



# On-The-Fly Range Reduction

Vincent Lefèvre, Jean-Michel Muller

► **To cite this version:**

Vincent Lefèvre, Jean-Michel Muller. On-The-Fly Range Reduction. [Research Report] RR-4043, LIP RR-2000-34, INRIA, LIP. 2000. inria-00072595

**HAL Id: inria-00072595**

**<https://hal.inria.fr/inria-00072595>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

## *On-the-fly Range Reduction*

Vincent Lefèvre , Jean-Michel Muller

**No 4043**

Novembre 2000

———— THÈME 2 ————

A large blue rectangular area at the bottom of the page. On the left side, there is a large, light grey stylized 'R' logo. To its right, the words 'Rapport de recherche' are written in a white serif font. A horizontal grey brushstroke underline is positioned below the text.

*Rapport  
de recherche*





## On-the-fly Range Reduction

Vincent Lefèvre\* , Jean-Michel Muller†

Thème 2 — Génie logiciel  
et calcul symbolique  
Projet Arénaire

Rapport de recherche n°4043 — Novembre 2000 — 9 pages

**Abstract:** In several cases, the input argument of an elementary function evaluation is given bit-serially, most significant bit first. We suggest a solution for performing the first step of the evaluation (namely, the range reduction) *on the fly*: the computation is overlapped with the reception of the input bits. This algorithm can be used for the trigonometric functions  $\sin$ ,  $\cos$ ,  $\tan$  as well as for the exponential function.

**Key-words:** Range reduction, Elementary functions, Computer arithmetic.

(Résumé : *tsvp*)

\* INRIA, Projet POLKA, UR INRIA Lorraine

† CNRS, Projet CNRS/ENS Lyon/INRIA ARENAIRE, Ecole Normale Supérieure de Lyon

## Réduction d'argument "au vol"

**Résumé :** Il arrive que l'opérande dont on doit calculer une fonction élémentaire soit disponible chiffre après chiffre, en série, en commençant par les poids forts. Nous proposons une solution permettant d'effectuer la première phase de l'évaluation (la réduction d'argument) *au vol*: le calcul et la réception des chiffres d'entrée se recouvrent. Cet algorithme peut être utilisé pour les fonctions trigonométriques sin, cos, tan, ainsi que pour l'exponentielle.

**Mots-clé :** Réduction d'argument, fonctions élémentaires, arithmétique des ordinateurs.

## 1 Introduction

The algorithms used for evaluating the elementary functions only give a correct result if the argument is within some bounded interval. To evaluate an elementary function  $f(x)$  (sine, cosine, exponential, ...) for any  $x$ , one must find some “transformation” that makes it possible to deduce  $f(x)$  from some value  $g(y)$ , where

- $y$ , called the *reduced argument*, is deduced from  $x$ ;
- $y$  belongs to the convergence domain of the algorithm implemented for the evaluation of  $g$ .

With the usual functions, the only cases for which reduction is not straightforward are the cases where  $y$  is equal to  $x - nC$ , where  $n$  is an integer and  $C$  a constant (for instance, for the trigonometric functions,  $C$  is a multiple of  $\pi/8$ ).

**Example 1 (Computation of the cosine function)** *Assume that we want to evaluate  $\cos(x)$ , and that the convergence domain of the algorithm used to evaluate the sine and cosine of the reduced argument contains  $[0, +\pi/4]$ . We choose  $C = \pi/4$ , and the computation of  $\cos(x)$  is decomposed in three steps:*

- compute  $y$  and  $n$  such that  $y \in [0, +\pi/4]$  and  $y = x - n\pi/4$ ;
- compute  $g(y, n) =$

$$\left\{ \begin{array}{ll} \cos(y) & \text{if } n \bmod 8 = 0 \\ \frac{\sqrt{2}}{2} (\cos(y) - \sin(y)) & \text{if } n \bmod 8 = 1 \\ -\sin(y) & \text{if } n \bmod 8 = 2 \\ -\frac{\sqrt{2}}{2} (\cos(y) + \sin(y)) & \text{if } n \bmod 8 = 3 \\ -\cos(y) & \text{if } n \bmod 8 = 4 \\ \frac{\sqrt{2}}{2} (-\cos(y) + \sin(y)) & \text{if } n \bmod 8 = 5 \\ \sin(y) & \text{if } n \bmod 8 = 6 \\ \frac{\sqrt{2}}{2} (\cos(y) + \sin(y)) & \text{if } n \bmod 8 = 7 \end{array} \right. \quad (1)$$

- obtain  $\cos(x) = g(y, n)$ .

