overset{*}{\nabla}_\beta \psi = \varepsilon \varepsilon_{\beta\alpha} u^\beta,$$

where $\varepsilon_{\beta\alpha}$ is the permutation tensor on S and $\varepsilon(x)$ the thickness of the stream layer.Substituting it into (1.3) we obtain

$$(1.5) \qquad \lambda \overset{*}{\Delta} (L\psi) + \varepsilon^{\alpha\beta}[\frac{1}{\varepsilon} \overset{*}{\nabla}_\alpha \psi \overset{*}{\nabla}_\beta (L\psi) + \overset{*}{\nabla}{}^\lambda (\frac{1}{\varepsilon} \overset{*}{\nabla}_\alpha \psi) \overset{*}{\nabla}_\lambda (\frac{1}{\varepsilon} \overset{*}{\nabla}_\beta \psi)] = (f - \lambda l(u)),$$

where

$$L\psi = \overset{*}{\nabla}_\alpha (\frac{a^{\alpha\beta}}{\varepsilon} \overset{*}{\nabla}_\beta \psi),$$

and $l(u)$ is defined by (3.27),$a^{\alpha\beta}$, $a_{\alpha\beta}$ are contravariant and covariant metric tensor on S, $\lambda = Re^{-1}$ and Re the Reynolds number of the flow.

If the flow is a plane, $\varepsilon(x)$ is constant and $l(u) = 0$,

$$\alpha^{\alpha\beta} = a_{\alpha\beta} = \delta_{\alpha\beta} = 0, \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\lambda} = 1, -1, 0.$$

It follows that

$$\varepsilon^{\alpha\beta} \overset{*}{\nabla}_\alpha \psi \overset{*}{\nabla}_\beta (L\psi) = -\frac{\partial(\psi, \Delta\psi)}{\partial(x^1, x^2)}, ; \quad L\psi = \frac{1}{b} \overset{*}{\Delta} \psi,$$

$$\varepsilon^{\alpha\beta} \overset{*}{\nabla}{}^\lambda (\frac{1}{\varepsilon} \overset{*}{\nabla}_\alpha \psi) \overset{*}{\nabla}_\lambda (\frac{1}{\varepsilon} \overset{*}{\nabla}_\beta \psi) = 0.$$

This ensure that (1.6) becomes (1.1).

Equation (1.6) includes inner geometry of 2-D manifold S. In Riemann manifold, the order of the covariant derivatives can not be commuted because of

$$\overset{*}{\nabla}_\alpha \overset{*}{\nabla}_\beta A^\lambda = \overset{*}{\nabla}_\beta \overset{*}{\nabla}_\alpha A^\lambda + R^\lambda_{.\alpha\beta\sigma} a^\sigma,$$

and the Riemann curvature $R^\lambda_{.\alpha\beta\sigma}$ is not equal to zero usualy. Therefore (1.6) is different from (1.1). Indeed, the observer is now in a Riemann space instead of an Euclidean space.

Second purpose of our paper is to develop a dimension splitting method for the 3-D Navier-Stokes equations.nnnnn

2 Stream Surface and Stream Layer

Firstly, let us introduce some new concepts of Stream Layer.

"Stream Surface": A two dimensional manifold S is called a stream surface if the velocity vector of the fluid at any point on S lies on its tangent space TS at this point. There exists an unique stream line passing a point on S and the interline between two stream surfaces is a stream line.

It is obvious that the stream surface which passes a given point in $\Omega \subset R^3$ occupied by viscous fluid is not unique. There are two sort of stream surfaces, one is with boundary which is a part of the boundary $\partial\Omega$, the other is without boundary.

"Stream Layer": The domain Ω^i_{i-1}, which is surrounded by two geodesic parallel stream surfaces(PGSS) S_{i-1} and S_i, is called a stream layer. For the first sort of stream surface, the boundary $\Gamma^i_{i-1} = \partial\Omega \cap \Omega^i_{i-1}$ is called lateral boundary of the stream layer, S_{i-1} and S_i are called lower and upper surface of stream layer respectively. The Geodesic distence between two PGSS is called thickness of stream layer and denoted by ε. In general $\varepsilon(x)$ is a fanction of point in S.

For every point in the stream layer, there is a unique geodesic parallel stream surface between S_{i-1} and S_i which passes this point.

Remark 1 we give an example to generate stream surfaces and compute its metric tensors.

Assume that $x = (x^1, x^2, x^3)$ is a curvilineal coordinate system in Euclidean space R^3. Let u be the velocity of fluid particle, and $x(t) = (x^1(t), x^2(t), x^3(t))$ be the path followed by fluid particle. It is well known

$$\frac{dx}{dt} = u(x(t),t) \quad x(0) = x_0 \tag{2.1}$$

A stream line L at fixed time is an integral curve of (2.1), i.e. if $x(s)$ is a stream line parametrized by s at the instant t, $x(s)$ satisfies

(2.2).
$$\frac{dx}{ds} = u(x(s),t) \quad \text{for fixed } t$$

A trajectory is the curve followed by a particle as time proceeds. Therefore, a trajectory is a solution of the differential equation (2.1) with suitable initial condition. If u is independent of t (i.e. $\frac{\partial u}{\partial t} = 0$), the stream line and trajectory coincide. In this case the flow is called stationary.

In particular, we will deal with stationary flow. So (2.1) can be rewritten as

$$\frac{dx}{ds} = u(x(s)), \quad x|_{s=0} = x_0(\tau) \tag{2.3}$$

Here we assume that the initial data x_0 depend upon one parameter τ. For fixed τ, (2.3) describe a stream line and therefore a trajectory.

Assume that $x(s,\tau) = x(s, x_0(\tau))$ is a solution of (2.3) and the Jacobian matrix reads

$$J(s,\tau) = \frac{\partial(x^1, x^2, x^3)}{\partial(s,\tau)}. \tag{2.4}$$

In sequence, we assume that the initial data ensure that $J(s,\tau)|_{s=0}$ is nonsingular, i.e.

$$\text{Rank}J(0,\tau) = 2. \tag{2.5}$$

Hence, we can choice $\xi^1 = s, \xi^2 = \tau$ as the coordinate system on S. Let g_{ij}, g^{ij} be the covariant and contravariant tensor of R^3 in local coordinate system x^i respectively in the neighborhood of x_0. Then the metric tensor $a_{\alpha\beta}$ of two dimensional manifold S is given as

$$a_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial x^j}{\partial \xi^\beta}. \tag{2.6}$$

In view of (2.5) ,$\frac{\partial x}{\partial \xi^\alpha}$ and $\frac{\partial x}{\partial \xi^\beta}$ are independent,therefore $det(a_{\alpha\beta}) \neq 0$.

Later on, for further simplification we shall adopt the so-called Einstein summation conventions in notation.

Combining (2.3)(2.6) leads to

$$\left\{ \begin{array}{l} a_{11} = g_{ij}\frac{\partial x^i}{\partial \xi^1}\frac{\partial x^j}{\partial \xi^2} = g_{ij}u^i u^j = |w|^2, \\ a_{12} = a_{21} = g_{ij}u^i \frac{\partial x^j}{\partial x_0^k}\frac{\partial x_0^k}{\partial \tau}, \quad a_{22} = g_{ij}\frac{\partial x^i}{\partial x_0^k}\frac{\partial x^j}{\partial x_0^l}\frac{\partial x_0^k}{\partial \tau}\frac{\partial x_0^l}{\partial \tau}. \end{array} \right. \tag{2.7}$$

where Jacobian matrix

$$[\varphi^m_k] = \frac{\partial(x^1, x^2, x^3)}{\partial(x_0^1, x_0^2, x_0^3)}, \quad x^i = x^i(s, x_0(\tau)) \tag{2.8}$$

satisfy the following linear system of variationl equations

$$\frac{\partial \varphi_k^m}{\partial s} = \frac{\partial u^m}{\partial x^l}\varphi_k^l, \quad \varphi_k^m|_{s=0} = \delta_k^m. \tag{2.9}$$

According to the ordinary differential equation and the definition of the manifold, we assert that the stream surface can be generated by the solution $x^i = x^i(s,\tau)$ of (2.3) locally.

3 Navier-Stokes Equation Under Semigeodesic Coordinate System

3.1 S-Coordinate System Based on the Stream Surface

In the sequel, we will study the N-S equations under a semigeodesic coordinate system based on stream surface.

Assume that S is a two dimensional surface,P is a point in the neighborhood of S. O is the original point.Therefore

$$\vec{R} = OP = \vec{r} + \xi\,\vec{n},\ \vec{r} = OP_o,\ P_oP = \xi\vec{n}$$

where P_o is the intersection point of normal to S to P,$\vec{n}$ is a unit vector along P_oP. Let (x^1,x^2) be a coordinate on S, P can be described uniquely by (x^1,x^2,ξ) which can be regarded as the coordinates of P and is called S-coordinate system. Its basic vectors are

$$\vec{e_\alpha} = \vec{R_\alpha} = \vec{r_\alpha} + \xi\vec{n_\alpha},\ \vec{e_3} = \frac{\partial \vec{R}}{\partial \xi} = \vec{n}.$$

In[5,Th.2.6-2] it shows that if $\vec{r}$:$R^2 \supset D \to R^3$ is a C^3–mapping,then the canonical extension $\vec{R}$ of the mapping $\vec{r}$ defined by $\vec{R}(x,\xi) = \vec{r}(x) + \xi\vec{n}(x), \forall(x,\xi) \in \Omega^\varepsilon$ is a C^1–diffeomorphism from Ω^ε onto $\vec{R}(\Omega^\varepsilon)$ and and three vectors $\vec{e_\alpha} = \vec{R}_\alpha, \vec{e_3} = \frac{\partial\vec{R}}{\partial\xi}$are linear independent at all point of Ω^ε,therefore (x,ξ) can be regarded as a coordinate in R^3 and called semigeodesic coordinate based on surface S (for simplicity,it is called S-coordinate System).

