Derivation of Viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation

Jean-Frédéric Gerbeau, Benoît Perthame

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 4084

Décembre 2000

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4084--FR+ENG

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Jean-Frédéric Gerbeau * , Benoît Perthame † *

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet M3N

Rapport de recherche n° 4084 — Décembre 2000 — 16 pages

Abstract: We derive the Saint-Venant system for the shallow waters including small friction, viscosity and Coriolis-Boussinesq factor departing from the Navier-Stokes system with a free moving boundary. This derivation relies on the hydrostatic approximation where we follow the role of viscosity and friction on the bottom. Numerical comparisons between the limiting Saint-Venant system and direct Navier-Stokes simulation allow to validate this derivation.

Key-words: Navier-Stokes equations, Saint-Venant equations, shallow water, free surface, asymptotic analysis, viscosity, friction

* INRIA Rocquencourt - Projet M3N
† Ecole Normale Supérieure - DMA

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Dérivation du système de Saint-Venant visqueux pour les écoulements laminaires en eaux peu profondes; validation numérique

Résumé : Nous établissons, à partir des équations de Navier-Stokes avec surface libre, le système de Saint-Venant pour les écoulements en eaux peu profondes en y incluant frottement, viscosité et coefficient de Coriolis-Boussinesq. Cette dérivation est fondée sur l'approximation hydrostatique. Le système obtenu est validé numériquement par comparaison avec le résultat de simulations directes des équations de Navier-Stokes.

Mots-clés : équations de Navier-Stokes, équations de Saint-Venant, surface libre, analyse asymptotique, viscosité, frottements

1 Introduction

In these notes, we recall how to derive the Saint-Venant system, intoduced in [7], from the Navier-Stokes system for incompressible flows with a free moving boundary. This derivation is classical when the viscosity is neglected (see Whitham [19], Johnson [12], Stoker [17]) but this is unsatisfactory (see Bernardi and Pironneau [4]). Indeed, for dam breaks or hydraulic jumps it does not allow to justify mathematically that the right jumps are those obtained using the momentum (and not the velocity) as the conservative variable which leads to different Rankyne-Hugoniot curves (see Dafermos [6], Serre [16]). Also, for numerical purpose it can be useful to understand precisely the derivation including viscosity in order to realize the coupling between the two models (see Formaggia, Gerbeau, Nobile and Quarteroni [8] for instance).
Here, we study the full derivation in two steps (i) we recover the viscous Saint-Venant system including a small friction term on the bottom (denoted by κ below), (ii) we derive a correction to the "motion by slices" (the horizontal velocity does not depend upon the vertical coordinate). This rises a "viscous Saint-Venant" system for shallow waters under the form

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial h\,u}{\partial x} = 0, \\ \dfrac{\partial h\,u}{\partial t} + \dfrac{\partial h\,u^2}{\partial x} + \dfrac{g}{2}\dfrac{\partial h^2}{\partial x} = -\dfrac{\kappa\,u}{1+\dfrac{\kappa\,h}{3\mu}} + 4\mu\dfrac{\partial}{\partial x}\left(h\,\dfrac{\partial u}{\partial x}\right). \end{cases} \tag{1}$$

This paper is organized as follows. In section 2, we recall the Navier-Stokes system and the closure problem and in section 3 the hydrostatic approximation. The Saint-Venant systems with friction and viscosity are derived in section 4 and numerical results are shown in section 5 to put in evidence the effect of the correction to the motion by slices.

2 The Navier-Stokes system and its closure

We depart from the Navier-Stokes system with a gravity (see Lions [13]) in which the z axis represents the vertical direction, and for the sake of simplicity, we restrict to two dimensions i.e. $x \in I\!R$ and the topography is not studied *i.e.* $Z(x) \equiv 0$. However, for specific applications, for instance flows in rivers (see *e.g.* Graf and Altinakar [9]), the viscosity tensor can take various forms in order to describe for instance turbulent viscosity. For simplicity, the density will be kept constant and equal to one throughout the paper. In a first step we therefore consider the following general case:

$$\begin{gathered} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial w\,u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z}, \\ \frac{\partial w}{\partial t} + \frac{\partial u\,w}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}, \end{gathered} \tag{2}$$

and we consider this system for

$$t > 0, \qquad x \in I\!R, \qquad 0 \le z \le h(t,x),$$

where $h(t,x)$ represents the height of the water. The explicit form of the viscosity tensor σ is not fundamental in the first steps of the derivation below, but one can keep in mind a formula like

$$\begin{aligned} \sigma_{xx} &= 2\,\mu\tfrac{\partial u}{\partial x}, \qquad & \sigma_{xz} &= \mu(\tfrac{\partial u}{\partial z} + \tfrac{\partial w}{\partial x}), \\ \sigma_{zz} &= 2\,\mu\tfrac{\partial w}{\partial z}, \qquad & \sigma_{zx} &= \mu(\tfrac{\partial u}{\partial z} + \tfrac{\partial w}{\partial x}). \end{aligned} \tag{3}$$

We define the total stress tensor

$$\sigma_T = -p\,Id + \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix}. \tag{4}$$

We complete this system with boundary conditions. On the free surface, we assume the two no-stress conditions; let n denotes the outward normal to the free surface

$$\sigma_T \cdot n = 0, \qquad \text{on} \quad z = h(t,x). \tag{5}$$

On the bottom, we consider a Navier condition with a friction coefficient κ and a no-penetration condition

$$(\kappa\ u - \sigma_{xz})|_{z=0} \ = 0, \qquad w|_{z=0} = 0. \tag{6}$$

For a viscous fluid, the no-slip condition, $u = 0$, is usually considered but it does fit in the analysis below. However, a wall-law like Navier's is necessary for practical computations and even more sophisticated models are used (see Mohammadi and Pironneau [15], Achdou, Pironneau and Valentin [1]).
As for the free surface, it is defined thanks to the indicator function for the fluid region

$$\varphi(t,x,z) = \begin{cases} 1 & \text{for } 0 \le z \le h(t,x), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

This fluid region is advected by the flow, which can be expressed, thanks to the incompressibility condition, by the relation

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi u}{\partial x} + \frac{\partial \varphi w}{\partial z} = 0. \tag{8}$$

The solution φ to this equation takes the values 0 and 1 only but it needs not be of the form (7) for all times. The analysis below is limited to the conditions where this form is preserved.
We finally notice that the normal to the free surface, directed toward the increasing z, is given in the (x,z) axis by

$$n(x, z = h(t,x)) = \frac{1}{\sqrt{1 + \frac{\partial h}{\partial x}^2}} \begin{pmatrix} -\frac{\partial h}{\partial x} \\ 1 \end{pmatrix}.$$

A remarkable property of the equation on the horizontal velocity is that it satisfies the following classical identities which express two basic conservation laws for the height of water and the momentum.

