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The Output Least Squares Identifiability of the Diffusion Coefficient from an H^1–Observation in a 2–D Elliptic Equation

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Abstract: Output least squares identifiability for the diffusion coefficient in an elliptic equation in dimension two is analyzed. This guarantees Lipschitz stability of the solution of the least squares formulation with respect to perturbations in the data independently of their attainability. The analysis takes into consideration the direction of the flow, and shows its influence on the parameter to be estimated. Identifiability is obtained at each scale of a multi-scale resolution of the unknown parameter.

Key-words: parameter estimation, diffusion coefficient, inverse problem, identifiability, least squares

(Résumé : tsvp)

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Identifiabilité au sens des Moindres Carrés du Coefficient de Diffusion dans une équation elliptique bidimensionnelle au vu d'une observation H^1

Résumé : Nous étudions l'identifiabilité au sens des moindres carrés du coefficient de diffusion dans une équation elliptique en dimension deux. Cette propriété garantit la stabilité Lipschitz de la solution du problème des moindres carrés par rapport aux perturbations sur les données, que ces dernières soient atteignables ou non. L'analyse montre l'influence de la direction de l'écoulement sur le paramètre à estimer. L'identifiabilité est obtenue pour chaque échelle d'une représentation multi échelle du paramètre inconnu.

Mots-clé : estimation de paramètre, coefficient de diffusion, problème inverse, identifiabilité, moindres carrés

1 Introduction

This paper is devoted to the analysis of output least squares identifiability of the diffusion coefficient in

$$- \text{div}\ (a \text{ grad } u) = f \text{ in } \Omega, \tag{1.1}$$

where Ω is a bounded two-dimensional domain and appropriate boundary conditions are added to (1.1). To identify a from data z a least squares formulation

$$\text{minimize } \frac{1}{2}\,|\mathcal{O}u_a - z|^2 \text{ over } C, \tag{1.2}$$

is used, where $\mathcal{O}$ denotes the observation operator, C stands for the set of admissible parameters and u_a denotes the solution to (1.1) as a function of a. Corresponding to C we define the set of attainable outputs by $D = \{\mathcal{O}u_a : a \in C\}$. The parameter $a \in C$ is called output least squares identifiable if there exists a neighborhood $\mathcal{V}$ of D such that for every $z \in \mathcal{V}$ the least squares problem (1.2) has no local minima and a unique solution in C depending Lipschitz-continuously on $z \in \mathcal{V}$. The precise definition of this concept is contained in Section 7. Note that output leasts squares identifiability is a stronger property than parameter identifiability which is the injectivity of the mapping $a \to \mathcal{O}u_a$, for $a \in C$, and also than parameter stability, which refers to the continuous dependence of the inverse of $a \to \mathcal{O}u_a$. Output least squares identifiability takes into account the practically relevant situation where due to errors in the data and in the model, the data z may not be contained in the attainable set D. As a consequence the problem of unique and continuous projection of z onto D must be considered. - Let us mention that our results can easily be extended to equations containing lower order terms.

In order to achieve these strong stability results, we shall limit ourselves to the case of a distributed H^1 observation, where $\mathcal{O}u_a = \text{grad } u_a$. Of course, it can seem unrealistic that one can observe or measure the gradient of u throughout Ω. However, the results of this paper can be combined with the technique of statespace regularization of [CK2] to handle the somewhat more realistic case of a distributed L^2 observation. Output least squares identifiability with boundary measurement is a completely open problem.

We briefly comment on some related works. In [$CK1$] output least squares identifiabilty for the one-dimensional version of (1.1) was considered. In that case one can take advantage of an explicit representation of the solution. Identifiability and stability of a in (1.1) was treated in several papers. We mention [R] and [CG], where the analysis is based on the method of characteristics for the hyperbolic equation for a, which arises from (1.1) when f and u are given functions. The analysis in [IK] is based on variational techniques. All these results refer to the case of distributed parameters. Many research efforts focused on identifiability in the case of boundary observations. Relevant references can be found in [I], for example.

2 The inverse problem

We consider a domain $\Omega \subset \mathbb{R}^2$ such that its boundary $\partial\Omega$ is partitioned into Γ_D, Γ_N, and Γ_i, $i = 1, \cdots, N$. Here Γ_i represent the boundaries of holes, which will be used to model source and sink terms, and Γ_D and Γ_N are a partition of the outer boundary of Ω corresponding to Dirichlet and Neumann boundary conditions.

We define the Hilbert space

$$\begin{aligned} &V = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0,\ v|_{\Gamma_i} = v_i = \text{ const },\, i = 1, \cdots, N\} \\ &\|v\|_V = |\nabla v|_{\mathbb{L}^2}, \end{aligned} \tag{2.1}$$

where in case $\Gamma_D = \phi$ the condition $v|_{\Gamma_D} = 0$ is replaced by $\int_\Omega v = 0$. It is supposed that Ω satisfies

$$\begin{aligned} &\Omega \text{ is bounded and connected; } \partial\Omega \in C^{1,1};\ \bar\Gamma_N, \bar\Gamma_D \text{ and} \\ &\bar\Gamma_i, i = 1, \cdots, N \text{ are pairwise disjoint .} \end{aligned} \tag{2.2}$$

On V we define the linear form L by

$$\begin{aligned} &L(v) = \textstyle\int_\Omega f\, v + \int_{\Gamma_N} g\, v + \sum_{i=1}^N Q_i v_i, \quad \text{for } v \in V, \\ &\text{where } f \in L^p(\Omega), g \in L^p(\Gamma_N), \text{for some } p > 2, Q_i \in \mathbb{R}, i = 1, \cdots, N, \end{aligned} \tag{2.3}$$

with the additional condition that

$$\int_\Omega f \ + \int_{\Gamma_N} g \ + \sum_{i=1}^N Q_i = 0 \ \text{ if } \ \Gamma_D = \phi.$$

Henceforth we denote by C the set of admissible parameters a. The precise conditions on C we be given further below. In particular they will allow to associate to every $a \in C$ the solution $u = u_a \in V$ defined by

$$\text{(Q)} \qquad \int_\Omega a \nabla u \nabla v = L(v) \ \text{ for all } \ v \in V,$$

which is the variational formulation of the elliptic equation

$$\left\{ \begin{array}{l} -\text{div }(a \text{ grad } u) = f \ \text{ in } \ \Omega \\ u|_{\Gamma_D} = 0,\ a\frac{\partial u}{\partial n}|_{\Gamma_N} = g \\ \int_{\Gamma_i} a\frac{\partial u}{\partial n} = Q_i,\ u = \text{ unknown constant on } \Gamma_i, i = 1, \cdots, N. \end{array} \right. \tag{2.4}$$

We shall be concerned with the inversion of the mapping $a \to \text{grad } u_a$ from $L^2(\Omega)$ to $\mathbb{L}^2(\Omega)$ in the least-squares sense:

$$\text{(P)} \qquad \text{minimize } \tfrac{1}{2}|\text{ grad } u_a - z|^2_{\mathbb{L}^2} \ \text{ over } \ a \in C,$$

where $z \in \mathbb{L}^2(\Omega)$ is a given observation. Our objective is to find conditions on C such that the parameter a is output least squares (OLS-)identifiable on C, i.e. that the nonlinear least squares problem (P)is quadratically (Q-) wellposed in the sense that:

$$D = \{ \text{ grad } u_a \in \mathbb{L}^2(\Omega) : a \in C\}$$

possesses a neighborhood $\mathcal{V}$ such that for every $z \in \mathcal{V}$ problem (P) has no local minima and admits a unique solution $\hat{a}$, such that the mapping $z \to \hat{a}$ is Lipschitz continuous from $\mathbb{L}^2(\Omega)$ to $L^2(\Omega)$. For this purpose we require that the parameters belong to the space

$$\mathcal{E} = \{a \in C^{0,1}(\overline{\Omega}) : a|_{\Gamma_i} = \text{ unknown constant } a_i = i = 1, \cdots, N\}, \tag{2.5}$$

equipped with the norm $\|\cdot\|_{C^{0,1}}$. The set of admissible parameters C is assumed to satisfy:

$$C \subset \{a \in \mathcal{E} : a_m \leq a(x) \text{ a.e. in } \Omega, \|a\|_{C^{0,1}} \leq a_M\}, \tag{2.6}$$

and

$$C \text{ is convex and closed in } L^2(\Omega), \tag{2.7}$$

where $0 < a_m \leq a_M$ are given constants. Note that the image in $C(\subset \mathcal{E})$ of the mapping $z \to \hat{a}$ is considered in the L^2-norm, whereas the set C is endowed with the norm of $\mathcal{E}$. Condition (2.6) ensures that (Q) has a unique solution for all $a \in C$. Moreover, the requirement that a is Lipschitz continuous, together with the modelling of source and sink terms by holes, and the regularity hypotheses (2.2), (2.3) for Ω, f, and g imply that $\{|u_a|_{W^{2,p}} : a \in C\}$ is bounded, (see $[T]$, pg. 180). Since $W^{2,p}(\Omega)$ is continuously embedded into $C^1(\overline{\Omega})$ for every $p > 2$, then exist u_M and γ_M such that

$$|u_a|_{L^\infty} \leq u_M, |\text{grad } u_a|_{\mathbb{L}^\infty} \leq \gamma_M \text{ for all } a \in C. \tag{2.8}$$

The hypothesis that the parameters satisfy $a|_{\Gamma_i} = a_i =$ unknown constant is the 2-D generalization of the hypothesis that a is constant on some neighborhood of each Dirac source term, which was required in the 1-D case to ensure OLS–identifiability $[CK1]$. It is also physically not too restrictive, as one can assume that the size of Γ_i's, which model the well boundaries, are small compared to the size of Ω. - The convexity and closedness condition (2.7) are required for the study of OLS–identifiability by the geometric techniques for nonlinear least squares developed in $[C1]$,$[C2]$.