**Example 2 (Computation of the exponential function)** *Assume that we want to evaluate  $e^x$  in a radix-2 number system, and that the convergence domain of the algorithm used to evaluate the exponential of the reduced argument contains  $[0, \ln(2)]$ . We can choose  $C = \ln(2)$ , and the computation of  $e^x$  is then decomposed in three steps:*

- compute  $y \in [0, \ln(2)]$  and  $n$  such that  $y = x - n \ln(2)$ ;
- compute  $g(y) = e^y$ ;
- compute  $e^x = 2^n g(y)$ .

Unless multiple-precision arithmetic is used during the intermediate calculations, a straightforward computation of  $y$  as  $x - nC$  is to be avoided, since this operation will lead to catastrophic cancellations (i.e., to very inaccurate estimates of  $y$ ) when  $x$  is large or close to an integer multiple of  $C$ . Many algorithms have been suggested for performing the range reduction accurately [1, 2, 3, 9, 11].

Now, there are many cases (on special-purpose systems) where the input argument of a calculation is generated most significant digit first. This happens, for instance, when this argument is the result of a division or a square root obtained through a digit-recurrence algorithm [7, 10], the output of an on-line algorithm [5, 12], or when it is generated by an analog-to-digital converter.

In the sequel of this paper, we present an adaptation of the Modular Range Reduction Algorithm [3, 8] that accepts such digit serial inputs and performs the range reduction “on the fly”: most of the computation is overlapped with the reception of the input bits, and the reduced argument is produced almost immediately after reception of the last input bit. On-the-fly arithmetic algorithms have already been proposed by Ercegovic and Lang for rounding or converting a number from redundant to non-redundant representation [4, 6].

## 2 Notations

In the sequel of the paper,  $x = x_h x_{h-1} \cdots x_0 . x_{-1} x_{-2} \cdots x_\ell$  is the input argument,  $C = 0.C_{-1} C_{-2} \cdots C_{-p}$  is the constant of the range reduction (with  $-p \leq \ell$ ), and  $y = 0.y_{-1} y_{-2} \cdots y_{-p}$  is the reduced argument. We assume  $1/2 \leq C < 1$ . These values satisfy:

- $0 \leq y < C$ ;
- $n = (x - y)/C$  is an integer.

We also define, for each  $i$ ,  $m_i$  (also called  $2^i \bmod C$ ) as the unique value between 0 and  $C$  such that  $(2^i - m_i)/C$  is an integer. These notations give some constraints on  $x$  and  $C$  (e.g.,  $C$  is less than 1,  $x$  is less than  $2^{h+1}$ ). One can easily adapt the algorithms given in the sequel of the paper to variables belonging to other domains. We chose these constraints to make the presentation of the algorithms simpler.

### 3 Non-redundant algorithm

Algorithm 1 is by far less efficient than the “redundant” algorithm given afterwards. We give it because it is simpler to understand, and because the other algorithm is derived from it. The basic idea is the following: at step  $i$  of the algorithm, when we receive input bit  $x_{h-i}$  of  $x$ , we add  $x_{h-i} \times (2^i \bmod C)$  to an accumulator. If the accumulated value becomes larger than  $C$ , we subtract  $C$  from it.

Let us call  $A_{i+1}$  the value obtained after this operation. One can easily check that  $0 \leq A_{i+1} < C$  and  $A_{i+1} - x_h x_{h-1} \cdots x_{h-i} \times 2^{h-i}$  is an integer multiple of  $C$ . Hence the final value stored in the accumulator is equal to the reduced argument  $y$ .

