Discrete Superstrategies

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Abstract: We study the minimal initial capital needed to super-replicate an European contingent claim in the Black-Scholes model in the following 'real' context: the hedger of the option will only trade at stopping times (which he may freely choose as the hedge ratios). In case the number of trading dates is fixed, we show that this capital corresponds to the buy-and-hold strategy (for a Call option, or the corresponding strategy for any option with a continuous payoff). In case the number may depend on the path of the underlying, we show that if the Black-Scholes delta of the contingent claim is itself a finite-variation process (which excludes standard options in general), this initial capital is the Black-Scholes price of the option. In other cases, e.g. standard options, even for the Call option, the question remains open.

Key-words: Discrete hedging, Black-Scholes model, Option pricing, Superhedging strategies, Stochastic integral.

(Résumé : tsvp)

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Sur-stratégies discrètes.

Résumé : On étudie la plus petite richesse initiale nécessaire pour surcouvrir une option Européenne dans le modèle de Black-Scholes dans le contexte réel suivant: le market-maker ne peut se couvrir qu'à des instants aléatoires de son choix. Dans le cas où le nombre de couverture est fixé, on montre que ce prix correspond à la stratégie buy-and-hold (pour un Call, ou la stratégie correspondante pour toute option avec un payoff continue). Dans le cas où le nombre peut dépendre de la trajectoire du spot et que le delta de l'option de Black-Scholes de l'actif contingent est un processus à variation finie (ce qui exclut toutes les options standards en général), on montre que le plus petit prix est le prix de Black-Scholes de l'option. Dans les autres cas, la question reste ouverte.

Mots-clé : Couverture discrète, Modèle de Black-Scholes, Evaluation d'options, Sur-stratégies, Intégrale stochastique.

1 Introduction

In the Black-Scholes option pricing model, hedging must occur continuously in time to have perfect replication of the claim. For obvious reasons, this is not realistic from a practical point of view and the transaction costs which result from rebalancing can be substantial. Therefore, many searchers have suggested a trading strategy based on a periodic revision of the portfolio. In this setting, Leland (in a market with transaction costs) used the Black-Scholes delta with an adjusted volatility and derived the Black-Scholes price when the level of transaction costs decreases to zero as the revision interval tends to zero. Let us cite also among others Zhang [6], Gobet and Temam [2], who have studied the variance of the replication error, and have evaluated the variance when the time number of rebalancing goes to infinity.

However, it does not appear reasonnable to hedge at deterministic times without consideration of the moves of the spot price. The problem of selecting the best hedging times and ratios for a quadratic criterion given a fixed number of trading times is solved in [3].

In this paper, we restrict ourself to the following subspace of the stochastic integrals:

$$E_\infty = \left\{ \begin{array}{c} X/\ \exists\,(0 = T_0 \leq T_1 \leq T_2 \leq \ldots \leq T_n \leq \ldots) \quad \text{an increasing} \\ \text{sequence of stopping times such that } T_n \uparrow \infty \text{ and} \\ (\Delta_{T_0}, \Delta_{T_1}, \Delta_{T_2}, \ldots, \Delta_{T_n}, \ldots)\ (\Delta_{T_i}\ \mathcal{F}_{T_i}\text{-measurable } \forall i) \\ \text{the corresponding hedging ratios such that } \forall t \geq 0 \\ \sum_{i=0}^{\infty} \Delta_{T_i} \left(S_{T_{i+1} \wedge t} - S_{T_i \wedge t} \right) = X_t \quad \text{a.s.} \end{array} \right\}$$

Here, S denotes the underlying value. To simplify the exposition we work throughout the paper in the context of the one-dimensional Black-Scholes model with no interest rate, specified moreover for commodity under the risk neutral probability, that is:

$$dS_t = \sigma S_t dB_t$$

with $S_0 > 0$ an almost surely, B a one dimensional standard Brownian motion. Since we deal with an almost sure criterion it is not a restriction to work under the risk neutral probability which we shall denote P^*, or whatever probability equivalent to P, for instance the one under which $S_t = e^{\sigma B_t}$ (cf section 3.3).

