Fractional Coloring of Bounded Degree Trees

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Abstract: We study the dipath-coloring problem in bounded degree and treewidth symmetric digraphs, in which one needs to color the dipaths with a minimum number of colors, in such a way that dipaths using the same arc have different colors. This classic combinatorial problem finds applications in the minimization of the number of wavelengths in wavelength division multiplexing (WDM) all optical networks.

In this paper, we prove that finding an optimal fractional coloring of such dipaths is polynomial. Our solution is constructive, i.e. we give an effective polynomial algorithm for the problem above. We also show some relationships between the integral and fractional problems, prove some polynomial instances of the coloring problem, and derive a $1 + 5/3e$ approximation for the WDM problem in symmetric directed trees, where e is the classical Neper constant, improving on previous results.

Finally we present computational results suggesting that fractional coloring is a good oracle for a *branch and bound* strategy for coloring dipaths in symmetric directed trees and cycles.

Key-words: fractional coloring, coloring, network, wdm, optical, fiber, linear programming, wavelength division multiple access, graph, tree, treewidth, NP, algorithm, approximation, bound

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Coloration fractionnaire des arbres de degré borné

Résumé : Ce rapport de recherche présente la preuve de polynomialité de la coloration fractionnaire optimale des sommets de graphes particuliers qui sont les graphes de conflits d'une famille de chemins orientés sur un graphe orienté symmétrique de degré et de largeur arborescente bornés.

Notre recherche est motivée par le problème de coloration de chemins orientés où il faut colorier les chemins orientés d'un graphe orienté avec un nombre de couleurs minimum en respectant la contrainte que deux chemins traversant le même arc doivent avoir des couleurs différentes. Ce deuxième problème est la version entière du premier et prend comme motivation pratique la minimisation du nombre de longueur d'ondes utilisées dans un réseau de fibres optiques à multiplexage en longueur d'ondes (WDM).

Notre solution est constructive puisque nous donnons un algorithme (que nous avons implémenté) pour le problème ci-dessus.Nous montrons aussi certaines relations entre les problèmes entier et fractionnaire, nous prouvons la polynomialité de certaines instances du problème de coloriage et nous démontrons une $1 + 5/3e$ approximation du problème dans les arbres, où e représente la constante de Néper.

Enfin nous présentons des résultats suggérant que la coloration fractionnaire est un bon oracle pour un coloriage de chemins orientés dans des arbres ou des anneaux avec une stratégie type *branch and bound*.

Mots-clés : coloration fractionnaire, réseau, optique, multiplexage en longueur d'onde, wdm, graphe, coloriage, approximation, fibre, programmation lineaire, algorithme, NP, arbre, largeur arborescente, borne

1 Introduction

Graph coloring is one of the most fundamental problems considered in Computer Science. Unfortunately it is very hard to solve to optimality or even to approximate. A related problem, the path coloring problem, consists in coloring a set Λ of paths in a given graph $\mathcal{G}$ so that two paths sharing an edge cannot have the same color. This problem is equivalent to coloring the corresponding conflict graph, where the conflict graph is defined as the graph whose vertices represent the paths of Λ and where there is an edge between vertices representing conflicting paths. Notice, however, that this problem was proven to be the same as standard graph vertex coloring in term of complexity or difficulty to approximate, since any n-vertex graph is the conflict graph of a set of dipaths of an $n \times n$ grid. Furthermore, Tarjan proved it to be $\mathcal{NP}$-hard even for trees [Tar85].

Recently, and motivated by applications in wavelength division multiplexed (WDM) optical networks and call scheduling, a renewed interest in dipath coloring of special instances of digraphs $\mathcal{G}$ has developed, especially for the directed version. We note that there are important differences between the directed and undirected cases. For instance, the problem is polynomial for directed symmetric stars, but is in $\mathcal{NP}$-hard for undirected star [Bea00].

In the WDM *problem*, one is given a digraph $\mathcal{G} = (V, E)$ and a multiset of communication requests (multisubset of $V \times V$) and must assign to each request a dipath and to each dipath a color (wavelength) so that *conflicting* dipaths (i.e. using the same arc) are assigned different colors. The goal is to minimize the number of colors used. The WDM problem has been widely studied. Not surprisingly it has been proved to be difficult (is $\mathcal{NP}$-hard even for cycle and binary trees), moreover one can show that there exist networks with $O(n^2)$ vertices and n requests on which it is hard to decide if the minimal number of colors is either 1 or n, thus in general the problem is also hard to approximate.

If one assumes that dipaths have already been assigned to requests, the problem is called *Wavelength Assignment Problem* (WAP) and is the directed version of the path coloring problem, described above. Hence, recent works have studied specific instances of the underlying networks: in [EJKP97], a polynomial algorithm was given to compute the WAP on symmetric trees with $\frac{5\pi}{3}$ colors (where π is the maximum load of an arc). In [Kar80], a $\frac{3}{2}$ approximation for the WAP on cycles was proposed, which was later improved to a $1 + \frac{1}{e}$ approximation ([Kum98]), where e is the classical Neper constant, by means of a linear relaxation of the circular arc graph coloring problem.