The first, the second and the third fundamental form of S are given by

$$\begin{cases} a_{\alpha\beta} = \vec{r}_\alpha\vec{r}_\beta,\ b_{\alpha\beta} = \vec{n}\vec{r}_{\alpha\beta} = -\frac{1}{2}(\vec{n}_\alpha\vec{r}_\beta + \vec{r}_\alpha\vec{n}_\beta), \quad c_{\alpha\beta} = \vec{n}_\alpha\vec{n}_\beta = a^{\lambda\sigma}b_{\alpha\lambda}b_{\beta\sigma}; \\ a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma, \quad \hat{b}^{\alpha\beta}\,b_{\beta\lambda} = \delta^\alpha_\lambda,\ \hat{c}^{\alpha\beta}\,c_{\beta\lambda} = \delta^\alpha_\lambda;\ b^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\sigma}b_{\lambda\sigma}, \quad c^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\sigma}c_{\lambda\sigma}. \end{cases}$$

Mean and Gaussian Curvature H and K of S

$$K = \frac{\det(b_{\alpha\beta})}{\det(a_{\alpha\beta})},\ H = \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta}.$$

Then it is easy to show following relation which will be frequently used

$$K\,a_{\alpha\beta} - 2H\,b_{\alpha\beta} + c_{\alpha\beta} = 0, \quad a^{\alpha\beta} - 2H\hat{b}^{\alpha\beta} + K\hat{c}^{\alpha\beta} = 0; \tag{3.1}$$

Assume that $\varepsilon^{\alpha\beta}, \varepsilon_{\alpha\beta}$ are the permutation tensor on S defined by

$$\varepsilon_{\alpha\beta} = \begin{cases} \sqrt{a}, \\ -\sqrt{a}, \\ 0, \end{cases} \quad \varepsilon_{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{a}},\ (\alpha,\beta) : \text{even permutation of (1,2)}, \\ -\frac{1}{\sqrt{a}},\ (\alpha,\beta) : \text{odd permutation of (1,2)}, \\ 0,\ \text{otherwise}, \end{cases}$$

Then following formulae are hold

$$\begin{cases} a^{\lambda\sigma} = \varepsilon^{\lambda\alpha}\varepsilon^{\sigma\beta}a_{\alpha\beta}, \quad K\hat{b}^{\lambda\sigma} = \varepsilon^{\lambda\alpha}\varepsilon^{\sigma\beta}b_{\alpha\beta}, \quad K^2\hat{c}^{\lambda\sigma} = \varepsilon^{\lambda\alpha}\varepsilon^{\sigma\beta}c_{\alpha\beta}, \\ \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\alpha\beta}b_{\lambda\sigma} = 2K, \quad \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}a_{\alpha\beta}b_{\lambda\sigma} = 2H, \\ \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}c_{\alpha\beta}c_{\lambda\sigma} = 2K^2, \quad \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\alpha\beta}c_{\lambda\sigma} = 2HK. \end{cases} \tag{3.2}$$

$$(3.3)\quad \begin{cases} \hat{b}^{\alpha\beta}c_{\alpha\beta}=2H, \quad \hat{c}^{\alpha\beta}=\hat{b}^{\alpha}_{\lambda}\hat{b}^{\beta\lambda}, \quad c^{\alpha}_{\lambda}=b^{\alpha}_{\beta}b^{\beta}_{\lambda}, \quad K\hat{b}^{\alpha}_{\alpha}=2H,\\ K^2a_{\alpha\beta}\hat{c}^{\alpha\beta}=4H^2-2K, \quad Kb_{\alpha\beta}\hat{c}^{\alpha\beta}=2H,\\ \hat{c}^{\beta\sigma}b^{\alpha}_{\beta}=\hat{b}^{\alpha\sigma}, \quad c_{\alpha\lambda}b^{\lambda}_{\beta}=2Hc_{\alpha\beta}-Kb_{\alpha\beta}, \hat{c}^{\beta\sigma}b_{\sigma\alpha}=\hat{b}^{\alpha}_{\beta},\\ \hat{b}^{\alpha\sigma}c_{\sigma\beta}=b^{\alpha}_{\beta}, \quad b^{\alpha\beta}b_{\alpha\beta}=a^{\lambda\sigma}c_{\lambda\sigma}=4H^2-2K, \quad b^{\alpha\beta}c_{\alpha\beta}=8H^3-6HK \end{cases}$$

Furthermore, we have the following lemma,the proof is omitted.

Lemma 1 Under this coordinate system, the metric tensor, Christoffel symbols and the covariant derivatives read

$$(3.4)\quad \begin{cases} g_{\alpha\beta}=a_{\alpha\beta}-2\xi\, b_{\alpha\beta}+\xi^2\, c_{\alpha\beta}; \quad g_{3\alpha}=g_{\alpha 3}=0, \quad g_{33}=1;\\ g^{\alpha\beta}=\theta^{-2}\,G^{\alpha\beta}; \quad g^{3\alpha}=g^{\alpha 3}=0,\ g^{33}=1; \quad g=\det(g_{ij})=\theta^2\,a, \quad a=\det(a_{\alpha\beta}) \end{cases}$$

where $\theta=1-2H\,\xi+K\,\xi^2$, $G^{\alpha\beta}=a^{\alpha\beta}-2K\,\xi\,\hat{b}^{\alpha\beta}+\xi^2\,K^2\,\hat{c}^{\alpha\beta}$.

$$(3.5)\quad \Gamma^{\gamma}_{\alpha\beta}=\overset{*}{\Gamma}{}^{\gamma}{}_{\alpha\beta}+\theta^{-1}R^{\gamma}_{\alpha\beta}, \Gamma^{\alpha}_{\beta 3}=\theta^{-1}I^{\alpha}_{\beta}, \quad \Gamma^{3}_{\alpha\beta}=J_{\alpha\beta}, \Gamma^{3}_{\beta 3}=\Gamma^{3}_{3\beta}=\Gamma^{\alpha}_{33}=\Gamma^{3}_{33}=0.$$

$$(3.6)\quad \begin{cases} \nabla_{\alpha}u^{\beta}=\overset{*}{\nabla}_{\alpha}u^{\beta}+\theta^{-1}(I^{\beta}_{\alpha}u^3+R^{\beta}_{\alpha\lambda}u^{\lambda}), \nabla_{\alpha}u^3=\frac{\partial u^3}{\partial x^{\alpha}}+J_{\alpha\beta}u^{\beta},\\ \nabla_3u^3=\frac{\partial u^3}{\partial\xi}, \nabla_3u^{\beta}=\frac{\partial u^{\beta}}{\partial\xi}+\theta^{-1}I^{\beta}_{\alpha}u^{\alpha}, \quad \operatorname{rot}u=(\operatorname{rot}u)^{\alpha}\vec{e_{\alpha}}+(\operatorname{rot}u)^3\vec{n},\\ \operatorname{div}u=\overset{*}{\operatorname{div}}\,u+\frac{\partial u^3}{\partial\xi}+u^{\alpha}\overset{*}{\nabla}_{\alpha}\ln\theta+u^3\frac{\partial\ln(\theta\sqrt{a})}{\partial\xi},\\ (\operatorname{rot}(u))^{\alpha}=\theta^{-1}\varepsilon^{\alpha\beta}(\overset{*}{\nabla}_{\beta}u^3-g_{\beta\lambda}\frac{\partial u^{\lambda}}{\partial\xi}), \quad (\operatorname{rot}(u))^3=\varepsilon^{\beta\sigma}g_{\beta\lambda}[\theta^{-1}\overset{*}{\nabla}_{\sigma}u^{\lambda}+\theta^{-2}R^{\lambda}_{\sigma\nu}u^{\nu}]. \end{cases}$$

The Laplace operator can be expressed by

$$\begin{aligned} \Delta u^{\alpha}=g^{ij}\nabla_i\nabla_ju^{\alpha}&=g^{\beta\sigma}\overset{*}{\nabla}_{\beta}\overset{*}{\nabla}_{\sigma}u^{\alpha}+\frac{\partial^2u^{\alpha}}{\partial\xi^2}+2\theta^{-1}[2K\delta^{\alpha}_{\beta}\xi-(b^{\alpha}_{\beta}+H\delta^{\alpha}_{\beta})]\frac{\partial u^{\beta}}{\partial\xi}\\ &+\theta^{-1}g^{\beta\sigma}[(2R^{\alpha}_{\beta\lambda}\delta^{\nu}_{\sigma}-R^{\nu}_{\beta\sigma}\delta^{\alpha}_{\lambda})\overset{*}{\nabla}_{\nu}u^{\lambda}+2I^{\alpha}_{\beta}\overset{*}{\nabla}_{\sigma}u^3+(\overset{*}{\nabla}_{\beta}I^{\alpha}_{\sigma}+\theta^{-1}(R^{\alpha}_{\beta\lambda}I^{\lambda}_{\sigma}-I^{\alpha}_{\lambda}R^{\lambda}_{\beta\sigma}))u^3]\\ &+(\theta^{-1}g^{\beta\sigma}[\overset{*}{\nabla}_{\beta}R^{\alpha}_{\sigma\lambda}+\theta^{-1}(R^{\alpha}_{\beta\nu}R^{\nu}_{\lambda\sigma}-R^{\alpha}_{\lambda\nu}R^{\nu}_{\beta\sigma})]+\theta^{-3}\delta^{\alpha}_{\lambda}(K-4HK\xi+2K(2H^2+K)\xi^2-4HK^2\xi^3))u^{\lambda}; \end{aligned}$$