Now, given a European contingent claim with maturity T, say H_T (with $H_T \geq 0$ a.s. and $\mathbb{E}^{P^*}[H_T] < \infty$), the least value of the super-hedging strategies is defined in a natural way by:

$$U_{\infty,H_T} = \inf \left\{ \begin{array}{c} x > 0/\ \exists X \in E_\infty \\ \text{such that } x + X_T \geq H_T \quad \text{a.s.} \end{array} \right\}$$

We also define U_{n,H_T} in the same way but with a fixed number $n \geq 1$ of hedges.

$$U_{n,H_T} = \inf \left\{ \begin{array}{c} x > 0/\ \exists\, (0 = T_0 \leq T_1 \leq T_2 \leq \ldots \leq T_n \leq T_{n+1} = T) \\ \text{stopping times and } (\Delta_{T_0}, \Delta_{T_1}, \Delta_{T_2}, \ldots, \Delta_{T_n}) \\ (\Delta_{T_i}\ \mathcal{F}_{T_i}\text{-measurable } \forall\ 0 \leq i \leq n) \\ \text{the corresponding hedging ratios such that} \\ x + \sum\limits_{i=0}^{n} \Delta_{T_i} \left(S_{T_{i+1}} - S_{T_i} \right) \geq H_T \quad \text{a.s.} \end{array} \right\}$$

The right question seems to be now: what is the value of U_{∞,H_T}, or in other words, is it true that U_{∞,H_T} is equal to the Black-Scholes price $\mathbb{E}^{P^*}[H_T]$ of the option? The difficulty comes from the almost sure demand on the super replication: even if it is obvious that the value at time T of any continuous stochastic integral which is in addition a martingale may be approximated to an arbitrary level of accuracy by the natural finite Riemann sum, this approximation is usually in a weak sense, e.g. L^2 so that typically the approximating sequence may oscillate around the asymptotic limit.

In fact, even for a Call option, we did not manage to answer to the above question. Nevertheless, in this paper, we show the following:

- If the number of stopping times is fixed in advance, then for standard options with a continuous payoff, $U_{n,g(S_T)} = \widehat{g}(S_0)$ where $\widehat{g}$ is the least concave function above g, or in other words $U_{n,g(S_T)} = U_{1,g(S_T)}$ since obviously $U_{1,g(S_T)} = \widehat{g}(S_0)$. This is done in section 2.

- If H_T writes $\mathbb{E}^{P^*}[H_T] + \int_0^T A_t dS_t$ where A is a finite-variation process which is absolutely continuous, then $U_{\infty,H_T} = \mathbb{E}^{P^*}[H_T]$. This result is not surprising since in such a case the stochastic integral $\int_0^T A_t dS_t$ may be defined pathwise as a Lebesgue-Stieltjes integral. This is done in section 3.

2 Super-strategies with n interventions

In this section, we consider European contingent claims defined by a terminal payoff $g(S_T)$, where g is a nonnegative continuous function. We fix the number n of trading dates and we consider $U_{n,g(S_T)}$.
Notice first:

Lemma 1 $U_{1,g(S_T)} = \widehat{g}\,(S_0)$ *where* $\widehat{g}$ *is the least concave function greater than* g.

Proof. Indeed one has:

$$\widehat{g}(S_0) = \inf \left\{ \begin{array}{c} x > 0/\ \exists \delta \in \mathbb{R} \text{ such that} \\ \forall z \in \mathbb{R}_+^* \quad x + \delta(z - S_0) \geq g(z) \end{array} \right\}$$

Now after the continuity of g and the fact that the law of S_T weights every open set of $\mathbb{R}_+^*$ this is equivalent to

$$\widehat{g}(S_0) = \inf \left\{ \begin{array}{c} x > 0/\ \exists \delta \in \mathbb{R} \text{ such that} \\ x + \delta(S_T - S_0) \geq g(S_T) \text{ a.s.} \end{array} \right\}$$

whence the result. $\square$

Theorem 2 $\forall n \in \mathbb{N}^*, \quad U_{n,g(S_T)} = U_{1,g(S_T)} = \widehat{g}\,(S_0)$.

We need the following lemma which is a direct consequence of the fact that S is a continuous process which satisfies

$$P(T_\epsilon > \lambda) > 0$$

for any $\lambda > 0$, where for $\epsilon > 0$, T_ϵ is defined by $T_\epsilon = \inf\{u/|S_u - S_0| > \epsilon\}$.