LEMMA 2.1 *As long as the solution φ to the equation (8) for the fluid domain admits the form (7)* i.e. *there is no folding, then we have*

$$\frac{\partial}{\partial t}\int_0^h u(t,x,\eta)\,d\eta + \frac{\partial}{\partial x}\int_0^h u^2(t,x,\eta)\,d\eta + \frac{\partial}{\partial x}\int_0^h p(t,x,\eta)\,d\eta$$

$$= -\kappa\ u(t,x,0) + \frac{\partial}{\partial x}\left(\int_0^h \sigma_{xx}(t,x,\eta)\,d\eta\right), \tag{9}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u(t,x,\eta)d\eta = 0, \tag{10}$$

$$\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} - w\right)|_{z=h} = 0. \tag{11}$$

Proof of Lemma 2.1. We integrate the equation for the horizontal velocity in z for $0 \leq z < h(t,x)$. It gives

$$\frac{\partial}{\partial t}\int_0^h u(t,x,\eta)\,d\eta + \frac{\partial}{\partial x}\int_0^h u^2(t,x,\eta)\,d\eta + \frac{\partial}{\partial x}\int_0^h p(t,x,\eta)\,d\eta$$

$$-u\Big(\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} - w\Big)|_{z=h} - u\,w|_{z=0}$$

$$= \frac{\partial}{\partial x}\left(\int_0^h \sigma_{xx}(t,x,\eta)\,d\eta\right) + \frac{\partial h}{\partial x}(x)\;(p-\sigma_{xx})|_{z=h} +$$

$$[\sigma_{xz}|_{z=h} - \sigma_{xz}|_{z=0}]$$

$$= \kappa\, u(t,x,z=0) + \frac{\partial}{\partial x}\left(\int_0^h \sigma_{xx}(t,x,\eta)\,d\eta\right).$$

Next, we use (i) the boundary conditions at $z=0$ and specifically that $\sigma_{xz}(z=0)=0$, (ii) the boundary condition (12) at $z=h$ and we obtain

$$\frac{\partial}{\partial t}\int_0^h u(t,x,\eta)\,d\eta + \frac{\partial}{\partial x}\int_0^h u^2(t,x,\eta)\,d\eta + \frac{\partial}{\partial x}\int_0^h p(t,x,\eta)\,d\eta$$

$$-u\Big(\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} - w\Big)|_{z=h}$$

$$= \frac{\partial}{\partial x}\left(\int_0^h \sigma_{xx}(t,x,\eta)\,d\eta\right).$$

And it remains to prove the identity (10) to conclude the derivation of (9). We consider the equation for the free surface (8). Integrating it in $z \geq 0$ gives the equation (10). On the other hand, integrating it in z for $0 \leq z < h(t,x)$ gives, with the same calculation as for the horizontal velocity,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u(t,x,\eta)d\eta - \Big(\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} - w\Big)(z=h) = 0.$$

From this, we deduce the last equation (11) for the free surface, and we complete the proof (the first term vanishes) of (9).$\diamondsuit$

Henceforth, in order to reduce the moving boundary Navier-Stokes system to the Saint-Venant form, it remains to justify a closure of the form

$$u(t,x,z) := u(t,x).$$

This can be achieved by the shallow water assumption.

3 Rescaled system and hydrostatic approximation

We introduce a "small parameter" $\varepsilon = H/L$, where H and L are two characteristic dimensions along the axis Oz and Ox respectively. We then introduce some characteristic dimensions: U for the horizontal velocity, $W = \varepsilon U$ for the vertical velocity, $T = L/U$ for the time, $P = U^2$ for the pressure (we recall that we suppose the density $\rho = 1$). We denote temporarily the dimensionless quantities by $\tilde{u} = u/U$, $\tilde{w} = w/W$, $\tilde{x} = x/L$, $\tilde{z} = z/H$, $\tilde{t} = t/T$, $\tilde{p} = p/P$. We denote the inverse of the Reynolds number by $\nu = \mu/(UL)$, the inverse of the Froude number by $G = gH/U^2$ and we set $\alpha = \kappa/U$. Then, using now the specific form of the stress tensor (3) and dropping the ˜ for the sake of clarity, the dimensionless Navier-Stokes system reads:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial w\, u}{\partial z} + \dfrac{\partial p}{\partial x} = \dfrac{\partial}{\partial x}\left(2\nu \dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\nu}{\varepsilon^2}\dfrac{\partial u}{\partial z} + \nu \dfrac{\partial w}{\partial x}\right) \\ \varepsilon^2\left(\dfrac{\partial w}{\partial t} + \dfrac{\partial u\, w}{\partial x} + \dfrac{\partial w^2}{\partial z}\right) + \dfrac{\partial p}{\partial z} = -G + \dfrac{\partial}{\partial x}\left(\nu \dfrac{\partial u}{\partial z} + \varepsilon^2 \nu \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial z}\left(2\nu \dfrac{\partial w}{\partial z}\right). \end{cases}$$

The definition of φ is unchanged and the equation (8) describing the free surface is not modified in its dimensionless form.
As for the boundary conditions they are modified since the normal to the free boundary is now

$$n(x, z = h(t,x)) = \frac{1}{\sqrt{1 + \varepsilon^2 \frac{\partial h}{\partial x}^2}} \begin{pmatrix} -\varepsilon \frac{\partial h}{\partial x} \\ 1 \end{pmatrix}.$$