As a first step towards OLS–identifiability we shall analyse in Section 3 inverse stability estimates of the form

$$\text{(S)} \qquad |(a-b) \text{ grad } u_a|_{\mathbb{L}^2} \leq k\, |b(\text{grad } u_a - \text{ grad } u_b)|_{\mathbb{L}^2},$$

for $k \geq 0$. As this stability estimate ought to hold for perturbations $a - b$ in any direction $b \in \mathcal{E}$, we attempt to prove (S) only at points $a \in C$ which are identifiable:

Definition 2.1 *The parameter $a \in C$ is identifiable in $\mathcal{E}$ if, for every $b \in \mathcal{E}$ which admits a solution $u_b \in V$ to (Q) one has*

$$u_b = u_a \text{ implies } b = a. \tag{2.9}$$

However, we shall see in Section 3 that one cannot find for any $k > 0$ an infinite set C satisfying (2.5), (2.6), (2.7) on which (S) holds uniformly. Therefore we reduce in Section 4 our attention to finite dimensional parameter sets: For this purpose let $\mathcal{E}_n$, $n \in \mathbb{N}$ be a family of subspaces such that

$$\begin{cases} \mathcal{E}_0 = \{\text{constant functions}\} \subset \mathcal{E}_1 \subset \mathcal{E}_2 \cdots \subset \mathcal{E} \subset C^{0,1}(\bar{\Omega}) \\ \dim \mathcal{E}_n < \infty \quad \text{for each} \quad n \\ \overline{\bigcup_{n \in \mathbb{N}} \mathcal{E}_n} = L^2(\Omega), \end{cases} \tag{2.10}$$

where the closure is taken in $L^2(\Omega)$, and define for all n:

$$C_n = C \cap \mathcal{E}_n. \tag{2.11}$$

In order to have a chance for (S) to hold uniformly on C_n we require that the data of the problem, i.e. $(\Omega, \Gamma_D, \Gamma_N, \Gamma_i, f, g, Q_i, a_m, a_M, \mathcal{E}_n)$, are chosen such that

$$\text{(H)} \qquad \begin{cases} \text{for all} \quad n \in \mathbb{N} \quad \text{one has} \\ a \in C_n \quad \text{implies that } a \text{ is identifiable in } \mathcal{E}_n. \end{cases}$$

For the definition of identifiability of $a \in C_n$ in $\mathcal{E}_n$ one simply replaces $C, \mathcal{E}$ in Definition 2.1 by $C_n, \mathcal{E}_n$. Under condition (H) we shall be able to prove in Section 4 the inverse stability estimate (S) on C_n for all $n \in \mathbb{N}$, for some $k = k_n$, with $\lim_{n \to \infty} k_n = \infty$, and to estimate, in Section 5, sensitivity, deflection and curvature of the $a \to \text{grad}\, u_a$ mapping. These estimates will be used to prove, in Section 6, that a is OLS–identifiable on C_n for each n, provided the diameter of C in $L^\infty(\Omega)$, denoted by $diam_\infty$, is small enough. The last section will be devoted to the analysis of the advantages and disadvantages of parameterizing the problem by $b = 1/a$ instead of a.

Before proceeding to the next section we make sure that our theory is not empty by giving an example for sufficient conditions which ensure that (H) holds. We require two technical lemmas.

Lemma 2.1 *The parameter $a \in C$ is identifiable in $\mathcal{E}$ if and only if*

$$h \in \mathcal{E} \;\; \textit{and} \;\; \int_\Omega h \; grad\; u_a \; grad v = 0 \;\; \textit{for all} \;\; v \in V \;\; \textit{implies} \;\; h = 0.$$

As a consequence we observe that if $a \in C$ is identifiable in $\mathcal{E}$ and $h \neq 0$, $h \in \mathcal{E}$, then necessarily h grad $u_a \neq 0$.

Lemma 2.2 *For $a \in C$ and $h \in \mathcal{E}$ define $u = u_a$ and $v = \frac{hu}{a}$. Then $v \in V$ and*

$$\int_\Omega h \; grad\; u \cdot \; grad\; v = \; \frac{1}{2}\int_\Omega \frac{h^2}{a} | \; grad\; u|^2 + \frac{1}{2}\int_\Omega \frac{h^2}{a^2}\, u\, f$$

$$+\frac{1}{2}\int_{\Gamma_N} \frac{h^2}{a^2}\, u\, g + \sum_{i=1}^{N} \frac{h_i^2}{a_i^2}\, u_i Q_i.$$

Proof. Since $C \subset \mathcal{E} \subset C^{0,1}(\bar{\Omega})$ one has $v = \frac{hu}{a} \in H^1(\Omega)$. Moreover v satisfies the boundary conditions defining V and hence $v \in V$. It follows that

$$\int_\Omega h \text{ grad } u \cdot \text{ grad } v = \int_\Omega \frac{h^2}{a} | \text{ grad } u|^2 + \frac{1}{2} \int_\Omega \frac{u}{a} \text{ grad } u \cdot \text{ grad } h^2 - \int_\Omega \frac{h^2 u}{a^2} \text{ grad } a \cdot \text{ grad } u.$$

Integrating by parts the second term on the right hand side implies

$$\int_\Omega h \text{ grad } u \cdot \text{ grad } v = \frac{1}{2} \int_\Omega \frac{h^2}{a} | \text{ grad } u|^2 - \frac{1}{2} \int_\Omega h^2 \left(\frac{u}{a} \Delta u + \frac{u}{a^2} \text{ grad } a \cdot \text{ grad } u \right) + \frac{1}{2} \int_{\Gamma_N} u\, g \frac{h^2}{a^2} + \frac{1}{2} \sum_{i=1}^N u_i Q_i \frac{h_i^2}{a_i^2},$$

which, utilizing $-a\Delta u -$ grad $a \cdot$ grad $u = f$ gives the desired result. □

Theorem 2.1 *Let the data of the problem satisfy*

- $f = g = 0$,
- $Q_i, i = 1, \cdots, N$ *are not all zero,*
- $0 < a_m \leq a_M$
- $|\Gamma_i|, i = 1, \cdots, N,$ *are sufficiently small,*

Then for all n, all $a \in C_n$ are identifiable in $\mathcal{E}_n$ and (H) holds.

Proof. Let $n \in \mathbb{N}$ be given. Let $a \in C_n$ and u_a denote the solution to (2.4). We argue that grad $u(a)$ cannot vanish on a set I of positive measure. Let γ denote a curve in Ω connecting the inner boundaries Γ_i to $\Gamma_D \cup \Gamma_N$ such that $\Omega \backslash \gamma$ is simply connected and meas $\gamma = 0$. Then $I_\gamma = (\Omega \backslash \gamma) \cap I$ satisfies meas $I_\gamma > 0$. From [AM], Theorem 2.1 and Remark it follows that either u_a is constant on $\Omega \backslash \gamma$ and hence on Ω or u_a has only isolated critical points, i.e. points z satisfying $\nabla u(z)$. But u_a cannot equal a constant over Ω since this violates the boundary conditions at the wells Γ_i at which $Q_i \neq 0$. On the other hand the number of isolated critical points in I_γ can be at most countable, and hence meas $I_\gamma = 0$. Consequently meas $\{x : \nabla u_a(x) = 0\} = 0$.
Suppose that Γ_i surrounds for each $i = 1, \cdots, N$, a fixed source/sink location x_i. If $|\Gamma_i| \to 0$, for all $i = 1, \cdots, N$, the solution u_a converges towards the weak solution associated to a right-hand side with Dirac source term $\sum_{i=1}^N Q_i \delta(x - x_i)$, which is singular at x_i. Hence

$u_a|_{\Gamma_i} = u_{a,i} \to \infty$ if $Q_i > 0$ and $u_a|\Gamma_i \to -\infty$ if $Q_i < 0$. Since C_n is compact and $a \to u_{a,i}$ is continuous, we conclude that for $|\Gamma_i|$ sufficiently small the solution u_a satisfies

$$u_{a,i}\, Q_i \geq 0 \ \text{ for } \ i = 1, \cdots, N, \ \text{ and all } \ a \in C_n.$$

Henceforth it is assumed that $|\Gamma_i|, i = 1, \cdots, N$, is sufficienty small. For each $a \in C_n$, Lemma 2.2 implies that

$$\begin{cases} \text{for each } \ h \in \mathcal{E}_n \ \text{ and } \ v = \frac{h u_a}{a} \\ \displaystyle\int_\Omega h \text{ grad } u_a \cdot \text{ grad } v \geq \frac{1}{2} \int_\Omega \frac{h^2}{a} |\text{grad } u_a|^2. \end{cases}$$

Since $|\text{grad } u_a(x)| > 0$ a.e. on Ω,

$$\int_\Omega h \ \text{ grad } u_a \cdot \ \text{ grad } w = 0 \ \text{ for all } \ w \in V$$

implies, by choosing $w = \frac{h u_a}{a}$, that $h^2 = 0$ a.e. on Ω and hence a is identifiable in $\mathcal{E}_n$ by Lemma 2.1.

3 Decomposition of $\mathbb{L}^2(\Omega)$

It will be convenient to denote by $(\cdot, \cdot)$ the scalar products in $L^2(\Omega)$ and $\mathbb{L}^2(\Omega)$. Then for every $a, b \in C$ we obtain from the variational formulation (Q) defining u_a and u_b that

$$((a - b) \text{ grad } u_a, \text{ grad } v) = (b(\text{ grad } u_b - \text{ grad } u_a), \text{ grad } v), \tag{3.1}$$

for all $v \in V$. This suggests to associate to V an equivalence relation $\sim$ of vectorfields in $\mathbb{L}^2(\Omega)$ according to

$$\vec{q} \sim \vec{q'} \ \text{ if } \ \vec{q} \cdot \text{grad } v = \vec{q'} \cdot \text{grad } v \ \text{ for all } \ v \in V, \tag{3.2}$$

to denote by

$$\begin{cases} G = L^2(\Omega)/_\sim & \text{the quotient space} \\ G^\perp & \text{the orthogonal complement.} \end{cases} \tag{3.3}$$

and by

$$P \text{ and } P^\perp \ \text{ the orthogonal projection in } \mathbb{L}^2(\Omega) \ \text{ onto } G \text{ and } G^\perp. \tag{3.4}$$

The decomposition

$$\mathbb{L}^2(\Omega) = G \oplus G^\perp$$

is, by construction, adapted to the elliptic problem (Q). We further introduce

$$W = \{\psi \in H^1(\Omega) \colon \psi|_{\Gamma_N} = 0\},$$

where the condition $\int_\Omega \psi = 0$ is added to the definition of W if $\Gamma_N = \phi$. For $\varphi \in H^1(\Omega)$ and $\vec{\psi} \in H^1(\Omega) \times H^1(\Omega)$ we define

$$\vec{\text{rot}}\ \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x_2} \\ \\ -\frac{\partial \varphi}{\partial x_1} \end{pmatrix} \quad \text{and rot } \vec{\psi} = \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2}.$$

The following representation for G and $G^\perp$ can be obtained.