---

**Algorithm 1** Non-redundant algorithm.

---

```

 $A_0 = 0$ 
for  $i = 0$  to  $h - \ell$  do
     $T_i = A_i + x_{h-i} m_{h-i}$ 
    if  $T_i < C$  then
         $A_{i+1} = T_i$ 
    else
         $A_{i+1} = T_i - C$ 
 $y = A_{h-\ell+1}$ 

```

---

A possible variant consists in computing  $U_i = A_i + x_{h-i} (m_{h-i} - C)$  in parallel with  $T_i$ , and then to choose  $A_{i+1}$  equal to  $U_i$  if  $U_i \geq 0$ , otherwise  $T_i$ .

### 4 Redundant algorithm

Now, to accelerate the reduction, we assume that we perform the accumulations with *carry-save* additions. The carry-save number system allows very fast, carry-free additions. On the other hand, its intrinsic redundancy makes comparisons somewhat more complex. The accumulator will store the values  $A_i$  in carry-save. In the previous algorithm, we needed “exact” comparisons between the  $A_i$ ’s and  $C$ . Having the  $A_i$ ’s stored in carry-save makes these “exact” comparisons difficult. Instead of that, we will perform comparisons based on the examination of the first three carry-save positions of  $A_i$  only. This will not allow to bound the  $A_i$ ’s by  $C$ . Nevertheless, we will show that the  $A_i$ ’s will be upper-bounded by



$C + \frac{1}{2}$  (therefore by  $\frac{3}{2}$ ), which will suffice for our purpose. We denote:

$$A_i = \left( (A_{i,0}^{(1)}, A_{i,0}^{(2)}); (A_{i,-1}^{(1)}, A_{i,-1}^{(2)}); (A_{i,-2}^{(1)}, A_{i,-2}^{(2)}); \dots; (A_{i,-p}^{(1)}, A_{i,-p}^{(2)}) \right)$$

where  $A_{i,j}^{(1)}$  and  $A_{i,j}^{(2)}$  are in  $\{0, 1\}$  and

$$A_i = \sum_{j=0}^p (A_{i,j}^{(1)} + A_{i,j}^{(2)}) \cdot 2^{-j}.$$

The variable  $T_i$  of the non-redundant algorithm is used again, and is also represented in carry-save form:

$$T_i = \left( (T_{i,0}^{(1)}, T_{i,0}^{(2)}); (T_{i,-1}^{(1)}, T_{i,-1}^{(2)}); (T_{i,-2}^{(1)}, T_{i,-2}^{(2)}); \dots; (T_{i,-p}^{(1)}, T_{i,-p}^{(2)}) \right)$$

This gives algorithm 2.

---

**Algorithm 2** Redundant algorithm.

---

$A_0 = 0 + 1$

**for**  $i = 0$  **to**  $h - \ell$  **do**

$T_i = A_i +_{\text{cs}} x_{h-i} m_{h-i}$

$\hat{T}_i = \left( (T_{i,0}^{(1)}, T_{i,0}^{(2)}); (T_{i,-1}^{(1)}, T_{i,-1}^{(2)}); (T_{i,-2}^{(1)}, T_{i,-2}^{(2)}) \right) - 1$

converted to non-redundant binary using a 3-bit adder

**if**  $\hat{T}_i < C$  **then**

$A_{i+1} = T_i$

**else**

$A_{i+1} = T_i -_{\text{cs}} C$  (or  $T_i +_{\text{cs}} (1 - C) - 1$ )

$B = A_{h-\ell+1} +_{\text{cs}} (1 - C)$

Convert  $A_{h-\ell+1}$  and  $B$  to non-redundant binary.

**if**  $B < 2$  **then**

$y = A_{h-\ell+1} - 1$

**else**

$y = B - 2$

---

In the loop, we do not want to waste time with a full comparison to know whether we need to subtract  $C$  from  $T_i$  or not. Thus we use a rough approximation  $\hat{T}_i$  to  $T_i$  based on the first three digits of  $T_i$ . Since

$$\left( (T_{i,-3}^{(1)}, T_{i,-3}^{(2)}); \dots; (T_{i,-p}^{(1)}, T_{i,-p}^{(2)}) \right) \leq 2 \cdot 2^{-3} + 2 \cdot 2^{-4} + \dots + 2 \cdot 2^{-p} < \frac{1}{2},$$

we have:

$$\widehat{T}_i \leq T_i < \widehat{T}_i + \frac{1}{2}$$

We want to ensure that  $A_i$  is always positive, that is,  $T_i - C$  does not lead to a negative number. Then, the subtraction is performed only when  $\widehat{T}_i \geq C$ . In this case,  $T_i - C \geq \widehat{T}_i - C \geq 0$ .