In this paper we show that such a linear relaxation can be generalized to the natural relaxation of vertex coloring into *fractional coloring* ([GLS81]), where the independent sets covering the vertices may have fractional weights (instead of weight 1 as in the normal coloring). Unfortunately, the fractional chromatic number is, in general, also hard to approximate, since a classical result states that any ρ approximation of the fractional chromatic number leads to a $\rho log(n)$ approximation of the chromatic number ([GLS81]). On the positive side, however, we present a polynomial algorithm to compute an optimal fractional coloring of a set of dipaths in a bounded degree and bounded treewidth network, proving that fractional coloring is in $\mathcal{P}$ for such networks. We also provide some universal fractional coloring of dipaths of a symmetric directed binary tree and we prove that the fractional chromatic number of a set of dipaths in such networks is at most $\frac{7\pi}{5}$, improving the results in [EJKP97]. This also implies that the maximum independent set in such networks is of size at least $\frac{5}{7\pi}$.

2 Fractional coloring

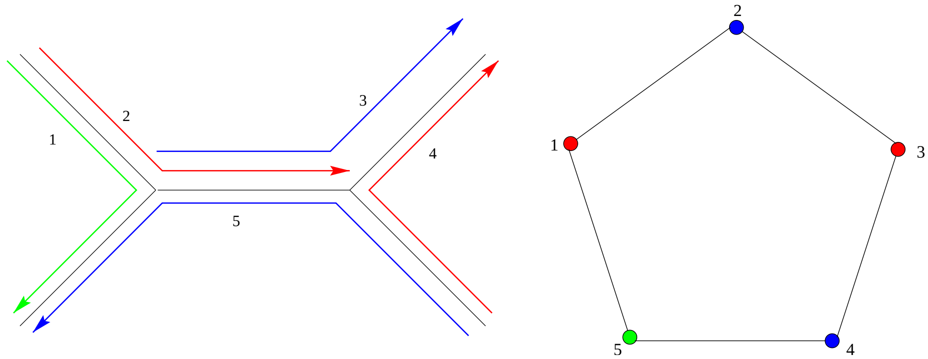


Figure 1: Example of a WDM network and the associated conflict graph.

The graph vertex-coloring problem can be considered as finding a minimum cost integral covering of the vertices by independent sets of cost 1. This means solving the following linear program with binary variables :

$$\begin{array}{l}\text{minimize } \sum_{I \in \mathcal{I}} x(I),\ \forall\ I,\ x(I) \in \{0,1\} \\ \text{subject to } \forall v \in V(\mathcal{G}),\ \sum_{I \in \mathcal{I},\ v \in I} x(I) \geq 1\end{array}$$

Where $\mathcal{I}$ denotes the set of the independent sets of $\mathcal{G}$.

This formulation as a standard minimum cost integral covering problem has a natural relaxation into a linear programming problem.

Definition 1 (Fractional coloring of a graph $\mathcal{G}$)

$$\begin{array}{l}\textit{minimize } \sum_{I \in \mathcal{I}} w(I) \\ \textit{subject to } \forall\ v \in V(\mathcal{G}), \sum_{I \in \mathcal{I}, v \in I} w(I) \geq 1\end{array}$$

This problem is called the fractional coloring problem *(see [GLS81]), and the optimal value of a solution is the* fractional chromatic number $\omega_f(\mathcal{G})$.

If w is a valid weight function over the independent sets of the graph $\mathcal{G}$, we note it $w(\mathcal{G})$ and call it a fractional coloring of $\mathcal{G}$. *The same symbol will denote the cost of $w(\mathcal{G})$.*

In general the fractional chromatic number is as hard to approximate as the chromatic number since a classical result of Lovász states that any ρ approximation of the fractional chromatic number leads to a $\rho log(n)$ approximation of the chromatic number. Indeed the size of the above described linear problem is exponential (about the number of independent sets of $\mathcal{G}$).

It is well-known that the dual of covering problem consists in finding a positive weight function w^* on the vertices such that [GLS81]

$$\begin{array}{l} \sum_{v \in V} w^*(v) \text{ is maximum} \\ \forall I \in \mathcal{I}, \sum_{v \in I} w^*(v) \leq 1 \end{array}$$

In this problem, a non trivial constraint is violated if and only if the weight of one independent set is > 1[1]; hence the maximum weighted independent set problem (MWIS) is a separation oracle for this last problem. According to the separation & optimization equivalence (WSEP = WOPT, see [GLS93], th. 4.2.7, p 106 and [GLS81]) the dual of fractional coloring and the MWIS are equivalent up to polynomial reduction; hence computing the fractional chromatic number is polynomialy equivalent to solve MWIS. However, this duality argument does not provide any effective fractional coloring algorithm but rather a way to compute the fractional chromatic number.