Proposition 3.1

$$\begin{aligned} G \quad &= \{grad\ \varphi : \ \varphi \in V\} \\ G^\perp \quad &= \{rot\ \psi : \ \psi \in W\}. \end{aligned}$$

Moreover, for every $\vec{q} \in \mathbb{L}^2(\Omega)$ *one has*

$$P\vec{q} = \ grad\ \varphi, \ P^\perp \vec{q} = \ \vec{rot}\ \psi,$$

where $\varphi \in V$ *and* $\psi \in W$ *are given by*

$$\begin{aligned} (grad\ \varphi,\ grad\ v) \quad &= (\vec{q},\ grad\ v) \quad && \text{for all } v \in V, \\ (\vec{rot}\ \psi,\ \vec{rot}\ v) \quad &= (\vec{q},\ \vec{rot}\ v) \quad && \text{for all } v \in W. \end{aligned}$$

Except for the atypical boundary conditions this decomposition is rather standard. For convenience an outline of the proof is given in the Appendix. The identifiability condition can now be reformulated as

Proposition 3.2 *A parameter* $a \in C$ *(respectively* C_n*) is identifiable in* $\mathcal{E}$ *(resp.* $\mathcal{E}_n$*) if and only if:*

$$h \neq 0 \text{ and } h \in \mathcal{E} \text{ (resp. } \mathcal{E}_n\text{) imply } P(h\ grad\ u_a) \neq 0.$$

The proposition follows directly from Proposition 3.1 and Lemma 2.1.

4 From which direction can an identifiable parameter be recovered in a stable way?

Let $a \in C$ be a given reference parameter and let $b \in C$ be a possibly different parameter. We investigate in this section conditions on b for which the stability estimate (S) holds.

From (3.1) we have

$$(a-b) \text{ grad } u_a \sim b(\text{ grad } u_b - \text{ grad } u_a), \tag{4.1}$$

so that

$$\begin{aligned} \|(a-b) \text{ grad } u_a\|_G &= \|b(\text{ grad } u_b - \text{ grad } u_a)\|_G \\ &\le |b(\text{ grad } u_b - \text{ grad } u_a|_{\mathbb{L}^2}. \end{aligned} \tag{4.2}$$

We decompose $(a-b)$ grad u_a on $G \oplus G^\perp$:

$$(a-b) \text{ grad } u_a = \text{ grad } \varphi + \vec{\text{rot}}\ \psi, \tag{4.3}$$

where $\varphi \in V$ and $\psi \in W$ are given according to Proposition 3.1 by

$$(\text{grad } \varphi,\ \text{grad } v) = ((a-b) \text{ grad } u_a,\ \text{grad } v) \ \text{ for all } \ v \in V, \tag{4.4}$$

$$(\vec{\text{rot}}\ \psi,\ \vec{\text{rot}}\ v) = ((a-b) \text{ grad } u_a,\ \vec{\text{rot}}\ v) \ \text{ for all } \ v \in W. \tag{4.5}$$

Clearly one has

$$\text{grad } \varphi = P((a-b) \text{ grad } u_a),\ \vec{\text{rot}}\ \psi = P^\perp((a-b) \text{ grad } u_a). \tag{4.6}$$

Moreover

$$\|(a-b) \text{ grad } u_a\|_G = |P((a-b) \text{ grad } u_a)|_{\mathbb{L}^2}$$

which together with (4.2) implies

$$|P((a-b) \text{ grad } u_a|_{\mathbb{L}^2} \le |b(\text{grad} u_b - \text{ grad } u_a)|_{\mathbb{L}^2}. \tag{4.7}$$

Defining for $M > 0$ the set

$$\begin{aligned} S_M(a) = \{b \in C : \quad & |P^\perp((b-a) \text{ grad } u_a)|_{\mathbb{L}^2} \le \\ & M|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}\}, \end{aligned} \tag{4.8}$$

we conclude that

$$|(b-a) \text{ grad } u_a|_{\mathbb{L}^2} \le (1+M^2)^{1/2}|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}, \tag{4.9}$$

for all $b \in S_M(a)$. Combining (4.7) and (4.9) implies

Proposition 4.1 *Let $M > 0$ and $a \in C$ be given. Then for all $b \in S_M(a)$ the stability estimate (S) holds with $k = (1+M^2)^{1/2}$:*

$$|(b-a) \textit{ grad } u_a|_{\mathbb{L}^2} \le (1+M^2)^{1/2}|b(\textit{grad } u_b - \textit{ grad } u_a)|_{\mathbb{L}^2}. \tag{4.10}$$

Hence the directions $b-a$, with $b \in S_M(a)$ are those from which the parameter a can be recovered within C with a stability constant $(1+M^2)^{1/2}$. We investigate now the shape of $S_M(a)$. The set $S_M(a)$ is the intersection of C with a wedge having its vertex at a. Clearly $a \in S_M(a)$. If a is in the $\mathcal{E}$–interior of C, $S_M(a)$ also contains parameters of the form $b = a + t\gamma(u_a)$ for t small enough, where γ is any regular function. In fact, in this case the gradients of $b-a$ and u_a are colinear so that $P^\perp((b-a) \text{ grad } u_a) = 0$ (see (4.14 below), and hence (S) holds with $M = 0$ and $k = 1$.

Proposition 4.2 *Let $a \in C$ be identifiable in $\mathcal{E}$. Then*

$$\bigcup_{M>0} S_M(a) = C. \tag{4.11}$$

Proof. Let $b \in C$ with $b \neq a$. As a is identifiable it follows from Proposition 3.2 that $P((b-a) \text{ grad } u_a) \neq 0$. Hence $b \in S_M(a)$ for M sufficiently large. □

We next interpret the quantities which enter the definition of $S_M(a)$.

Lemma 4.1 *For every $a \in C$ and $h \in \mathcal{E}$ we have*

$$\|div\ (h\ grad\ u_a)\|_{H^{-1}} \leq |P(h\ grad\ u_a)|_{\mathbb{L}^2} \leq |h\ grad\ u_a|_{\mathbb{L}^2}, \tag{4.12}$$

$$\begin{aligned} \|\vec{rot}\, h\ grad\ u_a\|_{W^*} &= |P^{\perp}(h\ grad\ u_a)|_{\mathbb{L}^2} \\ &\leq \min\{C_W|\vec{rot}\, h \cdot\ grad\ u_a|, |h\ grad\ u_a|\}, \end{aligned} \tag{4.13}$$

where C_W is the Poincare constant in W.

Proof. From Proposition 3.1 we have

$$|P(h \text{ grad } u_a)|_{\mathbb{L}^2} = |\text{grad } \varphi|_{\mathbb{L}^2} = \sup\{(\text{ grad } \varphi, \vec{q'}) : \vec{q'} \in \mathbb{L}^2, |\vec{q'}|_{\mathbb{L}^2} = 1\}.$$

But $\vec{q'} = \text{grad } v + \vec{\text{rot}}\ w$ with $v \in V$ and $w \in W$, so that

$$\begin{aligned} |P(h \text{ grad } u_a)|_{\mathbb{L}^2} &= \sup\{(\text{grad}\varphi,\ \text{grad } v + \vec{\text{rot}}\ w): \\ & v \in V, w \in W, |\text{ grad } v|^2 + |\vec{\text{rot}}\ w|^2 = 1\} \\ &= \sup\{(\text{grad}\varphi,\ \text{grad } v): v \in V, |\text{ grad } v| = 1\} \end{aligned}$$

and

$$|P(h \text{ grad } u_a)|_{\mathbb{L}^2} = \sup\left\{\int_\Omega h \text{ grad } u_a \cdot \text{grad} v : v \in V, |\text{ grad } v| = 1\right\}.$$

These estimates imply by the Cauchy–Schwartz the second inequality in (4.12). The first follows from $H_0^1(\Omega) \subset V$. From Proposition 3.1 we obtain as well that

$$\begin{aligned} &|P^{\perp}(h \text{ grad } u_a)|_{\mathbb{L}^2} = \\ &\quad \sup\left\{\int_\Omega h \text{ grad } u_a \cdot \vec{\text{rot}} w : w \in W,\ |\text{grad } w| = 1\right\}. \end{aligned} \tag{4.14}$$

By Green's formula we find:

$$\int_\Omega h \text{ grad } u_a \cdot \vec{\text{rot}}\ w = \int_\Omega \text{ rot } (h \text{ grad } u_a)\, w - \int_{\partial\Omega} h \frac{\partial u_a}{\partial \tau}\, w. \tag{4.15}$$

Since $u_a = 0$ on Γ_D and $u_a = \text{const}$ on each Γ_i, we have $\frac{\partial u_a}{\partial \tau} = 0$ on Γ_D and on $\Gamma_i, i = 1, \cdots, N$, and $w = 0$ on Γ_N. Thus all boundary terms vanish. From (4.14), (4.15) and the fact that $\text{rot}(h \text{ grad } u_a) = \vec{\text{rot}}\, h \cdot \text{ grad } u_a$ we find

$$|P^{\perp}(h \text{ grad } u_a)|_{\mathbb{L}^2} = \sup_{w \in W, |\text{ grad } w| = 1} \int_{\Omega} (\vec{\text{rot}}\, h \cdot \text{ grad } u_a) w,$$

which, by Poincarés inequality in W shows that

$$|P^{\perp}(h \text{ grad } u_a)|_{\mathbb{L}^2} = |\vec{\text{rot}}\, h \cdot \text{ grad } u_a|_{W^*} \leq C_W |\vec{\text{rot}}\, h \cdot \text{ grad } u_a|_{L^2}.$$

Combining (4.14) and the last estimate implies (4.13) and the lemma is proved. □

Lemma 4.1 implies that $S_M(a)$ contains all $b \in C$ such that $h = b - a$ satisfies

$$C_W |\vec{\text{rot}}\, h \text{ grad } u_a| \leq M |\text{ div } (h \text{ grad } u_a)|_{H^{-1}}.$$

Hence we see that the perturbations from which a can be stably recovered are, speaking loosely those whose gradient tends to be mostly oriented along the flow lines of u_a at each point $x \in \Omega$. In particular, a cannot be recovered stably from perturbations h such that $\text{div}(h \text{ grad } u_a) = 0$. When u_a is harmonic (e.g. if $f = 0$ and $a = \text{const}$), these unstable perturbations are such that $\text{grad } h \cdot \text{ grad } u_a = 0$. This corresponds to the intuition that the observation of the pressure field grad u_a gives little information on the diffusion parameter a orthogonal to flow lines.