Lemma 3 $\forall \varepsilon > 0,\ P\left(S_\tau \in]S_0 - \varepsilon, S_0 + \varepsilon[\right) > 0$ *for any bounded stopping time* τ .

Proof of the theorem. The proof is by induction over n. Suppose that $U_{n,g(S_T)} = \widehat{g}\,(S_0)$, let's show that $U_{n+1,g(S_T)} = \widehat{g}\,(S_0)$.

Obviously, since $m \mapsto U_{m,g(S_T)}$ is nonincreasing,

$$\widehat{g}\,(S_0) \geq U_{n+1,g(S_T)}.$$

Let x an initial capital, $(T_0 = 0 \leq T_1 \leq T_2 \leq \ldots \leq T_{n+2} = T)$ stopping times and $\left(\Delta_{T_0}, \Delta_{T_1}, \Delta_{T_2}, ..., \Delta_{T_{n+1}}\right)$ the corresponding adapted hedging ratios, such that

$$x + \sum_{i=0}^{n+1} \Delta_{T_i} \left(S_{T_{i+1}} - S_{T_i}\right) \geq g(S_T) \quad \text{a.s.}$$

or yet

$$C_{T_1} + \sum_{i=1}^{n+1} \Delta_{T_i} \left(S_{T_{i+1}} - S_{T_i}\right) \geq g(S_{T_1}\frac{S_T}{S_{T_1}}) \quad \text{a.s.}$$

where $C_{T_1} = x + \Delta_0\,(S_{T_1} - S_0)$.

Define

$$M_t = \frac{S_t}{S_0} = \exp\left\{-\frac{\sigma^2}{2}t + \sigma B_t\right\}$$

By the strong Markov property, the conditional law of $\frac{S_T}{S_{T_1}}$ with respect to $\mathcal{F}_{T_1}$ is that of M_{T-T_1}. Then using the recurrence hypothesis we obtain

$$x + \Delta_0\,(S_{T_1} - S_0) \geq \widehat{g}(S_{T_1}) \text{ a.s.}$$

Since $\widehat{g}$ is continuous, we deduce from lemma 3 that for any $\varepsilon > 0$ small enough

$$\forall y \in\,]S_0 - \varepsilon, S_0 + \varepsilon[\,, \quad x + \Delta_0\,(y - S_0) \geq \widehat{g}(y)$$

whence $x \geq \widehat{g}(S_0)$ which gives $U_{n+1} \geq \widehat{g}(S_0)$. The proof is complete. $\square$

In the case of a European Call option, the price $U_{n,g(S_T)}$ corresponds to the cost of the 'buy-and-hold' strategy of acquiring one share of the stock at $t = 0$ and holding it until $t = T$.

3 Approximation of Stochastic Integral with Finite Variation Integrand

In this section, we assume that the number n of trading dates may depend on the path of the underlying. We give an approximation of the stochastic integral $\int_0^t A_u dB_u$ by elementary stochastic integrals of the form

$$X_t = \sum_{i=0}^{\infty} A_{T_i} \left(B_{T_{i+1} \wedge t} - B_{T_i \wedge t} \right)$$

when A is a finite variation process, $(T_0 \leq T_1 \leq \ldots \leq T_i \leq \ldots)$ a sequence of stopping times, B a one-dimensional standard Brownian motion. Then we extend this approximation to the Black-Scholes model.

3.1 Absolutely continuous integrand with bounded derivative

Let A be an absolutely continuous process, i.e.

$$A_r = A_0 + \int_0^r V_u du \quad , \quad \forall r \in [0, t]$$

where V is an adapted process.

We suppose in this section that A is of bounded derivative, i.e.

$$|V_.| < M_V < \infty \text{ a.s}$$

where M_V is a positive constant.

Our approximation procedure is divided in 3 parts.