[1]the weight of an independent set I is the sum of the weights of the vertices in I : $w^*(I) = \sum_{v \in I} w^*(v)$

3 Fractional coloring for dipaths in cycles, trees

From now on we will consider only the problem of coloring dipaths in a symmetric digraph. We extend some terms of vertex graph coloring to path coloring.

Definition 2 (Path coloring and load of $\mathcal{G}, \Lambda$)
We define an independent set *of dipaths as a set of pairwise arc-disjoints dipaths*[2].

If $\mathcal{G}_c$ is the conflict graph of $(\mathcal{G}, \Lambda)$, we will denote $\omega(\mathcal{G}, \Lambda)$ ($\omega_f(\mathcal{G}, \mathcal{P})$) the (fractional) chromatic number of $\mathcal{G}_c$ and call them the (fractional) chromatic number of Λ on $\mathcal{G}$.

We will also call $\pi(\mathcal{G}, \Lambda)$ the load of $(\mathcal{G}, \Lambda)$, i.e. the maximum number of dipaths of Λ using any fixed arc of $\mathcal{G}$.

Since using dynamic programming one can solve the paths-MWIS problem in polynomial time and space on cycles [CLR90] and bounded degree trees [Gar94] [3], computing the fractional chromatic number of a set of dipaths is polynomial in these graphs. As often, the separation theorem indicates that a direct polynomial algorithm should exist.

Note that in the case of rings, the coloring problem is equivalent to the circular arc graph coloring problem; Tucker [Tuc75] proved that it is equivalent to solve a very specific integral multi-commodity flow problem. In [Kum98] Kumar used a randomized rounding of a fractional optimal solution of the multi-flow problem to prove that generally $\omega_f(C_n, \Lambda) \leq \omega(C_n, \Lambda) \leq \omega_f(C_n, \Lambda) + \frac{\pi(\mathcal{G}, \mathcal{P})}{e}$. Indeed we will show that this technique, which motivated our work, is equivalent to compute an optimal fractional coloring and round it randomly.

In the following we provide an algorithm to compute an optimal fractional coloring of a bounded degree tree, the technique is then easily applied to bounded degree & bounded treewidth networks. We prove some relation between ω_f and ω in binary trees and that $\forall \mathcal{G}$ fixed graph $\exists R(\mathcal{G})$ s.t. $\forall \Lambda$ family of paths, $\omega_f(\mathcal{G}, \Lambda) \leq \omega(\mathcal{G}, \Lambda) \leq \omega_f(\mathcal{G}, \Lambda) + R(\mathcal{G})$. We also prove that dipaths coloring in trees and cycles is polynomial if the number of non equivalent dipaths crossing an arc is bounded; as example the problem is polynomial in bounded trees, cycles.

[2] i.e. a set of paths whose corresponding vertices form an independent set of the conflict graph
[3] the original result is for the undirected version, but one can use the same for directed versions

4 An algorithm for fractional coloring of trees

In this section we study the Tree Dipath Fractional Coloring problem (TDFC). We assume that dipaths are labeled so that labels are unique, and given a family of dipaths we will denote $\Lambda(p)$ the path p of family Λ. The special label $\emptyset$ will be associated to a non-existent or void path.

4.1 Trace of a fractional coloring

Given a solution w for a TDFC that is a weight function w on the independent sets of the tree T, we define the *trace* of the fractional coloring on an edge e as the following vector:

Definition 3 (Trace of an edge)

- *Let $\mathcal{I}(p)$ be the family of independent sets containing $\Lambda(p)$, and $\mathcal{I}(p,q) = \mathcal{I}(p) \cap \mathcal{I}(q)$.*
- *for any couple of label p, q such that $\Lambda(p), \Lambda(q)$ use e in opposite directions let $X_{p,q} = \sum_{I \in \mathcal{I}(p,q)} w(I)$.*
- *for any label p such that $\Lambda(p)$ uses e, let $\mathcal{I}_e(p,\emptyset)$ be the set of independent sets of $\mathcal{I}(p)$ using e in only one direction, and let $X_{p,\emptyset} = \sum_{I \in \mathcal{I}(p,\emptyset), p \in I} w(I)$.*
- *Finally let $\mathcal{I}(\emptyset,\emptyset)$ be the family of independent set not using e.*

We will call trace of w on the edge e *the vector X.*

Definition 4 ($Sol((T,\Lambda),c)$)
Given a TDFC *(T,Λ) we denote $Sol((T,\Lambda),c)$ the set of all the fractional coloring of (T,Λ) with cost less than c.*

4.2 Split & Merge

Our algorithm is constructing inductively a polynomial size linear program whose solution provides a fractional coloring. Our induction is based on *merge & split* operations which allow to build any instance of TDFC starting from instances on star and merging them step by step. Consider a TDFC (T,Λ) where T is not a star and such that $e = [u,v]$ is a non-terminal edge of T, by **splitting T at** $[u,v]$ we mean defining two smaller TDFC (T_1,Λ_1) and (T_2,Λ_2) as follow (cf. 2) :

