Unbounded Volatility in the Uncertain Volatility Model

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Abstract: We work in the Uncertain Volatility Model setting of Avellaneda, Levy, Paras [1] and Lyons [10] (cf. also [11]). We first look at European options in a market with no interest rate and focus on the extreme case where the volatility has a lower bound but no upper bound. We show that the smallest riskless selling price of the claim is the Black-Scholes price (at volatility given by the lower bound) of an option with payoff the smallest concave function above the initial payoff. We next extend our results to the case with interest rate.

Key-words: European options, Hamilton-Jacobi-Bellman equation, Stochastic control, Superstrategies.

(Résumé : tsvp)

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Volatilité non bornée dans le modèle UVM

Résumé : Dans le cadre du modèle UVM d'Avellaneda, Levy, Paras [1], et Lyons [10] (cf. also [11]), on étudie le pricing d'options européennes dans le cas où la volatilité n'est pas bornée supérieurement. On montre que le plus petit prix de surcouverture est donné par le prix Black-Scholes (à volatilité la borne inférieure du modèle UVM) d'une option de payoff l'enveloppe concave supérieure du payoff initial.

Mots-clé : Contrôle stochastique, Equation d'Hamilton-Jacobi-Bellman, Options Européennes, Surstratégies.

1 Introduction

Avellaneda, Levy and Paras in [1] and Lyons in [10] consider the pricing of derivative securities in the context where the volatility of the underlying is unknown but assumed to lie in a compact interval $I = [\underline{\sigma}, \overline{\sigma}]$. They show that the smallest riskless selling price in an arbitrage-free world with constant interest rate ρ should be

$$\sup_{P \in \mathcal{P}_I} E^P \left[e^{-\rho t} \varphi \left(S_t \right) \right] \tag{1}$$

where $\mathcal{P}_I$ is the set of the risk-neutral probabilities which depends on the interval and $\varphi(S_t)$ the payoff of the option at hand with maturity $t > 0$. We write E^P for the expectation under a probability P.

They show that this quantity is also the solution of the Hamilton-Jacobi-Bellman equation

$$\rho \left(x \frac{\partial u}{\partial x} - u \right) + \sup_{\alpha \in [\underline{\sigma}, \overline{\sigma}]} \frac{1}{2} \alpha^2 x^2 \frac{\partial^2 u}{\partial x^2} = - \frac{\partial u}{\partial s} \tag{2}$$

with terminal condition $u(t, x) = \varphi(x)$, in case φ is smooth enough. In the two papers it is assumed that $\underline{\sigma} > 0$ and $\overline{\sigma} < +\infty$.

In [11], in a zero interest rate market, the second author has settled the case of continuous payoffs thanks on one hand to a sound definition of the selling price and on the other hand to the celebrated work of Krylov on controlled diffusion processes ([9]). In [12], he studied the case $\underline{\sigma} = 0$ and showed that the selling price (1) is actually the price of an American option in the Black-Scholes model at the upper bound of the volatility.

In this paper, we look at the other extreme case where $\overline{\sigma} = +\infty$. Of course, even if the volatility process may take arbitrarily high values, we require some integrability condition in order to define the underlying process in a sound way. By a direct study we show that the value function (1) is the price of a European option in the Black-Scholes model at the lower bound of the volatility $\underline{\sigma}$ with a payoff which is given by the smallest concave function above the initial payoff function. Then, we show that it is actually the smallest riskless selling price of the claim.

In [2], Cvitanic, Pham and Touzi obtain the same result in the context of a stochastic volatility model.
Lastly we extend our results and those of [11], [12] to the non-zero interest rate case.

2 Framework

We assume that the dynamic of the underlying S of the option is of the type:

$$\begin{aligned} S_0 &= x > 0 \\ dS_u &= \rho S_u du + \sigma_u S_u dW_u \ , u \geq 0 \end{aligned} \tag{3}$$

which is defined on a given filtered probability space, with W taken as a standard Brownian motion under some risk-neutral probability. Here ρ is the constant interest rate and the volatility process σ is supposed to be a progressive process with values in $I = [\underline{\sigma}, \overline{\sigma}]$. We then have

$$d \langle S \rangle_u = \sigma_u^2 S_u^2 du$$

By Itô's formula we get

$$S_u = x e^{\rho u} \exp \left\{ \int_0^u \sigma_v dW_v - \frac{1}{2} \int_0^u \sigma_v^2 dv \right\} \ \ a.s.$$

As the volatility process is unknown, and since the filtered probability space is not fixed, we shall deal with the image law of S on the canonical space. Therefore, as in [11], we introduce the following setting:

2.1 Modelling and notations

Let us denote $\Omega = C(\mathbb{R}^+, \mathbb{R}^+_*)$. We define ω_s the coordinate maps on Ω by

$$\omega_s : (x_u)_{u \in \mathbb{R}^+} \rightarrow x_s$$

The coordinate filtration is defined as $\mathcal{F}_u = \sigma(\omega_s, s \leq u)$, and $\mathcal{F} = \vee_u \mathcal{F}_u = \sigma(\omega_s, s \in \mathbb{R}^+)$.

We now consider the laws P on $\left(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\geq 0}\right)$ that satisfy:
(a) $(\omega_u)_{u\geq 0}$ is a P-martingale
(b) $d\langle\omega\rangle_u << du$ and $\sqrt{\frac{d\langle\omega\rangle_u}{\omega_u^2 du}} \in I \; P-a.s.$
(c) $\omega_0 = 1 \; P-a.s.$
Let $\mathcal{P}_I$ be this set of laws. Each law on $\left(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\geq 0}\right)$ of a process $\left(\frac{S_u}{xe^{\rho u}}\right)_{u\geq 0}$, with S satisfying the dynamic (3), also satisfies (a), (b) and (c). Conversely, with a representation theorem, we see that the dynamic of $(xe^{\rho u}\omega_u)_{u\geq 0}$ under a law satisfying (a), (b) and (c) is of type (3). Therefore, $\mathcal{P}_I$ is exactly the set of the laws on $\left(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u\geq 0}\right)$ of the processes $\left(\frac{S_u}{xe^{\rho u}}\right)_{u\geq 0}$ with volatility in I.

We define the family of $\mathcal{P}_I$−null sets by:

$$N^{\mathcal{P}_I} = \{A \in \mathcal{F} \,/\, \forall P \in \mathcal{P}_I, \, \exists B \in \mathcal{F}, \, A \subset B \text{ and } P(B) = 0\}$$

Let us set $(\mathcal{F}_t)_{t\geq 0}^{\mathcal{P}_I} \overset{def}{=} \left(\mathcal{F}_t \vee N^{\mathcal{P}_I}\right)_{t\geq 0}$. If $\Pr og^{\mathcal{P}_I}$ denote the set of progressive processes for $(\mathcal{F}_t)_{t\geq 0}^{\mathcal{P}_I}$, then obviously such processes are progressive for the filtration $(\mathcal{F}_t)_{t\geq 0}^{P}$ for any P.

We finally fix φ as a Borel function defined on $\mathbb{R}^+$, valued in $\mathbb{R}$ which will be the payoff function of the option we consider. Of course in the applications φ should be nonnegative, but we won't have to take care in general of the sign of the payoff function.

Let us set now a classical framework for stochastic control.

2.2 Krylov's setting of stochastic control

We use some notations of [9].

We assume that $(w_t, (\mathcal{F}'_t))$ is a 1-dimensional Brownian motion on the Wiener space. We define U as the space of $(\mathcal{F}'_t)$ −progressive processes with values in $I = [\underline{\sigma}, \overline{\sigma}]$.

We choose to set the following functions:

$$\begin{aligned} \sigma(\alpha,t,x) &= \alpha x \\ b(\alpha,t,x) &= \rho x \\ c(\alpha,t,x) &= \rho \\ f(\alpha,t,x) &= 0 \\ g(x) &= \varphi(x) \end{aligned} \tag{4}$$

The state processes $X^{\rho,\alpha,s,x}$ controlled by the process $\alpha \in U$ is defined by:

$$\begin{aligned} dX_u^{\rho,\alpha,s,x} &= \sigma\left(\alpha_u, s+u, X_u^{\rho,\alpha,s,x}\right) dz_u + b\left(\alpha_u, s+u, X_u^{\rho,\alpha,s,x}\right) du \\ X_0^{\rho,\alpha,s,x} &= x \end{aligned}$$

For $T > 0$ deterministic and $s \leq T$ we define

$$\begin{aligned} v^\alpha(s,x) &= E[\int_0^{T-s} f\left(\alpha_u, s+u, X_u^{\rho,\alpha,s,x}\right) e^{-\int_0^u c\left(\alpha_r, s+r, X_r^{\rho,\alpha,s,x}\right)dr} du \\ &\quad + e^{-\int_0^{T-s} c\left(\alpha_r, s+r, X_r^{\rho,\alpha,s,x}\right)dr} g\left(X_{T-s}^{\rho,\alpha,s,x}\right)] \end{aligned}$$

and

$$v(s,x) = \sup_\alpha v^\alpha(s,x)$$

We recall now the important remark 5.2 of [11].

Remark 1 *Of course here the situation is homogeneous in time, so that* $X_\cdot^{\rho,\alpha,s,x}$ *doesn't depend on s. In the same way* $v_{T_1}^\alpha$ *and* $v_{T_2}^\alpha$ *associated with 2 horizons* T_1 *and* T_2 *satisfy* $v_{T_1}^\alpha(s,x) = v_{T_2}^\alpha(T_2 - (T_1 - s), x)$.