(a) We have the simple integration by parts formula

$$\int_{T_i}^{T_{i+1}} A_u dB_u = A_{T_i} \left(B_{T_{i+1}} - B_{T_i} \right) + \int_{T_i}^{T_{i+1}} V_r \left(B_{T_{i+1}} - B_r \right) dr$$

Set

$$D_i \triangleq \int_{T_i}^{T_{i+1}} A_u dB_u - A_{T_i}\left(B_{T_{i+1}} - B_{T_i}\right) = \int_{T_i}^{T_{i+1}} V_r\left(B_{T_{i+1}} - B_r\right) dr$$

then

$$|D_i| \leq M_V \int_{T_i}^{T_{i+1}} \left|B_{T_{i+1}} - B_r\right| dr \text{ a.s.}$$

Let $\sum_{i\geq 0} \alpha_i$ a deterministic convergent serie with positive terms. Consider the sequence of stopping times

$$\begin{aligned} &T_0 = 0 \\ &T_{i+1} = \inf\left\{u > 0, \int_{T_i}^{u} |B_u - B_r|\, dr = \alpha_i\right\},\ i \geq 0 \end{aligned}$$

Obviously the random variables $\eta_i \triangleq (T_{i+1} - T_i)$ are independent, then by using the zero-one law (cf [4]) we have

$$P\left(\sum_{i=0}^{\infty} \eta_i = +\infty\right) \in \{0,1\}$$

(b) If we construct the sequence (α_i) such that

$$P\left(\sum_{i=0}^{\infty} \eta_i = +\infty\right) = 1 \tag{1}$$

we will obtain

$$\begin{aligned} -M_V \sum_{i=0}^{\infty} \alpha_i \quad &+ \quad \sum_{i=0}^{\infty} A_{T_i}\left(B_{T_{i+1}\wedge t} - B_{T_i \wedge t}\right) \\ &\leq \int_0^t A_u dB_u \leq \sum_{i=0}^{\infty} A_{T_i}\left(B_{T_{i+1}\wedge t} - B_{T_i\wedge t}\right) + M_V \sum_{i=0}^{\infty} \alpha_i \quad \text{a.s.} \end{aligned}$$

We then rescale our stopping times like:

$$T_0^\varepsilon = 0$$
$$T_{i+1}^\varepsilon = \inf\left\{ u > 0, \int_{T_i^\varepsilon}^{u} |B_u - B_r|\, dr = \varepsilon\alpha_i \right\},\ \varepsilon > 0$$

By the scaling property of the Brownian motion

$$\forall u > 0,\ \int_{T_i^\varepsilon}^{u} |B_u - B_r|\, dr \overset{(d)}{=} \int_0^{u-T_i^\varepsilon} \left|B_{u-T_i^\varepsilon} - B_r\right| dr$$

For any $a > 0$, we have

$$\int_0^v |B_v - B_r|\, dr \overset{(d)}{=} a^{\frac{3}{2}} \int_0^{\frac{v}{a}} \left|\widetilde{B}_{\frac{v}{a}} - \widetilde{B}_s\right| ds$$

where $\widetilde{B}_. = \frac{1}{\sqrt{a}} B_{a.}$ is a standard Brownian motion. With $a = \varepsilon^{\frac{2}{3}}$, we get

$$T_{i+1}^\varepsilon - T_i^\varepsilon \overset{(d)}{=} \varepsilon^{\frac{2}{3}}(T_{i+1} - T_i)$$

with also

$$P\left(\sum_{i=0}^{\infty} \eta_i^\varepsilon = +\infty\right) = 1 \quad \text{where} \quad \eta_i^\varepsilon \triangleq \left(T_{i+1}^\varepsilon - T_i^\varepsilon\right)$$

thus we get an approximation with an arbitrary accuracy.

(c) It remains to choose (α_i) such that

$$P\left(\sum_{i=0}^{\infty} \eta_i = +\infty\right) = 1$$

holds.

For all $r \in [0, u]$, we have

$$B_u - \sup_{0\le v\le u} B_v \le B_u - B_r \le B_u - \inf_{0\le v\le u} B_v$$

and

$$|B_u - B_r| \leq Z_u \triangleq \max\left(Z_u^i, Z_u^s\right)$$

where

$$Z_u^i \triangleq B_u - \inf_{0\leq v\leq u} B_v \quad \text{and} \quad Z_u^s \triangleq \sup_{0\leq v\leq u} B_v - B_u$$

Set

$$\eta_i^* \triangleq \inf\left\{u > 0, uZ_u = \alpha_i\right\}$$

then $\eta_i \geq \eta_i^*$ a.s.