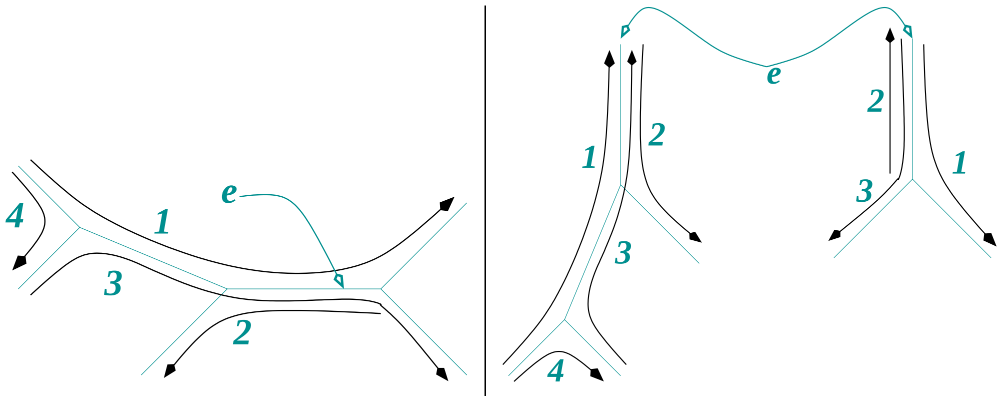


Figure 2: Splitting a tree on e.

- Let S_1 and S_2 be the two connected components of $T \setminus \{u, v\}$, and let $T_1 = S_1 \cup \{[u, v]\}$, $T_2 = S_2 \cup \{[u, v]\}$.
- Let $\Lambda_1 = \Lambda \cap T_1$, $\Lambda_2 = \Lambda \cap T_2$ (i.e in each subtree each original dipath is replaced by the dipath it induces with a label equal to the original one).

Note that the split operation can be easily reverted by a *merge* one if one keeps track of the dipaths labels and of the edge where the split occurred. In what follow we will always assume that (T, Λ) can be split on the edge e into (T_1, Λ_1) and (T_2, Λ_2).

Now, we prove that one can build $Sol((T, \Lambda), c)$ from $Sol((T_1, \Lambda_1), c)$ and $Sol((T_2, \Lambda_2), c)$.

Proposition 1 *The elements of $Sol((T, \Lambda), c)$ are obtained by merging elements of $Sol((T_1, \Lambda_1), c)$ and $Sol((T_2, \Lambda_2), c)$ having the same trace on e.*

Proof : First we remark that any fractional coloring of T with cost c induces fractional colorings on (T_1, Λ_1) and (T_2, Λ_2) with cost less than c from construction both have the same trace on e.

Conversely, given two fractional colorings $w_1 \in Sol((T_1, \Lambda_1), c)$ and $w_2 \in Sol((T_2, \Lambda_2), c)$ having the same trace on e we *merge* them into a fractional coloring in $Sol((T, \Lambda), c)$ by repeating the next procedure until $X_1(p, q) = 0, \forall p, q$:

- If for some couple of labels (p, q) (including the $\emptyset$ label) $X_1(p, q) > 0$, then since $X_1(p, q) = X_2(p, q)$ there exists $I_1 \in \mathcal{I}_1(p, q)$ and $I_2 \in \mathcal{I}_2(p, q)$ with $w_1(I_1) > 0$ and $w_2(I_2) > 0$.
- Let $w_{min} = min(w_1(I_1), w_2(I_2))$.
- Let $I = I_1 \setminus \{\Lambda_1(p), \Lambda_1(q)\} \cup I_2 \setminus \{\Lambda_2(p), \Lambda_2(q)\} \cup \{p, q\}$, and note that I is an independent set of T
- Increase $w(I)$ by w_{min} decrease $w_1(I_1)$ and $w_2(I_2)$ by w_{min}.

To verify the claim, just note that the procedure preserves the following invariant :

- trace equality $(X_1 = X_2)$
- Paths are covered either by w_1, w_2 or w
- $cost(w) + cost(w_1) = cost(w_2) + cost(w) = c$

It follows that, at the end, w is a fractional coloring of (T, Λ) with cost c; moreover, for $i = 1, 2$ the trace of w on any edge of T equals the one of w_i in T_i.

□

Corollary 2 *One can compute a polynomial size linear program whose solutions are valid traces of elements of $Sol((T, \Lambda), c)$*

Proof : We first assume that such a program does exist for bounded degree stars, a naive way to get one is to use *labeled independent set variables*, that is one variable for any possible labeled independent set in a bounded degree star (as dipaths are labeled, we must distinguish between similar dipath having different labels). Hence, if the load is π, at most $(\pi)^{2d}$ different labeled independent sets can correspond to an unlabeled independent set. Hence a pessimistic estimate count about $M(d)\pi^{2d}$ variables, where $M(d)$ is the number of perfect matchings in $K_{d,d}$. Fractional coloring is trivially described from this variables, then the trace variables are simple sum of subsets of the labeled independent set variables.