We can set:

$$V^\rho(I,t,\varphi,x) \stackrel{def}{=} v(T-t,x)$$

which is well-defined for $t \leq T$. Then

$$V^\rho(I,t,\varphi,x) = \sup_{\alpha \in U} E\left[e^{-\rho t}\varphi\left(X_t^{\rho,\alpha,0,x}\right)\right]$$

Let us now recall the results of [11], where it is assumed that $\rho = 0$.

3 European options with no interest rate

In this section, we suppose $\rho = 0$ and also:

(A) φ **is continuous and** $x \mapsto \frac{\varphi(x)}{1+x}$ **is bounded on** $]0, +\infty[$**.**

This assumption will allow us to make use of the results of stochastic control theory (see Krylov [9] chapter 3).

3.1 Bounded volatility with $\underline{\sigma} > 0$

We define the minimum riskless selling price $Q(I, t, \varphi, x)$ (or shorter $Q(t, x)$ if there is no risk of confusion about I or φ) of a European option at time 0 by:

$$Q(I, t, \varphi, x) = \inf \left\{ \begin{array}{c} c \in \overline{\mathbb{R}} / \exists L > 0 \ , \ \exists \Delta \in \Pr og^{\mathcal{P}_I} / \forall P \in \mathcal{P}_I : \\ (i) \int_0^t \Delta^2(\omega, u) \, d\langle \omega \rangle_u < \infty \ P \text{ a.s.} \\ (ii) \int_0^{\cdot} \Delta(\omega, u) \, d\omega_u > -L \ P \text{ a.s.} \\ (iii) \quad c + \int_0^t \Delta(\omega, u) \, d\omega_u \geq \varphi(x\omega_t) \ \ P \text{ a.s.} \end{array} \right\} \qquad (5)$$

where t is the maturity and φ the payoff function.

We shortly justify this definition. (i) is a classic integrability condition and can be replaced by $\int_0^t \Delta^2(\omega, u) \, \omega_u^2 du < \infty$ P a.s., since the volatility is bounded; (ii) avoids doubling strategies (see for example [7]); finally (iii) is the super-hedging condition. Those conditions have to be satisfied for every probability law of $\mathcal{P}_I$. Remark that we have dropped the dependence on x of the stochastic integral. This is possible because $\int_0^t \Delta(x\omega, u) \, d(x\omega_u) \overset{P-a.s.}{=} \int_0^t x\Delta(x\omega, u) \, d\omega_u$ and by replacing $x\Delta(x\omega, u)$ by $\Delta'(\omega, u)$ (see [11] section 2.2 for details).

We first recall a classic result (see for instance [7]). We denote $C_{am}^{\rho}(x, \varphi, \sigma, t)$ the price at time 0 of an American option in the Black-Scholes model with constant interest rate ρ, volatility σ, maturity t and payoff φ.

Proposition 2 *Fix* $\rho \geq 0$. *Let* $S_t = x \exp\left(\sigma W_t + \left(\rho - \frac{\sigma^2}{2}\right) t\right)$ *and* $\widetilde{S}_t = e^{-\rho u} S_t$. *If* φ *is continuous, nonnegative,*

$$C_{am}^{\rho}(x, \varphi, \sigma, t) = \sup_{\tau} E\left[e^{-\rho\tau} \varphi\left(x \exp\left(\sigma W_\tau + \left(\rho - \frac{\sigma^2}{2} \right) \tau \right) \right) \right]$$

where τ runs across all the stopping times of the filtration of the Brownian motion W with values in $[0,t]$ a.s. .

Moreover there exists a predictable process Δ^{am} such that for every time $u \in [0,t]$:

$$C^{\rho}_{am}(x,\varphi,\sigma,t) + \int_0^u \Delta^{am}_v d\widetilde{S}_v \geq e^{-\rho u}\varphi(S_u) \;\; a.s$$

where the stochastic integral is well-defined and uniformly bounded from below.

Remark 3 *The last statement is a direct consequence of Doob's decomposition for supermartingales altogether with the peculiar form of Brownian local martingales. In [11] the same statement is found, without reference, with a Borel function $\Delta_{am} : \mathbb{R}^+_* \times [0,t[\to \mathbb{R}$ composed with (v, S_v) for the integrand of the stochastic integral. In fact this particular form does not play any role in the proof (cf [11]). Since we did not manage to find out a precise reference, we prefer the above formulation.*

If we look at the Black-Scholes model with zero interest rate, volatility equal to 1, and consider the problem of pricing at time 0 an American option A with maturity $\overline{\sigma}^2 t$, payoff $\varphi\left(\widetilde{S}_u\right)$ and exercise window $[\underline{\sigma}^2 t, \overline{\sigma}^2 t]$, we get:

Corollary 4 *If φ is continuous, nonnegative, the price of the option A is given by*

$$c_{am}\left(x,\varphi,\underline{\sigma}^2 t, \overline{\sigma}^2 t\right) = \sup_{\tau} E\left[\varphi\left(x\exp\left(W_\tau - \frac{\tau}{2}\right)\right)\right]$$

where τ runs across all the stopping times of the filtration of the Brownian motion W with values in $[\underline{\sigma}^2 t, \overline{\sigma}^2 t]$ a.s.

Moreover there exists a predictable process δ^{am} such that for every time $u \in [0,t]$:

$$c_{am}(x,\varphi,\sigma,t) + \int_0^u \delta^{am}_v d\widetilde{S}_v \geq e^{-\rho u}\varphi(S_u) \;\; a.s$$

where the stochastic integral is well-defined and uniformly bounded from below.

Then, with a time change argument, we get:

Proposition 5 *([11] Property (P15)). If φ is continuous, nonnegative,*

$$Q\left(I,t,\varphi,x\right) \leq c_{am}\left(x,\varphi,\underline{\sigma}^2 t,\overline{\sigma}^2 t\right)$$

Now we denote

(A') the limits $\lim\limits_{x\to 0^+}\varphi(x)$ **and** $\lim\limits_{x\to+\infty}\frac{\varphi(x)}{1+x}$ **exist**

(A") φ **is** C^2 **with** φ'' **with polynomial growth**

Assumption (A') was introduced in [11] and will be justified by theorem 6.

Assumption (A") comes from the stochastic control theory and a similar assumption can be found for instance in [9], chapter 4, section 2.

The main result of [11] is:

Theorem 6 *Let*

$$G=\left\{g\in C\left(\mathbb{R}_*^+,\mathbb{R}\right)\ /\ \exists\lim_{x\to 0^+}g(x)\,,\ \exists\lim_{x\to\infty}\frac{g(x)}{1+x}\right\}$$

which is a Banach space for the norm $\|g\|=\sup_x\frac{|g(x)|}{1+x}$.

In a zero interest rate market,

(i) If φ satisfies (A) and (A"), then

$$Q\left(I,t,\varphi,x\right)=\sup_{P\in\mathcal{P}_I}E^P\left[\varphi\left(x\omega_t\right)\right]=V^0\left(I,t,\varphi,x\right)$$

Moreover,

$(t,x)\to Q(I,t,\varphi,x)$ *is continuous and* $W^{1,2}$ *in the sense of Krylov in* $]0,T[\times\mathbb{R}_*^+$,

$Q(I,t,\varphi,x)$ *satisfies the HJB equation* $\sup_{\alpha\in[\underline{\sigma},\overline{\sigma}]}\frac{1}{2}\alpha^2x^2\frac{\partial^2u}{\partial x^2}=\frac{\partial u}{\partial s}$,

The infimum in (5) is attained with $\Delta(\omega,u)=\frac{\partial V^0}{\partial y}\left(t-u,x\omega_u\right)$.

(ii) If φ is in G and satisfies (A"), then $V^0(I,t,\varphi,x)\in G$ *and*

$$\lim_{x\to 0^+}V^0\left(I,t,\varphi,x\right)=\lim_{x\to 0^+}\varphi(x)\,,\ \lim_{x\to\infty}\frac{V^0\left(I,t,\varphi,x\right)}{1+x}=\lim_{x\to\infty}\frac{\varphi(x)}{1+x}. \qquad (6)$$

(iii) If φ in G then $V^0(I,t,\varphi,x)\in G$ *and*

$$Q\left(I,t,\varphi,x\right)=\sup_{P\in\mathcal{P}_I}E^P\left[\varphi\left(x\omega_t\right)\right]=V^0\left(I,t,\varphi,x\right)$$

Moreover (6) is in force with Q in place of V^0.

3.2 Bounded volatility with $\underline{\sigma} = 0$

We use the notations of the previous subsection and we take the set $J = [0, \overline{\sigma}]$ instead of I in all the previous definitions. The following is shown in [12]. For the sake of completeness we shall provide a proof.

Theorem 7 *Under assumption (A), if φ is nonnegative,*

$$Q(J, t, \varphi, x) = c_{am}\left(x, \varphi, 0, \overline{\sigma}^2 t\right) = C^0_{am}(x, \varphi, \overline{\sigma}, t)$$

Proof. We go back to the proof of [12] in our setting.

The inequality $Q(J, t, \varphi, x) \leq c_{am}(x, \varphi, 0, \overline{\sigma}^2 t)$ is given by proposition 5 which is valid for $\underline{\sigma} = 0$.