$$\begin{aligned} \Delta u^3=g^{ij}\nabla_i\nabla_ju^3&=g^{\beta\sigma}\overset{*}{\nabla}_{\beta}\overset{*}{\nabla}_{\sigma}u^3+\frac{\partial^2u^3}{\partial\xi^2}+2\theta^{-1}(K\xi-H)\frac{\partial u^3}{\partial\xi}--\theta^{-1}[2I^{\nu}_{\lambda}\overset{*}{\nabla}_{\nu}u^{\lambda}+g^{\beta\sigma}R^{\lambda}_{\beta\sigma}\overset{*}{\nabla}_{\lambda}u^3]\\ &+g^{\beta\sigma}(\overset{*}{\nabla}_{\beta}J_{\sigma\lambda}+\theta^{-1}(J_{\beta\nu}R^{\nu}_{\sigma\lambda}-J_{\nu\lambda}R^{\nu}_{\beta\sigma}))u^{\lambda}+2\theta^{-2}(K-2H^2-2HK\xi+K^2\xi^2)u^3; \end{aligned}$$

$$\Delta T=\quad \begin{cases} \overset{*}{\Delta}T+\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^{\alpha}}(\sqrt{a}(-2K\hat{b}^{\alpha\beta}\xi+K^2\hat{c}^{\alpha\beta}\xi^2)\frac{\partial T}{\partial x^{\beta}})\\ +\theta^{-3}G^{\alpha\beta}(2\overset{*}{\nabla}_{\alpha}H\xi-\overset{*}{\nabla}_{\alpha}K\xi^2)\frac{\partial T}{\partial x^{\beta}}+\theta^{-1}(-H+K\xi)\frac{\partial T}{\partial\xi}+\frac{\partial^2T}{\partial\xi^2} \end{cases}$$

where $\overset{*}{\Gamma}{}^{\gamma}{}_{\alpha\beta}, \overset{*}{\Delta}, \overset{*}{\operatorname{div}}, \overset{*}{\nabla}_{\alpha}$ are the Christoffel symbols,Laplace Operator, *div* and ∇_{α} are on S,and

$$I^{\alpha}_{\beta}=-b^{\alpha}_{\beta}+K\xi\delta^{\alpha}_{\beta}, \quad J_{\alpha\beta}=b_{\alpha\beta}-\xi c_{\alpha\beta}, \quad R^{\alpha}_{\beta\sigma}=-\overset{*}{\nabla}_{\beta}b^{\alpha}_{\sigma}\xi+K\hat{b}^{\alpha}_{\mu}\overset{*}{\nabla}_{\beta}b^{\mu}_{\sigma}\xi^2.$$

In particular,$\xi=0$,and $u^3=0$ on S ,then

$$(3.7)\quad \begin{cases} g_{\alpha\beta}=a_{\alpha\beta}, \quad g_{\alpha 3}=g_{3\alpha}=0,\ g_{33}=1,\ g^{\alpha\beta}=a^{\alpha\beta}, \quad g^{\alpha 3}=g^{3\alpha}=0,\ g^{33}=1;\ g=a\\ \Gamma^{\gamma}_{\alpha\beta}=\overset{*}{\Gamma}{}^{\gamma}{}_{\alpha\beta}, \quad \Gamma^{\alpha}_{\beta 3}=-b^{\alpha}_{\beta}, \quad \Gamma^{3}_{\alpha\beta}=b_{\alpha\beta}, \quad \Gamma^{3}_{i3}=\Gamma^{i}_{33}=0, \end{cases}$$

$$(3.8)\quad \begin{cases} \nabla_{\alpha}u^{\beta}=\overset{*}{\nabla}_{\alpha}u^{\beta}, \quad \nabla_3u^{\beta}=\frac{\partial u^{\beta}}{\partial\xi}-b^{\beta}_{\alpha}u^{\alpha}, \quad \nabla_{\alpha}u^3=\frac{\partial u^3}{\partial x^{\alpha}}+b_{\alpha\beta}u^{\beta}, \nabla_3u^3=\frac{\partial u^3}{\partial\xi},\\ \operatorname{div}u=\overset{*}{\operatorname{div}}\,u+\frac{\partial u^3}{\partial\xi}, \quad \operatorname{rot}(u)^{\alpha}=-\varepsilon^{\alpha\beta}a_{\beta\lambda}\frac{\partial u^{\lambda}}{\partial\xi}, \quad \operatorname{rot}(u)^3=\varepsilon^{\beta\sigma}a_{\beta\lambda}\overset{*}{\nabla}_{\sigma}u^{\lambda} \end{cases}$$

$$(3.9)\qquad \begin{cases} \Delta u^\alpha = \overset{*}{\Delta} u^\alpha + \frac{\partial^2 u^\alpha}{\partial \xi^2} - 2(b^\alpha_\beta + H\delta^\alpha_\beta)\frac{\partial u^\beta}{\partial \xi} + K u^\alpha, \\ \Delta u^3 = \frac{\partial^2 u^\alpha}{\partial \xi^2} - 2H\frac{\partial u^3}{\partial \xi} + 2b^\nu_\lambda \overset{*}{\nabla}_\nu u^\lambda + 2u^\beta \overset{*}{\nabla}_\beta H, \\ \Delta T = \overset{*}{\Delta} T + \frac{\partial^2 T}{\partial \xi^2} - H\frac{\partial T}{\partial \xi}. \end{cases}$$

where we use

$$\overset{*}{\Delta} u^3 = a^{\alpha\beta} \overset{*}{\nabla}_\alpha \overset{*}{\nabla}_\beta u^3 = 0, \quad \overset{*}{\nabla}_\nu u^3 = 0 \quad \text{on} \quad S$$

by virtue of $u^3 = 0$ on S. In sequece we will use notation

$$(3.10)\qquad \overset{*}{\text{rot}}\, u = [\text{rot}(u)]^3|_S = \varepsilon^{\beta\sigma} a_{\beta\lambda} \overset{*}{\nabla}_\sigma u^\lambda = \varepsilon_{\alpha\beta} \overset{*}{\nabla}{}^\alpha u^\beta = \varepsilon^{\alpha\beta} \overset{*}{\nabla}_\alpha u_\beta$$

and it is called Curl operatornon on stream surface.

3.2 The Navier-Stokes Equations Under S-coordinate System

Let us consider stationary compressible viscous flow in turbomachinery. We employ coordinate system fixed with impeller rotating around axis z with angular velocity ω. Then govering equations are given by

$$(3.11)\qquad \begin{cases} \text{div}(\rho w) = 0, \quad \text{Continuous Eq.} \\ -\mu\Delta w + \rho w \nabla w + \nabla(p - \frac{\mu}{3}\text{div}\, w) + 2\rho\omega \times w = \rho\omega \times (\omega \times R) + f, \quad \text{Moment Eqs.} \\ \text{div}(\rho E w) + p\text{div}w - \text{div}(\kappa\, \text{grad}\, T) - \Phi = 0, \quad \text{Energy Eq.} \\ p = p(\rho, T), \quad \text{State Eq.} \end{cases}$$

where w is the relative velocity of the fluid with respect to rotating coordinate system, ρ density,p pressure,$E = C_v T$ interior energy per unit volume, C_v constant for ideal gas,T temperature,κ heat conduct coefficient, $\nu = \frac{\mu}{\varrho}$ dynamical viscosity constant; $2\omega \times w$ coriolis's force, $F = \frac{f}{\varrho} + \omega \times (\omega \times R)$, exterior and centrifugal forces per unit mass; and σ stress tensor,Φ diccipative function are given by

$$(3.12)\qquad \sigma^{ij} = (-p + \frac{2}{3}\mu \quad \text{div} \quad w)g^{ij} + 2\mu e^{ij}(w), \quad e^{ij}(w) = \frac{1}{2}(\nabla^i w^j + \nabla^j w^i).$$

$$(3.13)\qquad \Phi = 2\mu e^{ij}(w) e_{ij}(w) + \frac{2}{3}\mu(\text{div} w)^2.$$

Some time, instead of energy equation,one emploies entropy equation

$$(3.14)\qquad \frac{dS}{dt} = \frac{1}{WT}(\frac{\kappa}{\varrho}\Delta T + \Phi/\varrho).$$

where $S = R\log(T^{\frac{\gamma}{\gamma-1}}/p)$ is entropy and $W = g_{ij}w^i w^j$.

By lemma 1 innertia terms and Coriolis' force read :

$$(3.15)\qquad \begin{cases} w^j \nabla_j w^\alpha = w^\beta \overset{*}{\nabla}_\beta w^\alpha + w^3 \frac{\partial w^\alpha}{\partial \xi} + \theta^{-1}[2I^\alpha_\beta w^\beta w^3 + R^\alpha_{\beta\lambda} w^\beta w^\lambda] \\ w^j \nabla_j w^3 = w^\beta \overset{*}{\nabla}_\beta w^3 + w^3 \frac{\partial w^3}{\partial \xi} + J_{\alpha\beta} w^\alpha w^\beta, \\ (\omega \times w)^\alpha = \theta g^{\alpha\beta} \varepsilon_{\beta\lambda}(\omega^\lambda w^3 - \omega^3 w^\lambda); \quad (\omega \times w)^3 = \theta\varepsilon_{\alpha\beta}\omega^\alpha w^\beta. \end{cases}$$

where we used lemma1 and $\varepsilon_{3\alpha\beta} = \theta\varepsilon_{\alpha\beta}$. Then we obtain