We then use an inductive algorithm to generate a linear program whose solutions are traces of $Sol((T, \Lambda), c)$. Assume that T can be split into (T_1, Λ_1) and (T_2, Λ_2) and let S_i be the linear program for traces of $Sol((T_i, \Lambda_i), c)$, where we assume that the trace on e is associated to the variables $X^i(p, q)$. Then a system for (T, Λ) is :

$$S = S_1 \cup S_2 \cup \{X^1{}_{p,q} = X^2{}_{p,q} \mid \forall p, q\}$$

□

Proposition 3 TDFC *reduces to solve a polynomial size linear problem.*

Proof : We simply have to show how to compute an element of $Sol((T,\Lambda),c)$ from a trace $\mathcal{T}$ of an element of $Sol((T,\Lambda),c)$ (obtained from corollary 2). Again we proceed inductively : we start from stars and find for each one a fractional coloring having the trace that $\mathcal{T}$ induces on it. Then if (T,Λ) can be split into (T_1,Λ_1) and (T_2,Λ_2), proposition 1 provides a way to merge fractional colorings of (T_1,Λ_1) and (T_2,Λ_2) when their traces are equal, and this is the case since the global trace is valid. □

4.3 Reducing the problem size

Note that our first model induces very large systems since we could get $O(1)n\pi^6$ variables for binary trees, and $O(1)n\pi^8$ for ternary ones. In this section we show how to reduce the system size to $dM(d)\pi^2 n$, that is a size of order $n\pi^2 O(1)$ for bounded degree trees (with still a fast growing constant).

Proposition 4 TDFC *reduces to solve linear problem with size $M(d)\pi^2 dn$.*

Proof : First and without loss of generality we assume that the load is uniform (if not, we add length 1 dipaths which do not modify the fractional chromatic number).

Note that the system that describes valid traces can be alternatively considered as follow :

- Variables $X_{p,q,e}$ for $e \in \Lambda(p)$ or $p = \emptyset$ and $e \in \Lambda(q)$ or $q = \emptyset$ describe the trace itself. (note that for $p \neq \emptyset$ and $q \neq \emptyset$, $X_{p,q,e}$ do not depend on the edge considered).
- For each star of the tree the description of a local *constrained fractional coloring problem* which encode the fact that the star must be fractionally colored with a trace consistent with X.

We use a flow-like description for the problem which can be related to well-known properties :

a) the matching polytope of a bipartite graph is encoded by simple flow equations.

b) equiv. probability matrices are convex combination of permutation matrices.

As our problem is slightly different from matching we have to describe our construction : Let $\mathcal{M}$ be the set of load 1 independent sets (ie subsets of dipaths loading each arc exactly once). First we fix an independent set $M \in \mathcal{M}$ and we consider the following auxiliary flow problem F_M :

- For each dipath $\Lambda(p) \in M$ we add a **vertex** $V(p)$.
- For each couple of dipaths $\Lambda(i), \Lambda(j) \in M$ such that $\Lambda(i) = [a,b]$ and $\Lambda(j) = [b,c]$ we add an arc $(V(i), V(j))$.
- We define a flow function[4] f on the arcs which induces a flow function f' on the vertices.
- For each leaf a of the star we add the constraint

$$\sum\{\Lambda(p) \mid \Lambda(d) \text{ a dipath ending at } a\} f'(V(p)) = X_M$$

To any solution of the previous system corresponds a weight function on the labeled independent set isomorphic to M with total weight X_M, this induces a covering C_M of the dipaths with $C_M(p) = f'(p)$ and inducing a trace $\mathcal{T}_M(p,q)$ equals to $f(p,q)$.

The total system for a star is obtained as follow :

- The $F_M, M \in \mathcal{M}$ are considered all together.
- $X_{p,q} = \sum_{M \in \mathcal{M}} \mathcal{T}_M(p,q)$
- $\sum_{M \in \mathcal{M}} C_M(p) \geq 1$
- $cost \geq \sum_{M \in \mathcal{M}} X_M$

Due to the above mentioned properties of the matching polytope, this system describes the fractional coloring problem for a star with an encoding of the trace variables.

It follows that each constrained start fractional coloring can be described with at most $|\mathcal{M}|$ (one for each unlabeled full load independent set) systems of size $\pi^2 d$.

□

[4]by flow function we mean conservative for vertices

5 Consequences

5.1 Bounded treewidth

Proposition 5 *Dipath fractional coloring is polynomial for bounded treewidth & degree digraphs.*

Proof : As often the proof is the same than for trees, assuming that one cuts the network with k edges into some set of smaller networks, we use the same split operation than in trees. The only difference is that the trace must be now defined for each of the k edges leading to $(\pi^2)^k$ trace variables where k is bounded by the treewidth. □

5.2 Some polynomial cases

To conclude we mention some polynomial cases : Note that our approach can be considered as dynamic programming where one maintains a polynomial encoding of the *valid traces*, for coloring one only need to keep integral traces (e.g. permutation in the case of the cycle [Tuc75]). In some cases integral traces can also be encoded.