They are rescaled as

$$\begin{cases} \dfrac{\partial h}{\partial x} p - 2\nu \dfrac{\partial h}{\partial x}\dfrac{\partial u}{\partial x} + \dfrac{\nu}{\varepsilon^2}\dfrac{\partial u}{\partial z} + \nu \dfrac{\partial w}{\partial x} = 0, \qquad \text{on} \quad z = h(t,x), \\ p + \nu \dfrac{\partial h}{\partial x}\dfrac{\partial u}{\partial z} + \nu \varepsilon^2 \dfrac{\partial w}{\partial x}\dfrac{\partial h}{\partial x} - 2\nu \dfrac{\partial w}{\partial z} = 0, \qquad \text{on} \quad z = h(t,x), \\ \alpha\, u - \dfrac{\nu}{\varepsilon}\dfrac{\partial u}{\partial z} = 0, \quad \text{and} \quad w = 0, \qquad \text{on} \quad z = 0. \end{cases} \tag{12}$$

One possible route to go further in the derivation consists in the *hydrostatic approximation.* We take the formal limit as ε vanishes in the equation for the vertical velocity (see [5], [10], [13]) and we obtain the hydrostatic system:

$$\frac{\partial u_\varepsilon}{\partial x} + \frac{\partial w_\varepsilon}{\partial z} = 0, \tag{13}$$

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon^2}{\partial x} + \frac{\partial u_\varepsilon\, w_\varepsilon}{\partial z} + \frac{\partial p_\varepsilon}{\partial x} = \frac{\partial}{\partial x}\left(2\nu \frac{\partial u_\varepsilon}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\nu}{\varepsilon^2}\frac{\partial u_\varepsilon}{\partial z} + \nu \frac{\partial w_\varepsilon}{\partial x}\right) \tag{14}$$

$$\frac{\partial p_\varepsilon}{\partial z} = -G + \frac{\partial}{\partial x}\left(\nu \frac{\partial u_\varepsilon}{\partial z}\right) + \frac{\partial}{\partial z}\left(2\nu \frac{\partial w_\varepsilon}{\partial z}\right). \tag{15}$$

Although this system results from a limit as ε vanishes in the vertical velocity equation, we choose to denote its solutions by $(u_\varepsilon, w_\varepsilon, p_\varepsilon)$ to outline the fact that they actually depend on ε because of the terms ν/ε^2 in the horizontal velocity equation.
As for the boundary condition on $z = h_\varepsilon(t,x)$, relations (12) give firstly

$$\nu \frac{\partial u_\varepsilon}{\partial z}|_{z=h_\varepsilon} = -\varepsilon^2 \left(\frac{\partial h_\varepsilon}{\partial x}\, p_\varepsilon - 2\nu \frac{\partial h_\varepsilon}{\partial x}\frac{\partial u_\varepsilon}{\partial x} + \nu \frac{\partial w_\varepsilon}{\partial x}\right). \tag{16}$$

Notice that we cannot neglect the terms in ε^2 in this free surface condition because we also keep the terms in $1/\varepsilon^2$ in the horizontal velocity equation (14) and we are interested in computing a result at zeroth order. For the vertical velocity we can keep the mere zero order terms. From the second relation in (12), which reads

$$p|_{z=h} - 2\nu \frac{\partial w}{\partial z}|_{z=h} = -\nu \varepsilon^2 \frac{\partial w}{\partial x}\frac{\partial h}{\partial x} - \nu \frac{\partial u}{\partial z}\frac{\partial h}{\partial x} = \varepsilon^2 \left(\frac{\partial h}{\partial x}\right)^2 \left(p - 2\nu \frac{\partial u}{\partial x}\right),$$

we can therefore take the formal limit as ε vanishes and we obtain

$$p_\varepsilon|_{z=h_\varepsilon} \; - 2\nu \frac{\partial w_\varepsilon}{\partial z}|_{z=h_\varepsilon} = 0. \tag{17}$$

And it is also completed on the flat bottom $z = 0$ with the friction

$$\varepsilon \; \alpha \; u_\varepsilon|_{z=0} \; - \nu \frac{\partial u_\varepsilon}{\partial z}|_{z=0} \; = \; 0 \tag{18}$$

and non-penetration condition

$$w_\varepsilon|_{z=0} \; = \; 0. \tag{19}$$

This system still holds in the fluid domain defined by $0 \leq z \leq h_\varepsilon(t,x)$ and the equation on φ_ε is

$$\frac{\partial \varphi_\varepsilon}{\partial t} + \frac{\partial \varphi_\varepsilon u_\varepsilon}{\partial x} + \frac{\partial \varphi_\varepsilon w_\varepsilon}{\partial z} = 0. \tag{20}$$

This system (13)–(19) is considered in the following. But before all, we would like to insist on the choice of neglecting the right terms at order ε^2, it makes that the Lemma 2.1 still holds true for this hydrostatic system (see Lemma 3.1 below).
System (13)–(19) can be somewhat simplified. Especially, we can integrate the "hydrostatic equation" (15) for the pressure and use the condition (17) at $z = h_\varepsilon$. Then, for ν constant, we get

$$\begin{aligned} p_\varepsilon(t,x,z) \;&=\; G\left(h_\varepsilon(t,x) - z\right) + \int_{h_\varepsilon}^{z} \frac{\partial}{\partial x}\left(\nu \frac{\partial u_\varepsilon}{\partial z}(t,x,\eta)\right) d\eta + 2\nu \frac{\partial w_\varepsilon}{\partial z}(t,x,z) \\ &=\; G\left(h_\varepsilon(t,x) - z\right) + \nu \frac{\partial u_\varepsilon}{\partial x}(t,x,z) - \nu \frac{\partial u_\varepsilon}{\partial x}(t,x,h_\varepsilon) + 2\nu \frac{\partial w_\varepsilon}{\partial z}(t,x,z). \end{aligned}$$

And using the free divergence condition, we find

$$p_\varepsilon(t,x,z) = G\left(h_\varepsilon(t,x) - z\right) - \nu \frac{\partial u_\varepsilon}{\partial x}(t,x,z) - \nu \frac{\partial u_\varepsilon}{\partial x}(t,x,h_\varepsilon(t,x)). \tag{21}$$

We can also recover the vertical velocity equation, thanks to the integration of the divergence free condition and the condition $w_\varepsilon|_{z=0} = 0$,

$$w_\varepsilon(t,x,z) = -\int_0^z \frac{\partial u_\varepsilon}{\partial x}(t,x,\eta)\; d\eta. \tag{22}$$

Gathering these relations and proceeding as in the proof of Lemma 2.1, we obtain the following result.