We next show that if a can be recovered stably from b it can also be recovered stably, but with a larger constant, from all $c \in C$ in some L^2- neighborhood of b:

Theorem 4.1 *Let $a \in C$ be identifiable in $\mathcal{E}$, $b \in C$, $b \neq a$ be given, and define:*

$$0 \leq M = \frac{|P^{\perp}((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}}{|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}} < \infty.$$

Then for every $M' > M$ there exists $\epsilon > 0$ such that

$$S_{M'}(a) \supset C \cap \{c \in L^2(\Omega) \colon |c - b|_{L^2} \leq \epsilon\}. \tag{4.16}$$

Proof. The mappings $\Lambda(h) = P(h \text{ grad } u_a)$ and $\Lambda^{\perp}(h) = P^{\perp}(h \nabla u_a)$ are continuous from $\mathbb{L}^2(\Omega)$ into itself. Hence we get for $|c - b|_{L^2} \leq \epsilon$

$$|P^{\perp}((c-a) \text{ grad } u_a)| \leq |P^{\perp}((b-a) \text{ grad } u_a)| + \|\Lambda^{\perp}\| \epsilon,$$

$$|P((c-a) \text{ grad } u_a)| \geq |P((b-a) \text{ grad } u_a)| - \|\Lambda\| \epsilon.$$

Since a is identifiable we have $P((b-a) \text{ grad } u_a) \neq 0$. Hence for $M' > M$ there exists $\epsilon > 0$ such that $|P^{\perp}((c-a) \text{ grad } u_a)|\ |P((c-a) \text{ grad } u_a|^{-1} \leq M'$ as soon as $|c - b| \leq \epsilon$. This implies (4.16). Of course $c = a$ cannot belong to this neighborhood of b ! □

At this point the question arises whether it is possible for the stability estimate (S), or (4.10) to hold uniformly for some $k = (1+M^2)^{1/2}$ for all $a, b \in C$. In other terms we search for C satisfying (2.6), (2.7) and

$$a, b \in C \ \text{ implies } \ b \in S_M(a). \tag{4.17}$$

We give a negative answer in the sense that there is no infinite dimensional C with nonempty $\mathcal{E}$-interior satisfying the specified properties.

Suppose that such a C exists, and let a be an element of the $\mathcal{E}$-interior of C. Further let B denote a ball with center a and radius ϵ contained in C. Then (4.17) would imply that $B \subset S_M(a)$ and hence as $S_M(a)$ is the intersection of C with a wedge, that

$$|P^{\perp}(h \text{ grad } u_a)|_{\mathbb{L}^2} \leq M|P(h \text{ grad } u_a)|_{\mathbb{L}^2} \ \text{ for all } \ h \in \mathcal{E}. \tag{4.18}$$

We show on a simple example that (4.18) cannot be true in general. For this purpose consider (Q) with Ω the unit square in $\mathbb{R}^2$, $f = 0$, $g = 0$ on top and bottom, $g = -1$, on the left and $g = 1$ on the right lateral boundary, and no internal sources and sinks. The solution corresponding to $a = 1$ is given by $u_a(x, y) = x - \frac{1}{2}$.

- We check that $a = 1$ is identifiable in $\mathcal{E}$. For every $h \in \mathcal{E}$ we have from Lemma 2.2 that for $v = \frac{h u_a}{a} \in V$

$$\int_\Omega h \text{ grad } u_a \text{ grad } v = \frac{1}{2}\int_\Omega h^2 + \frac{1}{2}\int_{\Gamma_N} h^2 u\, g.$$

  Since $u\, g \geq 0$ on $\partial\Omega$ we see that

$$\int_\Omega h \text{ grad } u_a \text{ grad } v = 0 \ \text{ for all } \ v \in V \ \text{ implies } \ h = 0,$$

  and thus by Lemma 2.1 a is identifiable in $\mathcal{E}$.

- Next we consider parameter perturbations which are orthogonal to the flow lines. This results in choosing (x, y denote the coordinates in $\mathbb{R}^2$)

$$h(x, y) = h(y).$$

We shall require that

$$h \in C^{0,1}(0,1), \int_0^1 h = 0 \ \text{ and } \ h(0) = h(1) = 0. \tag{4.19}$$

Under these conditions we estimate lower and upper bounds to $|P^{\perp}(h \text{ grad } u_a)|$ and $|P(h$ grad $u_a)|$.

(i) From Lemma 4.1 we have

$$|P^{\perp}(h \text{ grad } u_a)| = \| \text{ rot } h \text{ grad } u_a\|_{W^*} = \|h'\|_{W^*},$$

and by (4.19)

$$\|h'\|_{W^*} = \sup \{\int_\Omega h\frac{\partial w}{\partial y}: w \in W, |\text{ grad } w| = 1\}. \tag{4.20}$$

Let H be the primitive of h:

$$H(y) = \int_0^y h(\tau)d\tau$$

and note that $H(0) = H(1) = 0$. We define

$$\widetilde{w}(x,y) = x(1-x)H(y),$$

so that $\widetilde{w} = 0$ on $\partial\Omega = \Gamma_N$, and hence $\widetilde{w} \in W$, with

$$|\text{ grad } \widetilde{w}|^2 = \tfrac{1}{3}(|H|^2 + \tfrac{1}{10}|h|^2).$$

Choosing $w = \frac{\widetilde{w}}{|\text{ grad } \widetilde{w}|}$ in (4.20) gives

$$\|h'\|_{W^*} \geq \frac{\sqrt{3}|h|^2}{6\sqrt{|H|^2 + \frac{1}{10}|h|^2}}.$$

Since $|H|^2 \leq \frac{1}{2}|h|^2$ we obtain

$$|P^{\perp}(h \text{ grad } u_a)| \geq \tfrac{2}{3}|h|. \tag{4.21}$$

(ii) From the proof of Lemma 4.1 we get

$$|P(h \text{ grad } u_a)| = \sup \{\int_\Omega h\frac{\partial v}{\partial x} : v \in V, |\text{ grad } v| = 1\},$$

and, integrating by parts with respect to x,

$$|P(h \text{ grad } u_a)| = \sup \{\int_0^1 h(y)(v(1,y) - v(0,y))dy : v \in V, |\text{grad } v| = 1\}.$$

Let $\gamma = \partial\Omega \cup \{x = 1\}$, the right lateral boundary of Ω, and denote by c the continuity constant from V to $H^{1/2}(\gamma)$. Then $|\text{ grad } v| = 1$ implies $\|v|_\gamma\|_{H^{1/2}(\gamma)} \leq c$, so that by symmetry

$$|P(h \text{ grad } u_a)| \leq 2\sup \left\{\int_0^1 h\,\xi : \xi \in H^{1/2}(\gamma), \|\xi\|_{H^{1/2}(\gamma)} \leq c\right\}.$$

Associating to h a function $H \in H^{1/2}(\gamma)$ satisfying

$$((H,\xi))_{H^{1/2}(\gamma)} = \int_0^1 h\,\xi, \quad \text{for all} \quad \xi \in H^{1/2}(\gamma), \tag{4.22}$$

we have

$$|P(h \text{ grad } u_a)| \le 2 \sup_{\|\xi\|_{H^{1/2}(\gamma)} \le c} ((H,\xi))_{H^{1/2}} = 2c\|H\|_{H^{1/2}}. \tag{4.23}$$

From (4.21), (4.23) we see that (4.18) would imply that

$$|h| \le 3Mc\, \|H\|_{H^{1/2}(\gamma)}, \tag{4.24}$$

for all $h \in \mathcal{E}$ satisfying (4.19). But we can choose a sequence of functions h_n satisfying

$$h_n \in C^{0,1}(0,1), \int_0^1 h_n = 0, h_n(0) = h_n(1) = 0,$$

and

$$|h_n|_{L^2} = \text{ const }, |h_n|_{H^{-1/2}(\gamma)} \to 0 \text{ for } n \to \infty.$$

From (4.22) we see that $\|H_n\|_{H^{1/2}(\gamma)} \to 0$ for $n \to \infty$. This contradicts (4.24) and consequently (4.18)as well. □

It is hence impossible to find an infinite dimensional set C with nonempty interior on which the stability estimate holds uniformly. This motivates the reduction to finite dimensional parameter sets in the remaining sections.