Proposition 6 *If the number of non isomorphic dipaths crossing any edge is bounded,* TDFC *is polynomial.*

Proof : Note that if we assume that at most k non isomorphic dipaths cross any edge and that the load is π, the trace can be encoded with integral k-uples with bounded sum. So in this particular case the number of different traces is π^k. □

5.3 Dipath Coloring in trees

Proposition 7 *There exist a* $1 + \frac{5}{3e} \sim 1.61$ *approximation to the Tree Dipath Coloring Problem.*

Proof : As in Kumar paper [Kum98], one can perform a randomized rounding of a fractional coloring w, taking the independent set I with probability $w(I)$. The expected cost is $w_f(T, \Lambda)$ and one can show that with high probability the cost will be $w_f(T, \Lambda) + o(w_f(T, \Lambda))$. Still some dipaths are not covered, but the load of the uncovered dipaths can be shown to be less than $\frac{\pi}{e}$. Using Kaklamanis & all algorithm [EJKP97] one can then color them using $\frac{5\pi}{3e}$ additional independent sets. On the whole we get a $1 + \frac{5}{3e}$ approximation. □

In the next section we give a *uniform* fractional coloring for 3-stars of cost $\frac{7\pi}{5}$ which allows us to say that the optimal fractional coloring of binary trees is at most $\frac{7\pi}{5}$.

6 Universal Fractional coloring of the 3-stars

The previous proposition shows that TDFC can be considered as coloring dipaths in several stars with some correlation equations.

It is hence natural to wonder if one can define a universal fractional coloring in the following sense : If, in a tree, one uses at each star an universal fractional coloring then the additional correlation constraints are always satisfied; note that such an universal coloring allows to color each star independently.

So our aim is to find an universal coloring, valid for any tree and having minimal cost (for a certain load of the tree). It turns out that the good notion is *balanced* fractional coloring.

6.1 Definition of a *balanced* fractional coloring

We say that a coloring is balanced if and only if its trace is balanced on every edges.

Definition 5 (Balanced trace)
A fractional coloring is $(x, 1-x)$ balanced if and only if on any edge the trace satisfies

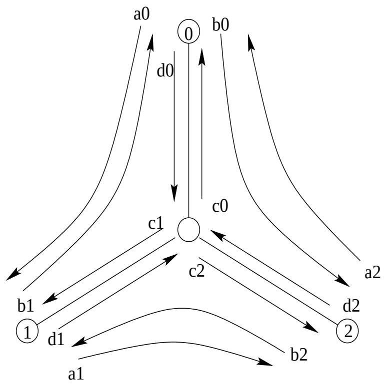
- $\forall$ *a, b paths going in opposite direction through the edge e,* $X_{a,b} = \frac{x}{\pi}$
- $\forall$ *a path going through the edge e,* $X_{a,\emptyset} = 1 - x$

First we reduces the number of parameters of our problem by coloring reduced 3-stars.

6.2 Reduction of the 3-star

A generic 3-star is described by a lot of parameters (see fig. 3). We recall a standard [ACKP00] argument which transform any a 3-star into a reduced 3-star which is described by only 3 parameters.

1. First we can assume that the load is uniform.
2. Second, we can decrease the number of length 1 paths by concatenating as much of them as possible. Two paths can been concatenated when one stops at the central node and the other starts from here and if they are not using the same arc in opposite directions.



$$\forall i,\ a_i + b_i + d_i = \pi_{\text{out},i}$$

$$a_2 + b_1 + c0 = \pi_{\text{in},0}$$

$$a_0 + b_2 + c_1 = \pi_{\text{in},1}$$

$$a_1 + b_0 + c_2 = \pi_{\text{in},2}$$

$$\forall i,\ \pi_{\text{in},i} \leq \pi,\ \pi_{\text{out},i} \leq \pi$$

Figure 3: The generic 3-star and its parameters.

After this reduction, paths of length 1 are found on only one link and there is the same amount of them on both arcs. This fact is due to a flow conservation law at the central vertex.

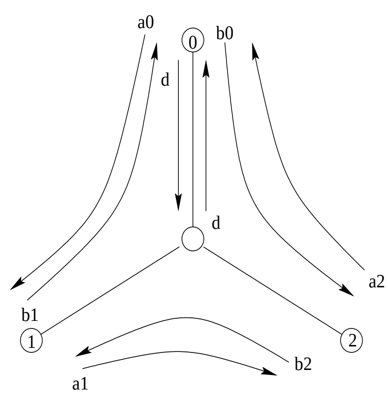


Figure 4: The generic reduced 3-star

The reduced 3-star is shown on picture 4.

Lemma 8 *The reduced 3-star can be described with 3 parameters* α, β *and* γ *with* $\alpha+\beta+\gamma = \pi$.

