Lyapunov Equation for the Stability of Linear Delay Systems of Retarded and Neutral Type

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Abstract: In this note is studied the delay-independent stability of delay systems. It is shown that the strong delay-independent stability is tantamount to the feasibility of certain linear matrix inequality, or equivalently to the existence of certain quadratic Lyapunov-Krasovskii functional, independent of the (nonnegative) value of the delay. This constitutes the analogue of some well-known properties of finite-dimensional systems. This result is then applied to study delay-independent stability of systems with commensurate delays, delay-independent stabilizability, and delay-independent stability of systems with polytopic uncertainties.

Key-words: linear delay systems, delay-independent stability, quadratic Lyapunov-Krasovskii functionals, linear matrix inequalities, structured singular values.

(Résumé : tsvp)

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Equation de Lyapunov pour la stabilité des systèmes linéaires à retard de type retardé et neutre

Résumé : L'objet de cette note est l'étude de la stabilité indépendante du retard (*delay-independent stability* en anglais) des systèmes à retard. On montre que la stabilité fortement indépendante du retard équivaut à la solvabilité d'une inégalité linéaire matricielle, ou encore à l'existence d'une fonctionnelle quadratique de Lyapunov-Krasovskii indépendante de la valeur (positive ou nulle) du retard. Ceci constitue un analogue d'une propriété bien connue pour les systèmes de dimension finie. Ce résultat est ensuite appliqué à l'étude de la stabilité indépendante du retard des systèmes avec retards commensuraux, à la stabilisabilité indépendante du retard, et à la stabilité indépendante du retard des systèmes avec incertitudes polytopiques.

Mots-clé : Systèmes linéaires à retard, stabilité indépendante du retard, fonctionnelles de Lyapunov-Krasovskii quadratiques, inégalités linéaires matricielles, valeurs singulières structurées.

1 Introduction

The stability of linear delay systems of retarded or neutral type is a field of intense research [15, 19]. A major difficulty lies in the fact that the delays are usually imperfectly known. A way to ensure stability robustness with respect to this uncertainty, is to employ stability criteria valid for any nonnegative value of the delays, that is *delay-independent results.* This assumption that no information on the value of the delay is known, is often coarse in practice: in general some estimates are available, and it is more appropriate (and sometimes unavoidable) to consider the stability of the systems obtained for the different values of the delays in the corresponding product of bounded intervals. However, the design of stability tests adapted to this task, both *numerically tractable* and *nonconservative*, seems still more complicated. This is the reason why the delay-independent results are still of interest.

Frequency domain criterion for delay-independent stability (or for *strong* delay-independent stability [18], see below), expressed by conditions on the zeros of a polynomial with two variables in the frequency domain, has been found for retarded type systems [12, 13, 14], and more recently for neutral type systems [9, 17, 20]. Chen *et al.* [6] have shown that checking this property amounts to verify conditions involving structured singular values with respect to complex uncertainties and having the form of small gain conditions appearing in robust stability analysis. This approach, however, does not give rise to easy-to-check conditions.

On the other hand, various easy-to-check stability conditions have appeared in the literature, based on time-domain techniques, see e.g. [11, 22, 21] and the references therein. Approaches by quadratic Lyapunov-Krasovskii functionals are intensively used, leading in particular to conditions expressed under the form of linear matrix inequalities (see [8, 26] for retarded type systems, [21] for neutral type), a class of problems for which widespread powerful numerical algorithms exist. However, these stability criteria are only sufficient, see [27].

In this paper is proposed a LMI condition *equivalent* to the (strong) delay-independent stability of neutral or retarded type delay differential systems. More precisely, one displays a family of LMIs of increasing size, each of them ensuring delay-independent stability. The key result is that, reciprocally, the strong delay-independent stability ensures that the LMIs are solvable beyond a certain rank.

The main idea, based on an improvement of the existing time-domain methods, consists in using, instead of the usual state variable, say $\{x(t+\tau)\ :\ -h\leq\tau\leq 0\}$, the augmented, nonminimal, state $\{x(t+\tau)\ :\ -kh\leq\tau\leq 0\}$, for some $k\in\mathbb{N}\backslash\{0\}$. A similar idea has been used in [2, 8] to get sufficient stability conditions. Of course, not any function in the new state space, even sufficiently smooth, can be part of a trajectory of the system under study when $k>1$, and the supplementary constraint is reintroduced when estimating the derivative of the candidate Lyapunov-Krasovskii functional. It turns out that the LMIs found by this method constitute *necessary and sufficient* conditions for strong delay-independent stability for large enough values of k. Incidentally, this furnishes a method for checking scaled small gain conditions as the ones depicted by Chen *et al.* [6], by use of LMIs. Moreover, there is hope that the method could be generalized to other related problems, namely stability of systems with noncommensurate delays, analysis of $\mathcal{H}_\infty$ performance and other. Work is in progress on these aspects.

In Section 2, the method is exposed and the main result is stated for a general discrete/continuous 2-D system (Theorem 1). In Section 3, the previous result is used to characterize the strong delay-independent stability of delay differential systems with one delay (Theorem 2). In Section 4, one provides various applications: to delay-independent stability of systems with commensurate delays (Theorem 3), to delay-independent stabilizability (Theorem 5) and to delay-independent stability of systems with polytopic uncertainties (Theorem 6). Finally, a complete proof of Theorem 1 is given in Section 5.

The notations are standard. I_n, $0_{m\times n}$ stand respectively for the identity matrix of size n and the null matrix of size $m\times n$ (simply abbreviated 0_n when $m=n$). The Kronecker product is denoted $\otimes$. By $\mathbb{C}^+$ is meant the set of complex numbers with nonnegative real part, and by $\mathbb{D}$ the closed unit disk $\{z \in \mathbb{C} \ : \ |z| \leq 1\}$. The spectrum, spectral radius and maximal singular value of a matrix M are respectively denoted $\sigma(M)$, $\rho(M)$ and $\|M\|$. Also, the conjugate and transconjugate of M, are denoted M^T and M^*. Last, for systems with a delay $h \geq 0$, x_t designates the function $x(t+\cdot)$, defined on $[-h, 0]$.

2 Internal stability criterion for discrete/continuous 2-D systems

As in [24, 1], the following discrete/continuous 2-D system is studied

$$\dot{x}_1(t,i) = Ax_1(t,i) + Bx_2(t,i), \tag{1a}$$
$$x_2(t,i+1) = Cx_1(t,i) + Dx_2(t,i) \ . \tag{1b}$$

In this section,

$$A \in \mathbb{R}^{n_1 \times n_1}, \ B \in \mathbb{R}^{n_1 \times n_2}, \ C \in \mathbb{R}^{n_2 \times n_1}, \ D \in \mathbb{R}^{n_2 \times n_2},$$

for certain $n_1, n_2 \in \mathbb{N} \setminus \{0\}$.