Lemma 2 The Navier-Stokes equations (3.12) under S-coordinate system can be readed (3.16a)

$$\begin{aligned} &-\mu[g^{\beta\sigma} \overset{*}{\nabla}_\beta \overset{*}{\nabla}_\sigma + \frac{\partial^2 w^\alpha}{\partial \xi^2} + 2\theta^{-1}(2K\delta^\alpha_\beta \xi - (b^\alpha_\beta + H\delta^\alpha_\beta))\frac{\partial w^\beta}{\partial \xi} \\ &+\theta^{-1} g^{\beta\sigma}((2\delta^\nu_\sigma R^\alpha_{\beta\lambda} - R^\nu_{\beta\sigma}\delta^\alpha_\lambda) \overset{*}{\nabla}_\nu w^\lambda + 2I^\alpha_\beta \overset{*}{\nabla}_\sigma w^3) + g^{\beta\sigma}(\theta^{-1} \overset{*}{\nabla}_\beta I^\alpha_\sigma + \theta^{-2}(R^\alpha_{\beta\lambda} I^\lambda_\sigma - I^\alpha_\lambda R^\lambda_{\beta\sigma}))w^3 \\ &+(\theta^{-3} K\delta^\alpha_\lambda(1 - 4H\xi + (4H^2 + 2K)\xi^2 - 4HK\xi^3) + \theta^{-1} g^{\beta\sigma} \overset{*}{\nabla}_\beta R^\alpha_{\sigma\lambda} + \theta^{-2} g^{\beta\sigma}(R^\alpha_{\beta\nu} R^\nu_{\lambda\sigma} - R^\alpha_{\lambda\nu} R^\nu_{\beta\sigma}))w^\lambda] \\ &+\rho w^\beta \overset{*}{\nabla}_\beta w^\alpha + \rho w^3 \frac{\partial w^\alpha}{\partial \xi} + \rho\theta^{-1}[2I^\alpha_\beta w^\beta w^3 + R^\alpha_{\beta\lambda} w^\beta w^\lambda] \\ &+g^{\alpha\beta} \overset{*}{\nabla}_\beta (p - \frac{\mu}{3}(\overset{*}{\text{div}} + \frac{\partial w^3}{\partial \xi} + w^\alpha \overset{*}{\nabla}_\alpha \ln\theta + w^3 \frac{\partial \ln(\theta\sqrt{(a)})}{\partial \xi})) + 2\rho\theta g^{\alpha\beta}\varepsilon_{\beta\lambda}(\omega^\lambda w^3 - \omega^3 w^\lambda) = F^\alpha; \end{aligned}$$

(3.16b)

$$-\mu[g^{\beta\sigma} \overset{*}{\nabla}_\beta \overset{*}{\nabla}_\sigma w^3 + \tfrac{\partial^2 w^3}{\partial \xi^2} + 2\theta^{-1}(K\xi - H)\tfrac{\partial w^3}{\partial \xi} - \theta^{-1}(2I^\nu_\lambda \overset{*}{\nabla}_\nu w^\lambda + g^{\beta\sigma} R^\lambda_{\beta\sigma} \overset{*}{\nabla}_\lambda w^3)$$
$$+g^{\beta\sigma}(\overset{*}{\nabla}_\beta J_{\sigma\lambda} + \theta^{-1}(J_{\beta\nu}R^\nu_{\sigma\lambda} - J_{\nu\lambda}R^\nu_{\beta\sigma}))w^\lambda + 2\theta^{-2}(K - 2H - 2HK\xi + K^2\xi^2)w^3]$$
$$\tfrac{\partial}{\partial \xi}(p - \tfrac{\mu}{3}(\overset{*}{\text{div}}\, w + \tfrac{\partial w^3}{\partial \xi} + w^\alpha \overset{*}{\nabla}_\alpha \ln\theta + w^3 \tfrac{\partial \ln(\theta\sqrt{a})}{\partial \xi}))$$
$$+\rho w^\beta \overset{*}{\nabla}_\beta w^3 + \rho w^3 \tfrac{\partial w^3}{\partial \xi} + \rho J_{\alpha\beta} w^\alpha w^\beta + +2\rho\theta g^{\alpha\beta}\varepsilon_{\beta\lambda}(\omega^\lambda w^3 - \omega^3 w^\lambda)) + 2\rho\theta\varepsilon_{\alpha\beta}\omega^\alpha w^\beta = F^3;$$

(3.16c)
$$\text{div}\,(\varrho w) = \overset{*}{\text{div}}\;(\varrho w) + \frac{\partial \varrho w^3}{\partial \xi} + \varrho w^\alpha \overset{*}{\nabla}_\alpha \ln\theta + \varrho w^3 \frac{\partial \ln(\theta\sqrt{a})}{\partial \xi} = 0;$$

Concerning energy equation (3.15) we recall lemma 1 and

$$\Phi = 2\mu(e^{\alpha\beta}(w)e_{\alpha\beta}(w) + 2e^{3\alpha}(w)e_{3\alpha}(w) + e^{33}(w)e_{33}(w)) + \frac{2}{3}\mu(\text{div} w)^2,$$

$$e_{\alpha\beta} = \frac{1}{2}(g_{\beta\lambda} \overset{*}{\nabla}_\alpha w^\lambda + g_{\alpha\lambda} \overset{*}{\nabla}_\beta w^\lambda) - \theta^2 J_{\alpha\beta} w^3 + \frac{1}{2}(g_{\alpha\lambda}R^\lambda_{\beta\nu} + g_{\beta\lambda}R^\lambda_{\alpha\nu})w^\nu.$$

Theorem 1 The Navier-Stokes equations (3.12) and (3.15) on S ($\xi = 0, u^3 = 0$) can be written as

(3.17a)
$$-\mu(\overset{*}{\Delta} w^\alpha + K w^\alpha) + \rho w^\beta \overset{*}{\nabla}_\beta w^\alpha + a^{\alpha\beta} \overset{*}{\nabla}_\beta (p - \frac{\mu}{3} \overset{*}{\text{div}}\, w)$$
$$-2\rho a^{\alpha\beta}\varepsilon_{\beta\lambda} w^\lambda \omega^3 = F^\alpha + \mu(-\frac{\partial^2 w^\alpha}{\partial \xi^2} + 2(H\delta^\alpha_\lambda + b^\alpha_\lambda)\frac{\partial w^\lambda}{\partial \xi}),$$

(3.17b)
$$-\mu(\frac{\partial^2 w^3}{\partial \xi^2} + 2H\frac{\partial w^3}{\partial \xi}) + \frac{\partial}{\partial \xi}(p - \frac{\mu}{3} \overset{*}{\text{div}}\, w)$$
$$= 2\rho\varepsilon_{\alpha\beta}\omega^\alpha w^\beta + 2\mu(w^\alpha \overset{*}{\nabla}_\beta H + b^\alpha_\beta \overset{*}{\nabla}_\alpha w^\beta) - \rho b_{\alpha\beta} w^\alpha w^\beta + F^3,$$

(3.17c)
$$\overset{*}{div}\;(\varrho w) + \frac{\partial(\varrho w)^3}{\partial \xi} = 0,$$

(3.17d)
$$(w \overset{*}{\nabla})S = \frac{1}{WT\rho}(\kappa(\overset{*}{\Delta} T + \frac{\partial^2 T}{\partial \xi^2} - 2H\frac{\partial T}{\partial \xi}) + \Phi_0).$$

where
$$\Phi_0 = 2\mu a^{\alpha\lambda} a^{\beta\sigma} \overset{*}{e}_{\alpha\beta} \overset{*}{e}_{\lambda\sigma} + \frac{1}{2}\mu a_{\alpha\beta}\frac{\partial w^\alpha}{\partial \xi}\frac{\partial w^\beta}{\partial \xi} - \frac{2}{3}\mu(\overset{*}{div}\;(w) + \frac{\partial (w)^3}{\partial \xi}).$$

Remark 2 Theorem 1 shows that a flow is a curve flow on the two dimensional manifold if only if

$$(a) \qquad \frac{\partial^2 w^i}{\partial \xi^2} = 0, \qquad \frac{\partial w^i}{\partial \xi} = 0, \forall i = 1, 2, 3;$$

and third componet of exterior force F^3 is not independent it is necessary to satisfy

$$(b) \qquad F^3 = -2\rho\varepsilon_{\alpha\beta}\omega^\alpha w^\beta - 2\mu(w^\alpha \overset{*}{\nabla}_\beta H + b^\alpha_\beta \overset{*}{\nabla}_\alpha w^\beta) + \rho b_{\alpha\beta} w^\alpha w^\beta.$$

We have following lemma concerning rule of operator commutation:
Lemma 3

(3.18)
$$\overset{*}{\text{rot}}\;(\overset{*}{\Delta}\, u + Ku) = \overset{*}{\Delta}\overset{*}{\text{rot}}\, u; \quad \overset{*}{\text{rot}}\;((u\cdot \overset{*}{\nabla})u) = (u\cdot \overset{*}{\nabla})\; \overset{*}{\text{rot}}\, u + \varepsilon_{\alpha\beta} \overset{*}{\nabla}^\alpha u^\lambda \overset{*}{\nabla}_\lambda u^\beta.$$

Proof Note
$$\overset{*}{\text{rot}}\overset{*}{\Delta}\, u=\varepsilon_{\lambda\alpha}a^{\lambda\nu}a^{\beta\gamma}\,\overset{*}{\nabla}_\nu\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\gamma\, u^\alpha.$$
We use the Riemann curvature tensor $R^\lambda_{\alpha\beta\gamma}$ to change the order of the derivatives:

$$\begin{cases}\overset{*}{\nabla}_\nu\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\gamma\, u^\alpha=\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\nu\overset{*}{\nabla}_\gamma\, u^\alpha+R^\alpha_{\eta\nu\beta}\,\overset{*}{\nabla}_\gamma\, u^\eta-R^\eta_{\gamma\nu\beta}\,\overset{*}{\nabla}_\eta\, u^\alpha\\ =\overset{*}{\nabla}_\beta\,(\overset{*}{\nabla}_\gamma\overset{*}{\nabla}_\nu\, u^\alpha+R^\alpha_{\eta\nu\gamma}u^\eta)+R^\alpha_{\eta\nu\beta}\,\overset{*}{\nabla}_\gamma\, u^\eta-R^\eta_{\gamma\nu\beta}\,\overset{*}{\nabla}_\eta\, u^\alpha\\ =\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\gamma\overset{*}{\nabla}_\nu\, u^\alpha+\overset{*}{\nabla}_\beta\,(R^\alpha_{\eta\nu\gamma})u^\eta+R^\alpha_{\eta\nu\gamma}\,\overset{*}{\nabla}_\beta\, u^\eta+R^\alpha_{\eta\nu\beta}\,\overset{*}{\nabla}_\gamma\, u^\eta-R^\eta_{\gamma\nu\beta}\,\overset{*}{\nabla}_\eta\, u^\alpha.\end{cases}$$