On the other way, we have $Q(J, t, \varphi, x) \geq \sup_{P \in \mathcal{P}_J} E^P[\varphi(x\omega_t)]$. On some probability space, let us define $\mathcal{S}_t$ the set of the stopping times with respect to the filtration generated by a Brownian motion W, with values in $[0, t]$ and take ν in $\mathcal{S}_t$. Let V be the set of progressive processes with respect to the Brownian filtration with values in J. Take γ in V. It is clear that the law on $\left(\Omega, \mathcal{F}, (\mathcal{F}_u)_{u \geq 0}\right)$ of the process $(\frac{Y_u}{x})$ where

$$\begin{aligned} dY_u &= \gamma_u Y_u dW_u \\ Y_0 &= x \end{aligned}$$

is in $\mathcal{P}_J$. Therefore,

$$\sup_{P \in \mathcal{P}_J} E^P[\varphi(x\omega_t)] \geq \sup_{\gamma \in V} E[\varphi(Y_t)]$$

In particular, we can take $\gamma_u = \overline{\sigma} 1_{(u \leq \nu)}$ which is also in V to get

$$\begin{aligned} \sup_{\gamma \in V} E[\varphi(Y_t)] &\geq \sup_{\nu \in \mathcal{S}_t} E\left[\varphi\left(x \exp\left\{\overline{\sigma} W_\nu - \frac{1}{2}\overline{\sigma}^2 \nu\right\}\right)\right] \\ &= C^0_{am}(x, \varphi, \overline{\sigma}, t) \\ &= c_{am}\left(x, \varphi, 0, \overline{\sigma}^2 t\right) \end{aligned}$$

□

3.3 Unbounded volatility

As we suppose here $\overline{\sigma} = +\infty$, the general setting of section 2 has to be modified. We have to consider the laws P satisfying (a), (c) and also :

(b') $d\langle\omega\rangle_u << du$, $\sqrt{\frac{d\langle\omega\rangle_u}{\omega_u^2 du}} \in K \overset{def}{=} [\underline{\sigma}, +\infty[$ $P - a.s.$

We will also need that $s \mapsto \int_0^s \sigma_u dB_u$ be a $P-$martingale (it was clear in the previous sections). So we add the following condition:

(d) $\forall t > 0 \quad E^P\left[\int_0^t \left(\sqrt{\frac{d\langle\omega\rangle_u}{\omega_u^2 du}}\right)^2 du\right] < +\infty.$

We denote $\mathcal{P}$ the set of law satisfying (a), (b'), (c) and (d). We also define the family of $\mathcal{P}-$null sets $N^{\mathcal{P}}$, the filtration $(\mathcal{F}_t)_{t\geq 0}^{\mathcal{P}}$, the set $\Pr og^{\mathcal{P}}$, as previously.

The definition of the minimum price $Q(K,t,\varphi,x)$ (or shorter $Q(t,x)$) is now:

$$Q(K,t,\varphi,x) = \inf\left\{\begin{array}{c} c \in \overline{\mathbb{R}} / \exists L > 0 \text{ , } \exists \Delta \in \Pr og^{\mathcal{P}} / \forall P \in \mathcal{P} : \\ (i) \int_0^t \Delta^2(\omega,u)\, d\langle\omega\rangle_u < \infty \ P \text{ a.s.} \\ (ii) \int_0^{\cdot} \Delta(\omega,u)\, d\omega_u > -L \ P \text{ a.s.} \\ (iii) \ \ c + \int_0^t \Delta(\omega,u)\, d\omega_u \geq \varphi(x\omega_t) \ \ P \text{ a.s.} \end{array}\right\} \tag{7}$$

Let us now turn to the quest of a closed formula for the value function:

$$C(t,x) \overset{def}{=} \sup_{P\in\mathcal{P}} E^P\left[\varphi(x\omega_t)\right]$$

Next we shall show that this value is the smallest initial value of a superstrategy for the payoff $\varphi(x\omega_t)$, i.e. $C(t,x) = Q(t,x)$.

What is the meaning of (2) in case $\overline{\sigma} = +\infty$? For $\rho = 0$, this equation may be rewritten:

$$\frac{1}{2}\overline{\sigma}^2 x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^+ - \frac{1}{2}\underline{\sigma}^2 x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^- = \frac{\partial u}{\partial s} \tag{8}$$

with initial condition $u(0,x) = \varphi(x)$. If we formally set $\overline{\sigma} = +\infty$, we should have

$$\frac{\partial^2 u}{\partial x^2}(s,x) \leq 0$$

This leads us to believe that $x \to C(t,x)$ is concave in x for $t>0$, and we (still formally) get

$$\frac{1}{2}\underline{\sigma}^2 x^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial s}$$

We remark that u satisfies the Black-Scholes PDE (where t is the time to maturity) with volatility $\underline{\sigma}$. Moreover, we obtain that $\frac{\partial u}{\partial s} \leq 0$. Hence we should have $t \to C(t,x)$ decreasing on $]0,+\infty[$. If we fix t in $]0,+\infty[$, it is then rather natural to introduce the set $\mathcal{A}$ of the functions defined on $[0,t]\times]0,+\infty[$ that satisfy:

(i) $\forall x>0, \quad f(0,x) \geq \varphi(x)$
(ii) f *is* $C^{1,2}$ in $]0,t]\times]0,+\infty[$, C^0 in $[0,t]\times]0,+\infty[$
(iii) $\forall u \in]0,t], \ \frac{\partial^2 f}{\partial x^2}(u,x) \leq 0$
(iv) $\forall x>0, \forall u \in]0,t], \quad \frac{1}{2}\underline{\sigma}^2 x^2 \frac{\partial^2 f}{\partial x^2}(u,x) - \frac{\partial f}{\partial u}(u,x) \leq 0$
(v) $f \geq 0$ everywhere

3.3.1 Properties of the value function C

Following the previous discussion, we shall show:

Proposition 8 *Under assumption (A), for every $x>0$, the map $t \to C(t,x)$ is nonincreasing on $]0,+\infty[$.*

Proof. Let P be a law in $\mathcal{P}$. For some Brownian motion B, with $\sigma_u = \sqrt{\frac{d\langle\omega\rangle_u}{\omega_u^2 du}}$, $\sigma_u \geq \underline{\sigma}$,

$$\omega_s = \exp\left\{\int_0^s \sigma_u dB_u - \frac{1}{2}\int_0^s \sigma_u^2 du\right\} \quad P-a.s.$$

Let us fix $0<r\leq s$. We first want to show

$$\begin{aligned} & E^P\left[\varphi\left(x\exp\left\{\int_0^s \sigma_u dB_u - \frac{1}{2}\int_0^s \sigma_u^2 du\right\}\right)\right] \\ = \ & E^P\left[\varphi\left(x\exp\left\{\int_0^r \alpha_v d\beta_v - \frac{1}{2}\int_0^r \alpha_v^2 dv\right\}\right)\right] \end{aligned} \tag{9}$$

for β Brownian motion under P with respect to a new filtration $(\mathcal{G}_t)$ and some process α adapted to the new filtration such that $\alpha_v \geq \underline{\sigma}$ for $v \in [0,r]$. This is done in two steps.

Since $\int_0^s \sigma_v^2 dv = \int_0^r \sigma_{\frac{s}{r}w}^2 \frac{s}{r}\, dw = \int_0^r \left(\sigma_{\frac{s}{r}w}\sqrt{\frac{s}{r}}\right)^2 dw$ it is natural to define α by $\alpha_w = \sigma_{\frac{s}{r}w}\sqrt{\frac{s}{r}}$. Since $s \geq r$, $\alpha_w \geq \underline{\sigma}$ for $w \in [0,r]$.

Now remark that $\beta_w \overset{def}{=} \sqrt{\frac{r}{s}} B_{\frac{s}{r}w}$ is a Brownian motion (by scaling). By fixing $\widetilde{N}_w \overset{def}{=} B_{\frac{s}{r}w}$, we have $P-a.s.$

$$\int_0^r \alpha_w d\beta_w = \int_0^r \sigma_{\frac{s}{r}w}\sqrt{\frac{s}{r}}\sqrt{\frac{r}{s}} d\widetilde{N}_w = \int_0^r \widetilde{\sigma}_w d\widetilde{N}_w$$

where $\widetilde{\sigma}_w \overset{def}{=} \sigma_{\frac{s}{r}w}$. Moreover $w \mapsto \widetilde{N}_w$ is a continuous martingale. As $\langle\beta\rangle_w = \frac{r}{s}\left\langle\widetilde{N}\right\rangle_w$, we have $\left\langle\widetilde{N}\right\rangle_w = \frac{s}{r}w$ and its inverse (as a function of w) is given by $T(v) = \frac{r}{s}v$. Then we can use a time change formula (see for example [6], proposition 3.4.8) to obtain $P-a.s.$

$$\int_0^r \widetilde{\sigma}_w d\widetilde{N}_w = \int_0^{\langle\widetilde{N}\rangle_r} \widetilde{\sigma}_{T(v)} d\left(\widetilde{N}_{T(v)}\right) = \int_0^s \sigma_v dB_v$$

Therefore

$$\int_0^s \sigma_v dB_v = \int_0^r \alpha_w d\beta_w \quad P-a.s.$$

and we get equation (9).