5 Finite stability estimates

We turn to the finite dimensional setting of Section 2 with $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}$ satisfying (2.11). We recall the definition $C_n = C \cap \mathcal{E}_n$ and suppose throughout this section that (H) holds. Identifiability of a in $\mathcal{E}_n$ implies by Proposition 3.2 that for all $a, b \in C_n$ with $b \neq a$ we have

$$P((b-a) \text{ grad } u_a) \neq 0. \tag{5.1}$$

Hence we can define for each $n \in \mathbb{N}$

$$M_n = \sup_{\substack{a,b \in C_n \\ a \neq b}} \frac{|P^{\perp}((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}}{|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}}. \tag{5.2}$$

From Proposition 3.2 we know that (5.1) holds with $b-a$ replaced by any $h \in \mathcal{E}_n$, $h \neq 0$, and hence

$$M_n \le \sup_{a \in C_n} \sup_{h \in \mathcal{E}_n, \|h\|_{\mathcal{E}}=1} \frac{|P^{\perp}(h \text{ grad } u_a)|_{\mathbb{L}^2}}{|P(h \text{ grad } u_a)|_{\mathbb{L}^2}}. \tag{5.3}$$

As the fraction in (5.3) is a continuous function of h and a, and the suprema are taken on compact sets it follows that M_n is finite for each n. Since $\mathcal{E}_0$ consists of constant functions $M_0 = 0$ by Lemma 4.1. Moreover since $\mathcal{E}_n \subset \mathcal{E}_{n+1}$, for all n, we have

$$0 = M_0 \leq M_1 \leq \cdots \leq M_n \cdots < \infty. \tag{5.4}$$

From the example at the end of the previous section we can generally expect that $\lim_{n\to\infty} M_n = \infty$.

Proposition 4.1 implies the following stability estimates.

Theorem 5.1 *Let (2.5)–(2.7), (2.10), (2.11) and (H) hold. Then for every* $n \in \mathbb{N}$

$$|(b-a)\ grad\ u_a|_{\mathbb{L}^2} \leq (1+M_n^2)^{1/2}\ |b(\ grad\ u_b -\ grad\ u_a)|_{\mathbb{L}^2},$$

for all $a, b \in C_n$*, where* M_n *is defined in (5.2).*

This theorem gives a rigorous justification to the numerical observation [L], [GM], [GKMN] that the sensitivity of the inversion of the $a \to u_a$ mapping is a decreasing function of the scale at which the parameter is estimated. This observation together with the analysis of the nonlinearity of the mapping $a \to u_a$, to be given in the next section, has motivated the introduction of successfull multiscale approaches to parameter estimation [CL], [L].

We close the section with an estimate of M_n in the case where (H) is satisfied by virtue of Theorem 2.1.

Theorem 5.2 *Let (2.5)–(2.7), (2.10), (2.11) hold, and suppose that the hypotheses of Theorem 2.1 are satisfied. Then*

$$M_n \leq \sup_{a\in C_n} \sup_{h\in\mathcal{E}_n, h\neq 0} M_n(a,h),$$

where, for $a \in C_n$ *and* $h \in \mathcal{E}_n$*,* $h \neq 0$*:*

$$M_n(a,h) = 2\left(\frac{a_M}{a_m}\right)^{1/2} \frac{|grad\ (\frac{h}{a})u_a|}{|\frac{h}{a} grad\ u_a|} \min\left\{1, C_W \frac{|\vec{rot}\, h\ grad\ u_a|}{|h\ grad\ u_a|}\right\}. \tag{5.5}$$

Proof. Choosing $v = \frac{hu_a}{a}|\text{ grad }(\frac{hu_a}{a})|_{\mathbb{L}^2}^{-1}$ on the right hand side of the equality above (4.14) we obtain by Theorem 2.1 and Lemma 2.2

$$|P(h \text{ grad } u_a)|_{\mathbb{L}^2} \geq \frac{1}{2} \frac{|\frac{h}{a^{1/2}} \text{ grad } u_a|_{\mathbb{L}^2}^2}{|\text{grad } (\frac{hu_a}{a})|_{\mathbb{L}^2}}$$

and hence

$$|P(h \text{ grad } u_a)|_{\mathbb{L}^2} \geq \frac{a_m^{1/2}}{2a_M^{1/2}} \frac{|\frac{h}{a} \text{ grad } u_a|\,|h \text{ grad } u_a|}{|\text{grad } (\frac{h}{a}u_a)}.$$

Combining this estimate with (4.13) of Lemma 4.1 and (5.3) implies the theorem. □

For $a, b \in C_n$ the stability constant $M_n(a, b-a)$ of (5.5) allows the following interpretation:

- the first factor is related to the size of C_n,
- the second factor tends to infinity when the dimension of $\mathcal{E}_n$ goes to infinity. It does not depend on the relative orientations of grad h and grad u_a,
- the third factor is bounded by 1 and tends to zero as grad h becomes colinear with grad u_a.

6 Finite dimensional sensitivity, deflection and curvature estimates

In this section we analyze the geometric quantities associated to the parameter–to–solution mapping $a \to u_a$ defined by (Q). For $a_0, a_1 \in C_n$ and $t \in [0,1]$ we set

$$h = a_1 - a_0, \tag{6.1}$$

$$a = (1-t)a_0 + t\, a_1. \tag{6.2}$$

The geometric quantities are related to the curve $t \in [0,1] \to \nabla u_a \in \mathbb{L}^2(\Omega)$ in the range of the mapping $a \to \nabla u_a$. We denote by η the velocity and ξ the accelleration, i.e. the first and second derivatives of u_a with respect to t. The equations characterizing $\eta \in V$ and $\xi \in V$ are found to be:

$$\int_\Omega a \text{ grad } \eta \cdot \text{ grad } v = -\int_\Omega h \text{ grad } u_a \text{ grad } v, \quad \text{for all } \; v \in V, \tag{6.3}$$

$$\int_\Omega a \text{ grad } \xi \cdot \text{ grad } v = -2\int_\Omega h \text{ grad } \eta \text{ grad } v, \quad \text{for all } \; v \in V. \tag{6.4}$$

For given $n \in \mathbb{N}$ the objective is to find $0 < \alpha_m \le \alpha_M$, $\Theta \ge 0$ and $R > 0$ such that the following inequalities hold uniformly for all $a_0, a_1 \in C_n$ and $t \in [0,1]$:

$$\alpha_m |h|_{L^2} \le |\text{grad } \eta|_{\mathbb{L}^2} \le \alpha_M |h|_{L^2}, \tag{6.5}$$

$$|\text{grad } \xi|_{\mathbb{L}^2} \le \Theta |\text{grad } \eta|_{\mathbb{L}^2}, \tag{6.6}$$

$$|\text{grad } \xi|_{\mathbb{L}^2} \le \frac{1}{R} |\text{grad } \eta|^2_{\mathbb{L}^2}. \tag{6.7}$$

These inequalities can be interpreted as follows:

- α_m and α_M are lower and upper bounds to the first derivative of $a \to \nabla u_a$. They are refered to as minimal and maximal sensitivity.
- Θ is an upper bound to the deflection along the curve $t \to \nabla u_a$ (i.e. to the angle between the tangents at two points of the curve).
- $\frac{1}{R}$ is an upper bound to the curvature along the curve.

Theorem 6.1 *Let (2.5)–(2.7), (2.10), (2.11) and (H) hold. Then, for every $n \in \mathbb{N}$, $a_0, a_1 \in C_n$ and $t \in [0,1]$, (6.5)–(6.7) hold with*

$$\alpha_m = \frac{\gamma_{m,n}}{a_M(1+M_n^2)^{1/2}}, \quad \alpha_M = \frac{\gamma_M}{a_m}, \tag{6.8}$$

$$\Theta = 2\ \frac{diam_\infty(C_n)}{a_m}, \tag{6.9}$$

$$\frac{1}{R} = 2\ K_n\ \frac{(1+M_n^2)^{1/2}}{\gamma_{m,n}}\ \frac{a_M}{a_m}, \tag{6.10}$$

where γ_M is defined in (2.8), and

$$\gamma_{m,n} = \inf_{a \in C_n} \quad \inf_{h \in \mathcal{E}_n,\ |h|_{L^2}=1} |h\ grad\ u_a|_{\mathbb{L}^2} > 0, \tag{6.11}$$

$$K_n = \sup_{h \in \mathcal{E}_n, |h|_{L^2}=1} |h|_{L^\infty}. \tag{6.12}$$

Proof. Let $t + dt \in [0,1]$ and denote by a_t and a_{t+dt} the corresponding values of a given by (6.2). Choosing $a = a_t$ and $b = a_{t+dt}$ in Theorem 5.1 and letting dt tend to zero implies that

$$|h \text{ grad } u_a|_{\mathbb{L}^2} \leq (1+M_n^2)^{1/2}\ |a \text{ grad } \eta|_{\mathbb{L}^2}. \tag{6.13}$$

Hence the left inequality of (6.5) is satisfied with α_m defined by (6.8), (6.11). The strict positivity of $\gamma_{m,n}$ follows from (H) which ensures that the argument of the inf is strictly positive, and hence the inf itself is strictly positive as it is taken over a compact set. The right inequality of (6.5) is obtained by choosing $v = \eta$ in (6.3) and using (2.9). Setting $v = \xi$ in (6.4) gives

$$|a^{1/2} \text{ grad } \xi|_{\mathbb{L}^2} \leq 2\ |\frac{h}{\sqrt{a}}|_{L^\infty}\ |\text{grad } \eta|,$$

which implies (6.9), and also (6.10) using (6.11)–(6.13). □

Let us discuss the behavior of the constants α_m, α_M, Θ and R as n tends to infinity. We notice first that $\gamma_{m,n}$ can be understood as a lower bound to the local mean value of grad u_a at scale n. In general u_a will have stationary points where grad $u_a(x) = 0$, in which case one expects that $\gamma_{m,n} \to 0$ for $n \to \infty$. In special cases, as for instance the example of Section 4, see also [IK], [R] for further examples, it can happen that $|\text{grad } u_a(x)| \geq \gamma_m > 0$ for all $a \in C$ and $x \in \Omega$, in which case $\gamma_{m,n} \geq \gamma_m > 0$ for all n.