Asymptotic stability of (1) is equivalent to [1]

$$\forall (s,z) \in \mathbb{C}^+ \times \mathbb{D}, \ \det \begin{pmatrix} sI_{n_1} - A & -B \\ -zC & I_{n_2} - zD \end{pmatrix} \neq 0 \ . \tag{2}$$

In the sequel, Lyapunov theory will be used. Rather than to prove the stability, it will intervene as a guideline, in order to find the adequate algebraic LMI conditions. Similarly with the technique used in [7] for the (discrete/discrete) 2-D systems written as Roesser models [25], one attempts to quantify the evolution of the state of system (1) by considering, for definite positive matrices $P \in \mathbb{R}^{n_1 \times n_1}$, $Q \in \mathbb{R}^{n_2 \times n_2}$, the quantity

$$\frac{d}{dt}[x_1(t,i)^T P x_1(t,i)] + x_2(t,i+1)^T Q x_2(t,i+1) - x_2(t,i)^T Q x_2(t,i) \ . \tag{3}$$

This expression is equal to

$$\begin{pmatrix} x_1(t,i) \\ x_2(t,i) \end{pmatrix}^T R \begin{pmatrix} x_1(t,i) \\ x_2(t,i) \end{pmatrix} \ \text{with } R \stackrel{\text{def}}{=} \begin{pmatrix} A^T P + PA & PB \\ B^T P & 0_{n_1} \end{pmatrix} + \begin{pmatrix} C^T QC & C^T QD \\ D^T QC & D^T QD - Q \end{pmatrix} , \tag{4}$$

and the feasibility of the so-called *2-D Lyapunov equation* [1]

$$P = P^T > 0, Q = Q^T > 0, R < 0 \text{ where } R \text{ is given in (4)} , \tag{5}$$

is indeed sufficient to have (2), see [1]. However, this is *not* in general a necessary condition. The reason [1] is that (5) is equivalent to:

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \min_{M \text{ invertible}} \sup_{s \in \mathbb{C}^+} \|M(C(sI-A)^{-1}B + D)M^{-1}\| < 1 ,$$

whereas (2) is equivalent to

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \sup_{s \in \mathbb{C}^+} \min_{M \text{ invertible}} \|M(C(sI-A)^{-1}B + D)M^{-1}\| < 1 .$$

The previous form may be written as a *frequency dependent 1-D Lyapunov equation* [1].

Our method is based on an improvement of the previous one. Consider now instead, for $k \in \mathbb{N}\backslash\{0\}$,

$$\mathcal{X}_{1,k}(t,i) \stackrel{\text{def}}{=} \begin{pmatrix} x_1(t,i) \\ x_1(t,i-1) \\ \vdots \\ x_1(t,i-k+1) \end{pmatrix}, \ \mathcal{X}_{2,k}(t,i) \stackrel{\text{def}}{=} \begin{pmatrix} x_2(t,i) \\ x_2(t,i-1) \\ \vdots \\ x_2(t,i-k+1) \end{pmatrix} . \tag{6}$$

The vectors $\mathcal{X}_{1,k}$, $\mathcal{X}_{2,k}$ are elements of $\mathbb{R}^{kn_1}$, $\mathbb{R}^{kn_2}$ respectively. From (1) one deduces that

$$\begin{aligned} \dot{\mathcal{X}}_{1,k}(t,i) &= (I_k \otimes A)\mathcal{X}_{1,k}(t,i) + (I_k \otimes B)\mathcal{X}_{2,k}(t,i), \\ \mathcal{X}_{2,k}(t,i+1) &= (I_k \otimes C)\mathcal{X}_{1,k}(t,i) + (I_k \otimes D)\mathcal{X}_{2,k}(t,i) . \end{aligned} \tag{7}$$

Take definite positive matrices $P_k \in \mathbb{R}^{kn_1 \times kn_1}$, $Q_k \in \mathbb{R}^{kn_2 \times kn_2}$ and verify that (compare with (3))

$$\begin{aligned} &\frac{d}{dt}[\mathcal{X}_1(t,i)^T P_k \mathcal{X}_1(t,i)] + \mathcal{X}_2(t,i+1)^T Q_k \mathcal{X}_2(t,i+1) - \mathcal{X}_2(t,i)^T Q_k \mathcal{X}_2(t,i) \\ &= \begin{pmatrix} \mathcal{X}_{1,k}(t,i) \\ \mathcal{X}_{2,k}(t,i) \end{pmatrix}^T \left[\begin{pmatrix} (I_k \otimes A)^T P_k + P_k(I_k \otimes A) & P_k(I_k \otimes B) \\ (I_k \otimes B)^T P_k & 0_{kn_2} \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} (I_k \otimes C)^T Q_k (I_k \otimes C) & (I_k \otimes C)^T Q_k (I_k \otimes D) \\ (I_k \otimes D)^T Q_k (I_k \otimes C) & (I_k \otimes D)^T Q_k (I_k \otimes D) - Q_k \end{pmatrix} \right] \begin{pmatrix} \mathcal{X}_{1,k}(t,i) \\ \mathcal{X}_{2,k}(t,i) \end{pmatrix} . \end{aligned}$$

One is only interested in the stability of some special trajectories of system (7) in the space $\mathbb{R}^{kn_1} \times \mathbb{R}^{kn_2}$: those for which there exist x_1, x_2 such that (6) holds. The key point now, is the fact that, due to (1b), $\mathcal{X}_{1,k}$ and $\mathcal{X}_{2,k}$ are constrained by the following identity

$$\mathcal{X}_{2,k}(t,i) = F_k \mathcal{X}_{1,k}(t,i) + f_k x_2(t,i-k+1) ,$$

where $F_k \in \mathbb{R}^{kn_2 \times kn_1}$, $f_k \in \mathbb{R}^{kn_2 \times n_2}$ are defined by induction by

$$f_1 \stackrel{\text{def}}{=} I_{n_2}, \ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}D \\ I_{n_2} \end{pmatrix}, \tag{8}$$

$$F_1 \stackrel{\text{def}}{=} 0_{n_2 \times n_1}, \ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1}C \\ 0_{n_2 \times (k-1)n_1} & 0_{n_2 \times n_1} \end{pmatrix} . \tag{9}$$

As an example,

$$f_2 = \begin{pmatrix} D \\ I_{n_2} \end{pmatrix}, \ f_3 = \begin{pmatrix} D^2 \\ D \\ I_{n_2} \end{pmatrix}, \ F_2 = \begin{pmatrix} 0_{n_2 \times n_1} & C \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_1} \end{pmatrix}, \ F_3 = \begin{pmatrix} 0_{n_2 \times n_1} & C & DC \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_1} & C \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_1} & 0_{n_2 \times n_1} \end{pmatrix} \dots$$

One deduces that, for any trajectory of (1) with $\mathcal{X}_{1,k}, \mathcal{X}_{2,k}$ defined as in (6), one has

$$\begin{aligned} &\frac{d}{dt}[\mathcal{X}_1(t,i)^T P_k \mathcal{X}_1(t,i)] + \mathcal{X}_2(t,i+1)^T Q_k \mathcal{X}_2(t,i+1) - \mathcal{X}_2(t,i)^T Q_k \mathcal{X}_2(t,i) \\ &\qquad = \begin{pmatrix} \mathcal{X}_{1,k}(t,i) \\ x_2(t,i-k+1) \end{pmatrix}^T R_k \begin{pmatrix} \mathcal{X}_{1,k}(t,i) \\ x_2(t,i-k+1) \end{pmatrix} , \end{aligned}$$

where

$$\begin{aligned} R_k \stackrel{\text{def}}{=} &\begin{pmatrix} I_{kn_1} & 0_{kn_1 \times n_2} \\ F_k & f_k \end{pmatrix}^T \left[\begin{pmatrix} (I_k \otimes A)^T P_k + P_k(I_k \otimes A) & P_k(I_k \otimes B) \\ (I_k \otimes B)^T P_k & 0_{kn_2} \end{pmatrix} \right. \\ &\left. + \begin{pmatrix} (I_k \otimes C)^T Q_k (I_k \otimes C) & (I_k \otimes C)^T Q_k (I_k \otimes D) \\ (I_k \otimes D)^T Q_k (I_k \otimes C) & (I_k \otimes D)^T Q_k (I_k \otimes D) - Q_k \end{pmatrix} \right] \begin{pmatrix} I_{kn_1} & 0_{kn_1 \times n_2} \\ F_k & f_k \end{pmatrix} . \end{aligned} \tag{10}$$

One is hence naturally led to study the solvability of the following LMIs, defined for any $k \in \mathbb{N}\backslash\{0\}$.

$$\begin{gathered} P_k \in \mathbb{R}^{kn_1 \times kn_1}, Q_k \in \mathbb{R}^{kn_2 \times kn_2}, P_k = P_k^T > 0, Q_k = Q_k^T > 0,\ R_k < 0, \\ \textit{where } R_k \in \mathbb{R}^{(kn_1+n_2)\times(kn_1+n_2)} \textit{is defined by } (8),(9),(10)\ . \end{gathered} \tag{11_k}$$

The case $k = 1$ reduces to (5).