Since $\alpha_w \geq \underline{\sigma}$, $w \in [0,r]$, we have:

$$\begin{aligned} C(s,x) &= \sup_{P\in\mathcal{P}} E^P\left[\varphi\left(x\exp\left\{\int_0^s \sigma_u dB_u - \frac{1}{2}\int_0^s \sigma_u^2 du\right\}\right)\right] \\ &\leq \sup_{P\in\mathcal{P}} E^P\left[\varphi\left(x\exp\left\{\int_0^r \alpha_v d\beta_v - \frac{1}{2}\int_0^r \alpha_v^2 dv\right\}\right)\right] = C(r,x) \end{aligned}$$

□

Remark 9 *This result is not valid in the case of bounded volatility because the process α would not be bounded from above by $\overline{\sigma}$, therefore the fact that $\overline{\sigma} = +\infty$ is crucial here.*

Proposition 10 *Suppose (A). If φ is Lipchitz continuous then for every $t > 0$, $x \to C(t,x)$ is Lipschitz continuous on $]0,+\infty[$.*

Proof. Let $P \in \mathcal{P}$. For some Brownian motion B, with $\sigma_u = \sqrt{\frac{d\langle\omega\rangle_u}{\omega_u^2 du}}$, we can write $P - a.s.$

$$\omega_t = \exp\left\{\int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du\right\}$$

Define then

$$v(t,x,P) \stackrel{def}{=} E^P[\varphi(x\omega_t)]$$

We consider the family $\{x \to v(t,x,P)\}_{P\in\mathcal{P}}$. $\forall P$, $\forall x,y>0$ we have

$$\begin{aligned} & |v(t,x,P) - v(t,y,P)| \\ \leq\ & E^P[|\varphi(x\omega_t) - \varphi(y\omega_t)|] \\ \leq\ & K\ \ |x-y|\ \ E^P[\omega_t] = K\ \ |x-y| \end{aligned}$$

where K is the Lipschitz constant of φ and does not depend on P. Thus, the family $\{x \to v(t,x,P)\}_P$ is equicontinuous with respect to P.

Let (P_n^x), respectively (P_n^y), be a maximizing sequence in $\mathcal{P}$ such that

$$v(t,x,P_n^x) \to \sup_{P\in\mathcal{P}} v(t,x,P) = C(t,x)$$

respectively

$$v(t,y,P_n^y) \to \sup_{P\in\mathcal{P}} v(t,y,P) = C(t,y)$$

Then the difference $D(x,y,t) = C(t,x) - C(t,y)$ is bounded from above by $A = \limsup_{n\to\infty}[v(t,x,P_n^x) - v(t,y,P_n^x)]$ and bounded from below by $B = \liminf_{n\to\infty}[v(t,x,P_n^y) - v(t,y,P_n^y)]$. Hence

$$|D(x,y,r)| \leq \max(|A|,|B|) \leq 2K|x-y|$$

whence the result. □

3.3.2 Stochastic control results

We introduce the sets $\mathcal{P}_n$ defined for n large enough by $\mathcal{P}_n \overset{def}{=} \mathcal{P}_{[\underline{\sigma},n]}$. Clearly $\mathcal{P}_n \subset \mathcal{P}_{n+1} \subset \mathcal{P}$.

We use here notations of section 2.2. Let us define $v_n(t,x) \overset{def}{=} \sup_{\alpha \in U_n} E\left[\varphi\left(X_t^{0,\alpha,0,x}\right)\right] = V^0([\underline{\sigma},n],t,\varphi,x)$ where U_n is the space of $(\mathcal{F}'_t)$-progressive processes with values in $[\underline{\sigma},n]$. Under assumptions (A) and (A'), by theorem 6,

$$v_n(t,x) = \sup_{P \in \mathcal{P}_n} E^P\left[\varphi(x\omega_t)\right]$$

Set now $\mathcal{P}_{\mathbb{N}} \overset{def}{=} \cup_n \mathcal{P}_n$. The next theorem is the key result for this section.

Theorem 11 *If φ is a measurable function such that*

$$\sup_{P \in \mathcal{P}} E^P\left[\varphi(x\omega_t)\right] \quad < \quad + \quad \infty$$

(in particular under assumption (A)), we have

$$\sup_{P \in \mathcal{P}} E^P\left[\varphi(x\omega_t)\right] = \sup_{P \in \mathcal{P}_{\mathbb{N}}} E^P\left[\varphi(x\omega_t)\right]$$

Therefore, under assumptions (A) and (A'),

$$\lim_{n \to \infty} \uparrow v_n(t,x) = C(t,x) \tag{10}$$

Proof. Clearly, since $\mathcal{P}_{\mathbb{N}} \subset \mathcal{P}$, $\sup_{P \in \mathcal{P}} E^P\left[\varphi(x\omega_t)\right] \geq \sup_{P \in \mathcal{P}_{\mathbb{N}}} E^P\left[\varphi(x\omega_t)\right]$.

Let us fix P in $\mathcal{P}$, σ being the associated volatility process. We define ω and ω^n by

$$\begin{aligned}
\omega_t &= \exp\left\{\int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du\right\} \\
\omega_t^n &= \exp\left\{\int_0^t (\sigma_u \wedge n)\, dB_u - \frac{1}{2}\int_0^t (\sigma_u \wedge n)^2\, du\right\}
\end{aligned}$$

Notice that the law of (ω_t^n) under P belongs to $\mathcal{P}_n$.

(1) First step. We show that we can find a subsequence $\left(\omega_t^{g(n)}\right)_n$ which converges P-almost surely to ω_t.

We first show that $\int_0^t (\sigma_u \wedge n)\, dB_u \overset{L^2(P)}{\rightarrow_n} \int_0^t \sigma_u dB_u$. We can write

$$E^P\left[\left(\int_0^t (\sigma_u \wedge n)\, dB_u - \int_0^t \sigma_u dB_u\right)^2\right] = E^P\left[\int_0^t ((\sigma_u \wedge n) - \sigma_u)^2\, du\right]$$

Moreover $((\sigma_u \wedge n) - \sigma_u)^2 \;\rightarrow_n\; 0 \quad \lambda \otimes P - a.s$ where λ is the Lebesgue measure on $[0, t]$. Also

$$((\sigma_u \wedge n) - \sigma_u)^2 \leq 2\left((\sigma_u \wedge n)^2 + \sigma_u^2\right) \leq 4\sigma_u^2$$

Hence, by Lebesgue theorem, since $\forall t > 0 \;\; E^P\left[\int_0^t \sigma_u^2 du\right] < +\infty$ by definition (d) of $\mathcal{P}$, we get $E^P\left[\left(\int_0^t (\sigma_u \wedge n) - \sigma_u dB_u\right)^2\right] \rightarrow 0$. In the same way $\int_0^t (\sigma_u \wedge n)^2\, du \overset{L^1(P)}{\rightarrow_n} \int_0^t \sigma_u^2 du$. At this step we have obtained

$$\int_0^t (\sigma_u \wedge n)\, dB_u - \frac{1}{2}\int_0^t (\sigma_u \wedge n)^2\, du \overset{L^1(P)}{\rightarrow} \int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du$$

So, we can find a subsequence, $g : \mathbb{N} \rightarrow \mathbb{N}$, such that

$$\int_0^t (\sigma_u \wedge g(n))\, dB_u - \frac{1}{2}\int_0^t (\sigma_u \wedge g(n))^2\, du \overset{P-a.s.}{\rightarrow} \int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du$$

Hence, $\omega_t^{g(n)} \overset{P-a.s.}{\rightarrow} \omega_t$ and since φ is measurable,

$$\varphi\left(x\omega_t^{g(n)}\right) \overset{P-a.s.}{\rightarrow} \varphi(x\omega_t)$$

(2) Second step: We show the result when φ is bounded.

By Lebesgue theorem we then have

$$E^P\left[\varphi\left(x\omega_t^{g(n)}\right)\right] \rightarrow E^P\left[\varphi(x\omega_t)\right]$$

Since the law of $\left(\omega_t^{g(n)}\right)$ under P is a law in $\mathcal{P}_{g(n)}$, we can find a probability $Q_n \in \mathcal{P}_{g(n)} \subset \mathcal{P}_{\mathbb{N}}$ such that $E^P\left[\varphi\left(x\omega_t^{g(n)}\right)\right] = E^{Q_n}\left[\varphi\left(x\omega_t\right)\right]$. Hence

$$\lim_n E^{Q_n}\left[\varphi\left(x\omega_t\right)\right] = E^P\left[\varphi\left(x\omega_t\right)\right] \tag{11}$$

Let us take now a maximizing sequence P^q such that

$$E^{P^q}\left[\varphi\left(x\omega_t\right)\right] \uparrow_q \sup_{P\in\mathcal{P}} E^P\left[\varphi\left(x\omega_t\right)\right]$$

Thanks to (11) we have with transparent notations $E^{P^q}\left[\varphi\left(x\omega_t\right)\right] = \lim_n E^{Q_n^q}\left[\varphi\left(x\omega_t\right)\right]$, thus

$$C(t,x) = \lim_q \lim_n E^{Q_n^q}\left[\varphi\left(x\omega_t\right)\right] \leq \sup_{P\in\mathcal{P}_{\mathbb{N}}} E^P\left[\varphi\left(x\omega_t\right)\right]$$

whence the result.