Due to the above mentionned matching polytope properties reduced 3 stars are convex combinations of reduced independent sets, and there is only three of them. The number of circuits going counter-clockwise (α), circuits going clockwise (β) and a set of 2 length 2 circuits (γ). We can identify these parameters with the previous ones :

$$a_0 = a_2 = \alpha, b_0 = b_1 = \beta, d = \gamma, a_1 = \alpha + \gamma, b_1 = \beta + \gamma$$

Appendix A describes a balanced fractional coloring of such a reduced 3-star :

Lemma 9 *Any (α, β, γ) 3-star admits a $(\frac{4}{5}, \frac{1}{5})$ balanced fractional coloring with cost*

$$\frac{4\pi}{5} + \frac{4\max(\alpha\beta + \alpha\gamma + \beta\gamma, \frac{\gamma\pi}{2})}{5\pi} + \frac{2}{5}(\alpha + \beta)$$

This implies the next proposition :

Proposition 10 *Any reduced 3-star admits a balanced $(\frac{4}{5}, \frac{1}{5})$ coloring with cost at most $\frac{7\pi}{5}$.*

Proof :

- If $\frac{\gamma\pi}{2} \leq \alpha\beta + \alpha\gamma + \beta\gamma$ then $\Gamma = \frac{4}{5}(\pi + \frac{\alpha\beta+\gamma(\pi-\gamma)}{\pi}) + \frac{2}{5}(\pi - \gamma)$ which is maximum for $\alpha = \beta = \frac{\pi}{2}, \gamma = 0 : \Gamma \leq \frac{7\pi}{5}$.
- Otherwise $\Gamma = \frac{4\pi}{5} + \frac{2\gamma}{5} + \frac{2}{5}(\pi - \gamma) = \frac{6\pi}{5}$.

Hence, $\forall\ \alpha + \beta + \gamma = \pi,\ \Gamma \leq \frac{7\pi}{5}$. □

Note that this result seems highly related to Auletta, Caragiannis, Kaklamanis, and Persiano [ACKP00] randomized algorithm which colors binary trees with $\frac{7}{5}\pi + o(\pi)$ colors, but till now no formal equivalence have been proven.

Corollary 11 *For binary trees $w_f(T, \Lambda) \leq \frac{7\pi}{5}$, and so the maximum independent set is at least $\frac{7}{5\pi} \cdot |\Lambda|$*

7 Conclusion

Our research in this paper was motivated by questions related to the design of wavelength division multiplexed optical networks.

We developed a new approximation tool for WDM networks by using the classical fractional coloring. We also proved the polynomiality of this method on bounded degree symmetric directed trees, then on bounded degree and treewidth symmetric digraphs. Furthermore, our polynomial algorithms could be used to improve some existing approximation results. Many applications of our techniques to WDM networks can be foreseen, as in branch and bound methods, randomized rounding, or even in the design of multifiber networks .

One intriguing remaining question is the size of the gap between integer and fractional coloring on such networks. We proved that this gap is bounded and we conjecture that it is *small.*

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Contents

A $\frac{7\pi}{5}$ coloring of reduced 3-star

A.1 The used independent sets

A fractional coloring is a collection of weighted independent sets. This collection has to cover every paths with a weight at least 1 which means :

$$\forall a, \ \sum_b X_{a,b} \ + \ X_{a,\emptyset} \geq 1$$

As far as we are exhibiting a balanced fractional coloring, we only have to be sure that we cover all *doublons* with weight $\frac{w}{\pi}$ and the pair $(a, \emptyset)$ with weight ϵ.

Definition 6 (Doublon)
On an edge, a doublon *is a couple of paths going through this edge in opposite directions.*

In order to cover the doublons, we will use the following independent set :

- The cycles A and B. We say that a doublon covered by those sets is of type A or B.

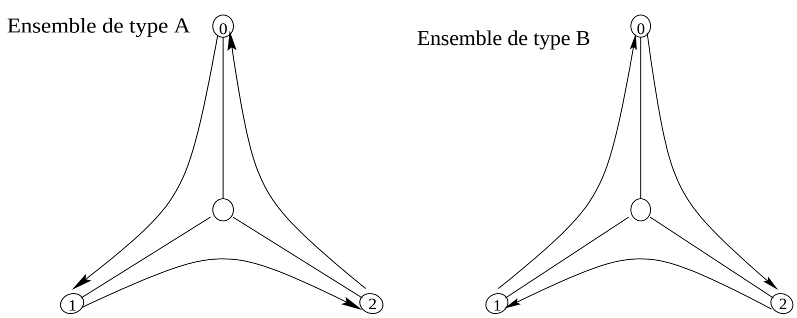

Figure 5: A and B sets

- The *back and forth* D_i, $i = 0, 1, 2$.

  A set D_i is a couple of parallel paths which don't touch node i.

- The C sets

  A set C is a set D_0 plus two paths of length 1.

- The sets E_i, $i = 1..4$.

  The E_i sets cover a doublon of type A on node 1 (E_1) and 2 (E_2) or type B (E_3 on node 1, E_4 on node 2).

  They also cover doublons composed of a length 1 path and a length 2 path, those doublon are said of type E_i.