LEMMA 3.1 *For a constant viscosity, the free surface "viscous hydrostatic system" (13)–(19) is equivalent to:*

$$(HS_\varepsilon) \left\{ \begin{aligned} \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon^2}{\partial x} + \frac{\partial u_\varepsilon w_\varepsilon}{\partial z} + \frac{\partial p_\varepsilon}{\partial x} \;&=\; \frac{\partial}{\partial x}\left(2\nu \frac{\partial u_\varepsilon}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\nu}{\varepsilon^2}\frac{\partial u_\varepsilon}{\partial z} + \nu \frac{\partial w_\varepsilon}{\partial x}\right), \\ w_\varepsilon(t,x,z) \;&=\; -\int_0^z \frac{\partial u_\varepsilon}{\partial x}(t,x,\eta)\, d\eta, \\ p_\varepsilon(t,x,z) \;&=\; G\left(h_\varepsilon(t,x) - z\right) - \nu \frac{\partial u_\varepsilon}{\partial x}(t,x,z) \\ &\quad - \nu \frac{\partial u_\varepsilon}{\partial x}(t,x,h_\varepsilon(t,x)), \end{aligned} \right.$$

completed with the equation (20) on $h_\varepsilon(t,x)$ and the boundary conditions (16), (18).
Moreover, a solution to this system satisfies

$$\frac{\partial}{\partial t}\int_0^{h_\varepsilon} u_\varepsilon(t,x,\eta)\, d\eta + \frac{\partial}{\partial x}\int_0^{h_\varepsilon} u_\varepsilon^2(t,x,\eta)\, d\eta + \frac{\partial}{\partial x}\int_0^{h_\varepsilon} p_\varepsilon(t,x,\eta)\, d\eta$$

$$= -\frac{\alpha}{\varepsilon}\, u_\varepsilon(t,x,0) + \frac{\partial}{\partial x}\int_0^{h_\varepsilon} 2\nu \frac{\partial u_\varepsilon}{\partial x}(t,x,\eta)\, d\eta, \tag{23}$$

and

$$\frac{\partial h_\varepsilon}{\partial t} + \frac{\partial}{\partial x}\int_0^{h_\varepsilon} u_\varepsilon(t,x,\eta) d\eta = 0. \tag{24}$$

REMARK 1 *When neglecting the viscosity in the viscous hydrostatic system (HS_ε), we obtain the so-called Benney equations [3]. They are much simpler since the boundary conditions disappear. They can be considered as an infinite hyperbolic system (see Teshukov [18] and the references therein, Brenier [5] for rigid roofs) which exhibits singularities in finite time especially related to nonconvex vertical velocity profiles ([5]). These are also related to instable boundary layers studied for instance in Grenier [10] at the viscous level. Keeping dominant viscous terms, as we do it in the next section, leads to a higher determination in the asymptotics and to constant vertical velocity profiles. One can conjecture that these are very stabilizing terms leading to well posedness of the Saint-Venant system (Lions, Perthame and Souganidis [14]). It also has the advantage of giving a clear conservation of momentum.*

4 The Saint-Venant system

We may now perform an asymptotic analysis of the hydrostatic system (HS_ε). We recall that we denote its solution by $(u_\varepsilon(t,x,z),\, w_\varepsilon(t,x,z),\, p_\varepsilon(t,x,z))$. In order to keep the exact averaged quantities, we denote

$$\bar{u}_\varepsilon(t,x) = \frac{1}{h_\varepsilon(t,x)}\int_0^{h_\varepsilon} u_\varepsilon(t,x,\eta)\; d\eta,$$

$$\overline{u_\varepsilon^2}(t,x) = \frac{1}{h_\varepsilon(t,x)}\int_0^{h_\varepsilon} (u_\varepsilon(t,x,\eta))^2\; d\eta.$$

These quantities are exactly those arising in the equation for the vertical velocity in the Lemma 3.1. For instance, the equation (24) on h_ε is rewritten as

$$\frac{\partial h_\varepsilon}{\partial t} + \frac{\partial\, h_\varepsilon \bar{u}_\varepsilon}{\partial x} = 0.$$

In the sequel, we shall suppose that we are in the following asymptotic regime:

$$\nu = \varepsilon\nu_0, \qquad \text{and} \qquad \alpha = \varepsilon\alpha_0. \tag{25}$$

Next, we perform the asymptotic analysis. Equations (14), (16) and (18) give

$$\frac{\partial^2 u_\varepsilon}{\partial z^2} = O(\varepsilon), \qquad \frac{\partial u_\varepsilon}{\partial z}|_{z=h_\varepsilon} = O(\varepsilon), \qquad \frac{\partial u_\varepsilon}{\partial z}|_{z=0} = O(\varepsilon). \tag{26}$$

This simply means that $u_\varepsilon(t,x,z) = u_\varepsilon(t,x,0) + O(\varepsilon)$, *i.e.* the so-called "motion by slices" of the usual Saint-Venant system. Thus, $u_\varepsilon(t,x,z) = \bar{u}_\varepsilon(t,x) + O(\varepsilon)$, and at first order, $\overline{u_\varepsilon^2}(t,x) = (\bar{u}_\varepsilon(t,x))^2$. Then, equation (22) on w_ε gives

$$w_\varepsilon(t,x,z) = -z\,\frac{\partial \bar{u}_\varepsilon}{\partial x}(t,x) + O(\varepsilon). \tag{27}$$

The pressure (21) reduces to

$$p_\varepsilon(t,x,z) = G\; (h_\varepsilon(t,x) - z)\; - 2\nu\frac{\partial \bar{u}_\varepsilon}{\partial x}(x,t) + O(\varepsilon),$$

and thus, recalling (25),

$$p_\varepsilon(t,x,z) = G\; (h_\varepsilon(t,x) - z) + O(\varepsilon). \tag{28}$$