(3) Third step. We show the result in the general case. If φ is not bounded we define $\varphi_m = \varphi \wedge m$ for $m \in \mathbb{N}$. By the monotonous convergence theorem,

$$\lim_{m\to+\infty} \uparrow E^P\left[\varphi_m\left(x\omega_t\right)\right] = E^P\left[\varphi\left(x\omega_t\right)\right]$$

But, thanks to the previous step, $E^P\left[\varphi_m\left(x\omega_t\right)\right] = \lim_{n\to+\infty} E^{Q_n^m}\left[\varphi_m\left(x\omega_t\right)\right]$.

We have $E^{Q_n^m}\left[\varphi_m\left(x\omega_t\right)\right] \leq E^{Q_n^m}\left[\varphi\left(x\omega_t\right)\right]$ and taking a maximizing sequence P^q such that $E^{P^q}\left[\varphi\left(x\omega_t\right)\right] \uparrow_q \sup_{P\in\mathcal{P}} E^P\left[\varphi\left(x\omega_t\right)\right]$, we can write

$$\begin{aligned} E^{P^q}\left[\varphi\left(x\omega_t\right)\right] &= \lim_{m\to+\infty} E^{P^q}\left[\varphi_m\left(x\omega_t\right)\right] \\ &= \lim_{m\to+\infty} \lim_{n\to+\infty} E^{Q_n^{q,m}}\left[\varphi_m\left(x\omega_t\right)\right] \\ &\leq \liminf_m \liminf_n E^{Q_n^{q,m}}\left[\varphi\left(x\omega_t\right)\right] \end{aligned}$$

Hence,

$$C(t,x) \leq \liminf_q \liminf_m \liminf_n E^{Q_n^{q,m}}\left[\varphi\left(x\omega_t\right)\right] \leq \sup_{P\in\mathcal{P}_{\mathbb{N}}} E^P\left[\varphi\left(x\omega_t\right)\right]$$

The second part of the theorem is clear since under (A) and (A'),

$$\sup_{P\in\mathcal{P}_{\mathbb{N}}} E^P\left[\varphi\left(x\omega_t\right)\right] \leq \lim_{n\to\infty} \uparrow v_n(t,x) \leq C(t,x)$$

□

Let us now recall the result:

Theorem 12 *([9] theorem 3.1.5, lemma 3.3.7). Under assumption (A),* $(t,x) \to v_n(t,x)$ *is continuous on* $\mathbb{R}_*^+ \times \mathbb{R}_*^+$.

So, if in addition we suppose (A'), $(t,x) \to \sup_{P \in \mathcal{P}_n} E^P[\varphi(x\omega_t)]$ is continuous on $\mathbb{R}_*^+ \times \mathbb{R}_*^+$ and therefore:

Corollary 13 *Suppose (A) and (A'). The function* $(t,x) \to C(t,x)$ *is lower semicontinuous on* $]0,+\infty[\times]0,+\infty[$.

We can now establish the Bellman principle for v_n. Using theorem 3.1.6 in [9], we obtain:

Theorem 14 *Under assumptions (A) and (A'),* $v_n(t,x)$ *satisfies, for* $u \in [0,t]$

$$v_n(t,x) = \sup_{P \in \mathcal{P}_n} E^P[v_n(t-u, x\omega_u)]$$

Proof. With the notations of section 2, the Bellman principle writes (see theorem 3.1.6 or 3.3.6 in [9]):

For $0 \le s \le T$ and $0 \le u \le T - s$:

$$v(s,x) = \sup_{\alpha \in U} E\left[v\left(s+u, X_u^{0,\alpha,0,x}\right)\right]$$

Hence, by definition, we get for every $u \in [0,t]$:

$$v_n(t,x) = \sup_{\alpha \in U_n} E\left[v_n\left(t-u, X_u^{0,\alpha,0,x}\right)\right]$$

which is valid for $t \le T$ and also for $t > T$, thanks to time homogeneity (see remark 1). Remark now that for every $t > 0$, $x \to v_n(t,x)$ satisfies (A) since φ does (the continuity comes from theorem 12 and the bound comes from the bound of φ and the fact that $E^P[\omega_t] = 1$). Moreover, thanks to theorem 6, $v_n(t,x)$ satisfies (A'). Therefore we can apply (again !) theorem 6 to the function $x \to v_n(t-u,x)$ (instead of φ) to get

$$\sup_{\alpha \in U_n} E\left[v_n\left(t-u, X_u^{0,\alpha,0,x}\right)\right] = \sup_{P \in \mathcal{P}_n} E^P[v_n(t-u, x\omega_u)]$$

□

Corollary 15 *Suppose (A) and (A'). For every* $u \in [0,t]$,

$$C(t,x) \leq \sup_{P\in\mathcal{P}} E^P \left[C(t-u, x\omega_u)\right]$$

Proof. We can write $v_n(t,x) = \sup_{P\in\mathcal{P}_n} E^P \left[v_n(t-u, x\omega_u)\right]$ for $u \in [0,t]$ by theorem 14. Thus

$$v_n(t,x) \leq \sup_{P\in\mathcal{P}_n} E^P \left[C(t-u, x\omega_u)\right] \leq \sup_{P\in\mathcal{P}} E^P \left[C(t-u, x\omega_u)\right]$$

Taking the limit, we have $C(t,x) \leq \sup_{P\in\mathcal{P}} E^P \left[C(t-u, x\omega_u)\right]$. □

Let us turn to the reverse inequality:

Theorem 16 *Suppose (A) and (A'). Set* $P \in \mathcal{P}$ *and* $u \in [0,t]$. *We have*

$$E^P \left[C(t-u, x\omega_u)\right] \leq C(t,x)$$

Proof. Set P in $\mathcal{P}$ and take the sequence $\omega_t^{g(n)}$ of theorem 11 which goes to ω_t $P-a.s.$ when n goes to $+\infty$. Let us fix $N \in \mathbb{N}$ large enough. We have seen that there exists a law $Q_N \in \mathcal{P}_{g(N)}$ such that

$$E^P \left[v_{g(N)+j}\left(t-u, x\omega_t^{g(N)}\right)\right] = E^{Q_N} \left[v_{g(N)+j}(t-u, x\omega_t)\right]$$

for $j = 1, 2, ...$ and $u \in [0,t]$. Moreover we have (theorem 14)

$$C(t,x) \geq v_{g(N)+j}(t,x) \geq E^{Q_N} \left[v_{g(N)+j}(t-u, x\omega_t)\right]$$

Hence,

$$\begin{aligned}
\liminf_{N\to\infty} \liminf_{j\to\infty} v_{g(N)+j}(t,x) &\geq \liminf_{N\to\infty} \liminf_{j\to\infty} E^P \left[v_{g(N)+j}\left(t-u, x\omega_t^{g(N)}\right)\right] \\
&\geq E^P \left[\liminf_{N\to\infty} C\left(t-u, x\omega_u^{g(N)}\right)\right] \\
&\geq E^P \left[C(t-u, x\omega_u)\right]
\end{aligned}$$

using (10) from theorem 11 and the lower semicontinuity of C, which completes the proof. □

Remark 17 *We have actually obtained the Bellman principle for C.*

3.3.3 Concavity of $x \mapsto C(t,x)$

From proposition 8 and theorem 16 we get:

Corollary 18 *Suppose (A) and (A'). Let $\sigma \geq \underline{\sigma}$. We have:* $\forall x > 0 \quad \forall u \in [0,t]$

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{y^2}{2(t-u)}}}{\sqrt{2\pi(t-u)}} C\left(t, x\exp\left(\sigma y - \sigma^2 \frac{t-u}{2}\right)\right) dy \leq C\left(t,x\right) \tag{12}$$

This can be rewritten using the semigroup notation:

$$P^{\sigma}_{t-u}\left[C\left(t,.\right)\right]\left(x\right) \leq C\left(t,x\right) \tag{13}$$

where $P^{\sigma}_{s} f\left(z\right) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{y^2}{2s}}}{\sqrt{2\pi s}} f\left(z\exp\left(\sigma y - \sigma^2 \frac{s}{2}\right)\right) dy$ *is the Black-Scholes semi-group with volatility σ.*

Proof. Thanks to theorem 16, we have

$$\forall P \in \mathcal{P} \quad \forall s \in [0,t] \quad E^P\left[C\left(s, x\omega_{t-s}\right)\right] \leq C\left(t,x\right)$$

Now choose P in $\mathcal{P}$ corresponding to a constant volatility process σ (with $\sigma \geq \underline{\sigma}$). We can write for some Brownian motion B,

$$\omega_{t-s} = \exp\left(\sigma B_{t-s} - \sigma^2 \frac{(t-s)}{2}\right)$$

Since $v \to C\left(v,x\right)$ is decreasing and $s \leq t$

$$E^P\left[C\left(t, x\omega_{t-s}\right)\right] \leq E^P\left[C\left(s, x\omega_{t-s}\right)\right]$$

and then

$$\begin{aligned} E^P\left[C\left(t, x\omega_{t-s}\right)\right] &= \int_{-\infty}^{+\infty} \frac{e^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi\left(t-s\right)}} C\left(t, x\exp\left(\sigma y - \sigma^2 \frac{(t-s)}{2}\right)\right) dy \\ &\leq C\left(t,x\right) \end{aligned}$$

□

This corollary and proposition 13 leads to:

Theorem 19 *Under assumptions (A) and (A'), for every $t > 0$, the map $x \to C(t,x)$ is concave on $]0,+\infty[$. In particular, it is continuous.*

Proof. Let f a lower semi continuous function on $\mathbb{R}_*^+$ satisfying (13). We first prove the proposition for $x \to f(x)$ sufficiently smooth. We have, according to (13)

$$\lim_{u \uparrow t} \frac{P_{t-u}^{\sigma} f(x) - f(x)}{t-u} \leq 0$$

Yet this limit is by definition: $Af(x) \overset{def}{=} \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(x)$. Thus for every $x > 0$, $\frac{\partial^2 f}{\partial x^2}(x) \leq 0$ and the result.