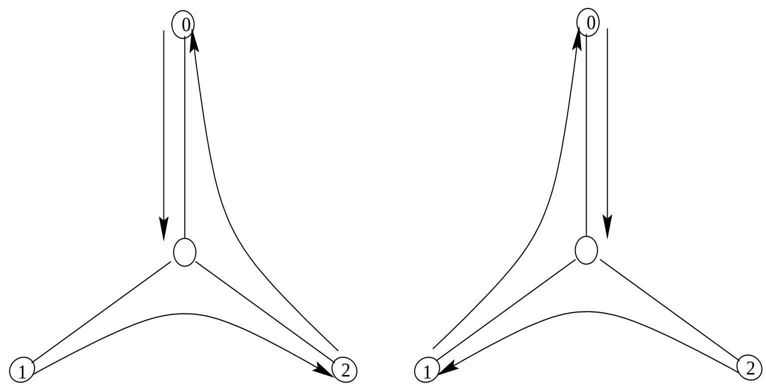

Figure 6: E_i sets (up to orientation change).

Those sets are enough to cover all the doublons. But they don't cover all the *half-color* (ie. the doublons $a, \emptyset$).

In order to cover them, we use the following sets :

- The F_i sets, $i = 1..4$.

  They are composed of only one path going to or from node 0.

- The C_i sets, $i = 0, 1$.

  They are subparts of the C set : there is only one length 1 path

A.2 The coloring

Number of each type of doublon :

Doublons of type A and B :

Type	Node 0	Nodes 1, 2
A	α^2	$\alpha(\alpha+\gamma)$
B	β^2	$\beta(\beta+\gamma)$

Other doublons :

Type	Number
D_0, D_2	$\alpha\beta$
D_1	$(\alpha+\gamma)(\beta+\gamma)$
E_1, E_2	$\alpha\gamma$
E_3, E_4	$\beta\gamma$
C	γ^2

Considering the covering constraints, we have that we have to use the followings number of sets :

e_1, e_2	$\alpha\gamma$
e_3, e_4	$\beta\gamma$
d_1, d_2	$\alpha\beta$
d_0	$\alpha\beta + \alpha\gamma + \beta\gamma$
a	α^2
b	β^2
c	γ^2

We then have to cover the half-colors. We have :

Type	Node 0	Node 1	Node 2
$A, \emptyset$	$\alpha\epsilon$	$\epsilon(\alpha+\gamma)$	$\epsilon\alpha$
$\emptyset, A$	$\alpha\epsilon$	$\epsilon\alpha$	$\epsilon(\alpha+\gamma)$
$B, \emptyset$	$\beta\epsilon$	$\epsilon(\beta+\gamma)$	$\epsilon\beta$
$\emptyset, B$	$\beta\epsilon$	$\epsilon\beta$	$\epsilon(\beta+\gamma)$
$C, \emptyset$	$\gamma\epsilon$		
$\emptyset, C$	$\gamma\epsilon$		

The E_i sets cover $\alpha\gamma w$ half-color of type $A, \emptyset$ on node 1, $\emptyset, A$ on node 2, and $\beta\gamma w$ of type $B, \emptyset$ on node1 and $\emptyset, B$ on node 2.

We need to use $\alpha\epsilon$ sets of type F_1 and F_2 and $\beta\epsilon$ of type F_3 and F_4 to cover all half-color on node 0. They cover the remaining half-color on node 0 and 1.

The balanced attribute of our coloring give us a constraint between w and ϵ :

$$\alpha\gamma w \leq (\alpha+\gamma)\epsilon \text{ and } \beta\gamma w \leq (\beta+\gamma)\epsilon$$

which leads to

$$\epsilon \geq \frac{\alpha\gamma}{\beta+\gamma}w \text{ and } \epsilon \geq \frac{\beta\gamma}{\beta+\gamma}w$$

Hence $\epsilon = \frac{w}{4} = \frac{1}{5}$ and $w = \frac{4}{5}$ is always a valid choice and is some times the best one (when $\alpha = \beta = \gamma$).

We still have to cover the half-color of type $C, \emptyset$ and $\emptyset, C$ on node 0. In order to do that, we will replace some sets of type D_0 by sets of type C_i and, if we need more half-color than the number of sets D_0, we will use sets composed of only one length 1 path.

We need $n_1 = 2\gamma\epsilon = \frac{\gamma}{2}$ half-color and we have $n_2 = w(\alpha\beta + \alpha\gamma + \beta\gamma)$ sets of type D_0 and $n_1 \geq n_2 \rightarrow \gamma \geq \frac{1}{2}$. Hence covering the D_0 doublons and the C half-color will need $w \max(\alpha\beta + \alpha\gamma + \beta\gamma, \frac{\gamma}{2})$ colors.

A.3 Cost of a balanced fractional coloring of the 3-star

We just have to add everything with weight $\frac{w}{\pi}$ or ϵ :

$$\Gamma = \frac{(\alpha+\beta+\gamma)^2 w}{\pi} + \frac{w \max(\alpha\beta + \alpha\gamma + \beta\gamma, \frac{\gamma}{2})}{\pi} + \epsilon(\alpha+\beta)$$

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