The following result tells that the solvability of (11_k) becomes *equivalent* to the asymptotic stability of (1) when k goes to infinity.

Theorem 1 (Stability criterion for discrete/continuous 2-D systems). *The following properties are equivalent.*

1. *System* (1) *is asymptotically stable.*
2. $\forall (s,z) \in \mathbb{C}^+ \times \mathbb{D},\ \det \begin{pmatrix} sI_{n_1} - A & -B \\ -zC & I_{n_2} - zD \end{pmatrix} \neq 0.$
3. $\rho(D) < 1$ *and* $\forall z \in \mathbb{D},\ \operatorname{Re} \sigma(A + zB(I_{n_2} - zD)^{-1}C) < 0.$
4. $\operatorname{Re} \sigma(A) < 0$ *and* $\sup_{s\in\mathbb{C}^+} \rho(C(sI_{n_1} - A)^{-1}B + D) < 1.$
5. *There exists* $k \in \mathbb{N} \setminus \{0\}$ *such that* (11_k) *is feasible.*
6. *There exists* $k^* \in \mathbb{N} \setminus \{0\}$ *such that, for any* $k \geq k^*$, (11_k) *is feasible.*

■

Theorem 1 gives a formal analogue, for discrete/continuous 2-D systems, to the equivalence between the spectral characterization of the asymptotic stability of $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$:

$$\forall s \in \mathbb{C}^+,\ \det(sI_n - A) \neq 0\ ,$$

and the solvability of the Lyapunov inequation

$$\exists P \in \mathbb{R}^{n\times n}, P = P^T > 0, A^T P + PA < 0\ .$$

Theorem 1 furnishes a family of LMI criteria, of arbitrary precision. The precision may be estimated as follows. One may check from the proof of Theorem 1, that a sufficient condition for solvability of (11_k) is

$$\operatorname{Re} \sigma(A) < 0 \text{ and } \sup_{s\in\mathbb{C}^+} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| < 1\ ,$$

whereas, using condition 4. of Theorem 1, stability appears to be equivalent to

$$\operatorname{Re} \sigma(A) < 0 \text{ and } \lim_{k\to+\infty} \sup_{s\in\mathbb{C}^+} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| = 0\ .$$

This constitutes a way of approximating the structured singular values with respect to one complex uncertainty [28]. For a similar result for the structured singular values associated to discrete systems, see [5].

3 LMI characterization of the strong delay-independent stability for delay differential systems

Theorem 1 is applied here to the following delay differential system.

$$\dot{x}(t) - E\dot{x}(t-h) = Ax(t) + Bx(t-h) \ , \tag{12}$$

$A, B, E \in \mathbb{R}^{n\times n}$. This is a delay differential equation of neutral type when $E \neq 0$, of retarded type when $E = 0$. The asymptotic stability of system (12) is equivalent to [9, 20]

$$\forall s \in \mathbb{C}^+, \ \det \begin{pmatrix} sI_n - A & -(AE+B) \\ -e^{-sh}I_n & I_n - e^{-sh}E \end{pmatrix} \neq 0 \ .$$

The notion of *delay-independent stability* has been introduced (see [12, 13, 14] for retarded systems, [9, 17, 20] for neutral systems): by definition, system (12) is (weakly) delay-independently stable if

$$\forall h \geq 0, \forall s \in \mathbb{C}^+, \ \det \begin{pmatrix} sI_n - A & -(AE+B) \\ -e^{-sh}I_n & I_n - e^{-sh}E \end{pmatrix} \neq 0 \ ,$$

or equivalently

$$\rho(E) < 1 \text{ and } \forall h \geq 0, \forall s \in \mathbb{C}^+, \ \det(s(I_n - e^{-sh}E) - A - e^{-sh}B) \neq 0 \ .$$

Delay-independent stability of system (12) may be proved to be equivalent to [9, 10]

$$\rho(E) < 1 \text{ and } \forall (s,z) \in \mathbb{C}^+ \times \mathbb{D}, s \neq 0 \text{ or } s = 0, z = 1 \Rightarrow \det(s(I_n - zE) - A - zB) \neq 0 \ .$$

A slightly stronger property may be introduced, as in [18] for retarded type systems: system (12) is called *strongly* delay-independently stable if

$$\rho(E) < 1 \text{ and } \forall (s,z) \in \mathbb{C}^+ \times \mathbb{D}, \ \det(s(I_n - zE) - A - zB) \neq 0 \ .$$

Clearly, this appears as a particular case of system (1), obtained by replacing

$$A, B, C, D \text{ by } A, AE+B, I_n, E \tag{13}$$

respectively. The variables $x_1(t,i)$, $x_2(t,i)$ are then changed into $x(t) - Ex(t-h)$, $x(t-h)$.

One may show easily as in [3] for systems of retarded type, that the strong delay-independent stability is a property robust with respect to perturbations of the matrices A, B, E, whereas the delay-independent stability is not. Indeed, infinitely close (in the sense of the distance induced by the maximal singular value on the product space of the triplets (A, B, E)) to any system fulfilling the weak property but not the strong one, there exist unstable systems. More precisely, the set of the triplets corresponding to strongly delay-independently systems is the *interior* of the set of the triplets corresponding to (weakly) delay-independently stable systems.

Generalizations of the Lyapunov method to delay differential equations have been proposed. In particular, a class of quadratic Lyapunov-Krasovskii functionals [16, 8] has been used early for this purpose, afterwards generalized to neutral type systems [21] under the form

$$V(x_t) = (x(t) - Ex(t-h))^T P(x(t) - Ex(t-h)) + \int_{t-h}^{t} x^T(\tau)Qx(\tau)\, d\tau \ . \tag{14}$$

It is remarkable that, along the trajectories of (12),

$$\frac{d[V(x_t)]}{dt} = \begin{pmatrix} x(t) - Ex(t-h) \\ x(t-h) \end{pmatrix}^T \left(\begin{pmatrix} A^TP + PA & P(AE+B) \\ (AE+B)^TP & 0_n \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} Q & QE \\ E^TQ & E^TQE - Q \end{pmatrix} \right) \begin{pmatrix} x(t) - Ex(t-h) \\ x(t-h) \end{pmatrix} \ ,$$

so the use of this class of Lyapunov-Krasovskii functionals leads exactly to the same LMI (5) with the choice (13).

Application of Theorem 1 with the change (13) gives the following result.