If now $x \to f(x)$ is not smooth enough, let us fix $t > \varepsilon > 0$. The map $x \to P_{t-\varepsilon}^{\sigma} f(x)$ is C^{∞}. As f satisfies (13), we get (because $P_{t-u}^{\sigma} P_{t-\varepsilon}^{\sigma} = P_{t-\varepsilon}^{\sigma} P_{t-u}^{\sigma}$)

$$P_{t-u}^{\sigma}\left(P_{t-\varepsilon}^{\sigma} f\right)(x) \leq P_{t-\varepsilon}^{\sigma} f(x)$$

So $x \to P_{t-\varepsilon}^{\sigma} f(x)$ is concave (first step).

Let us write $S_u = \exp\left(\sigma B_u - \sigma^2 \frac{u}{2}\right)$ for some standard Brownian motion B, the Black-Scholes stock price at volatility $\sigma \geq \underline{\sigma}$. Then, by definition, $P_{t-\varepsilon}^{\sigma} f(x) = E\left[f(xS_{t-\varepsilon})\right]$. We can write:

$$\begin{aligned} f(x) &= \liminf_{\varepsilon \to t} f(x) \geq \liminf_{\varepsilon \to t} E\left[f(xS_{t-\varepsilon})\right] \\ &\geq E\left[\liminf_{\varepsilon \to t} f(xS_{t-\varepsilon})\right] \geq E\left[f(xS_0)\right] = f(x) \end{aligned}$$

where we use first (13), next Fatou's lemma and the fact that f is lower semicontinuous for the last inequality.

In particular we have shown that

$$f(x) = \liminf_{\varepsilon \to t} E\left[f(xS_{t-\varepsilon})\right] = \liminf_{\varepsilon \to t} P_{t-\varepsilon}^{\sigma} f(x)$$

Finally, as a limit inf of a sequence of concave functions is a concave function, f is concave. We complete the proof with corollary 13 and corollary 18. □

3.3.4 Main Theorem

Theorem 20 *Let us fix* (t,x) *in* $]0,+\infty[\times]0,+\infty[$ *and let* $\underline{P}$ *be the probability law of* $\mathcal{P}$ *such that* $\sqrt{\frac{d\langle\omega\rangle_u}{\omega_u^2 du}} \equiv \underline{\sigma} \quad \underline{P}-a.s.$ *for every* u *in* $]0,t[$. *We also denote by* $\widehat{\varphi}$ *the upper concave envelope of* φ, *i.e. the smallest concave function above* φ.

(i) Under assumption (A),

$$\inf\left\{f\left(t,x\right), f\in\mathcal{A}\right\} = E^{\underline{P}}\left[\widehat{\varphi}\left(x\omega_t\right)\right] \geq C\left(t,x\right)$$

(ii) Under assumptions (A) and (A'),

$$C(t,x) \geq E^{\underline{P}}\left[\widehat{\varphi}\left(x\omega_t\right)\right]$$

Remark 21 *If* φ *satisfies (A),* $\widehat{\varphi}$ *does.*

Remark 22 *If* φ *is nonnegative,* $E^{\underline{P}}\left[\widehat{\varphi}\left(x\omega_t\right)\right]$ *is the price at time* 0 *of a European option with payoff* $\widehat{\varphi}\left(S_t\right)$ *in the Black-Scholes model at volatility* $\underline{\sigma}$ *without interest rate.*

Proof. **(i)** Let $f\in\mathcal{A}$ and $P\in\mathcal{P}$. Applying Itô formula to the process $u\to f\left(t-u,x\omega_u\right)$ between 0 and $t-\varepsilon$ for $\varepsilon\in]0,t]$, gives

$$\begin{aligned}
f\left(\varepsilon,x\omega_{t-\varepsilon}\right) &= f\left(t,x\omega_0\right)+\int_0^{t-\varepsilon}-\frac{\partial f}{\partial u}\left(t-u,x\omega_u\right)du \\
&\quad+\int_0^{t-\varepsilon}\frac{1}{2}\frac{\partial^2 f}{\partial y^2}\left(t-u,x\omega_u\right)\ x^2 d\left\langle\omega\right\rangle_u \\
&\quad+\int_0^{t-\varepsilon}x\frac{\partial f}{\partial y}\left(t-u,x\omega_u\right)d\omega_u
\end{aligned}$$

if we set $\sigma_u = \sqrt{\dfrac{d\left\langle\omega\right\rangle_u}{\omega_u^2 du}}$ then

$$f\left(\varepsilon,x\omega_{t-\varepsilon}\right) = f\left(t,x\right)+\int_0^{t-\varepsilon}\left[\frac{1}{2}\frac{\partial^2 f}{\partial y^2}\left(t-u,x\omega_u\right)\left(\sigma_u^2-\underline{\sigma}^2\right)x^2\omega_u^2\right]du$$

$$+\int_0^{t-\varepsilon}\left[-\frac{\partial f}{\partial u}(t-u,x\omega_u)+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t-u,x\omega_u)\,\underline{\sigma}^2(x\omega_u)^2\right]du$$
$$+\int_0^{t-\varepsilon}x\frac{\partial f}{\partial y}(t-u,x\omega_u)\,d\omega_u$$

Because of the definitions (iii) and (iv) within $\mathcal{A}$, both integrals with finite variation are nonpositive. Hence, $P-a.s.$

$$0\leq f(\varepsilon,x\omega_{t-\varepsilon})\leq f(t,x)+\int_0^{t-\varepsilon}x\frac{\partial f}{\partial y}(t-u,x\omega_u)\,d\omega_u$$

Suppose for a while (this will be shown later):

$$\forall\varepsilon\in]0,t],\quad E^P\left[\int_0^{t-\varepsilon}x\frac{\partial f}{\partial y}(t-u,x\omega_u)\,d\omega_u\right]\leq 0 \tag{14}$$

By Fatou's lemma we obtain

$$\begin{aligned}E^P[\varphi(x\omega_t)] &\leq E^P[f(0,x\omega_t)]=E^P\left[\lim_{\varepsilon\to 0}f(\varepsilon,x\omega_{t-\varepsilon})\right]\\ &\leq \liminf_{\varepsilon\to 0}E^P[f(\varepsilon,x\omega_{t-\varepsilon})]\leq f(t,x)\end{aligned}$$

Hence,

$$f(t,x)\geq\sup_{P\in\mathcal{P}}E^P[\varphi(x\omega_t)]=C(t,x)$$

Now let $\widehat{\varphi}$ be the upper concave envelope of φ. As f is concave in x on $]0,t]$, $\lim_{\varepsilon\to 0}f(\varepsilon,x)$ is concave as a limit of concave functions. By continuity, $\lim_{\varepsilon\to 0}f(\varepsilon,x)=f(0,x)\geq\varphi(x)$, so the minimality of $\widehat{\varphi}$ leads to

$$f(0,x)\geq\widehat{\varphi}(x)\geq\varphi(x)$$

Hence (by the same reasoning with $\widehat{\varphi}$ in place of φ), for any $f\in\mathcal{A}$,

$$f(t,x)\geq E^P[\widehat{\varphi}(x\omega_t)]\geq E^P[\varphi(x\omega_t)] \tag{15}$$

Set $f^*(t,x)\overset{def}{=}E^{\underline{P}}[\widehat{\varphi}(x\omega_t)]$ which belongs to $\mathcal{A}$. Applying (15) with f^*, we get (see also [3] for instance):

$$\sup_{P\in\mathcal{P}}E^P[\widehat{\varphi}(x\omega_t)]=f^*(t,x)$$

Therefore, for $x > 0$, $f(t,x) \geq f^*(t,x) \geq C(t,x)$.

Finally, because this proof is valid for any f in $\mathcal{A}$, we obtain (i).

It remains to show (14). Fix $s \in [0,t[$, and define the process

$$M_s = \int_0^s x \frac{\partial f}{\partial y}(t-u, x\omega_u)\, d\omega_u$$

M is a $P-$local martingale. Let $\tau_n = \inf\{s \geq 0, |M_s| = n\}$ for $n \geq 1$. $(M_{s\wedge\tau_n})_{s\geq 0}$ is a $P-$martingale. In particular $E^P[M_{s\wedge\tau_n}] = E^P[M_0] = 0$. Now, by Itô's formula we have

$$-f(t,x) \leq \int_0^{s\wedge\tau_n} x\frac{\partial f}{\partial y}(t-u, x\omega_u)\, d\omega_u = M_{s\wedge\tau_n} \quad P-a.s.$$

Because of this lower bound, we can apply Fatou's lemma to obtain

$$\begin{aligned} E^P\left[\int_0^s x\frac{\partial f}{\partial y}(t-u, x\omega_u)\, d\omega_u\right] &= E^P[M_s] \\ &= E^P\left[\liminf_n M_{s\wedge\tau_n}\right] \\ &\leq \lim_n E^P[M_{s\wedge\tau_n}] = 0 \end{aligned}$$

and (14) with $s = t - \varepsilon$.