Let now a positive sequence (β_i). We have by scaling

$$\begin{aligned} P\left(\eta_i^* > \beta_i\right) &= P\left(Z_{\beta_i} < \frac{\alpha_i}{\beta_i}\right) \\ &= P\left(\sqrt{\beta_i} Z_1 < \frac{\alpha_i}{\beta_i}\right) \\ &= P\left(Z_1 < \frac{\alpha_i}{\beta_i^{\frac{3}{2}}}\right) \end{aligned}$$

and consequently, as soon as $\sum_i \beta_i = +\infty$

$$\begin{aligned} P\left(\sum_i \eta_i^* = +\infty\right) &\geq \prod_i P\left(\eta_i^* > \beta_i\right) \\ &= \prod_i P\left(Z_1 < \frac{\alpha_i}{\beta_i^{\frac{3}{2}}}\right) \\ &= \prod_i \left(1 - P\left(Z_1 \geq \frac{\alpha_i}{\beta_i^{\frac{3}{2}}}\right)\right) \end{aligned}$$

To conclude, we have to find the sequences (α_i) and (β_i) such that:

$$
\begin{array}{ll}
(C_1) & \sum_i \alpha_i < +\infty \\
(C_2) & \sum_i \beta_i = +\infty \\
(C_3) & \dfrac{\alpha_i}{\beta_i^{\frac{3}{2}}} \to +\infty \\
(C_4) & \sum_i P\left(Z_1 \geq \dfrac{\alpha_i}{\beta_i^{\frac{3}{2}}}\right) < +\infty
\end{array}
$$

since by a classical result the conditions (C_3) and (C_4) ensure that

$$\prod_i \left(1 - P\left(Z_1 \geq \frac{\alpha_i}{\beta_i^{\frac{3}{2}}}\right)\right) > 0$$

We can write

$$
\begin{aligned}
Z_1 &= \max\left\{\sup_{0\leq s\leq 1}(B_s - B_1), \sup_{0\leq s\leq 1}(B_1 - B_s)\right\} \\
&= \sup_{0\leq s\leq 1}|B_1 - B_s| \\
&= \sup_{0\leq r\leq 1}|B_1 - B_{1-r}| \\
&= \sup_{0\leq r\leq 1}|W_r|
\end{aligned}
$$

where (W_r) is a standard Brownian motion.

Using the inequality

$$P\left(\sup_{0\leq r\leq 1} \mid W_r \mid \geq c\right) \leq 2e^{-\frac{c^2}{2}}$$

we get

$$P\left(Z_1 \geq c\right) \leq 2e^{-\frac{c^2}{2}}$$

Therefore (C_4) is granted as soon as

$$(C_4') \qquad \sum_{i=0}^{\infty} e^{-\frac{\alpha_i^2}{2\beta_i^3}} < +\infty$$

holds.

It is now easy to find two sequences (α_i) ,(β_i) satisfying the conditions $(C_1), (C_2), (C_3)$ and (C_4'). We can choose for example

$$\alpha_i = \frac{1}{i^{\frac{5}{4}}} \quad \text{and} \quad \beta_i = \frac{1}{i}, \ \forall\, i \geq 1$$

3.2 Absolutely continuous integrand with unbounded derivative

Suppose now A is a stochastic process with possibly unbounded derivative, which nevertheless satisfies

$$\sup_{0\leq u\leq T} \mid V_u \mid < \infty \text{ a.s.}$$

Then we are going to use results of 3.1 thanks to a localization procedure.

Let

$$V_r^n = (V_r \wedge n) \vee -n \quad \text{and} \quad A_r^n = A_0 + \int_0^r V_u^n du, \qquad r \in [0,t]\,, \quad n \in I\!N^* \tag{2}$$

We define the non-decreasing sequence of stopping times

$$\begin{aligned} \theta_0 &= 0 \\ \theta_n &= \inf\{u > 0; |V_u| > n\} \wedge t,\ n \geq 1 \end{aligned} \tag{3}$$

By (2) and (3)

$$\int_0^t A_u dB_u = \sum_{n=1}^{+\infty} \int_{\theta_{n-1}}^{\theta_n} A_u dB_u = \sum_{n=1}^{+\infty} \int_{\theta_{n-1}}^{\theta_n} A_u^n dB_u$$

Since $A^n_\cdot$ is a finite variation process with bounded derivative, we can approximate $\int\limits_{\theta_{n-1}}^{\theta_n} A^n_u dB_u$ as in the previous part.