Theorem 2. *The strong delay-independent stability of system* (12) *is equivalent to any of the following properties.*

1. $\forall (s,z)\in\mathbb{C}^+\times\mathbb{D},\ \det\begin{pmatrix} sI_n - A & -(AE+B)\\ -zI_n & I_n - zE\end{pmatrix}\neq 0.$

2. *There exists $k\in\mathbb{N}\setminus\{0\}$ such that the following LMI is feasible:*

$$P_k\in\mathbb{R}^{kn\times kn}, Q_k\in\mathbb{R}^{kn\times kn}, P_k=P_k^T>0, Q_k=Q_k^T>0,\ R_k<0, \tag{15_k}$$

where $R_k\in\mathbb{R}^{(k+1)n\times(k+1)n}$ is defined by

$$\begin{aligned} R_k &\stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n}\\ F_k & f_k\end{pmatrix}^T \left[\begin{pmatrix} (I_k\otimes A)^T P_k + P_k(I_k\otimes A) & P_k(I_k\otimes(AE+B))\\ (I_k\otimes(AE+B))^T P_k & 0_{kn}\end{pmatrix}\right.\\ &\quad \left.+\begin{pmatrix} Q_k & Q_k(I_k\otimes E)\\ (I_k\otimes E)^T Q_k & (I_k\otimes E)^T Q_k (I_k\otimes E) - Q_k\end{pmatrix}\right]\begin{pmatrix} I_{kn} & 0_{kn\times n}\\ F_k & f_k\end{pmatrix},\\ f_1 &\stackrel{\text{def}}{=} I_n,\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}E\\ I_n\end{pmatrix}\in\mathbb{R}^{kn\times n},\\ F_1 &\stackrel{\text{def}}{=} 0_n,\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1}\\ 0_{n\times(k-1)n} & 0_n\end{pmatrix}\in\mathbb{R}^{kn\times kn}\ . \end{aligned}$$

3. *There exists $k^*\in\mathbb{N}\setminus\{0\}$ such that, for any $k\geq k^*$, (15_k) is feasible.*

■

Recalling the analysis conducted in Section 2, one sees that system (12) is strongly delay-independently stable *if and only if* it possesses, for a certain $k\in\mathbb{N}\setminus\{0\}$, a quadratic Lyapunov-Krasovskii functional, valid for *any nonnegative value of the delay* $h\geq 0$, of the form:

$$\begin{pmatrix} x(t)-Ex(t-h)\\ x(t-h)-Ex(t-2h)\\ \vdots\\ x(t-(k-1)h)-Ex(t-kh)\end{pmatrix}^T P_k \begin{pmatrix} x(t)-Ex(t-h)\\ x(t-h)-Ex(t-2h)\\ \vdots\\ x(t-(k-1)h)-Ex(t-kh)\end{pmatrix} + \int_{t-h}^t \begin{pmatrix} x(\tau)\\ x(\tau-h)\\ \vdots\\ x(\tau-(k-1)h)\end{pmatrix}^T Q_k \begin{pmatrix} x(\tau)\\ x(\tau-h)\\ \vdots\\ x(\tau-(k-1)h)\end{pmatrix} d\tau\ ,$$

where P_k, Q_k are positive definite matrices from $\mathbb{R}^{kn\times kn}$ (compare with (14)). Also, if system (12) is (weakly) delay-independently stable, but does not possess a Lyapunov-Krasovskii functional of the previous type for a certain $k\in\mathbb{N}\setminus\{0\}$, then infinitesimal parametric perturbations make it unstable for some $h\geq 0$.

For systems of retarded type ($E=0$), one may show more precisely that $\begin{pmatrix} F_k & f_k\end{pmatrix} = \begin{pmatrix} 0_{kn\times n} & I_{kn}\end{pmatrix}$, and that the feasibility of (15_k) implies the same property for the rank $k+1$, see [4].

Let us give a numerical example of utilization of Theorem 2. Consider the matrices

$$A = \begin{pmatrix} -4 & 10 & 5.7 & -6.5 & -2 \\ -0.2 & 1.2 & 1.5 & -1.9 & -0.6 \\ 15 & -16 & -7 & -3.6 & -6.4 \\ 8.7 & 5.5 & 6.1 & -5.9 & -7.7 \\ 7.6 & -0.9 & 1.9 & -7.2 & -7.5 \end{pmatrix}, \quad B = \begin{pmatrix} 9.1 & 0 & 4.1 & 6.9 & 5.1 \\ -2.6 & -1.8 & 1.6 & 1.1 & 2.7 \\ -0.5 & -9.9 & -0.4 & 7.5 & 3.2 \\ -6 & 3.5 & 7.8 & 0.8 & 6.4 \\ 1.1 & -12 & 4.4 & -7.5 & 6 \end{pmatrix} .$$

One studies the strong delay-independent stability of the system

$$\dot{x} = Ax(t) + \alpha Bx(t-h) \ , \tag{16}$$

for various values of the real parameter α.

One checks that $\operatorname{Re}\sigma(A) < 0$, so system (16) is asymptotically stable for $\alpha = 0$. Also, $\sup\{\alpha \geq 0 : \sigma(A+\alpha B) \cap \mathbb{C}^+ = \emptyset\} = 0.1726$: for this value, system (16) is unstable when $h = 0$. One wishes to estimate the largest $\alpha > 0$ such that (16) is strongly delay-independently stable. This value, denoted α_∞, is computed analytically as

$$\alpha_\infty \stackrel{\text{def}}{=} \left(\sup_{\omega \in \mathbb{R}} \rho((j\omega I_5 - A)^{-1}B) \right)^{-1} \simeq 0.1647 \ .$$

Denote α_k the largest value of α ensuring solvability of (15_k) in Theorem 2, with B replaced by αB. The computations are achieved by the SCILAB package `lmitool`[1]. One finds

$$\alpha_1 = 0.1488, \ \alpha_2 = \alpha_3 = 0.1647\ldots$$

It comes from the proof of Theorem 2 that the margins α_k are not less than

$$\alpha'_k \stackrel{\text{def}}{=} \|[(sI_5 - A)^{-1}B]^k\|_\infty^{-1/k} \ .$$

These constants may be computed as

$$\alpha'_1 = 0.05175, \ \alpha'_2 = 0.1147, \ \alpha'_3 = 0.1361, \ \alpha'_4 = 0.1431\ldots$$

4 Applications

We present in the sequel three immediate applications of Theorem 2.

4.1 Stability of delay systems with commensurate delays

The previous result is here generalized to the following class of systems:

$$\dot{x} - \sum_{l=1}^{L} E_l \dot{x}(t-lh) = \sum_{l=0}^{L} A_l x(t-lh) \ , \tag{17}$$

where $L \in \mathbb{N}$ and $A_l, E_l \in \mathbb{R}^{n\times n}, l = \overline{0,L}$. The idea to handle system (17), is to transform it into a new, augmented, system having a unique delay. More precisely, defining

$$\mathcal{X}(t) \stackrel{\text{def}}{=} \begin{pmatrix} x(t) \\ x(t-h) \\ \vdots \\ x(t-(L-1)h) \end{pmatrix} ,$$