By virtue of
$$\varepsilon_{\alpha\beta}b^{\alpha\beta}=0,\quad ,\overset{*}{\nabla}_\eta\,\varepsilon_{\alpha\beta}=\overset{*}{\nabla}_\eta\, a_{\alpha\beta}=0$$
and

$$R_{\alpha\beta\lambda\gamma}=b_{\beta\lambda}b_{\alpha\gamma}-b_{\alpha\lambda}b_{\beta\gamma};\qquad b^\alpha_\lambda b^\beta_\nu\varepsilon_{\alpha\beta}=K\,\varepsilon_{\lambda\nu}\tag{3.19}$$

we derive that

$$\varepsilon_{\lambda\alpha}a^{\lambda\nu}a^{\beta\gamma}\,\overset{*}{\nabla}_\beta\,(R^\alpha_{\eta\nu\gamma})=\overset{*}{\nabla}_\beta\,(\varepsilon_{\lambda\alpha}a^{\lambda\nu}a^{\beta\gamma}a^{\alpha\sigma}(b_{\eta\nu}b_{\sigma\gamma}-b_{\sigma\nu}b_{\eta\gamma}))$$
$$=\overset{*}{\nabla}_\beta\,(\varepsilon_{\lambda\alpha}b^\lambda_\eta b^{\alpha\beta}-\varepsilon_{\lambda\alpha}b^{\lambda\alpha}b^\beta_\eta)=\overset{*}{\nabla}_\beta\,(K\varepsilon^\beta_\eta).$$
$$\varepsilon_{\lambda\alpha}a^{\lambda\nu}a^{\beta\gamma}R^\alpha_{\eta\nu\gamma}=K\varepsilon^\beta_\eta,\quad \varepsilon_{\lambda\alpha}a^{\lambda\nu}a^{\beta\gamma}R^\alpha_{\eta\nu\beta}=K\varepsilon^\gamma_\eta,$$
$$\varepsilon_{\lambda\alpha}a^{\lambda\nu}a^{\beta\gamma}R^\eta_{\gamma\nu\beta}=\varepsilon_{\lambda\alpha}a^{\lambda\nu}a^{\beta\gamma}a^{\eta\sigma}(b_{\gamma\nu}b_{\sigma\beta}-b_{\sigma\nu}b_{\gamma\beta})$$
$$=\varepsilon_{\lambda\alpha}(c^{\lambda\eta}-2Hb^{\lambda\eta})=\varepsilon_{\lambda\alpha}a^{\lambda\sigma}a^{\eta\beta}(-Ka^{\beta\sigma})=-K\varepsilon^\eta_\alpha.$$
>From this, it follows that

$$\overset{*}{\text{rot}}\overset{*}{\Delta}\, u=\overset{*}{\Delta}\overset{*}{\text{rot}}\, u+\varepsilon_{\alpha\beta}\,\overset{*}{\nabla}{}^\beta\, Ku^\alpha-K\,\overset{*}{\text{rot}}\, u=\overset{*}{\Delta}\overset{*}{\text{rot}}\, u-\overset{*}{\text{rot}}\,(Ku).$$

Now we consider the inertia term.

$$\overset{*}{\text{rot}}\,(u^\beta\,\overset{*}{\nabla}_\beta\, u)=\varepsilon_{\lambda\alpha}a^{\lambda\nu}\,\overset{*}{\nabla}_\nu\,(u^\beta\cdot\overset{*}{\nabla}_\beta\, u^\alpha)=u^\beta\varepsilon_{\lambda\alpha}a^{\lambda\nu}\,\overset{*}{\nabla}_\nu\overset{*}{\nabla}_\beta\, u^\alpha+\varepsilon_{\lambda\alpha}a^{\lambda\nu}\,\overset{*}{\nabla}_\nu\,(u^\beta)\,\overset{*}{\nabla}_\beta\, u^\alpha.$$

Applying the Riemann tensor and (3.19), we have

$$u^\beta\varepsilon_{\lambda\alpha}a^{\lambda\nu}\,\overset{*}{\nabla}_\nu\overset{*}{\nabla}_\beta\, u^\alpha=u^\beta\,\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\nu\,(\varepsilon_{\lambda\alpha}a^{\lambda\nu}u^\alpha)+u^\beta\varepsilon_{\lambda\alpha}a^{\lambda\alpha}R^\alpha_{\eta\nu\beta}u^\eta=u^\beta\,\overset{*}{\nabla}_\beta\overset{*}{\text{rot}}\, u,$$

and

$$u^\beta\varepsilon_{\lambda\alpha}a^{\lambda\alpha}R^\alpha_{\eta\nu\beta}u^\eta=u^\beta\varepsilon_{\alpha\lambda}a^{\lambda\nu}a^{\alpha\mu}(b_{\mu\beta}b_{\eta\nu}-b_{\mu\nu}b_{\eta\beta})u^\beta u^\eta=\varepsilon\alpha\lambda(b^\alpha_\beta b^\lambda_\eta-b^{\lambda\alpha}b_{\eta\beta})u^\beta u^\eta=0.$$

Therefore,
$$\overset{*}{\text{rot}}\,(u^\beta\,\overset{*}{\nabla}_\beta\, u)=u^\beta\,\overset{*}{\nabla}_\beta\overset{*}{\text{rot}}\, u+\varepsilon_{\lambda\alpha}\,\overset{*}{\nabla}{}^\lambda\, u^\beta\,\overset{*}{\nabla}_\beta\, u^\alpha.$$
The proof is completed.

Theorem 2 The Navier-Stokes equations(3.17) on S can be rewritten as

$$\begin{cases}-\mu\,\overset{*}{\Delta}\overset{*}{\text{rot}}\, w+\rho(w\cdot\overset{*}{\nabla})\,\overset{*}{\text{rot}}\, w+\rho\varepsilon_{\alpha\beta}\,\overset{*}{\nabla}{}^\alpha\, w^\lambda\,\overset{*}{\nabla}_\lambda\, w^\beta+\varepsilon_{\alpha\beta}(w\cdot\overset{*}{\nabla})w^\beta\,\overset{*}{\nabla}{}^\alpha\,\rho\\ +2\,\overset{*}{\nabla}_\lambda\,(\rho\omega^3w^\lambda)=\overset{*}{\text{rot}}\,(F-l(w)),\\ \mu(-\frac{\partial^2w^3}{\partial\xi^2}+2H\frac{\partial w^3}{\partial\xi})=(F^3-\tau\sqrt{\overset{*}{\Delta}\, H+4H^3-2HK}-\rho b_{\alpha\beta}w^\alpha w^\beta)\\ -2\rho\varepsilon_{\alpha\beta}\omega^\alpha w^\beta+2\mu(w^\beta\,\overset{*}{\nabla}_\beta\, H+b^\alpha_\beta\,\overset{*}{\nabla}_\alpha\, w^\beta),\\ \overset{*}{div}\,(\varepsilon\varrho w)=0,\end{cases}\tag{3.20}$$

The energy equation on S reads

$$-\kappa(\overset{*}{\Delta} T + \frac{\partial^2 T}{\partial \xi^2} - 2H\frac{\partial T}{\partial \xi}) + c_v \rho w^\alpha \overset{*}{\nabla}_\alpha T + p(\overset{*}{div}\, w + \frac{\partial w^3}{\partial \xi})$$

$$-2\mu \overset{*}{e^{\alpha\beta}} (w) \overset{*}{e}_{\alpha\beta} (w) - \frac{1}{2}\mu a^{\alpha\beta}\frac{\partial w^\alpha}{\partial \xi}\frac{\partial w^\beta}{\partial \xi} + \frac{2}{3}\mu(\overset{*}{div}\, w + \frac{\partial w^3}{\partial \xi})^2 = 0.$$

Here we use $\overset{*}{\Delta} w^3 = 0$ on S owing to $w^3 = 0$ and ε and $l(w)$ are defined by

$$w^\alpha \frac{\partial \ln \varepsilon}{\partial x^\alpha} = \frac{\partial w^3}{\partial x^3} \quad \text{on} \quad \text{S} \tag{3.21}$$

and

$$l^\alpha(w) = -\frac{\partial^2 w^\alpha}{\partial \xi^2} + 2(H\delta^\alpha_\lambda + b^\alpha_\lambda)\frac{\partial w^\lambda}{\partial \xi}. \tag{3.22}$$

respectively.

Remark 3 ε defined by (3.21) has clear mechanical meaning. It is a related thickness between two streams(seeing in[5])which is called the related thicknees of stream layer in sequel.