Finally the equation (23) on u_ε becomes

$$\frac{\partial h_\varepsilon \bar{u}_\varepsilon}{\partial t} + \frac{\partial h_\varepsilon \bar{u}_\varepsilon^2}{\partial x} + \frac{G}{2}\frac{\partial h_\varepsilon^2}{\partial x} = -\alpha_0\, u_\varepsilon(t,x,0) + O(\varepsilon)$$

and therefore

$$\frac{\partial h_\varepsilon \bar{u}_\varepsilon}{\partial t} + \frac{\partial h_\varepsilon \bar{u}_\varepsilon^2}{\partial x} + \frac{G}{2}\frac{\partial h_\varepsilon^2}{\partial x} = -\alpha_0\, \bar{u}_\varepsilon + O(\varepsilon). \tag{29}$$

Dropping the $O(\varepsilon)$, and multiplying by HU^2/L in order to recover the variables with dimension, we have proved the following result.

PROPOSITION 4.1 *If κ denotes the coefficient in the wall law (6) for the Navier-Stokes equations, then the "Saint-Venant system with friction", defined by*

$$(SV)\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} &= 0, \\ \dfrac{\partial hu}{\partial t} + \dfrac{\partial hu^2}{\partial x} + \dfrac{g}{2}\dfrac{\partial h^2}{\partial x} &= -\kappa\, u, \end{cases}$$

results from an approximation in $O(\varepsilon)$ of the viscous hydrostatic system, and therefore of the Navier-Stokes equations.

But, as will be numerically illustrated in the next section, we can improve the precision of the results given by the Saint-Venant system (SV) if we determine the first correction depending on z in the expansion of $u_\varepsilon(t,x,z)$. To do so, we come back to equation (14) and, using relations (28) and (29), we get

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{\nu_0}{\varepsilon}\frac{\partial u_\varepsilon}{\partial z}\right) &= \frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon\frac{\partial u_\varepsilon}{\partial x} + w_\varepsilon\frac{\partial u_\varepsilon}{\partial z} + \frac{\partial p_\varepsilon}{\partial x} - \frac{\partial}{\partial x}\left(\varepsilon\nu_0\frac{\partial u_\varepsilon}{\partial x}\right) \\ &= \frac{\partial \bar{u}_\varepsilon}{\partial t} + \bar{u}_\varepsilon\frac{\partial \bar{u}_\varepsilon}{\partial x} + G\frac{\partial h_\varepsilon}{\partial x} + O(\varepsilon) \\ &= -\frac{\alpha_0}{h_\varepsilon}u_\varepsilon(t,x,0) + O(\varepsilon). \end{aligned}$$

Integrating from 0 to z and taking into account the boundary condition (18), we deduce

$$\frac{\nu_0}{\varepsilon}\frac{\partial u_\varepsilon}{\partial z} = \alpha_0\left(1 - \frac{z}{h_\varepsilon}\right)u_\varepsilon(t,x,0) + O(\varepsilon),$$

and we obtain the following formula which gives a more detailed view of the vertical velocity through a parabolic correction

$$u_\varepsilon(t,x,z) = \left(1 + \frac{\alpha_0\,\varepsilon\, z}{\nu_0}\left(1 - \frac{z}{2h_\varepsilon}\right)\right)u_\varepsilon(t,x,0) + O(\varepsilon^2). \tag{30}$$

Then, integrating from 0 to h_ε and dividing by h_ε, we obtain

$$\bar{u}_\varepsilon = \left(1 + \frac{\alpha_0\varepsilon h_\varepsilon}{3\nu_0}\right)u_\varepsilon(t,x,0) + O(\varepsilon^2).$$

Moreover,

$$u_\varepsilon^2(t,x,z) = u_\varepsilon^2(t,x,0)\left(1 + \frac{2\alpha_0\varepsilon z}{\nu_0}\left(1 - \frac{z}{2h_\varepsilon}\right)\right) + O(\varepsilon^2),$$

which yields

$$\begin{aligned} \frac{1}{h_\varepsilon}\int_0^{h_\varepsilon} u_\varepsilon^2(t,x,\eta)\,d\eta &= u_\varepsilon^2(t,x,0)\left(1 + \frac{2\alpha_0\varepsilon h_\varepsilon}{3\nu_0}\right) + O(\varepsilon^2) \\ &= \bar{u}_\varepsilon^2\left(1 - \frac{2\alpha_0\varepsilon h_\varepsilon}{3\nu_0}\right)\left(1 + \frac{2\alpha_0\varepsilon h_\varepsilon}{3\nu_0}\right) + O(\varepsilon^2) \\ &= \bar{u}_\varepsilon^2 + O(\varepsilon^2). \end{aligned}$$

Finally, equation (21) gives

$$p_\varepsilon(t,x,z) = G\ (h_\varepsilon(t,x) - z)\ - 2\nu_0\varepsilon\frac{\partial \bar{u}_\varepsilon}{\partial x}(x,t) + O(\varepsilon^2).$$

Therefore, (23) reads

$$\frac{\partial h_\varepsilon \bar{u}_\varepsilon}{\partial t} + \frac{\partial h_\varepsilon \bar{u}_\varepsilon^2}{\partial x} + \frac{G}{2}\frac{\partial h_\varepsilon^2}{\partial x} = -\frac{\alpha_0}{1+\dfrac{\alpha_0\varepsilon h_\varepsilon}{3\nu_0}}\bar{u}_\varepsilon + \frac{\partial}{\partial x}\left(4\nu_0\varepsilon h_\varepsilon\frac{\partial \bar{u}_\varepsilon}{\partial x}\right) + O(\varepsilon^2).$$

Dropping the $O(\varepsilon^2)$ and switching to the variables with dimension, we have therefore established the following result.