(ii) Let $P \in \mathcal{P}$. According to theorem 16 we have for every u in $[0,t]$

$$C(t,x) \geq E^P[C(t-u, x\omega_u)]$$

Then Fatou's lemma gives $C(t,x) \geq E^P\left[\liminf_{u\uparrow t} C(t-u, x\omega_u)\right]$. Therefore, we need to show that

$$\liminf_{u\uparrow t} C(t-u, x\omega_u) \geq \widehat{\varphi}(x\omega_t) \quad P-a.s.$$

And it is sufficient to get $\liminf_{u\uparrow t} C(t-u,x) \geq \widehat{\varphi}(x)$. Because of the concavity of $x \to C(t-u,x)$ for $u < t$ (proposition 19) and the definition of $\widehat{\varphi}$, this amounts to show

$$\forall x > 0 \qquad \liminf_{u\uparrow t} C(t-u,x) \geq \varphi(x) \tag{16}$$

Yet, $\varphi(x) = C(0, x)$ and (16) is obtained thanks to the lower semi-continuity of C. □

Remark 23 *In particular, in case φ is nonnegative, the value function C belongs to $\mathcal{A}$, with equality in (iv).*

3.3.5 Connection with the selling price

Up to now, we have discussed the value function $C(t, x)$. In this brief subsection we shall see that this is actually the minimum price of the superstrategies for a European option in our framework of the Uncertain Volatility Model where $\overline{\sigma} = +\infty$.

Theorem 24 *Under assumptions (A) and (A'), if φ is nonnegative, $Q(t, x) = C(t, x)$ on $]0, +\infty[\times]0, +\infty[$.*

Proof. Thanks to the constraints (ii) and (iii) of definition (7):

$$Q(t, x) \geq \sup_{P \in \mathcal{P}} E^P [\varphi(x\omega_t)] = C(t, x)$$

Conversely, since $C(t, x) = P_t^{\underline{\sigma}} \widehat{\varphi}(x)$ is smooth enough, we get from Itô's formula:

$$C(t, x) + \int_0^t \frac{\partial C}{\partial y}(t - u, x\omega_u)\, x d\omega_u = \widehat{\varphi}(x\omega_t) \geq \varphi(x\omega_t) \geq 0 \quad P - a.s$$

We obtain (ii) with $L = C(t, x)$. (i) comes from the application of Itô's rule. Hence the converse inequality and the result. □

We have finally shown that, in a no interest rate market, under appropriate assumptions, the price of the minimum strategy of a standard European option of maturity t and payoff $\varphi(S_t)$, when volatility is unknown but assumed to be bounded from below by a constant $\underline{\sigma}$, is the price of a standard European option in the Black and Scholes model with the same maturity, volatility $\underline{\sigma}$ and as a payoff function, the upper concave envelop of φ. Moreover, the hedge amount is given by $\frac{\partial C}{\partial y}(t - u, S_u)$.

Remark 25 *In [11], a similar hedging strategy in the case where the volatility is bounded is obtained (theorem 6).*

Remark 26 *Our result remains valid if* $\underline{\sigma} = 0$.

4 Extension to non zero interest rate

We denote ρ the constant interest rate of the market and also:

(A) φ **is continuous and** $x \mapsto \frac{\varphi(x)}{1+x}$ **is bounded on** $]0, +\infty[$.

(A') the limits $\lim_{x\to 0^+} \varphi(x)$ **and** $\lim_{x\to+\infty} \frac{\varphi(x)}{1+x}$ **exist**

4.1 Bounded volatility with $\underline{\sigma} > 0$

We use the notations and definitions of section 2 and 3. In particular we use the same set of law $\mathcal{P}_I$.

Let us see how the definition of the price has to be modified. If at time u, one has the wealth X_u and invests the amount Δ_u in the stock price S_u, the amount of the investment in the no risky asset of price R is $\zeta_u = \frac{X_u - \Delta_u S_u}{R_u}$. The self financing condition is satisfied if

$$dX_u = \zeta_u dR_u + \Delta_u dS_u$$

The solution of this equation is given by

$$X_u = R_u \left(X_0 + \int_0^u \Delta_u d(\frac{S_u}{R_u}) \right), \quad u \in [0, t]$$

Mixing this with the previous definition, the price of a European option is defined by:

$$Q^\rho(I, t, \varphi, x) = \inf \left\{ \begin{array}{c} c \in \overline{\mathbb{R}} / \exists L > 0 \ , \ \exists \Delta \in \Pr og^{\mathcal{P}_I} / \forall P \in \mathcal{P}_I : \ P \text{ a.s.} \\ (i) \int_0^t \Delta^2 (e^{\rho u}\omega, u) \, \omega_u^2 du < \infty \\ (ii) \int_0^{\cdot} \Delta (e^{\rho u}\omega, u) \, d\omega_u > -L \\ (iii) \ \ c + \int_0^t \Delta (e^{\rho u}\omega, u) \, d\omega_u \geq e^{-\rho t} \varphi (x e^{\rho t} \omega_t) \end{array} \right\}$$

Remark that ω stands here for the discounted underlying.

We have the clear homogeneity property:

Proposition 27 $\forall\lambda \geq 0,\ \ Q^{\rho}(I,t,\lambda\varphi,x) = \lambda Q^{\rho}(I,t,\varphi,x)$. *In particular:*

$$Q^{\rho}(I,t,\varphi,x) = e^{-\rho t}Q^{\rho}\left(I,t,e^{\rho t}\varphi,x\right)$$

Proof. If $\lambda = 0$, it is straightforward. If $\lambda > 0$,

$$\lambda c + \int_0^t \lambda\Delta\left(e^{\rho u}\omega,u\right)xd\omega_u \geq e^{-\rho t}(\lambda\varphi)\left(xe^{\rho t}\omega_t\right)\ \ P \text{ a.s.}$$

then $\lambda Q^{\rho}(I,t,\varphi,x) \geq Q^{\rho}(I,t,\lambda\varphi,x)$. On the other way,

$$c + \int_0^t \Delta\left(e^{\rho u}\omega,u\right)xd\omega_u \geq e^{-\rho t}(\lambda\varphi)\left(xe^{\rho t}\omega_t\right)\ \ P \text{ a.s}$$

leads to

$$\frac{c}{\lambda} + \int_0^t \frac{\Delta\left(e^{\rho u}\omega,u\right)x}{\lambda}d\omega_u \geq e^{-\rho t}\varphi\left(xe^{\rho t}\omega_t\right)\ \ P \text{ a.s}$$

so, $\frac{Q^{\rho}(I,t,\lambda\varphi,x)}{\lambda} \geq Q^{\rho}(I,t,\varphi,x)$. $\square$

Let us now give a useful result which allows to make use of the results of the previous section.

Proposition 28

$$Q^0\left(I,t,\varphi\left(e^{\rho t}.\right),x\right) = Q^0\left(I,t,\varphi,xe^{\rho t}\right) = Q^{\rho}\left(I,t,e^{\rho t}\varphi,x\right)$$

Proof. The first equality comes from [11] (property (P11)) (here we explicit the dependence in the initial condition) :

$$\begin{aligned}
& c + \int_0^t \Delta(\omega,u)\,xd\omega_u \geq \varphi\left(xe^{\rho t}\omega_t\right)\ P \text{ a.s.} \\
\Leftrightarrow\ \ & c + \int_0^t \frac{\Delta(\omega,u)}{e^{\rho t}}xe^{\rho t}d\omega_u \geq \varphi\left(\left(xe^{\rho t}\right)\omega_t\right)\ P \text{ a.s.} \\
\Leftrightarrow\ \ & c + \int_0^t \Delta'(\omega,u)\left(xe^{\rho t}\right)d\omega_u \geq \varphi\left(\left(xe^{\rho t}\right)\omega_t\right)\ P \text{ a.s.}
\end{aligned}$$

And we can write

$$\begin{aligned} & c+\int_0^t \Delta(\omega,u)\, x d\omega_u \geq \varphi\left(x e^{\rho t}\omega_t\right)\ P \text{ a.s.} \\ \Leftrightarrow\ & c+\int_0^t \Delta'\left(e^{\rho u}\omega,u\right) x d\omega_u \geq \varphi\left(\left(x e^{\rho t}\right)\omega_t\right)\ P \text{ a.s.} \end{aligned}$$

hence the second equality. □

Remark 29 *Both propositions are valid whatever the set of laws and the interval.*

From proposition 5, we get:

Proposition 30 *Under assumption (A), if φ is nonnegative,*

$$Q^{\rho}(I,t,\varphi,x) \leq e^{-\rho t} c_{am}\left(x e^{\rho t},\varphi,\underline{\sigma}^2 t,\overline{\sigma}^2 t\right)$$

Proof. We have $Q^0(I,t,\varphi,x) \leq c_{am}(x,\varphi,\underline{\sigma}^2 t,\overline{\sigma}^2 t)$ and $Q^{\rho}(I,t,\varphi,x) = e^{-\rho t} Q^0\left(I,t,\varphi,x e^{\rho t}\right)$. □