Let $\varepsilon > 0$ and $(T_i^{\varepsilon,n})_i$ a sequence of stopping times defined by

$$\begin{aligned} T_0^{\varepsilon,n} &= 0 \\ T_{i+1}^{\varepsilon,n} &= \inf\left\{ u > 0;\ \int_{T_i^{\varepsilon,n}}^{u} |B_u - B_r|\, dr = \alpha_i \frac{\varepsilon}{n^3} \right\} \end{aligned}$$

where $\sum \alpha_i$ is a convergent serie satisfying (1).

We have

$$\begin{aligned} - \quad & n\frac{\varepsilon}{n^3} \sum_{i=0}^{\infty} \alpha_i + \sum_{i=0}^{\infty} A^n_{\left(T_i^{\varepsilon,n}\wedge\theta_n\right)\vee\theta_{n-1}} \left(B_{\left(T_{i+1}^{\varepsilon,n}\wedge\theta_n\right)\vee\theta_{n-1}} - B_{\left(T_i^{\varepsilon,n}\wedge\theta_n\right)\vee\theta_{n-1}} \right) \\ \leq \quad & \int\limits_{\theta_{n-1}}^{\theta_n} A^n_u dB_u \\ \leq \quad & \sum_{i=0}^{\infty} A^n_{\left(T_i^{\varepsilon,n}\wedge\theta_n\right)\vee\theta_{n-1}} \left(B_{\left(T_{i+1}^{\varepsilon,n}\wedge\theta_n\right)\vee\theta_{n-1}} - B_{\left(T_i^{\varepsilon,n}\wedge\theta_n\right)\vee\theta_{n-1}} \right) + n\frac{\varepsilon}{n^3} \sum_{i=0}^{\infty} \alpha_i \quad \text{a.s.} \end{aligned}$$

finally we obtain by summing over n

$$\begin{aligned} & -C\ \varepsilon \sum_{i=0}^{\infty} \alpha_i + \sum_{i=0}^{\infty} A_{\widetilde{T}_i^\varepsilon} \left(B_{\widetilde{T}_{i+1}^\varepsilon\wedge t} - B_{\widetilde{T}_i^\varepsilon\wedge t} \right) \\ \leq \ & \int\limits_0^t A_u dB_u \leq \sum_{i=0}^{\infty} A_{\widetilde{T}_i^\varepsilon} \left(B_{\widetilde{T}_{i+1}^\varepsilon\wedge t} - B_{\widetilde{T}_i^\varepsilon\wedge t} \right) +\ C\ \varepsilon \sum_{i=0}^{\infty} \alpha_i \text{ a.s.} \end{aligned}$$

where the $\widetilde{T}_i^\varepsilon$ are the above stopping times obtained by picking the $\left(T_j^{\varepsilon,n} \wedge\ \theta_n\right) \vee \theta_{n-1}$ on the interval $]\theta_{n-1}, \theta_n[$.

Therefore we get an approximation of $\int\limits_0^t A_u dB_u$ with an arbitrary accuracy.

3.3 Approximation of stochastic integral with respect to S

In this part, we replace the Brownian Motion B by the underlying S of the Black-Scholes model and we give an approximation of the stochastic integral $\int_0^t A_u dS_u$ by elementary stochastic integrals of the form

$$X_t = \sum_{i=0}^{\infty} A_{T_i} \left(S_{T_{i+1}\wedge t} - S_{T_i \wedge t}\right)$$

when A is a finite variation process, $(T_0 \leq T_1 \leq \ldots \leq T_i \leq \ldots)$ a sequence of stopping times, S the price of the spot given by

$$S_t = e^{B_t}$$

By the argument of localization used in 3.2, it is enough to handle the case where A is an absolutely continuous process of bounded derivative i.e.

$$|V_.| < M_V < \infty \text{ a.s}$$

where M_V is a positive constant.