PROPOSITION 4.2 *If κ denotes the coefficient in the wall law (6) for the Navier-Stokes equations, we define a modified friction coefficient for the Saint-Venant system by*

$$\kappa_{vsv}(h) = \frac{\kappa}{1+\dfrac{\kappa\, h}{3\mu}}.$$

Then, the "viscous Saint-Venant" system, defined by

$$(VSV)\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} = 0, \\ \dfrac{\partial hu}{\partial t} + \dfrac{\partial hu^2}{\partial x} + \dfrac{g}{2}\dfrac{\partial h^2}{\partial x} = -\kappa_{vsv}(h)\, u + 4\mu\dfrac{\partial}{\partial x}\left(h\dfrac{\partial u}{\partial x}\right), \end{cases}$$

results from an approximation in $O(\varepsilon^2)$ of the viscous hydrostatic system, and therefore of the Navier-Stokes equations. We emphasize that, at this order, the Coriolis-Boussinesq coefficient $\beta = \overline{u_\varepsilon^2}/\bar{u}_\varepsilon^2$ is equal to 1.

REMARK 2 *Note that an analogous derivation could be performed for narrow cylindrical mobile domains where the radius is related to the pressure (see* e.g. *[8] for the deformation of arteries by blood pressure).*

5 Numerical assessments

In order to assess the previous relations, we consider the standard dam-break problem and we aim at comparing the solutions given by the Saint-Venant systems (SV) and (VSV) and those obtained by direct simulations of the Navier-Stokes equations with free surface. Let us mention that direct simulations of the dam-break problem was also presented by Huerta and Liu [11].
Without entering into the details, we just mention that our Navier-Stokes solver is based on an arbitrary Euler-Lagrange formulation and it uses stabilized Q1/Q1 finite elements. The algorithm for the (SV) and (VSV) systems is a finite volume kinetic scheme of first order (see Audusse, Bristeau and Perthame [2]).
In the sequel, we consider four dam-break configurations with the following values for the water depth before the dam $h_l^{(1)} = 4$, $h_l^{(2)} = 2$, $h_l^{(3)} = 1$, $h_l^{(4)} = 0.5$ and $h_r^{(1)} = 2$, $h_r^{(2)} = 1$, $h_r^{(3)} = 0.5$, $h_r^{(4)} = 0.25$ after the dam. In all the cases we keep constant the gravity $g = 2$, the density $\rho = 1$ and the length of the computational domain $L = 100$. We set $\varepsilon_i = \frac{h_l^{(i)} - h_r^{(i)}}{L}$, thus $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.01$, $\varepsilon_3 = 0.005$, $\varepsilon_4 = 0.0025$. In order to consider the same asymptotic regime as in the previous section, the viscosity is $\mu_i = \varepsilon_i\mu_0$, and the friction coefficient is $\kappa = \varepsilon_i\kappa_0$. In the sequel $\mu_0 = 1$ and κ_0 takes the values 0, 0.1, 1 or 10.
First, we assume pure slip boundary conditions on the bed, *i.e.* $\kappa = 0$ in (6), thus the system (SV) coincides with the usual Saint-Venant system. Figure 1 shows the finite element mesh at

time $t = 0$ and $t = 14$ in the case $\varepsilon = \varepsilon_2$ (the scales on the vertical and the horizontal directions are different for the sake of clarity). Figure 2 shows, for $\varepsilon = \varepsilon_2$, the free surface elevations given by the usual Saint-Venant system (whose solution is analytically known) and by the Navier-Stokes simulations for various times. It is worth noticing that the shock speed and the intermediate water depth are quite similar in both cases.

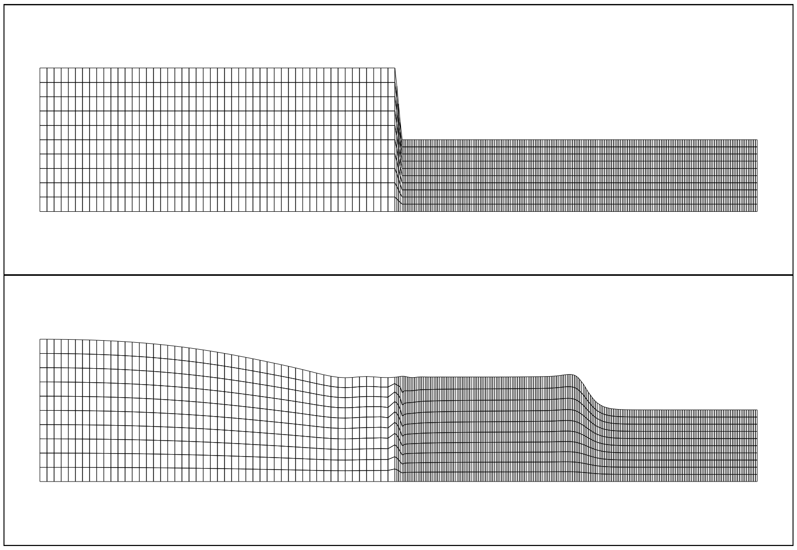

Figure 1: The mesh at time $t = 0$ and $t = 14$

Next, to illustrate the influence of the friction coefficient κ on the velocity profile, we show on Figure 3 the velocity $u(z)$ obtained with the Navier-Stokes solver for $\kappa_0 = 0, 0.1, 1., 10$ at time $t_0 = 14$ and $x = 9.3$ in the case $\varepsilon = \varepsilon_2$ (upper graph) and $x = 6.6$ in the case $\varepsilon = \varepsilon_4$ (lower graph). In both cases, the straight lines correspond to the exact values given by the Saint-Venant system without friction, and the lowest curves correspond to the Navier-Stokes solution with no-slip boundary conditions.

Finally, we investigate the dependence on ε. We denote by $\bar{u}_{ns}(t,x)$ the quantity $1/h(t,x)\int_0^{h(t,x)} u_{ns}(t,x,\eta)\,d\eta$, where u_{ns} is the result of the Navier-Stokes simulations with friction on the bed, by $u_{vsv}(t,x)$ the solution to the system (VSV) of Proposition 4.2. To emphasize the importance of the second order correction, we also compute the solution, denoted by $u_{sv}(t,x)$, given by the system (SV) of Proposition 4.1.