[1] SCILAB is a free software, distributed with its source code, see the homepage at `http://www-rocq.inria.fr/scilab/`

one may "rewrite" system (17) as:

$$\mathcal{E}_0\dot{\mathcal{X}}(t) - \mathcal{E}_1\dot{\mathcal{X}}(t-Lh) = \mathcal{A}\mathcal{X}(t) + \mathcal{B}\mathcal{X}(t-Lh) \ ,$$

where

$$\mathcal{A} \stackrel{\text{def}}{=} \sum_{l=0}^{L} J^l \otimes A_l,\ \mathcal{B} \stackrel{\text{def}}{=} \sum_{l=0}^{L} J^{(L-l)T} \otimes A_l,\ \mathcal{E}_0 \stackrel{\text{def}}{=} I_{Ln} - \sum_{l=1}^{L} J^l \otimes E_l,\ \mathcal{E}_1 \stackrel{\text{def}}{=} \sum_{l=1}^{L} J^{(L-l)T} \otimes E_l, \quad \text{(18a)}$$

$$J \in \mathbb{R}^{L\times L},\ J_{ij} \stackrel{\text{def}}{=} 1 \text{ if } i+1=j, 0 \text{ otherwise} \ , \quad \text{(18b)}$$

that is ($\mathcal{E}_0$ is invertible)

$$\dot{\mathcal{X}}(t) - \mathcal{E}_0^{-1}\mathcal{E}_1\dot{\mathcal{X}}(t-Lh) = \mathcal{E}_0^{-1}\mathcal{A}\mathcal{X}(t) + \mathcal{E}_0^{-1}\mathcal{B}\mathcal{X}(t-Lh) \ . \quad (19)$$

It is clear that the trajectories of system (17) may be obtained as projections of some trajectories of system (19); asymptotic stability of (19) hence implies asymptotic stability of (17). It turns out that strong delay-independent stability of (19) is indeed *equivalent* to strong delay-independent stability of (17). This leads to the following result.

Theorem 3. *The strong delay-independent stability of system* (17) *is equivalent to any of the following properties.*

1. $\forall (s,z) \in \mathbb{C}^+ \times \mathbb{D}$,

$$\det\left(I_n - \sum_{l=1}^{L} z^l E_l\right) \neq 0 \text{ and } \det\left(s\left(I_n - \sum_{l=1}^{L} z^l E_l\right) - \sum_{l=0}^{L} z^l A_l\right) \neq 0 \ .$$

2. *There exists $k \in \mathbb{N}\setminus\{0\}$ such that the following LMI is feasible:*

$$P_k \in \mathbb{R}^{kLn\times kLn}, Q_k \in \mathbb{R}^{kLn\times kLn}, P_k = P_k^T > 0, Q_k = Q_k^T > 0,\ R_k < 0, \quad (20_k)$$

where $R_k \in \mathbb{R}^{(k+1)Ln\times(k+1)Ln}$ is defined by

$$R_k \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix}^T \left[\begin{pmatrix} (I_k\otimes\mathcal{E}_0^{-1}\mathcal{A})^T P_k + P_k(I_k\otimes\mathcal{E}_0^{-1}\mathcal{A}) & P_k(I_k\otimes\mathcal{E}_0^{-1}(\mathcal{A}\mathcal{E}_0^{-1}\mathcal{E}_1+\mathcal{B})) \\ (I_k\otimes\mathcal{E}_0^{-1}(\mathcal{A}\mathcal{E}_0^{-1}\mathcal{E}_1+\mathcal{B}))^T P_k & 0_{kLn} \end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} Q_k & Q_k(I_k\otimes\mathcal{E}_0^{-1}\mathcal{E}_1) \\ (I_k\otimes\mathcal{E}_0^{-1}\mathcal{E}_1)^T Q_k & (I_k\otimes\mathcal{E}_0^{-1}\mathcal{E}_1)^T Q_k (I_k\otimes\mathcal{E}_0^{-1}\mathcal{E}_1) - Q_k \end{pmatrix}\right] \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix},$$
$$f_1 \stackrel{\text{def}}{=} I_{Ln},\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}\mathcal{E}_0^{-1}\mathcal{E}_1 \\ I_{Ln} \end{pmatrix} \in \mathbb{R}^{kLn\times Ln},$$
$$F_1 \stackrel{\text{def}}{=} 0_{Ln},\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1} \\ 0_{Ln\times(k-1)Ln} & 0_{Ln} \end{pmatrix} \in \mathbb{R}^{kLn\times kLn} \ .$$

3. *There exists $k^* \in \mathbb{N}\setminus\{0\}$ such that, for any $k \geq k^*$, (20_k) is feasible.*

■

Proof. After applying Theorem 2 to system (19), it remains to compare the two determinants given in 1. with $\det(I_{Ln} - z^L\mathcal{E}_0^{-1}\mathcal{E}_1)$ and $\det(s(I_{Ln} - z^L\mathcal{E}_0^{-1}\mathcal{E}_1) - \mathcal{E}_0^{-1}\mathcal{A} - z^L\mathcal{E}_0^{-1}\mathcal{B})$ respectively. One has:

$$(I_{Ln} - z^L\mathcal{E}_0^{-1}\mathcal{E}_1) = \mathcal{E}_0^{-1}(\mathcal{E}_0 - z^L\mathcal{E}_1) = \mathcal{E}_0^{-1}\left(I_{Ln} - \sum_{l=1}^{L}(J^l + z^L J^{(L-l)T})\otimes E_l\right) \ .$$

Now, for any $z \in \mathbb{C}$, define r and N as follows:

$$r = r(L) \stackrel{\text{def}}{=} e^{-\frac{2i\pi}{L}},$$
$$N = N(z, L) \in \mathbb{R}^{L\times L},\ N_{ij} \stackrel{\text{def}}{=} (r^{j-1}z)^{i-1}\ .$$

The next lemma is proved easily.

Lemma 4. *The following formula is true, for any $l = \overline{0, L}$:*

$$J^l + z^L J^{(L-l)T} = N\ \mathrm{diag}\{z^l, (rz)^l, \dots, (r^{L-1}z)^l\} N^{-1}\ .$$

■

The key point is that r and N do not depend upon l. One deduces from Lemma 4, that

$$\begin{aligned}\det \mathcal{E}_0\ \det(I_{Ln} - z^L \mathcal{E}_0^{-1}\mathcal{E}_1) &= \det\left((N \otimes I_n)\left(I_{Ln} - \sum_{l=1}^{L} \mathrm{diag}\{z^l, (rz)^l, \dots, (r^{L-1}z)^l\} \otimes E_l\right)(N^{-1}\otimes I_n)\right)\\ &= \prod_{l'=0}^{L-1} \det\left(I_n - \sum_{l=1}^{L} (r^{l'}z)^l E_l\right)\ .\end{aligned}$$

One shows similarly that

$$\det(s(I_{Ln} - z^L\mathcal{E}_0^{-1}\mathcal{E}_1) - \mathcal{E}_0^{-1}\mathcal{A} - z^L\mathcal{E}_0^{-1}\mathcal{B}) = \prod_{l'=0}^{L-1} \det\left(s\left(I_n - \sum_{l=1}^{L}(r^{l'}z)^l E_l\right) - \sum_{l=0}^{L}(r^{l'}z)^l A_l\right)\ .$$

As $|r| = 1$, the preceding formulae show that 1. is *equivalent* to

$$\forall (s,z) \in \mathbb{C}^+ \times \mathbb{D},\ \det(I_{Ln} - z^L\mathcal{E}_0^{-1}\mathcal{E}_1) \neq 0 \text{ and } \det(s(I_{Ln} - z^L\mathcal{E}_0^{-1}\mathcal{E}_1) - \mathcal{E}_0^{-1}\mathcal{A} - z^L\mathcal{E}_0^{-1}\mathcal{B}) \neq 0\ .$$

This permits to conclude the proof of Theorem 3. □

4.2 Delay-independent stabilizability

From Theorem 2 one deduces directly the following result, on strong delay-independent stabilizability of delay systems by static output feedback. Similar results may be obtained for dynamic feedback of given order. Detectability may be treated as well.