Proof Multiplying and contracted index

$$\varepsilon_{\lambda\alpha} a^{\lambda\nu} \overset{*}{\nabla}_\nu$$

on both sides of (3.17) and using

$$\varepsilon_{\lambda\alpha} a^{\lambda\nu} \overset{*}{\nabla}_\nu = \overset{*}{\text{rot}}, \quad \varepsilon_{\lambda\alpha} a^{\lambda\nu} a^{\alpha\gamma} \overset{*}{\nabla}_\nu \overset{*}{\nabla}_\gamma (p - \frac{\mu}{3} \overset{*}{\text{div}}\, w) = 0,$$

and

$$\varepsilon_{\lambda\alpha} a^{\lambda\nu} \overset{*}{\nabla}_\nu (2\rho a^{\alpha\beta}\varepsilon_{\beta\gamma} w^\gamma \omega^3) = 2\varepsilon_{\lambda\alpha} a^{\lambda\nu} a^{\alpha\beta}\varepsilon_{\beta\gamma} \overset{*}{\nabla}_\nu (\rho w^\gamma \omega^3) = 2\varepsilon^{\nu\beta}\varepsilon_{\beta\gamma} \overset{*}{\nabla}_\nu (\rho w^\gamma \omega^3) = -2 \overset{*}{\nabla}_\nu (\rho w^\nu \omega^3)$$

we obtain

$$-\mu\, \overset{*}{\text{rot}}\, (\overset{*}{\Delta} w + Kw) + \overset{*}{\text{rot}}\, (\rho(w \cdot \overset{*}{\nabla})w) + 2 \overset{*}{\nabla}_\nu (\rho w^\nu \omega^3) = \overset{*}{\text{rot}}\, (f - l(w)).$$

Applying lemma 4 leads to the first of (3.20).

Next we prove the second of (3.20). It is well known that the components of the stress tensor along normal to the stream surface S is proportional to mean curvature H of S. In S-Coordinate System we find $\sigma^{33}(w) = -\tau H$, where τ is a proportional divisor, and

$$\sigma^{33}(w) = 2\mu e^{33} - g^{33}(p - \frac{\mu}{3} \overset{*}{\text{div}}) = 2\mu\frac{\partial u^3}{\partial \xi} - g^{33}(p - \frac{\mu}{3} \overset{*}{\text{div}}\, w).$$

Therefore

$$2\mu\frac{\partial w^3}{\partial \xi} - (p - \frac{\mu}{3} \overset{*}{\text{div}}\, w) = -\tau H.$$

We assert

$$\frac{\partial}{\partial \xi}(p - \frac{\mu}{3} \overset{*}{\text{div}}\, w) = -\tau\frac{\partial H}{\partial \xi} + 2\mu\frac{\partial^2 w^3}{\partial \xi^2}.$$

On the other hand (see[3])

$$\frac{\partial H}{\partial \xi} = \sqrt{\overset{*}{\Delta} H + 4H^3 - 2HK}.$$

It follows that

$$\frac{\partial}{\partial \xi}(p - \frac{\mu}{3} \overset{*}{\text{div}}\, w) = 2\mu\frac{\partial^2 w^3}{\partial \xi^2} + \tau\sqrt{\overset{*}{\Delta} H + 4H^3 - 2HK}.$$

Combining above results, we derive

$$\mu(-\frac{\partial^2 w^3}{\partial \xi^2}+2H\frac{\partial w^3}{\partial \xi})=(F^3-\tau\sqrt{\overset{*}{\Delta}H+4H^3-2HK}-\rho b_{\alpha\beta}w^\alpha w^\beta)$$

$$-2\rho\varepsilon_{\alpha\beta}\omega^\alpha w^\beta+2\mu(w^\beta\overset{*}{\nabla}_\beta H+b^\alpha_\beta\overset{*}{\nabla}_\alpha w^\beta).$$

This is the second of (3.20).

According to the theory of the first order partial differential equations, there is an integral factor $\varepsilon(x)$ such that

$$\varrho w^\alpha\frac{\partial \ln\varepsilon}{\partial x^\alpha}=\frac{\partial(\varrho w^3)}{\partial x^3}\quad \text{on}\quad \text{S}.$$

By substituting above into (3.17c), the continuous equation can be expressed by

$$\overset{*}{\nabla}_\alpha(\varepsilon\varrho w^\alpha)=0\quad\text{on}\quad\text{S}.$$

This leads to the last equation of (3.20).Applying lemma 3 complete our proof.

4 Stream Function Equation On the Stream Surface

In this section we will define the stream function ψ on S and derive the equation satisfied by ψ.

>From the third equation of (3.20), we can define the stream function as follows

$$\varrho\varepsilon w^\beta=\varepsilon^{\beta\alpha}\overset{*}{\nabla}_\alpha\psi,\quad \overset{*}{\nabla}_\alpha\psi=\varrho\varepsilon\varepsilon_{\beta\alpha}w^\beta, \tag{4.1}$$

where $\varepsilon_{\beta\alpha}$ is the determinant tensor on S which is defined by (3.4).

In view of (4.1), we assert that

$$\overset{*}{\text{rot}}\,w=\overset{*}{\nabla}_\beta(\varepsilon_{\lambda\alpha}a^{\lambda\beta}w^\alpha)=-\overset{*}{\nabla}_\beta(\frac{a^{\beta\gamma}}{\varepsilon\rho}\overset{*}{\nabla}_\gamma\psi)=-L\psi. \tag{4.2}$$

Substituting (4.1) and (4.7) into the first equation of (3.23) leads to an equation for stream function:

$$\mu\overset{*}{\Delta}(L\psi)+\frac{1}{\varepsilon}\varepsilon^{\alpha\beta}\overset{*}{\nabla}_\alpha\psi\overset{*}{\nabla}_\beta(L\psi)+\rho\varepsilon^{\alpha\beta}\overset{*}{\nabla}_\beta(\frac{1}{\rho\varepsilon}\overset{*}{\nabla}_\lambda\psi)\overset{*}{\nabla}^\lambda(\frac{1}{\rho\varepsilon}\overset{*}{\nabla}_\alpha\psi)$$

$$+\frac{\varepsilon^{\alpha\beta}}{\rho\varepsilon}\overset{*}{\nabla}_\alpha\psi\overset{*}{\nabla}_\beta(\frac{1}{\rho\varepsilon}\overset{*}{\nabla}_\lambda\psi)\overset{*}{\nabla}^\lambda\rho+2\varepsilon^{\alpha\beta}\overset{*}{\nabla}_\alpha(\frac{\omega^3}{\varepsilon}\overset{*}{\nabla}_\beta\psi)=\overset{*}{\text{rot}}(f-\mu l). \tag{4.3}$$

Furthermore we must compute dissipative function via stream function. To do this recall

$$\overset{*}{\nabla}_\alpha\varepsilon^{\beta\gamma}=\overset{*}{\nabla}_\alpha a^{\beta\gamma}=0,\quad \overset{*}{\nabla}_\alpha\varepsilon_{\beta\gamma}=\overset{*}{\nabla}_\alpha a_{\beta\gamma}=0, \tag{4.4}$$

and by elemental calculation, it follows in view of (4.1) that

$$\overset{*}{e^{\alpha\beta}}(w)=\frac{1}{2}(a^{\alpha\gamma}\overset{*}{\nabla}_\gamma w^\beta+a^{\beta\gamma}\overset{*}{\nabla}_\gamma w^\alpha)=\frac{1}{2}A^{\alpha\beta\lambda\gamma}\overset{*}{\nabla}_\lambda(\varepsilon^{-1}\rho^{-1}\overset{*}{\nabla}_\gamma\psi); \tag{4.5}$$

(4.6)

$$\overset{*}{e^{\alpha\beta}}(w)\overset{*}{e}_{\alpha\beta}(w)=a_{\alpha\nu}a_{\beta\mu}\overset{*}{e^{\alpha\beta}}(w)\overset{*}{e^{\nu\mu}}(w)=\frac{1}{2}B^{\lambda\delta\gamma\sigma}\overset{*}{\nabla}_\lambda((\rho\varepsilon)^{-1}\overset{*}{\nabla}_\gamma\psi)\overset{*}{\nabla}_\delta((\rho\varepsilon)^{-1}\overset{*}{\nabla}_\sigma\psi^*),$$

where

$$A^{\alpha\beta\lambda\gamma}=a^{\alpha\lambda}\varepsilon^{\beta\gamma}+a^{\beta\lambda}\varepsilon^{\alpha\gamma};\quad B^{\lambda\delta\gamma\sigma}=a^{\lambda\delta}a^{\gamma\sigma}+\varepsilon^{\delta\gamma}\varepsilon^{\lambda\sigma}. \tag{4.7}$$

In addition,by vitue of (4.1),

$$\text{div} w|_S = \varrho^{-2}\varepsilon^{-1}\varepsilon^{\alpha\beta} \overset{*}{\nabla}_\alpha \psi \overset{*}{\nabla}_\beta \rho. \tag{4.8}$$

The density ρ of the fluid is a function of $\nabla\psi$ which is defined at each point by the relations for perfect gas:

$$\rho^{\gamma+1} - \rho^2 + \kappa q = 0, \quad q \triangleq b^{-2}\nabla\psi\nabla\psi = \frac{a^{\alpha\beta}}{\varepsilon^2}\nabla_\alpha\psi\nabla_\beta\psi \tag{4.9}$$

where ρ denote $\rho/\rho_s, \kappa = \frac{1}{2}(\gamma-1)a_s^2\rho_s^2, \rho_s, a_s$ denote the stagnant density and stagnant speed of sound at the reference point.

(4.3) is a fourth order of nonlinear elliptic equation with respect to unknow ψ. We have to impose boundary conditions. Suppose boundary $\Gamma = \Gamma_0 \bigcup \Gamma_s \bigcup \Gamma_p$ of S consist of three components where Γ_0 is the inlet and exit of the flow, Γ_s intersecting between S and the walls, and Γ_p an artificial boundary. The boundary conditations are described as follows:

$$\begin{cases} \psi|_{\Gamma_s} = \psi_s, \quad \frac{\partial\psi}{\partial\tau}|_{\gamma_s} = 0, \quad \frac{1}{\varepsilon\rho}a^{\alpha\beta} \overset{*}{\nabla}_\beta \psi\tau_\alpha|_{\Gamma_0} = w^\alpha\tau_\alpha = g_0, \\ \psi|_{\Gamma_{p1}} = \psi|_{\Gamma_{p2}} + \overset{*}{G}, \quad \frac{\partial\psi}{\partial\tau}|_{\gamma_{p1}} = \frac{\partial\psi}{\partial\tau}|_{\gamma_{p2}}; \\ \frac{1}{\varepsilon\rho}a^{\alpha\beta} \overset{*}{\nabla}_\beta \psi\tau_\alpha|_{\Gamma_{p1}} = \frac{1}{\varepsilon\rho}a^{\alpha\beta} \overset{*}{\nabla}_\beta \psi\tau_\alpha|_{\Gamma_{p2}} \end{cases} \tag{4.10}$$

where$\Gamma_p = \Gamma_{p1} \bigcup \Gamma_{p2}, \tau$ is a unit vector normal to the boundary lying in the tangent plane of S,$\overset{*}{G}$ is the mass flow.