We report on Table 1 the values measured at $t = 14$ in the middle of the zone located between the initial position of the dam, $x = 0$, and the current position of the shock (this roughly corresponds to $x = 18.7$ for $\varepsilon = \varepsilon_1$, $x = 13.2$ for $\varepsilon = \varepsilon_2$, $x = 9.3$ for $\varepsilon = \varepsilon_3$ and $x = 6.6$ for $\varepsilon = \varepsilon_4$). The results of Table 1 deserve some comments. First of all, the discretisation error in our simulations is about 10^{-3}. Thus, the numbers are presented with only three significative digits and a difference of the order of 10^{-3} is considered as not meaningful. According to this remark, we see that the quantities $|\bar{u}_{ns} - u_{vsv}|$ and $|\bar{u}_{ns} - u_{sv}|$ are dominated by the discretisation error for all the values of ε when the friction is $\kappa_0 = 0$ and 0.1. On the one hand, these results confirm the validity of the Saint-Venant model in that case, but on the other hand, they are not very useful to assess our corrections. Conversely, for $\kappa_0 = 1$ and 10, the influence of ε is more evident, and in that case, the (VSV) solutions are outstandingly closer to the Navier-Stokes solutions than the (SV) ones. To illustrate this point, we have represented on Figure 4 the quantities $|\bar{u}_{ns} - u_{vsv}|$ and $|\bar{u}_{ns} - u_{sv}|$ versus ε in a log-log scale when $\kappa_0 = 10$.

Remark 3 *In the results presented above, the meaningful differences between u_{sv} and u_{vsv} essentially come from the correction to the friction coefficient. Indeed, the term $4\mu\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right)$ in (VSV) have almost no influence on the results, at least in that case.*

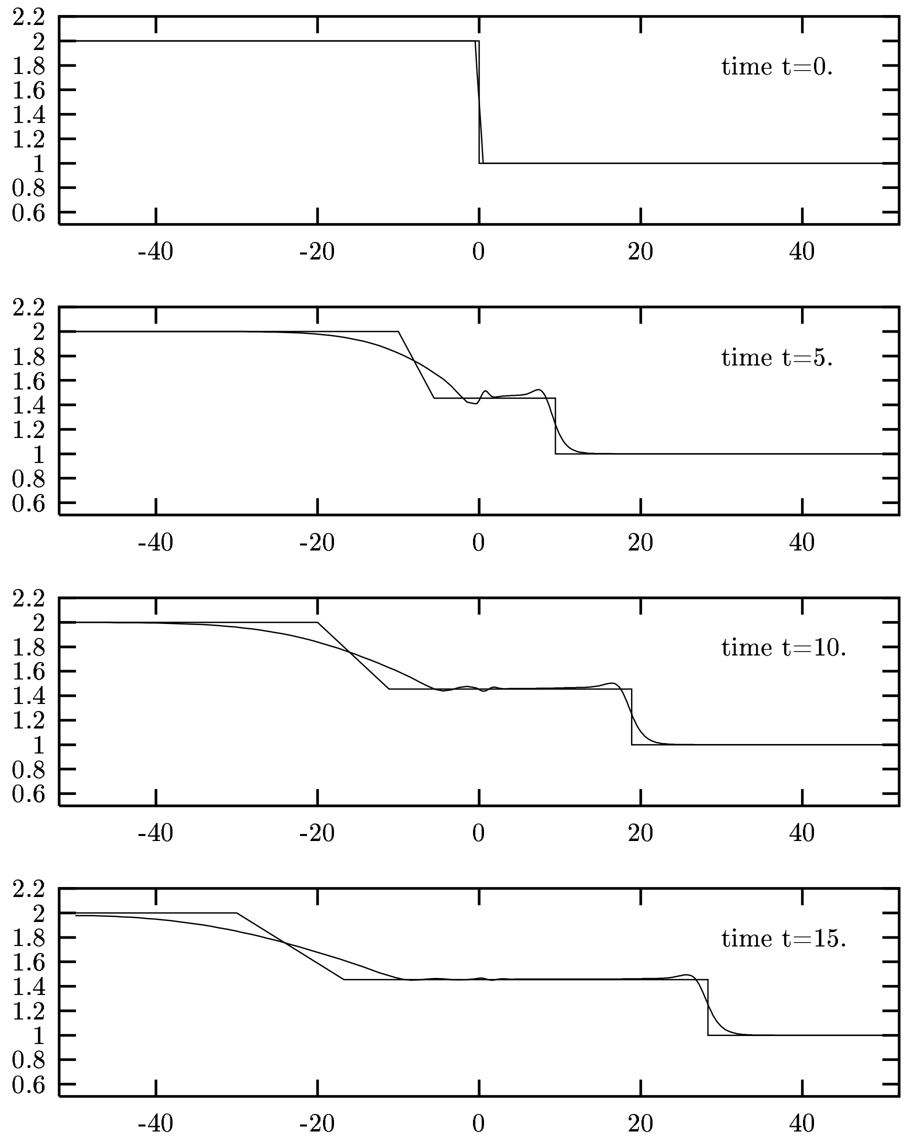

Figure 2: Free surface in the dam-break problem: theoretical solution to the Saint-Venant system and numerical solution to the Navier-Stokes equations. The similarity of the shock speeds and the intermediate water depths is striking