Remark 31 *Frey in [5] shows a similar result using the optional decomposition of El Karoui and Quenez ([4]) and Kramkov ([8]).*

Our following result extends theorem 6. First, we review some notations of section 2. The state processes $X^{\rho,\alpha,0,x}$ controlled by the process α was defined by:

$$\begin{aligned} dX_u^{\rho,\alpha,0,x} &= \rho X_u^{\rho,\alpha,t,x} du + \alpha_u X_u^{\rho,\alpha,0,x} dw_u \\ X_0^{\rho,\alpha,t,x} &= x \end{aligned}$$

and we had set $V^{\rho}(I,t,\varphi,x) \stackrel{def}{=} \sup_{\alpha\in U} E\left[e^{-\rho t}\varphi\left(X_t^{\rho,\alpha,0,x}\right)\right]$

With those notations, we obtain:

Theorem 32 *Under assumptions (A) and (A'),*

$$Q^{\rho}(I,t,\varphi,x) = \sup_{P\in\mathcal{P}_I} E^P\left[e^{-\rho t}\varphi\left(xe^{\rho t}\omega_t\right)\right] = V^{\rho}(I,t,\varphi,x)$$

Proof. Using theorem 6, we get, since $\mathcal{P}_I$ does not depend on the initial condition x,

$$Q^0\left(I,t,\varphi,xe^{\rho t}\right) = \sup_{P\in\mathcal{P}_I} E^P\left[\varphi\left(xe^{\rho t}\omega_t\right)\right]$$

Moreover, thanks to the propositions 27 and 28, we have

$$\begin{aligned} Q^{\rho}(I,t,\varphi,x) &= e^{-\rho t}Q^{\rho}\left(I,t,e^{\rho t}\varphi,x\right) \\ &= e^{-\rho t}Q^0\left(I,t,\varphi,xe^{\rho t}\right) \overset{thm\ 3.3}{=} e^{-\rho t}V^0\left(I,t,\varphi,xe^{\rho t}\right) \\ &= e^{-\rho t}\sup_{P\in\mathcal{P}_I} E^P\left[\varphi\left(xe^{\rho t}\omega_t\right)\right] \\ &= \sup_{P\in\mathcal{P}_I} E^P\left[e^{-\rho t}\varphi\left(xe^{\rho t}\omega_t\right)\right] \end{aligned}$$

since $\mathcal{P}_I$ does not depend on ρ.

Now, we can remark (by Itô's formula) that $X_t^{0,\alpha,0,xe^{\rho t}} \overset{a.e.}{=} X_t^{\rho,\alpha,0,x}$. So,

$$\begin{aligned} V^0\left(I,t,\varphi,xe^{\rho t}\right) &= \sup_{\alpha\in U} E\left[\varphi\left(X_t^{0,\alpha,0,xe^{\rho t}}\right)\right] = \sup_{\alpha\in U} E\left[\varphi\left(X_t^{\rho,\alpha,0,x}\right)\right] \\ &= e^{-\rho t}V^{\rho}(I,t,\varphi,x) \end{aligned}$$

The result follows. □

4.2 Bounded volatility with $\underline{\sigma} = 0$

Recall that $J = [0,\overline{\sigma}]$. $Q^{\rho}(J,t,\varphi,x)$ has the same properties than $Q^{\rho}(I,t,\varphi,x)$. Hence with theorem 7:

Theorem 33 *Under assumption (A), if φ is nonnegative,*

$$Q^{\rho}(J,t,\varphi,x) = e^{-\rho t}Q^0\left(J,t,\varphi,xe^{\rho t}\right) = e^{-\rho t}C^0_{am}\left(xe^{\rho t},\varphi,\overline{\sigma},t\right)$$

Remark 34 *A similar result has been shown by Frey in [5] in a framework of a stochastic volatility model. Moreover, although he works with a larger class of payoff functions than ours, his payoff has to be a function of the discounted price process.*

4.3 Unbounded volatility

We use here the notations and definitions of section 4. The riskless price at time 0 of the European option is now defined by:

$$Q^{\rho}(K,t,\varphi,x)=\inf\left\{\begin{array}{c} c\in\overline{\mathbb{R}}\,/\exists L>0\ ,\ \exists\Delta\in\Pr og^{\mathcal{P}}\,/\forall P\in\mathcal{P}:\ P\text{ a.s.}\\ (i)\int_0^t\Delta^2\left(e^{\rho u}\omega,u\right)d\left\langle\omega\right\rangle_u<\infty\\ (ii)\int_0^{\cdot}\Delta\left(e^{\rho u}\omega,u\right)d\omega_u>-L\\ (iii)\ \ c+\int_0^t\Delta\left(e^{\rho u}\omega,u\right)d\omega_u\geq e^{-\rho t}\varphi\left(xe^{\rho t}\omega_t\right)\end{array}\right\}$$

With propositions 27, 28 and remark 29, we get

Theorem 35 *Under assumptions (A) and (A'),*

$$Q^{\rho}(K,t,\varphi,x)=\sup_{P\in\mathcal{P}}E^{P}\left[e^{-\rho t}\varphi\left(xe^{\rho t}\omega_t\right)\right]=E^{\underline{P}}\left[e^{-\rho t}\widehat{\varphi}\left(xe^{\rho t}\omega_t\right)\right]$$

the last quantity being the price at time 0 *of a European option in the Black-Scholes model, with interest rate* ρ*, volatility* $\underline{\sigma}$ *and payoff* $\widehat{\varphi}(S_t)$*.*

Proof. First, since $\mathcal{P}$ does not depend neither on ρ nor on the initial condition x, as previously we have

$$\begin{aligned} Q^{\rho}(K,t,\varphi,x) &= e^{-\rho t}Q^{\rho}\left(K,t,e^{\rho t}\varphi,x\right)=e^{-\rho t}Q^{0}\left(K,t,\varphi\left(e^{\rho t}.\right),x\right)\\ &= e^{-\rho t}\sup_{P\in\mathcal{P}}E^{P}\left[\varphi\left(xe^{\rho t}\omega_t\right)\right]=\sup_{P\in\mathcal{P}}E^{P}\left[e^{-\rho t}\varphi\left(xe^{\rho t}\omega_t\right)\right]\end{aligned}$$

Secondly, if we define $\psi(.)=\varphi\left(e^{\rho t}.\right)$, thanks to theorem 20 ($\psi$ satisfies (A) and (A')),

$$\sup_{P\in\mathcal{P}}E^{P}\left[\varphi\left(xe^{\rho t}\omega_t\right)\right]=\sup_{P\in\mathcal{P}}E^{P}\left[\psi\left(x\omega_t\right)\right]=E^{\underline{P}}\left[\widehat{\psi}\left(x\omega_t\right)\right]$$

Lastly, it remains to show that on $]0,+\infty[$, $\widehat{\psi}(.) = \widehat{\varphi}(e^{\rho t}.)$. The function $\widehat{\varphi}(e^{\rho t}.)$ is concave. So, since $\widehat{\varphi}(e^{\rho t}.) \geq \varphi(e^{\rho t}.)$, by the minimality of the concave envelope, we get

$$\widehat{\varphi}\left(e^{\rho t}.\right) \geq \widehat{\varphi\left(e^{\rho t}.\right)} = \widehat{\psi}(.)$$

On the other way, $\varphi(x) = \psi(xe^{-\rho t})$. Because of the concavity of $\widehat{\psi}(e^{-\rho t}.)$, we get, for every x in $]0,+\infty[$

$$\widehat{\psi}\left(xe^{-\rho t}\right) \geq \widehat{\varphi}(x)$$

Let us fix $y = xe^{-\rho t}$. We have $\widehat{\psi}(y) \geq \widehat{\varphi}(ye^{\rho t})$ for every y in $]0,+\infty[$, which gives the converse inequality and completes the proof. □

Remark 36 *The results of this section can be extended to a deterministic interest rate function.*

5 Conclusion

In this paper we study the Uncertain Volatility Model (UVM), introduced in [1] and of [10], for payoffs assumed to be only continuous. Let us first summarize our results. Suppose the interest rate ρ is nonnegative. If φ is the (nonnegative) payoff function of an option, let us set:

(A) φ **is continuous and** $x \mapsto \frac{\varphi(x)}{1+x}$ **is bounded on** $]0,+\infty[$**.**

(A') the limits $\lim\limits_{x\to 0^+} \varphi(x)$ **and** $\lim\limits_{x\to+\infty} \frac{\varphi(x)}{1+x}$ **exist**

- If the volatility process takes values in $[\underline{\sigma},\overline{\sigma}]$ where $\underline{\sigma} > 0$ and $\overline{\sigma} < +\infty$, under assumptions (A) and (A'), the UVM-selling price of a European option is the UVM-selling price of the European option with no interest rate, discounted by $e^{-\rho t}$, with underlying initial value $xe^{\rho t}$ instead of x.

- If $\underline{\sigma} = 0$, under assumption (A) ((A') is not requested), the same result is true. Moreover, the price is also the Black-Scholes American price at volatility $\overline{\sigma}$ and no interest rate (with the same transformation).

- If $\overline{\sigma} = +\infty$ ($\underline{\sigma} \geq 0$) under assumptions (A) and (A'), the UVM-selling price of a European option is the price of a Black-Scholes European option with volatility $\underline{\sigma}$ and as a payoff, the smallest concave function above the initial payoff.

In future works, we plan to study the case of American options and also barrier options.

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