Theorem 5. *Consider the open-loop system*

$$\dot{x} - E\dot{x}(t-h) = A_0x(t) + A_1x(t-h) + Bu(t),\ y(t) = C_0x(t) + C_1x(t-h)\ , \tag{21}$$

for $A_0, A_1, E \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C_0, C_1 \in \mathbb{R}^{q\times n}$, $n, p, q \in \mathbb{N}\setminus\{0\}$. There exists a static output feedback $u(t) = Ky(t)$ ensuring strong delay-independent stability of the corresponding closed-loop system if and only if there exist $K \in \mathbb{R}^{p\times q}$ and $k \in \mathbb{N}\setminus\{0\}$, for which the following LMI is feasible.

$$P_k, Q_k \in \mathbb{R}^{kn\times kn}, P_k = P_k^T > 0, Q_k = Q_k^T > 0,\ R_k < 0\ ,$$

where, denoting $A_K \stackrel{\text{def}}{=} A_0 + BKC_0$, $B_K \stackrel{\text{def}}{=} A_1 + BKC_1$, *the matrix* $R_k \in \mathbb{R}^{(k+1)n\times(k+1)n}$ *is defined by*

$$R_k \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix}^T \left[\begin{pmatrix} (I_k\otimes A_K)^T P_k + P_k(I_k\otimes A_K) & P_k(I_k\otimes(A_KE+B_K)) \\ (I_k\otimes(A_KE+B_K))^T P_k & 0_{kn} \end{pmatrix} + \begin{pmatrix} Q_k & Q_k(I_k\otimes E) \\ (I_k\otimes E)^T Q_k & (I_k\otimes E)^T Q_k (I_k\otimes E) - Q_k \end{pmatrix} \right] \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix},$$

$$f_1 \stackrel{\text{def}}{=} I_n,\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}E \\ I_n \end{pmatrix} \in \mathbb{R}^{kn\times n},$$

$$F_1 \stackrel{\text{def}}{=} 0_n,\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1} \\ 0_{n\times(k-1)n} & 0_n \end{pmatrix} \in \mathbb{R}^{kn\times kn}\ .$$

■

4.3 Delay-independent stability of systems with polytopic uncertainties

One considers here, for fixed $A_l, B_l \in \mathbb{R}^{n\times n}$, $l = \overline{1,L}$, the convex class of the systems (12) such that

$$(A,B) \in \left\{ \sum_{l=1}^{L} \lambda_l (A_l, B_l)\ :\ 0 \leq \lambda_l \leq 1, \sum_{l=1}^{L} \lambda_l = 1 \right\}\ . \tag{22}$$

As for the finite-dimensional systems, a way to ensure the stability of all the previous systems is to exhibit a common Lyapunov-Krasovskii functional. This leads to the following result, whose residual conservatism is due only to the method of simultaneous stability itself (as is the case for finite-dimensional systems).

Theorem 6. *Suppose there exist* $k \in \mathbb{N}\setminus\{0\}$ *and positive matrices* $P_k, Q_k \in \mathbb{R}^{kn\times kn}$ *such that, for all* $l = \overline{1,L}$,

$$R_{l,k} \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix}^T \left[\begin{pmatrix} (I_k\otimes A_l)^T P_k + P_k(I_k\otimes A_l) & P_k(I_k\otimes(A_lE+B_l)) \\ (I_k\otimes(A_lE+B_l))^T P_k & 0_{kn} \end{pmatrix} + \begin{pmatrix} Q_k & Q_k(I_k\otimes E) \\ (I_k\otimes E)^T Q_k & (I_k\otimes E)^T Q_k (I_k\otimes E) - Q_k \end{pmatrix} \right] \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix} < 0\ ,$$

where

$$f_1 \stackrel{\text{def}}{=} I_n,\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}E \\ I_n \end{pmatrix} \in \mathbb{R}^{kn\times n},$$

$$F_1 \stackrel{\text{def}}{=} 0_n,\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1} \\ 0_{n\times(k-1)n} & 0_n \end{pmatrix} \in \mathbb{R}^{kn\times kn}\ .$$

Then, the system (12) *with constraint* (22) *is strongly delay-independently stable.* ■

The proof proceeds directly from the remark that, for any choice of $\lambda = (\lambda_1, \dots, \lambda_L)$ as in (22), R_k as defined in Theorem 2 is such that

$$R_k = \sum_{l=1}^{L} \lambda_l R_{l,k} < 0\ .$$

The stability of system (12), (22) for any nonnegative value of h is also ensured when the coefficients λ_l are time-dependent (e.g. state-dependent). To prove this fact, show the decrease along the trajectories of the quadratic Lyapunov-Krasovskii functional constructed with P_k, Q_k, and use Lyapunov-Krasovskii theory [16]. Special care has to be taken for neutral systems, see [8].

5 Proof of Theorem 1

The equivalence between 1., 2., 3. and 4. is known [1], and the implication 6. ⇒ 5. is straightforward. One will show that 5. implies 3., and then that 4. implies 6.

5.1 Proof of the implication 5. ⇒ 3.

Consider first that the feasibility of (11_k) implies that

$$0 > \begin{pmatrix} 0_{kn_1\times n_2} \\ I_{n_2} \end{pmatrix}^* R_k \begin{pmatrix} 0_{kn_1\times n_2} \\ I_{n_2} \end{pmatrix} = f_k^T[(I_k\otimes D)^T Q_k (I_k\otimes D) - Q_k] f_k = D^T f_k^T Q_k f_k D - f_k^T Q_k f_k \ ,$$

so D fulfills a (discrete-time) Lyapunov equation, and $\rho(D)<1$.

Define now, for any $z\in\mathbb{C}$ and for $k\in\mathbb{N}\setminus\{0\}$ the matrices $v_{1,k}(z)\in\mathbb{R}^{kn_1\times n_1}$, $v_{2,k}(z)\in\mathbb{R}^{kn_2\times n_2}$, $w_k(z)\in\mathbb{R}^{(kn_1+n_2)\times n_1}$ by

$$v_{1,k}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I_{n_1} \\ zI_{n_1} \\ \vdots \\ z^{k-1}I_{n_1} \end{pmatrix},\quad v_{2,k}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I_{n_2} \\ zI_{n_2} \\ \vdots \\ z^{k-1}I_{n_2} \end{pmatrix},\quad w_k(z) \stackrel{\text{def}}{=} \begin{pmatrix} v_{1,k}(z) \\ z^k(I_{n_2}-zD)^{-1}C \end{pmatrix} \ .$$

Then,

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} w_k(z) = \begin{pmatrix} v_{1,k}(z) \\ F_k v_{1,k}(z) + z^k f_k (I-zD)^{-1}C \end{pmatrix} \ ,$$

and, using (8), (9),

$$F_k v_{1,k}(z) + z^k f_k(I-zD)^{-1}C = \begin{pmatrix} F_{k-1}v_{1,k-1}(z) + z^{k-1}f_{k-1}(I-zD)^{-1}C \\ z^k(I-zD)^{-1}C \end{pmatrix} \ .$$

One then proves by induction that

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} w_k(z) = \begin{pmatrix} I_{kn_1} \\ zI_k\otimes(I_{n_2}-zD)^{-1}C \end{pmatrix} v_{1,k}(z) \ .$$