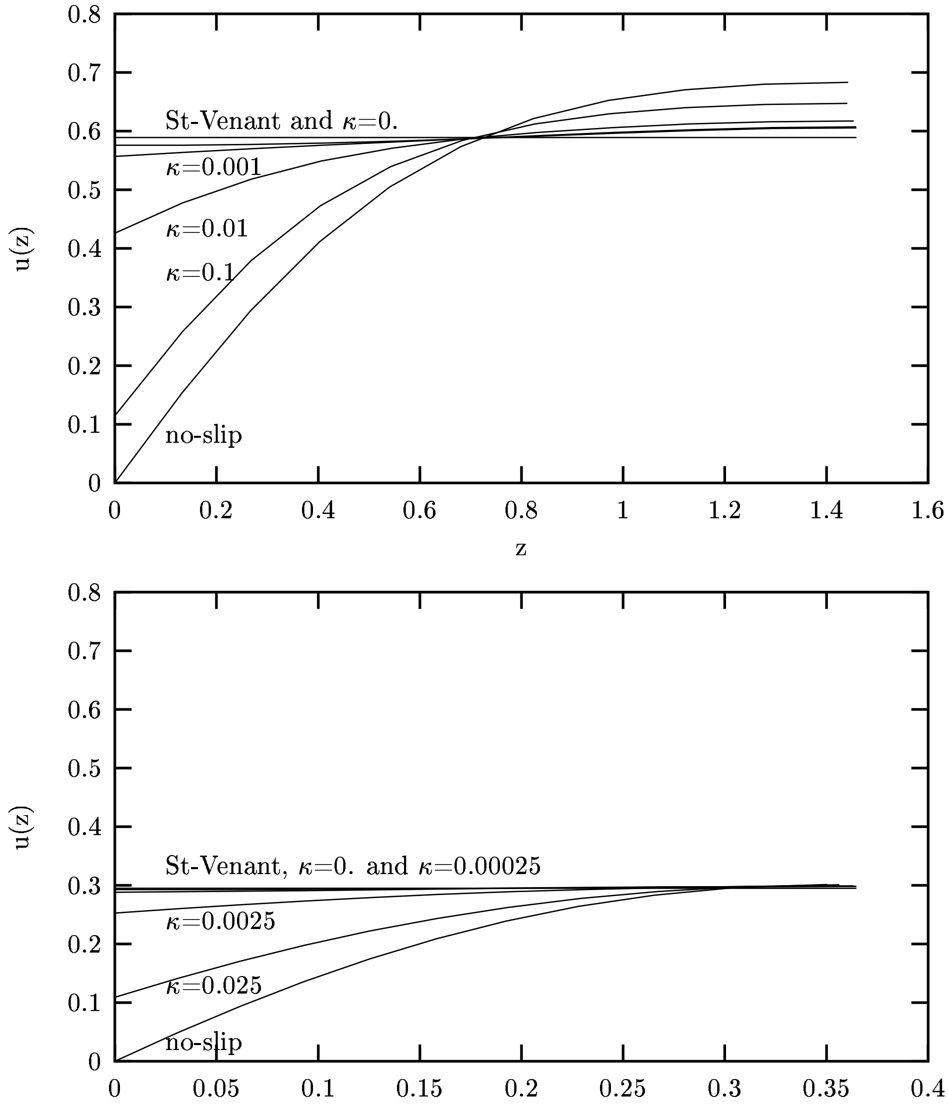

Figure 3: Horizontal velocity versus the depth (upper: $\varepsilon = \varepsilon_2 = 0.01$, lower: $\varepsilon = \varepsilon_4 = 0.0025$)

ε	κ_0	$\bar{u}_{ns}$	u_{vsv}	u_{sv}	$\lvert\bar{u}_{ns} - u_{vsv}\rvert$	$\lvert\bar{u}_{ns} - u_{sv}\rvert$
0.02	0.	0.833	0.833	0.833	0.000	0.000
	0.1	0.829	0.829	0.829	0.000	0.000
	1.	0.806	0.814	0.796	0.008	0.010
	10.	0.761	0.798	0.544	0.037	0.217
0.01	0.	0.589	0.589	0.589	0.000	0.000
	0.1	0.587	0.586	0.586	0.001	0.001
	1.	0.569	0.571	0.562	0.002	0.007
	10.	0.521	0.544	0.385	0.023	0.136
0.005	0.	0.418	0.417	0.417	0.001	0.001
	0.1	0.416	0.415	0.415	0.001	0.002
	1.	0.402	0.401	0.398	0.001	0.004
	10.	0.354	0.364	0.272	0.010	0.082
0.0025	0.	0.295	0.294	0.294	0.001	0.001
	0.1	0.294	0.293	0.293	0.001	0.001
	1.	0.283	0.282	0.281	0.001	0.002
	10.	0.238	0.240	0.192	0.002	0.046

Table 1: Navier-Stokes solution $\bar{u}_{ns}$ compared to Saint-Venant solutions u_{sv} and u_{vsv}

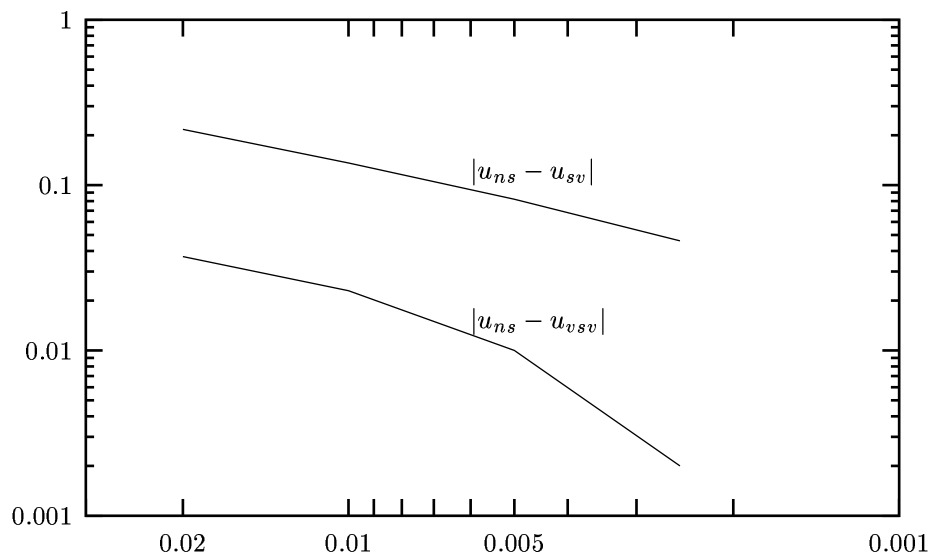

Figure 4: Difference between Saint-Venant and Navier-Stokes solutions versus ε in a log-log scale in the case $\kappa_0 = 10$

6 Conclusion

We have performed an asymptotic analysis of the Navier-Stokes system with a free moving boundary in the shallow water limit.
We have first considered the only hydrostatic approximation with friction boundary conditions, this allows to obtain the right conservative form with viscosity. Next, a correction to the motion by slices gives the "viscous Saint-Venant system" (VSV) with a modified friction coefficient and a

viscosity four times as large as the Navier-Stokes viscosity. Numerical comparisons with a direct Navier-Stokes solution have put in evidence the better accuracy of this system compared to the classical Saint-Venant equations.

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Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
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ISSN 0249-6399