If flow is three dimensional then

$$\frac{\partial w^\alpha}{\partial\xi} \neq 0, \quad \text{hence} \quad l(w) \neq 0.$$

(4.8) will couple with a one dimensional equation.

5 Finite Element Solution

Let

$$V = \{u|u \in H^2(\Omega), u|_{\Gamma_s} = \frac{\partial u}{\partial\tau}|_{\Gamma_s} = 0, u|_{\Gamma_{p1}} = u|_{\Gamma_{p2}}, \frac{\partial u}{\partial\tau}|_{\Gamma_{p1}} = \frac{\partial u}{\partial\tau}|_{\Gamma_{p2}}\}$$

be a Hilbert space. Let

$$a(\rho;\psi,\psi^*) = \mu\int_S \frac{1}{\varepsilon.\rho} \overset{*}{\Delta} \psi \overset{*}{\Delta} \psi^* dS, \quad a_1(\rho;\psi,\psi^*) = \mu\int_S a^{\alpha\beta} \overset{*}{\nabla}_\alpha \frac{1}{\varepsilon\rho} \overset{*}{\nabla}_\beta \psi \overset{*}{\Delta} \psi^* dS. \tag{5.1}$$

$$B(\rho;\psi,\psi^*) = \int_S [L\psi\varepsilon^{\alpha\beta} \overset{*}{\nabla}_\beta \psi \overset{*}{\nabla}_\alpha (\frac{\psi^*}{\varepsilon}) + \tag{5.2}$$

$$\rho \overset{*}{\nabla}_\lambda (\frac{\varepsilon^{\beta\sigma}}{\rho\varepsilon} \overset{*}{\nabla}_\sigma \psi) \overset{*}{\nabla}_\beta (\frac{a^{\lambda\gamma}}{\varepsilon\rho} \overset{*}{\nabla}_\gamma \psi)\psi^* + \frac{\varepsilon^{\alpha\beta}}{\varepsilon\rho} \overset{*}{\nabla}_\lambda \rho \overset{*}{\nabla}_\alpha \psi \overset{*}{\nabla}_\beta (\frac{a^{\lambda\gamma}}{\varepsilon\rho} \overset{*}{\nabla}_\gamma \psi)\psi^*]dS,$$

$$C(\psi,\psi^*) = 2\int_\Omega \frac{\omega^3}{\varepsilon}\varepsilon^{\lambda\gamma} \overset{*}{\nabla}_\lambda \psi \overset{*}{\nabla}_\gamma \psi^* dS, \quad (f,\psi^*) = (\omega)^2\int_\Omega \rho\varepsilon_{\alpha\lambda}a^{\alpha\beta}r^\lambda \overset{*}{\nabla}_\beta \psi^*. \tag{5.3}$$

Note that

$$\int_S \frac{1}{\varepsilon}\varepsilon^{\alpha\beta} \overset{*}{\nabla}_\alpha \psi \overset{*}{\nabla}_\beta (L\psi)\psi^* dS = \int_S L\psi\varepsilon^{\alpha\beta} \overset{*}{\nabla}_\beta \psi \overset{*}{\nabla}_\alpha (\frac{\psi^*}{\varepsilon})dS.$$

In addition,assume that there exist a $\psi_0 \in H^2(\Omega)$ such that ψ_0 satisfies boundarary condation (4.10).Then variatiational formulation for (4.3) reads

$$(5.4)\qquad \begin{cases} \text{find} \quad \psi \in V \quad \text{such that} \\ a(\rho;\psi_0+\psi,\psi^*)+a_1(\rho;\psi_0+\psi,\psi^*)+B(\rho;\psi_0+\psi,\psi^*)+C(\psi_0+\psi,\psi^*) \\ =\{f,\psi^*\}, \quad \forall\psi^* \in V \end{cases}$$

Suppose that $V_N=\{\phi_i, i=1,2,\cdots,N\}$ is a conforming finite elemnet subspace which is dence in V. let $\psi_N=\psi_0+c^i\phi_i \in \psi_0+V_N$. Then finite element approximation is given by

$$(5.5)\qquad \begin{cases} \text{find} \quad \psi_N \in \{V+\psi_0\} \quad \text{such that} \\ a(\rho_N;\psi_0+\psi_N,\psi_N^*)+a_1(\rho_N;\psi_0+\psi_N,\psi_N^*) \\ +B(\rho_N;\psi_0+\psi_N,\psi_N^*)+C(\psi_0+\psi_N,\psi_N^*)=\{f,\psi_N^*\}, \quad \forall\psi_N^* \in V \end{cases}$$

where $\rho_N=\rho(\psi_0+\psi_N)$.The corresponding algebraic equation is

$$(5.7)\qquad A_{ki}(C)C^k-F_i(C)=0$$

where

$$A_{ki}=a(\rho_N;\psi_0+\phi_k,\phi_i)+a_1(\rho_N;\psi_0+\phi_k,\phi_i)+C(\psi_0+\phi_k,\phi_i),$$
$$F_i(C)=B(\rho_N;\psi_0+\psi_N,\phi_i)-(f,\phi_i).$$

Following lemmae show the relationship between density ρ and flow Mach number M (see in [1]):

lemma 4 In the compressible flow of perfect gas, $\rho'=\frac{\partial\rho}{\partial q}$ and M satisfy

$$2\rho\rho'=\frac{2\kappa}{\gamma+1}(1+\frac{\gamma-1}{2}M^2)/(M^2-1), \quad -2\sigma\sigma'=\frac{2\kappa}{\gamma+1}(1+\frac{\gamma-1}{2}M^2)/(M^2-1),$$

where $\sigma=\rho^{-1}, \kappa=\frac{\gamma-1}{2a_s^2\rho_s^2}$.

lemma 5 ρ and σ are descreasing and increasing monotone function with respect to q respectvely

$$(\rho(q_1)-\rho(q_2))(q_1-q_2)<0, \quad (\rho(q_1)-\rho(q_2))(q_1-q_2)>0, \quad \forall q_1 \neq q_1.$$

lemma 6 The quasilinear form $a(\cdot;\cdot,\cdot)$ defined by (5.1) satisfies

$$(5.10)\qquad a(\rho(\psi);\psi,\psi-\phi)-a(\rho(\phi);\phi,\psi-\phi)\geq\delta\|\psi-\phi\|_{2,\Omega}, \forall\psi,\phi\in H^2(\Omega), \psi-\phi\in V.$$

where δ is a conctant indepnendent of ψ,ϕ,ρ.

By similar manner in [1] we can prove following error estimate:

Theorem 3 Assume that ψ, ψ_N are the solutions of (5.5)(5.6) respectively. Furthermore, finite element subspace V_N is comforming.Then, for subsonic flow, following error estimate holds

$$\|\psi-\psi_N\|_{2,\Omega}\leq(c_1+c_2\frac{M^2}{1-M^2}) \inf_{\phi\in V_N} \|\psi_0+\psi-\phi\|_{2,\Omega},$$

6 Dimensional Split Algorithm

Before considering the split of dimension, we need following lemma (see in [1]).

Lemma 7 The value of the stream function along any streamline remains constant.

Lemma 8 Suppose that L_1, L_2 are two streamlines on the stream surface S. The difference between values of the stream function on L_1 and L_2 is equal to the mass flow through the stream layer with thickness ε between L_1 and L_2.

Above two lemmas show that ε is a relative thickness between two stream surfaces. The space between the two surfaces with thickness b is called the stream layer which

enclosed by the upper and lower stream surfaces S_1, S_2, the lateral blade's wall Γ_s, the artificial boundary Γ_p along the stream line, and the inlet and exit boundary Γ_i and Γ_o.

The following "Dimension Split Method" is proposed according to the above discussion.

1). Partition of the Domain Ω.

A partition of the domain Ω is assumed to be $\Omega = \cup_i \Omega_{i-1}^i$, where the subdomain Ω_{i-1}^i is a stream layer and its upper and lower boundary surfaces S_i and $S_{i-1}$are the geodesic supper parallel stream surfaces, the thickness $\varepsilon(x)$ of the stream layer Ω_{i-1}^i is very small. S_i can be represented locally by a diffeomorphism

$$F_i : R^2 \supset D \to F_i(D) = S_i \subset R^3.$$

Indeed, we assume that there exist a family of maps $F(\cdot,\xi)$ satisfying

$$F(\cdot,\xi) = S(\xi), \quad \sigma^{ij} n_j = -H(\xi) n^i, \quad \text{on } S(\xi), \tag{5.1}$$

where σ^{ij} is the stress tensor on $S(\xi)$, $H(\xi)$ is the mean curvature of $S(\xi)$ and $S_i = S(\xi_i)$.