From the solvability of (11_k) one hence deduces that, for any $z\in\mathbb{C}$ such that $I-zD$ is invertible,

$$\begin{aligned}
0 &> w_k(z)^* R_k w_k(z) \\
&= \left[\begin{pmatrix} I_{kn_1} \\ zI_k\otimes(I-zD)^{-1}C \end{pmatrix} v_{1,k}(z)\right]^* \left[\begin{pmatrix} (I_k\otimes A)^T P_k + P_k(I_k\otimes A) & P_k(I_k\otimes B) \\ (I_k\otimes B)^T P_k & 0_{kn_2} \end{pmatrix}\right. \\
&+ \left.\begin{pmatrix} (I_k\otimes C)^T Q_k (I_k\otimes C) & (I_k\otimes C)^T Q_k (I_k\otimes D) \\ (I_k\otimes D)^T Q_k (I_k\otimes C) & (I_k\otimes D)^T Q_k (I_k\otimes D) - Q_k \end{pmatrix}\right] \begin{pmatrix} I_{kn_1} \\ zI_k\otimes(I-zD)^{-1}C \end{pmatrix} v_{1,k}(z) \\
&= v_{1,k}(z)^* \left[(I_k\otimes(A+zB(I-zD)^{-1}C))^* P_k + P_k(I_k\otimes(A+zB(I-zD)^{-1}C))\right] v_{1,k}(z) \\
&+ (1-|z|^2)\left[(I_k\otimes(I-zD)^{-1}C)v_{1,k}(z)\right]^* Q_k (I_k\otimes(I-zD)^{-1}C)v_{1,k}(z) \\
&= (A+zB(I-zD)^{-1}C)^* v_{1,k}(z)^* P_k v_{1,k}(z) + v_{1,k}(z)^* P_k v_{1,k}(z)(A+zB(I-zD)^{-1}C) \\
&+ (1-|z|^2)[v_{2,k}(z)(I-zD)^{-1}C]^* Q_k v_{2,k}(z)(I-zD)^{-1}C \ .
\end{aligned}$$

In particular, if $|z|\leq 1$ (and then $I-zD$ invertible), this yields

$$(A+zB(I-zD)^{-1}C)^* v_{1,k}(z)^* P_k v_{1,k}(z) + v_{1,k}(z)^* P_k v_{1,k}(z)(A+zB(I-zD)^{-1}C) < 0 \ .$$

As the matrix $v_{1,k}(z)^* P_k v_{1,k}(z)$ is positive definite, one deduces that for any z such that $|z|\leq 1$, the matrix $A+zB(I-zD)^{-1}C$ fulfills a Lyapunov equation, so $\operatorname{Re}\sigma(A+zB(I-zD)^{-1}C)<0$. This achieves the proof of the implication 5. ⇒ 3.

5.2 Proof of the implication 4. ⇒ 6.

• One first transforms condition 4. It is well-known that, for any square matrix M,

$$\rho(M) < 1 \Leftrightarrow \lim_{k\to+\infty} \|M^k\| = 0 \Leftrightarrow \limsup_{k\to+\infty} \|M^k\| < 1$$

(where the second equivalence is obtained using the fact that the matrix norm induced by the euclidian norm is submultiplicative). One hence deduces that condition 4. is indeed *equivalent* to

$$\mathrm{Re}\,\sigma(A) < 0 \text{ and } \sup_{s\in\mathbb{C}^+} \limsup_{k\to+\infty} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| < 1 \ . \tag{23}$$

• One now transforms (11_k). Developing the first term in R_k leads to the identity

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix}^T \begin{pmatrix} (I_k\otimes A)^T P_k + (I_k\otimes A)P_k & P_k(I_k\otimes B) \\ (I_k\otimes B)^T P_k & 0_{kn_2} \end{pmatrix} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix}$$
$$= \begin{pmatrix} ((I_k\otimes A)+(I_k\otimes B)F_k)^T P_k + P_k((I_k\otimes A)+(I_k\otimes B)F_k) & P_k(I_k\otimes B)f_k \\ ((I_k\otimes B)f_k)^T P_k & 0_{n_2} \end{pmatrix} .$$

Written under this form, one may apply Kalman-Yakubovich-Popov lemma (reproduced in Appendix) to (11_k), taking into account the fact that $\sigma((I_k\otimes A)+(I_k\otimes B)F_k) = \sigma(A) \subset \mathbb{C}\setminus\mathbb{C}^+$. Denoting

$$\mathcal{S}_k = \mathcal{S}_k(s) \stackrel{\text{def}}{=} (sI_{kn_1} - (I_k\otimes A) - (I_k\otimes B)F_k)^{-1} ,$$

a *sufficient* (indeed equivalent) condition for solvability of (11_k) appears to be

$$\exists Q_k = Q_k^T > 0, \forall s\in\mathbb{C}^+\cup\{j\infty\},\ \det(sI_{kn_1} - (I_k\otimes A) - (I_k\otimes B)F_k) \neq 0 \text{ and}$$
$$\left[\cdot\right]^* \begin{pmatrix} (I_k\otimes C)^T Q_k (I_k\otimes C) & (I_k\otimes C)^T Q_k (I_k\otimes D) \\ (I_k\otimes D)^T Q_k (I_k\otimes C) & (I_k\otimes D)^T Q_k (I_k\otimes D) - Q_k \end{pmatrix} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} \begin{pmatrix} \mathcal{S}_k(s)(I_k\otimes B)f_k \\ I_{n_2} \end{pmatrix} < 0 ,$$

where the dot in the brackets has to be replaced by the last two matrices. The introduction of the infinite value $j\infty$ means that the 2nd inequality in the previous formula has to remain true at the limit when $s\in j\mathbb{R}$, $|s|\to\infty$. Simplifications yield: $\exists Q_k = Q_k^T > 0, \forall s\in\mathbb{C}^+\cup\{j\infty\},\ \det(sI_{n_1} - A) \neq 0$ and

$$[(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k + (I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k)]^* Q_k [(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k + (I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k)]$$
$$< [F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k]^* Q_k [F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k] .$$

Let us simplify these expressions. From (8), (9), one gets that

$$\mathcal{S}_k(s) = \begin{pmatrix} sI_{(k-1)n_1} - (I_{k-1}\otimes A) - (I_{k-1}\otimes B)F_{k-1} & -(I_{k-1}\otimes B)f_{k-1}C \\ 0_{n_1\times(k-1)n_1} & sI_{n_1} - A \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \mathcal{S}_{k-1} & \mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}C(sI_{n_1} - A)^{-1} \\ 0_{n_1\times(k-1)n_1} & (sI_{n_1} - A)^{-1} \end{pmatrix} ,$$
$$(I_k\otimes B)f_k = \begin{pmatrix} (I_{k-1}\otimes B)f_{k-1}D \\ B \end{pmatrix} .$$

This permits to establish that

$$\mathcal{S}_k(I_k\otimes B)f_k = \begin{pmatrix} \mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}(C(sI_{n_1} - A)^{-1}B + D) \\ (sI_{n_1} - A)^{-1}B \end{pmatrix} ,$$

from which it is deduced that

$$F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k=\begin{pmatrix}(F_{k-1}\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}+f_{k-1})(C(sI_{n_1}-A)^{-1}B+D)\\ I_{n_2}\end{pmatrix}$$

and

$$\begin{aligned}&(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k+(I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k)\\ &=\begin{pmatrix}((I_{k-1}\otimes C)\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}+(I_{k-1}\otimes D)(F_{k-1}\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}+f_{k-1}))(C(sI_{n_1}-A)^{-1}B+D)\\ C(sI_{n_1}-A)^{-1}B+D\end{pmatrix}.\end{aligned}$$