In case (H) is satisfied through the assumptions of Theorem 2.1 an upper bound to M_n can obtained by Theorem 5.2:

$$M_n \leq 2 \left(\frac{a_M}{a_m} \right)^{1/2} \left(1 + \frac{u_M}{\gamma_{m,n}} \sup_{a \in C_n} \sup_{h \in \mathcal{E}_n, |\frac{h}{a}|_{L^2}=1} |\text{grad } \frac{h}{a}| \right). \tag{6.14}$$

In case the finite dimensional spaces $\mathcal{E}_n$ are obtained from a regular family of triangulations of Ω by elements of size Δx, the right hand side of (6.14) is of the order $\frac{1}{\Delta x}$ for $n \to \infty$. In this situation the continuity constant K_n of the $L^2 \to L^\infty$ injection (for $h \in \mathcal{E}_n$) is also of the order $\frac{1}{\Delta x}$ as $n \to \infty$. These considerations imply the following

Corollary 6.1 *Under the conditions of Theorem 6.1, as the scale parameter $n \to \infty$ we have*

- *the minimal sensitivity $\alpha_m \to 0$,*
- *the maximal sensitivity α_M remains bounded,*
- *the deflection Θ remains bounded,*
- *the curvature $\frac{1}{R} \to \infty$.*

In case (H) is satisfied through the assumption of Theorem 2.1 and $\mathcal{E}_n$ is constructed based on a regular triangulation of Ω by elements characterized by meshsize Δx, one has

- $\alpha_m \geq const \Delta x \, \gamma_{m,n}$
- $\frac{1}{R} \leq const/(\Delta x^2 \, \gamma_{m,n})$.

7 Output least squares identifiability of a.

We study in this section the structure of the nonlinear least squares problem (P). We have seen in Section 4 that there is no hope for (P) itself to be quadratically wellposed, or equivalently for a to be OLS – identifiable on C. Therefore we choose finite dimensional subspaces $\mathcal{E}_n$ satisfying (2.10)–(2.11) and consider for each $n \in \mathbb{N}$ the finite dimensional estimation problems

$$(\mathrm{P}_n) \qquad \text{minimize } \frac{1}{2} |\text{grad } u_a - z|^2_{\mathbb{L}^2} \text{ over } C_n.$$

The geometric estimates of Section 6 imply that the corresponding output sets $D_n = \{\text{ grad } u_a \in \mathbb{L}^2(\Omega): a \in C_n\}$ are strictly quasiconvex if $\Theta \leq \frac{\pi}{2}$, [C1],[C2]. Hence the following theorem holds:

Theorem 7.1 *Let (2.5)–(2.7), (2.10), (2.11) and (H) hold and suppose that C satisfies*

$$\Theta = \frac{2}{a_m} \ \ diam_\infty C \leq \frac{\pi}{2}. \tag{7.1}$$

Then (P_n) is quadratically wellposed on

$$\mathcal{V}_n = \{\vec{q} \in \mathbb{L}^2(\Omega): \ dist\ (\vec{q}, D_n) < R_n = \frac{\gamma_{m,n}}{2K_n(1+M_n^2)^{1/2}}\} \tag{7.2}$$

for all $n \in \mathbb{N}$, that is

- *for every $z \in \mathcal{V}_n$, (P_n) has a unique solution $\hat{a}_n$,*
- *for every $z \in \mathcal{V}_n$, the function $a \rightarrow \frac{1}{2}\ |grad\ u_a - z|^2_{\mathbb{L}^2}$ has no local minima,*
- *the mapping $z \rightarrow \hat{a}_n$ is Lipschitz continuous i.e. for all $z_0, z_1 \in \mathcal{V}_n$ satisfying $|z_0 - z_1|_{\mathbb{L}^2} + \max\limits_{j=0,1} d(z_i, D_n) \leq d_n < R_n$,*

  $$\alpha_m |\hat{a}_{n,0} - \hat{a}_{n,1}|_{L^2} \leq L \leq (1 - \frac{d_n}{R_n})^{-1} |z_0 - z_1|_{\mathbb{L}^2},$$

  where L is the arclength in $\mathbb{L}^2(\Omega)$ of the curve $t \in [0,1] \rightarrow u_a \in \mathbb{L}^2(\Omega)$, and a is defined by (6.2) with $a_j = \hat{a}_{n,j}, j = 0, 1$,
- *every minimizing sequence of (P_n) converges to $\hat{a}_n$.*

8 Parameterizing by $b = \frac{1}{a}$.

As suggested by the 1–D case [CK1] where the mapping $b = \frac{1}{a} \rightarrow u_b$ is "quasilinear" we investigate in this section the question of using $b = \frac{1}{a}$ in place of a as optimization variable. Additional motivations are given by the facts that $b \rightarrow u_b$ is affine when b is constant and that $a \rightarrow \frac{1}{a}$, which is in some sense contained in $a \rightarrow u_a$, is not twice differentiable on $L^2(\Omega)$.

In order to see whether $b \rightarrow u_b$ is less non–linear and its inverse is less illposed than $a \rightarrow u_a$, one can carry the analysis of the previous sections with the appropriate modifications. In analogy to (2.6)–(2.7) the parameter set $\widetilde{C}$ for b is chosen such that

$$\widetilde{C} \subset \{b \in \mathcal{E}: b_m \leq b(x) \ \text{ in } \ \Omega, \ \|b\|_{C^{0,1}} \leq b_M\}, \tag{8.1}$$

$$\widetilde{C} \ \text{ is convex and closed in } \ L^2(\Omega), \tag{8.2}$$

where $0 < b_m \leq b_M$. As in (2.11) we set

$$\widetilde{C}_n = \widetilde{C} \cap \mathcal{E}_n, \tag{8.3}$$

and define identifiability of $b \in \widetilde{C}_n$ by analogy with Lemma 2.1:

Definition 8.1 *The parameter $b \in \widetilde{C}$ is identifiable in $\mathcal{E}_n$ if:*

$$h \in \mathcal{E}_n \quad \textit{and} \quad \int_\Omega \frac{h}{b^2} \ \textit{grad}\ u_b \cdot \ \textit{grad}\ v = 0 \ \ \textit{for all} \ \ v \in V \quad \textit{implies} \ \ h = 0.$$

This leads to replace (H) by

$$(\widetilde{H}) \qquad \begin{cases} \text{for all} \ \ b \in \mathbb{N} \\ b \in \widetilde{C}_n \ \ \text{implies that } b \text{ is identifiable in } \mathcal{E}_n. \end{cases}$$

Similar to Section 6 for the $a \to u_a$ mapping we now investigate the geometrical quantities related to $b \to u_b$. For $b_0, b_1 \in \widehat{C}_n$ and $t \in [0,1]$ we set

$$h = b_1 - b_0, \tag{8.4}$$

$$b = (1-t)b_0 + t\, b_1, \tag{8.5}$$

and search for $\tilde{\alpha}_m, \tilde{\alpha}_M, \widetilde{\Theta}$, and $\frac{1}{\tilde{R}}$ satisfying (6.5)–(6.7), where now the velocity $\tilde{\eta} \in V$ and the accelleration $\tilde{\xi} \in V$ are defined by

$$\int_\Omega \frac{1}{b} \ \text{grad}\ \tilde{\eta} \cdot \ \text{grad}\ v = \int_\Omega \frac{h}{b^2} \ \text{grad}\ u_b \cdot \ \text{grad}\ v \ \ \text{for all} \ \ v \in V, \tag{8.6}$$

$$\int_\Omega \frac{1}{b} \ \text{grad}\ \tilde{\xi} \cdot \ \text{grad}\ v = \int_\Omega \left(\frac{h}{b^2} \ \text{grad}\ \tilde{\eta} - \frac{h^2}{b^3} \ \text{grad}\ u_b \right) \cdot \ \text{grad}\ v, \tag{8.7}$$

for all $v \in V$. The following theorem provides the analog to (6.13) for the lower bound on the linearization of $b \to u_b$.

Theorem 8.1 *Let (8.1)–(8.3) and $(\tilde{H})$ hold. Then for every $n \in \mathbb{N}$, $b \in \widetilde{C}_n$ and $h \in \mathcal{E}_n$:*

$$|\frac{h}{b^2} \ \textit{grad}\ u_b|_{\mathbb{L}^2} \leq (1 + \widetilde{M}_n^2)^{1/2} \, |\frac{1}{b} \ \textit{grad}\ \tilde{\eta}|_{\mathbb{L}^2}, \tag{8.8}$$

where

$$\widetilde{M}_n = \sup_{b \in \hat{C}_n} \ \sup_{0 \neq h \in \mathcal{E}_n} \frac{|P^\perp(\frac{h}{b^2} \ \textit{grad}\ u_b)|_{\mathbb{L}^2}}{|P(\frac{h}{b^2} \ \textit{grad}\ u_b)|_{\mathbb{L}^2}}.$$

Proof. The proof of (8.8) is similar to that of Proposition 4.1 and Theorem 5.1. It is based on the decomposition of $\frac{h}{b^2}$ grad u_b on G and $G^\perp$ with a non - zero component on G due to $(\tilde{H})$. □

Remark 8.1 *When $(\tilde{H})$ is satisfied by virtue of Theorem 2.1 one has, analogous to Theorem 5.2:*

$$\widetilde{M}_n \leq \sup_{b\in\hat{C}_n} \sup_{0\neq h\in\mathcal{E}_n} \widetilde{M}_n(b,h),$$

where,

$$\widetilde{M}_n(b,h) = 2\left(\frac{b_M}{b_m}\right)^{1/2} \frac{|grad\ (\frac{h}{b}u_b)|_{\mathbb{L}^2}}{|\frac{h}{b}\ grad\ u_b|_{\mathbb{L}^2}} \min\left\{1, C_W\ \frac{|\vec{rot}\ \frac{h}{b^2}\cdot\ grad\ u_b|}{|\frac{h}{b^2}\ grad\ u_b|_{\mathbb{L}^2}}\right\}. \quad \square$$

Theorem 8.2 *Let (8.1)–(8.3) hold. Then for every $b \in \widetilde{C}$ and $h \in \mathcal{E}$ the following estimates hold:*

$$|\frac{1}{b^{1/2}}\ grad\ \tilde{\eta}|_{\mathbb{L}^2}\ \leq\ |\frac{h}{b^{3/2}}\ grad\ u_b|_{\mathbb{L}^2}, \tag{8.9}$$

$$|\frac{1}{b^{1/2}}\ grad\ \tilde{\xi}|_{\mathbb{L}^2}\ \leq\ 2\,|\frac{h}{b}|_{L^\infty}|\frac{h}{b^{3/2}}\ grad\ u_b|_{\mathbb{L}^2}, \tag{8.10}$$

$$|\frac{1}{b^{1/2}}\ grad\ \tilde{\xi}|_{\mathbb{L}^2}\ \leq\ \frac{K_c}{b_m}\ |b^{1/2}\vec{rot}\ \frac{h}{b}|_{\mathbb{L}^2}\ |\vec{rot}\ \frac{h}{b}\cdot\ grad\ u_b|, \tag{8.11}$$

where K_c depends on Ω, b_m and b_M.