2). Assume that all information on S_{i-1} are known. Then the information on S_i are determined by

a) Initial guess of velocity u on S_i; let us recall lemma 2 ,we rewrite (3.16):

$$\frac{\partial^2 u^\alpha}{\partial \xi^2} - 2(b_\beta^\alpha + H\delta_\beta^\alpha)\frac{\partial u^\alpha}{\partial \xi} = L_0^\alpha(u) + L_1^\alpha(u)\xi + o(\xi^2). \tag{5.2}$$

here $L_0^\alpha(u) = 0$ is leading term in view of first approximation.Using Goddazzi formula we have

$$L_1^\alpha(u) = -(4Ha^{\beta\sigma} + 2K\hat{b}^{\beta\sigma}) \overset{*}{\nabla}_\beta \overset{*}{\nabla}_\sigma u^\alpha + 2a^{\beta\nu}(\overset{*}{\nabla}_\beta b_\lambda^\alpha - \overset{*}{\nabla}_\beta H\delta_\lambda^\alpha) \overset{*}{\nabla}_\nu u^\lambda$$

$$+(\overset{*}{\Delta} b_\lambda^\alpha + 4Kb_\lambda^\alpha - 6HK\delta_\lambda^\alpha)u^\lambda - 2Hu^\beta \overset{*}{\nabla}_\beta u^\alpha - u^\lambda u^\beta \overset{*}{\nabla}_\beta b_\lambda^\alpha - (4Ha^{\alpha\beta} + 2K\hat{b}^{\alpha\beta}) \overset{*}{\nabla}_\beta p.$$

Let

$$Y^\alpha = \frac{\partial u^\alpha}{\partial \xi}, \quad A_\beta^\alpha = 2(b_\beta^\alpha + H\delta_\beta^\alpha),$$

then (5.2) reads

$$\mu(\frac{\partial Y^\alpha}{\partial \xi} - A_\beta^\alpha Y^\beta) = L_1^\alpha(u)\xi. \tag{5.3}$$

The corresponding eigenvalue problem is

$$|\lambda I - A_\beta^\alpha| = 2|(\lambda/2 - H)\delta_\beta^\alpha - b_\beta^\alpha| = 2|\hat{\lambda}\delta_\beta^\alpha - b_\beta^\alpha| = 0. \tag{5.4}$$

It is well known that the eigenvalues of (5.4) are the principal normal curvatures $k_1,\ i = 1,2$ of S, i.e.

$$\hat{\lambda}_i = k_i,\ i = 1,2\ (K = k_1 k_2;\ H = \frac{1}{2}(k_1 + k_2)).$$

Therefore

$$\lambda_1 = 2H + 2k_1 = 4H + 2\sqrt{H^2 - K}; \quad \lambda_2 = 2H + 2k_2 = 4H - 2\sqrt{H^2 - K}. \tag{5.5}$$

Assume that $s^{(i)} = (s_1^{(i)}, s_2^{(i)})$ are corresponding eigenvectors. Then the fundamental solution to the homogeneous equation (5.3) are given by

$$\Phi(\xi) = \begin{bmatrix} \exp(\lambda_1\xi)s_1^{(1)}, & \exp(\lambda_2\xi)s_1^{(2)} \\ \exp(\lambda_1\xi)s_2^{(1)}, & \exp(\lambda_2\xi)s_2^{(2)} \end{bmatrix}.$$

Hence the general solution of (5.3) can be expressed as

$$Y = (Y^1, Y^2)^t = \Phi(\xi)\Phi^{-1}(0)Y(0) + \int_0^{\xi} s\Phi(\xi)\Phi^{-1}(s)dsL_1(u). \tag{5.6}$$

(5.6) shows that $\frac{\partial u^\alpha}{\partial \xi}$ **strongly depend upon the Gaussian and mean curvatures of the stream surface.**

Applying (5.5) and (5.6), we can get the approximate solutions

$$u^\alpha|_{S_i} = u^\alpha|_{S_{i-1}} + (Y^\alpha(\triangle\xi) + Y^\alpha(0))/2\triangle\xi, \tag{5.7}$$

$$Y(\triangle\xi) = \Phi(\triangle\xi)\Phi(0)^{-1}(Y(0)+\frac{1}{2}\triangle\xi^2 F_1|_{S_{i-1}}) = (I+(\lambda_1'(0)+\lambda_2'(0))\triangle\xi)[Y(0)+\frac{1}{2}\triangle\xi^2 F_1|_{S_{i-1}}] \tag{5.8}$$

where we used

$$\Phi(0)' = ((\lambda_1'(0) + \lambda_2'(0))\Phi(0).$$

Here we need calculate the derivatives of the principal normal curvatures by using following formulae which will be proven elsewhere:

(i) $H' = \frac{\partial H}{\partial \xi} = \sqrt{\overset{*}{\Delta} H + (4H^2 - 2K)H},$

(ii) $\frac{\partial (H^2-K)}{\partial \xi} = (\frac{\partial H}{\partial \xi})^{-1}[\overset{*}{\Delta} (H^2 - K) + 4(2H^2 - K)(H^2 - K)$
$-(4H^2|\nabla H|^2 + \frac{1}{2}|\nabla K|^2 + 2K|\nabla H|^2 - 4H|\nabla H||\nabla K|)/(2H^2 - K)],$

(iii) $\lambda_1' = (\frac{\partial H}{\partial \xi})^{-1}(4H' - (H^2 - K)'/\sqrt{H^2 - K}), \quad \lambda_2' = (\frac{\partial H}{\partial \xi})^{-1}(4H' + (H^2 - K)'/\sqrt{H^2 - K}).$

b) Calculate $\varepsilon(x)$ **on** S_i **via (3.21). Let us come back to lemma 5. We rewrite second equation in lemma 4 into following**

$$\frac{\partial X}{\partial \xi} + 2HX = G(w), \quad \text{With two point boundare value or periodic condition} \tag{5.9}$$

where $X = \frac{\partial w^3}{\partial \xi}$**,**

$$G(w) = 2\mu(w^\beta \overset{*}{\nabla}_\beta H + b_\beta^\alpha \overset{*}{\nabla}_\alpha w^\beta) + (f^3 - \rho b_{\alpha\beta} w^\alpha w^\beta - \tau\sqrt{\overset{*}{\Delta} H + 4H^3 - 2HK}). \tag{5.10}$$

c) Geometric Position and Metric Tensor of S_i**.**

Assume that

$$r_{i-1}(x) = F(\xi_{i-1})$$

is an equation describing S_{i-1}**, then**

$$r_i(x) = r_{i-1}(x) + \varepsilon(x)n$$

is an equation of S_i**.**

Assume that (y^1, y^2, y^3) **is the Descartes coordinate and**

$$r_{i-1}(x) = y^1_{i-1}(x)i + y^2_{i-1}(x)j + y^3_{i-1}(x)k; \quad r_i(x) = y^1_i(x)i + y^2_i(x)j + y^3_i(x)k.$$

Therefore

$$y^k_i(x) = y^k_{i-1} + \varepsilon(x)A^k/a$$

where

$$A^1 = \frac{\partial(y^2_{i-1}, y^3_{i-1})}{\partial(x^1, x^2)}, \quad A^2 = \frac{\partial(y^3_{i-1}, y^1_{i-1})}{\partial(x^1, x^2)}, \quad A^3 = \frac{\partial(y^1_{i-1}, y^2_{i-1})}{\partial(x^1, x^2)}.$$

The metric tensor is

$$a_{\alpha\beta} = \sum_{k=1}^{k=3} \frac{\partial y^k_i}{\partial x^\alpha}\frac{\partial y^k_i}{\partial x^\beta}$$

On the other hand, owing to (3.4) we assert

$$a^{\alpha\beta}|_{S_i} = a^{\alpha\beta}|_{S_{i-1}} - 2\varepsilon b_{\alpha\beta} + \varepsilon^2 c^{\alpha\beta}.$$

Above two formulae are eqivalent.

7 Numerical Computational Examples

Fist example on the impellers of T1102 and ARC No.3 are provided. Numerical results which fit the experiment data very well are shown in Figs.1 and 2.

Thcond example is Gostelow's compressor. Fig.3 shows that the numerical results of the velocity distribution W is quite close to the exact values, and better than the results obtained by the finite difference method,especially at the leading edge and the trailing edge.

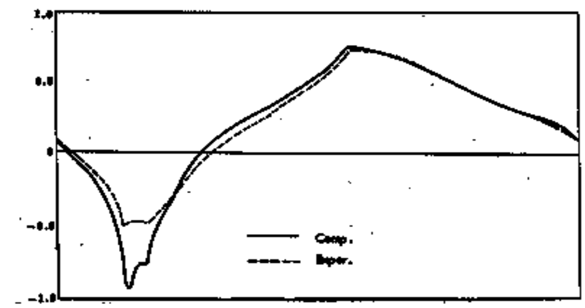

FIG.1 1102-blade $t = 0.7986, \beta y = 59.7$, static pressure distribution on blade faces

FIG.2 ARC No.3-blade,static pressure distribution on blade faces

FIG.3 Variation of the relation velocity along blade faces

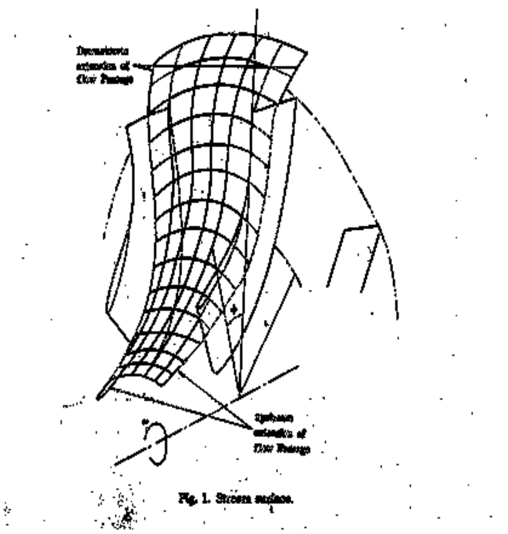

FIG.4 finit element mesh on stream surface   FIG.5 Unshrouded Impeller with spinner

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