The same expressions appear in the left-hand and right-hand sides, with adjacent ranks. One proves recursively that

$$F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k=\begin{pmatrix}[C(sI_{n_1}-A)^{-1}B+D]^{k-1}\\ \vdots\\ I_{n_2}\end{pmatrix},$$

$$(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k+(I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k)=\begin{pmatrix}[C(sI_{n_1}-A)^{-1}B+D]^{k}\\ \vdots\\ [C(sI_{n_1}-A)^{-1}B+D]\end{pmatrix}.$$

Taking $Q_k=I_{kn_2}$, one finally gets that a *sufficient* condition for realization of 6. is

$$\operatorname{Re}\sigma(A)<0 \text{ and } \limsup_{k\to+\infty}\sup_{s\in\mathbb{C}^+}\|[C(sI_{n_1}-A)^{-1}B+D]^k\|<1\ . \tag{24}$$

• In view of (23) and (24), it remains, in order to prove that 4. implies 6., to show that one may choose in (23) the index k *uniformly* with respect to $s\in\mathbb{C}^+$. The final argument, based on compactness, is itself decomposed into two parts. It will be shown first that (23) implies

$$\operatorname{Re}\sigma(A)<0 \text{ and } \exists k\in\mathbb{N}\setminus\{0\},\ \sup_{s\in\mathbb{C}^+}\|[C(sI_{n_1}-A)^{-1}B+D]^k\|<1\ , \tag{25}$$

and then that (25) implies (24).
• For $k\in\mathbb{N}\setminus\{0\}$, let

$$K_k\stackrel{\text{def}}{=}\{s\in\mathbb{C}^+\ :\ \|[C(sI_n-A)^{-1}B+D]^k\|\geq 1\}\ .$$

By continuity, the sets K_k are closed. Moreover,

$$s\in K_{2k}\Rightarrow 1\leq\|[C(sI_n-A)^{-1}B+D]^{2k}\|\leq\|[C(sI_{n_1}-A)^{-1}B+D]^k\|^2\Rightarrow s\in K_k\ .$$

Hence $K_{2k}\subset K_k$, for any $k\in\mathbb{N}\setminus\{0\}$.

Assume now that (25) does *not* hold. If $\operatorname{Re}\sigma(A)\not<0$, then (23) does not hold. If $\rho(D)\not<1$, then (23) does not hold either, as necessarily one would have $\limsup_{k\to+\infty}\|D^k\|<1$. Suppose now that $\operatorname{Re}\sigma(A)<0$ and $\rho(D)<1$. Then, for any $k\in\mathbb{N}\setminus\{0\}$, the sets K_k are nonempty and bounded (as $\rho(D)<1$). The sequence K_{2^k} is thus a nested sequence of nonempty compact sets. In particular,

$$\exists s_0\in\bigcap_{k\in\mathbb{N}}K_{2^k}\ ,$$

that is

$$\exists s_0\in\mathbb{C}^+,\forall k\in\mathbb{N},\ \|[C(s_0I-A)^{-1}B+D]^{2^k}\|\geq 1\ .$$

Hence,

$$\forall k^* \in \mathbb{N} \setminus \{0\},\ \sup_{k \geq k^*} \|[C(s_0 I - A)^{-1} B + D]^k\| \geq 1 \ ,$$

and (23) does not hold either. One has hence proved by contradiction that (23) implies (25).

• Let us prove now that (25) implies (24). Suppose that (25) holds, and let $k^* \in \mathbb{N} \setminus \{0\}$ and $c_1 > 0$ be such that

$$\sup_{s \in \mathbb{C}^+} \|[C(sI_{n_1} - A)^{-1} B + D]^{k^*}\| = c_1 < 1 \ .$$

Define also

$$c_2 \stackrel{\text{def}}{=} \sup \left\{ \sup_{s \in \mathbb{C}^+} \|[C(sI_{n_1} - A)^{-1} B + D]^k\| \ : \ k \in \{0, \dots, k^* - 1\} \right\} \ .$$

Then c_2 is finite.

Now, fix $k^{**} \in \mathbb{N} \setminus \{0\}$ such that

$$k^{**} > \left(-\frac{\log c_1}{\log c_2} + 3 \right) k^* \ ,$$

and let $s \in \mathbb{C}^+$ and $k \in \mathbb{N}$ such that $k \geq k^{**}$. Denote q and r the quotient and the rest of the euclidian division of k by k^*, that is:

$$q \in \mathbb{N},\ \ r \in \{0, 1, \dots, k^* - 1\},\ \ k = qk^* + r \ .$$

Remark that $k \geq k^{**}$ implies

$$q \geq -\frac{\log c_1}{\log c_2} + 2 > -\frac{\log c_1}{\log c_2} + 1 \ . \tag{26}$$

Then,

$$\|[C(sI_{n_1} - A)^{-1} B + D]^k\| \leq \|[C(sI_{n_1} - A)^{-1} B + D]^{k^*}\|^q \|[C(sI_{n_1} - A)^{-1} B + D]^r\| \leq c_1^q c_2 < c_1 < 1 \ ,$$

due to (26) and the fact that $c_1 < 1$. From (25), one has hence deduced the existence of k^{**} such that

$$\forall k \in \mathbb{N}, k \geq k^{**},\ \sup_{s \in \mathbb{C}^+} \|[C(sI_{n_1} - A)^{-1} B + D]^k\| < c_1 < 1 \ ,$$

so one gets

$$\operatorname{Re} \sigma(A) < 0 \text{ and } \limsup_{k \to +\infty} \sup_{s \in \mathbb{C}^+} \|[C(sI_{n_1} - A)^{-1} B + D]^k\| < 1 \ ,$$

which is nothing but (24).

To summarize, it has been successively shown that: Condition 4. ⇔ (23), (24) ⇒ Condition 6., (23) ⇒ (25), (25) ⇒ (24). This shows finally that condition 4. implies condition 6., and concludes the proof of Theorem 1.

A Appendix – Kalman-Yakubovich-Popov lemma

We use the statement as stated e.g. in [23]. Denote $\overline{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty\}$. Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, M = M^T \in \mathbb{R}^{(n+p)\times(n+p)}$.

Lemma 7. *If* $\det(sI_n - A) \neq 0$ *for* $s \in j\mathbb{R}$*, then the two following properties are equivalent.*

1. *For any* $s \in j\overline{\mathbb{R}}$*,*

$$\begin{pmatrix} (sI_n - A)^{-1}B \\ I_p \end{pmatrix}^* M \begin{pmatrix} (sI_n - A)^{-1}B \\ I_p \end{pmatrix} < 0 \ .$$

2. *There exists* $P = P^T \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{pmatrix} A^TP + PA & PB \\ B^TP & 0_p \end{pmatrix} + M < 0 \ .$$

■

When

$$\begin{pmatrix} I_n & 0_{n\times p} \end{pmatrix} M \begin{pmatrix} I_n \\ 0_{p\times n} \end{pmatrix} \geq 0 \ ,$$

one may equivalently replace in Lemma 7 $s \in j\overline{\mathbb{R}}$ in condition 1. by: $s \in j\overline{\mathbb{R}} \cup \mathbb{C}^+$.

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