Proof. Estimate (8.10) follows from (8.6) with $v = \tilde{\eta}$ and the Cauchy–Schwarz inequality. We turn to (8.11). From (8.6) and (8.7) we notice, using the decomposition $\mathbb{L}^2(\Omega) = G \oplus G^\perp$ that

$$P\left(\frac{1}{b}\ \mathrm{grad}\ \tilde{\eta} - \frac{h}{b^2}\ \mathrm{grad}\ u_b\right) = 0,$$

and

$$P\left(\frac{1}{b}\ \mathrm{grad}\ \tilde{\xi} - 2\ \frac{h}{b}\ \left(\frac{1}{b}\ \mathrm{grad}\ \tilde{\eta} - \frac{h}{b^2}\ \mathrm{grad}\ u_b\right)\right) = 0.$$

Hence there exist ψ_η and ψ_ξ in W such that:

$$\tfrac{1}{b}\ \mathrm{grad}\ \tilde{\eta} - \tfrac{h}{b^2}\ \mathrm{grad}\ u_b = \vec{\mathrm{rot}}\ \psi_\eta,$$

$$\tfrac{1}{b}\ \mathrm{grad}\ \tilde{\xi} - 2\ \tfrac{h}{b}\ \mathrm{rot}\ \psi_\eta = \vec{\mathrm{rot}}\ \psi_\xi.$$

Using Proposition 3.1 equation (8.7) for $\tilde{\xi}$ can be expressed as

$$\int_\Omega \frac{1}{b}\ \mathrm{grad}\ \tilde{\xi}\cdot\ \mathrm{grad}\ v = 2\int_\Omega \frac{h}{b}\ \vec{\mathrm{rot}}\ \psi_\eta\cdot\ \mathrm{grad}\ v, \quad \text{for all}\ \ v\in V, \tag{8.12}$$

where ψ_η is given by

$$\int_\Omega b\ \vec{\mathrm{rot}}\ \psi_\eta\cdot\ \mathrm{rot}\ w = \int_\Omega \frac{h}{b}\ \mathrm{grad}\ u_b\cdot\ \vec{\mathrm{rot}}\ w \ \ \text{for all}\ \ w\in V. \tag{8.13}$$

Choosing $v = \tilde{\xi}$ in (8.12) and $w = \psi_\eta$ in (8.13) implies

$$|\frac{1}{\sqrt{b}} \text{ grad } \tilde{\xi}|_{\mathbb{L}^2} \leq 2\, |\frac{h}{b}|_{L^\infty}\ |\sqrt{b}\ \vec{\text{rot}}\ \psi_\eta|_{\mathbb{L}^2},$$

and

$$|\sqrt{b}\ \vec{\text{rot}}\ \psi_\eta|_{\mathbb{L}^2} \leq |\frac{h}{b^{3/2}} \text{ grad } u_b|_{\mathbb{L}^2},$$

which implies (8.9). Using Green's formula on the right hand sides of (8.12), (8.13) we obtain

$$\int_\Omega \frac{1}{b} \text{ grad } \tilde{\xi} \cdot \text{ grad } v = -2 \int_\Omega \psi_\eta\ \vec{\text{rot}}\ \frac{h}{b} \cdot \text{ grad } v, \quad \text{for all } \ v \in V, \tag{8.14}$$

$$\int_\Omega b\ \vec{\text{rot}}\ \psi_\eta \cdot \vec{\text{rot}}\ w = - \int_\Omega \vec{\text{rot}}\ \frac{h}{b} \cdot \text{ grad } u_b\ w, \quad \text{for all } \ w \in W, \tag{8.15}$$

The term $\vec{\text{rot}}\ \frac{h}{b} \cdot$ grad u_b on the right hand side of (8.15) is in $L^2(\Omega)$. Due to the regularity hypotheses on Ω and b, the solution ψ_η of (8.15) is in $H^2(\Omega) \subset C(\bar{\Omega})$. Hence there exists a constant K_c depending only on Ω, b_m and b_M such that

$$2\, b_m |\psi_\eta|_{L^\infty} \ \leq\ K_c |\vec{\text{rot}}\ \frac{h}{b} \cdot \ \text{grad } u_b|_{L^2}. \tag{8.16}$$

Setting $v = \tilde{\xi}$ in (8.14) implies

$$|\frac{1}{\sqrt{b}} \text{ grad } \tilde{\xi}|_{\mathbb{L}^2} \ \leq\ 2\, |\psi_\eta|_{L^\infty} |\sqrt{b}\ \vec{\text{rot}}\ \frac{h}{b}|_{\mathbb{L}^2},$$

which, combined with (8.16) proves (8.11). □

Remark 8.2 The estimates (8.9)–(8.11) are valid in infinite dimensions. Moreover (8.11) shows that | grad $\tilde{\xi}|_{\mathbb{L}^2}$ tends to zero

- like $|\text{grad } \frac{h}{b}|^2$ when the coefficients tend to be constants
- like $|\vec{\text{rot}}\ \frac{h}{b} \cdot \text{ grad } u_b|$ when the $\frac{1}{b}$–weighted perturbation h tends to be orthogonal to the flow lines of u_b.

These observations are a consequence of the fact that $b \to u_b$ is linear for constant b. □

Combining Theorems 8.1 and 8.2 with finite dimensionality of $\widetilde{C}_n$ the following sensitivity, deflection and curvature estimates are obtained.

Theorem 8.3 *Assume that (8.1)–(8.3) and $(\widetilde{H})$ hold. Then for every $n \in \mathbb{N}$ with $b_0, b_1 \in \widetilde{C}_n$ and $t \in [0,1]$, estimates (6.5)–(6.7) hold with*

$$\tilde{\alpha}_m = \frac{b_m\ \tilde{\gamma}_{m,n}}{b_M^2 (1 + \widetilde{M}_n^2)^{1/2}}, \quad \tilde{\alpha}_M = \left(\frac{b_M}{b_m}\right)^{1/2} \frac{\gamma_M}{b_m}, \tag{8.17}$$

$$\widetilde{\Theta} = 2\ \frac{diam_\infty \widetilde{C}_n}{b_m^2}\ b_M \delta_n\ (1+\widetilde{M}_n^2)^{1/2}, \tag{8.18}$$

where $0 = \delta_0 \le \delta_1 \le \cdots \le 1$,

$$\frac{1}{\widetilde{R}} = 2\,\widetilde{K}_n \quad \frac{1+\widetilde{M}_n^2}{\tilde{\gamma}_{m,n}} \quad \left(\frac{b_M}{b_m}\right)^2 \ \mu_n, \tag{8.19}$$

where $0 = \mu_0 \le \mu_1 \le \cdots \le 1$,

$$\tilde{\gamma}_{m,n} = \inf_{b\in\widetilde{C}_n} \quad \inf_{h\in\mathcal{E}_n, |\frac{h}{b}|_{L^2}=1} \ |\frac{h}{b}\ grad\ u_b|_{\mathbb{L}^2} > 0,$$

and

$$\widetilde{K}_n = \sup_{b\in\widetilde{C}_n} \quad \sup_{h\in\mathcal{E}_n, |\frac{h}{b}|_{L^2}=1} \ |\frac{h}{b}|_{L^\infty}.$$

Proof. Recall the definition of γ_M in (2.8), and compare the definitions of $\gamma_{m,n}, K_n$ in (6.11), (6.12). Theorem 8.1 implies (8.17). Hypothesis $(\widetilde{H})$ ensures that $P\left(\frac{h}{b^2}\ \text{grad}\ u_b\right)$ and hence also $\frac{h}{b}$ grad u_b are nonzero if $h \neq 0$. Consequently $\tilde{\gamma}_{m,n}$ is positive. To verify (8.18) we deduce from (8.10) and (8.11) that

$$\begin{aligned}|\text{grad}\ \tilde{\xi}|_{\mathbb{L}^2} \le 2\,b_M \quad \min\Big\{ & |\tfrac{h}{b}|_{L^\infty} |\tfrac{h}{b^2}\ \text{grad}\ u_{b}|_{\mathbb{L}^2}\ , \\ & \tfrac{K_c}{b_m}\ |\vec{\text{rot}}\ \tfrac{h}{b}|_{\mathbb{L}^2} |\vec{\text{rot}}\ \tfrac{h}{b}\ \text{grad}\ u_b|_{\mathbb{L}^2} \Big\},\end{aligned} \tag{8.20}$$

which, together with (8.9) implies (8.19), by setting

$$\delta_n = \min\left\{1, \frac{b_M K_c}{\text{diam}_\infty \widetilde{C}_n} \quad \sup\ \frac{|\text{grad}\ \frac{h}{b}|_{\mathbb{L}^2} |\vec{\text{rot}}\ \frac{h}{b}\cdot\ \text{grad}\ u_b|}{|\frac{h}{b}\ \text{grad}\ u_b|_{\mathbb{L}^2}}\right\},$$

where the sup is taken over $b_0, b_1 \in \widetilde{C}_n$, $t\in[0,1]$, $h = b_1 - b_0$, and $b = (1-t)b_0 + t\,b_1$. Using the definitions of $\tilde{\gamma}_{m,n}$ and $\widetilde{K}_n$ we obtain from (8.20):