(a) We have the simple integration by parts formula

$$\int_{T_i}^{T_{i+1}} A_u dS_u = A_{T_i} \left(S_{T_{i+1}} - S_{T_i}\right) + \int_{T_i}^{T_{i+1}} V_r \left(S_{T_{i+1}} - S_r\right) dr$$

Set

$$D_i \triangleq \int_{T_i}^{T_{i+1}} A_u dS_u - A_{T_i} \left(S_{T_{i+1}} - S_{T_i}\right) = \int_{T_i}^{T_{i+1}} V_r \left(S_{T_{i+1}} - S_r\right) dr$$

then

$$|D_i| \leq M_V \int_{T_i}^{T_{i+1}} \left|S_{T_{i+1}} - S_r\right| dr \text{ a.s.}$$

(b) We define now a non-decreasing sequence of stopping times

$$\begin{aligned}\tau_0 &= 0\\ \tau_{n+1} &= \inf\{t \geq \tau_n / |B_t| \geq n+1\} \wedge t, n \geq 0\end{aligned}$$

We have

$$\int_0^t A_u dS_u = \sum_{n=1}^{+\infty} \int_{\tau_n}^{\tau_{n+1}} A_u dS_u$$

Obviously there exists a constant M_n such that

$$|e^{B_u} - e^{B_r}| \leq M_n |B_u - B_r| \text{ a.s.} \forall u, r \in [\tau_n, \tau_{n+1}]$$

(c) Let $(T_i^n)_i$ a sequence of stopping times defined by

$$\begin{aligned}T_0^n &= 0\\ T_{i+1}^n &= \inf\{u \geq T_i^n / \int_{T_i^n}^{u} |B_u - B_r| dr = \alpha_i^n\}\end{aligned}$$

We have

$$\begin{aligned}&-M_V \sum_{i=0}^{\infty} M_n \alpha_i^n + \sum_{i=0}^{\infty} A_{T_i^n} \left(S_{(T_{i+1}^n \wedge \tau_{n+1}) \vee \tau_n} - S_{(T_i^n \wedge \tau_{n+1}) \vee \tau_n}\right)\\ \leq &\int_{\tau_n}^{\tau_{n+1}} A_u dS_u \leq \sum_{i=0}^{\infty} A_{T_i^n} \left(S_{(T_{i+1}^n \wedge \tau_{n+1}) \vee \tau_n} - S_{(T_i^n \wedge \tau_{n+1}) \vee \tau_n}\right) + M_V \sum_{i=0}^{\infty} M_n \alpha_i^n \quad \text{a.s.}\end{aligned}$$

(d) Finally by choosing $\alpha_i^n = \dfrac{\alpha_i}{M_n} \dfrac{\varepsilon}{n^2}$ and by summing over n, we obtain

$$\begin{aligned}&-C\varepsilon \sum_{i=0}^{\infty} \alpha_i + \sum_{i=0}^{\infty} A_{T_i} \left(S_{(T_{i+1} \wedge t)} - S_{(T_i \wedge t)}\right)\\ \leq &\int_0^t A_u dS_u \leq \sum_{i=0}^{\infty} A_{T_i} \left(S_{(T_{i+1} \wedge t)} - S_{(T_i \wedge t)}\right) + C\varepsilon \sum_{i=0}^{\infty} \alpha_i \quad \text{a.s.}\end{aligned}$$

where the T_i are the above stopping times obtained by picking the $\left(T_j^n \wedge \tau_{n+1}\right) \vee \tau_n$ on the interval $]\tau_n, \tau_{n+1}[$.

Therefore we get an approximation of $\int\limits_0^t A_u dS_u$ with an arbitrary accuracy. As a corollary:

Corollary 4 *If H_T writes $\mathbb{E}^{P^*}[H_T] + \int_0^T A_t dS_t$ where A is a finite-variation process which is absolutely continuous with an almost surely finite derivative, then $U_{\infty,H_T} = \mathbb{E}^{P^*}[H_T]$.*

Remark 5 *In the case of infinite-variation integrand (e.g. standard options), the question remains open. The difficulty is that we require almost sure rather than just approximate hedge. The investor has to be able to super-replicate the claim at maturity whatever the path of the underlying. In this direction, let us state the following question:*

Is there an infinite variation process A such that

$$U_{\infty, c+\int_0^T A_t dB_t} = c \text{ for } c \in \mathbb{R}? \tag{4}$$

Notice that even in the simplest case $(A_t)_{t\geq 0} = (B_t)_{t\geq 0}$, we don't know if 4 holds.

4 Conclusion

We study the minimal initial capital needed to super-replicate some given European contingent claim when the number of trading dates is fixed. We show that the price is the value of the concave envelope of the payoff function at the initial stock price.

We also construct an approximation of the stochastic integral when the integrand is a finite variation process. If the Black-Scholes delta is a finite variation process, the least super-replication cost is the Black-Scholes price.

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