$$\begin{aligned}|\text{grad}\ \xi|_{\mathbb{L}^2} \le 2\,b_M \quad \min\Big\{ & \tfrac{\widetilde{K}_n}{\tilde{\gamma}_{m,n}} |\tfrac{h}{b}\ \text{grad}\ u_b|\ |\tfrac{h}{b^2}\ \text{grad}\ u_b|\ , \\ & \tfrac{K_c}{b_m}\ |\text{grad}\ \tfrac{h}{b}|\ |\text{rot}\ \tfrac{h}{b}\cdot\ \text{grad}\ u_b| \Big\},\end{aligned}$$

and by (8.8)

$$|\text{grad}\ \xi|_{\mathbb{L}^2} \le \quad 2\,\widetilde{K}_n\ \tfrac{1+\widetilde{M}_n^2}{\tilde{\gamma}_{m,n}}\ \left(\tfrac{b_M}{b_m}\right)^2 \min\left\{1, \tfrac{b_M}{b_m}\,K_c\ \tfrac{\tilde{\gamma}_{m,n}}{\widetilde{K}_n}\ \frac{|\text{grad}\ \frac{h}{b}|\,|\text{rot}\ \frac{h}{b}\cdot\ \text{grad}\ u_b|}{|\frac{h}{b}\ \text{grad}\ u_b|^2}\right\}\ |\text{grad}\ \tilde{\eta}|^2,$$

which implies (8.19) with

$$\mu_n = \min\left\{1,\ \frac{b_M}{b_m}\ \frac{K_c\,\tilde\gamma_{m,n}}{\widetilde K_n}\ \sup\ \frac{|\text{grad}\ \frac{h}{b}|\,|\vec{\text{rot}}\ \frac{h}{b}\cdot\ \text{grad}\ u_b|^2}{|\frac{h}{b}\ \text{grad}\ u_b|^2}\right\},$$

where the sup is taken over $b_0, b_1 \in \widetilde C_n$, and $t \in [0,1]$. □

Let us compare Theorem 6.1 and Theorem 8.3 to assess the relative advantages of the $b = \frac{1}{a}$ and the a parametrizations. Though the definitions are slightly different we shall suppose that the constants $\widetilde M_n, \tilde\gamma_{m,n}, \widetilde K_n$ and $M_n, \gamma_{m,n}, K_n$ behave similarly as functions of n. Then we see that:

- The two parametrizations have the same behavior with respect to sensitivity :

  $\tilde\alpha_m$ and $\alpha_m \to 0$ at similar rates when $n \to \infty$, $\tilde\alpha_M$ and α_M are both bounded with respect to n.

- At coarse scales, the b–parametrization is advantageous for deflection and curvature:

  $\widetilde\Theta$ and $\frac{1}{\widetilde R} \to 0$ for $n \to 0$.

  This reflects the fact that the problem becomes "less nonlinear" for the b–parametrization as the scale gets coarser.

- At fine scales the a–parametrization is advantageous as

  Θ remains bounded whereas $\widetilde\Theta \to \infty$ with $n \to \infty$,

  for fixed value of $\text{diam}_\infty C$;

  $\frac{1/\widetilde R}{1/R} \sim \left(1 + \widetilde M_n^2\right)^{1/2} \to \infty$ with $n \to \infty$.

The fact that $\widetilde\Theta \to \infty$ with $n \to \infty$ is a big drawback as this will require to deduce the size of $\widetilde C_n$ when $n \to \infty$, if one wants to ensure the Q–well posedness over $\widetilde C_n$ for the b–parametrization. Of course, $\widetilde\Theta$ is only an upper bound to the maximum deflection. We do not know if the maximum deflection over $\widetilde C_n$ actually tends to infinity with the scale n.

Appendix

Proof of Proposition 3.1

Step 1. Following [GR], Chapter 1 we define

$$H = \{\vec q \in \mathbb{L}^2(\Omega):\ \text{div}\ \vec q = 0, \vec q\cdot n|\Gamma_N = 0, \int_{\Gamma_i} \vec q\cdot n = 0\},$$

where div is understood in the variational sense. As in [GR] one argues that H is a closed subspace of $\mathbb{L}^2(\Omega)$, and hence we have the decomposition

(A.1) $$\mathbb{L}^2(\Omega) = H \oplus H^{\perp}.$$

Step 2. We argue that $H^{\perp} = G$. Since G is closed in $\mathbb{L}^2(\Omega)$ it suffices to show that $H = G^{\perp}$. For this purpose choose and fix $\vec{q} \in H$ arbitrarily. Then for every $\varphi \in V$ we have $(\vec{q}, \nabla\varphi) = \sum_{i=1}^{N} \varphi_i \int_{\Gamma_i} \vec{q} \cdot n = 0$ and hence $\vec{q} \in G^{\perp}$. Conversely if $\vec{q} \in G^{\perp}$, then $(\vec{q}, \nabla\varphi) = 0$ for all $\varphi \in V$, in particular for all $\varphi \in \mathcal{D}(\Omega)$ and hence div $\vec{q} = 0$. Chossing $\varphi \in V$ implies $\vec{q} \cdot n|\Gamma_N = 0$ and $\int_{\Gamma_i} \vec{q} \cdot n = 0$, for $i = 1, \cdots, N$. Hence $\vec{q} \in H$ and $H^{\perp} = G$.

Step 3. We show that $H = \{ \vec{\text{rot}}\, \psi : \psi \in W\}$. For $\psi \in W$ we have div $\vec{\text{rot}}\, \psi$ in the variational sense and $\int_{\Gamma_i} \vec{\text{rot}}\, \psi \cdot n = \int_{\Gamma_i} \nabla\psi \cdot \tau = 0$, where τ denotes the tangent to Γ_i. Moreover $\vec{\text{rot}}\, \psi \cdot n = \nabla\psi \cdot \tau = 0$ on Γ_N and hence $\vec{\text{rot}}\, \psi \in H$. Conversely, if $\vec{q} \in H$, then by the arguments in [GR], pg. 36 there exists $\psi \in H^1(\Omega)$ such that $\vec{\text{rot}}\, \psi = \vec{q}$. Since $\vec{q} \cdot n = \vec{\text{rot}}\, \psi \cdot n = \nabla\psi \cdot \tau = 0$ on Γ_N it follows that $\psi =$ const a.e. on Γ_N and without loss of generality we may take this constant to be 0. Hence $\vec{\text{rot}}\, \psi = \vec{q}$ with $\psi \in W$.

Step 4. Let $\vec{q} \in \mathbb{L}^2(\Omega)$. Then the elliptic problem

(A.2) $$(\text{grad}\, \varphi,\ \text{grad}\, v) = (\vec{q},\ \text{grad}\, v) \ \text{ for all } \ v \in V$$

has a unique solution in V. Consider $\vec{q} -$ grad $\varphi \in \mathbb{L}^2(\Omega)$, and note that div $(\vec{q} - \text{grad}\, \varphi) = 0$. Moreover $(\vec{q} - \text{grad}\, \varphi) \cdot n|\Gamma_N = 0$ and $\int_{\Gamma_i} (\vec{q} - \text{grad}\, \varphi) \cdot n = 0$, for $i = 1, \cdots, M$. It follows that $\vec{q} -$ grad $\varphi \in H$ and hence there exists $\psi \in W$ such that $\vec{\text{rot}}\, \psi = \vec{q} -$ grad φ in H. Consequently

$$(\vec{\text{rot}}\, \psi,\ \vec{\text{rot}}\, v) = (\vec{q} - \text{grad}\, \varphi,\ \vec{\text{rot}}\, v) \ \text{ for all } \ v \in W,$$

and utilizing div $\varphi = 0$ and boundary conditions for v and φ

(A.3) $$(\vec{\text{rot}}\, \psi,\ \vec{\text{rot}}\, v) = (\vec{q},\ \vec{\text{rot}}\, v), \quad \text{for all } \ v \in W.$$

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References

[C1] G. Chavent: New size x curvature conditions for strict quasi convexity of sets, SIAM J. on Control and Optimization.

[C2] G. Chavent: Quasi convex sets and size x curvature condition, application to nonlinear inversion, J. on Applied Mathematics and Optimization, **24** (1991), 129–169.

[CK1] G. Chavent and K. Kunisch: A geometric theory for the L^2–stability of the inverse problem in a 1–D elliptic equation from an H^1–observation, Appl. Math. and Optim. **27** (1993), 231–260.

[CK2] G. Chavent and K. Kunisch: State-space regularization: geometric theory: Appl. Math. and Optim. 37 (1998), 243-267.

[CL] G. Chavent and J. Liu: Multiscale parametrization for the estimation of a diffusion coefficient in elliptic and parabolic problems, in Fifth IFAC Symposium on Control of Distributed Parameter Systems, Perpignan, France, 1989.

[CG] C. Chicone and J. Gerlach: A note on the identifiability of distributed parameters in elliptic systems, SIAM J.Math.Anal., **18** (1987), 13781–384.

[GR] V. Girault and P. A. Raviart: Finite Element Methods for Navier–Stokes Equations, Springer–Verlag, Berlin, 1979.

[GKMN] A. Grimstad, K. Kolltveit, T. Mannseth and J. Nordtvedt: Assessing the validity of a linearized error analysis for a nonlinear parameter estimation problem, preprint.

[GM] A. Grimstad and T. Mannseth: Nonlinearity, scale, and sensitivity for parameter estimation problems, preprint.

[I] V. Isakov: Inverse Problems for Partial Differential Equations, Springer–Verlag, Berlin, 1998.

[IK] K. Ito and K. Kunisch: On the injectivity and linearization of the coefficient to solution mapping for elliptic boundary value problems, J. Math. Anal. and Appl. **188** (1994), 1040–1066.

[L] J. Liu: A multiresolution method for distributed parameter estimation, SIAM J. on Scientific and Stat. Comp. 14 (1993), 389-405.

[R] G. R. Richter: An inverse problem for the steady state diffusion equation, SIAM J. Appl. Math. **4** (1981), 210–221.

[T] G. M. Troianiello: Elliptic Differential Equations and Obstacle Problems, Plenum Press, New York, 1987.

[AM] G. Alessandrini and R. Magnanini: Elliptic Equations in Divergence Form, Geometric Critical Points of Solutions, and Stekloff Eigenfunctions, Siam J. Math. Anal. **25-5** (1994), 1259-1268

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