Now, we want to find an upper bound on all the  $A_i$ 's (and one on the  $T_i$ 's). Suppose that for a given  $i$ , we have  $A_i \leq M$ . Thus  $T_i \leq M + C$ . If  $\widehat{T}_i < C$ , then  $A_{i+1} = T_i < \widehat{T}_i + \frac{1}{2} < C + \frac{1}{2}$ ; otherwise,  $A_{i+1} = T_i - C \leq M$ . If we choose  $M = C + \frac{1}{2}$ , then  $A_{i+1} \leq M$  in both cases. By induction,  $A_i \leq C + \frac{1}{2}$  and  $T_i \leq 2C + \frac{1}{2}$  for all  $i$ .

The final value of  $y$  is converted to non-redundant representation using a conventional (i.e., non-redundant) addition. Another, faster, solution is to convert it on-the-fly, during the second loop of the algorithm, using Ercegovac and Lang's on-the-fly algorithm [4, 6] for conversion from redundant to non-redundant representation.

## 5 An example: computation of $\cos(1010.111)$ .

We choose  $C = \pi/4 \approx 0.1100101$  ( $p = 7$ ). Since  $x = 1010.111$ , we have  $h = 3$  and  $\ell = -3$ ).

$$\text{The values of the } m_i \text{'s are: } \begin{cases} m_3 & = 2^3 \pmod{\pi/4} \approx 0.0010011 \\ m_2 & = 2^2 \pmod{\pi/4} \approx 0.0001001 \\ m_1 & = 2^1 \pmod{\pi/4} \approx 0.0110111 \\ m_0 & = 2^0 \pmod{\pi/4} \approx 0.0011011 \\ m_{-1} & = 2^{-1} \pmod{\pi/4} = 0.1 \\ m_{-2} & = 2^{-2} \pmod{\pi/4} = 0.01 \\ m_{-3} & = 2^{-3} \pmod{\pi/4} = 0.001 \end{cases}$$

The carry-save representations of the variables  $T_i$  and  $A_i$  generated by the redundant algorithm are

$x_3 = 1$	$T_0 = \begin{cases} 1.0010011 \\ 0.0000000 \end{cases}$	$0 < C$	$A_0 = \begin{cases} 1.0010011 \\ 0.0000000 \end{cases}$
$x_2 = 0$	$T_1 = \begin{cases} 1.0010011 \\ 0.0000000 \end{cases}$	$0 < C$	$A_1 = \begin{cases} 1.0010011 \\ 0.0000000 \end{cases}$
$x_1 = 1$	$T_2 = \begin{cases} 1.0100100 \\ 0.0100110 \end{cases}$	$0.1 < C$	$A_2 = \begin{cases} 1.0100100 \\ 0.0100110 \end{cases}$
$x_0 = 0$	$T_3 = \begin{cases} 1.0000010 \\ 0.1001000 \end{cases}$	$0.1 < C$	$A_3 = \begin{cases} 1.0000010 \\ 0.1001000 \end{cases}$
$x_{-1} = 1$	$T_4 = \begin{cases} 1.0001010 \\ 1.0000000 \end{cases}$	$1 \geq C$	$A_4 = \begin{cases} 1.0010001 \\ 0.0010100 \end{cases}$
$x_{-2} = 1$	$T_5 = \begin{cases} 1.0100101 \\ 0.0100000 \end{cases}$	$0.1 < C$	$A_5 = \begin{cases} 1.0100101 \\ 0.0100000 \end{cases}$
$x_{-3} = 1$	$T_6 = \begin{cases} 1.0010101 \\ 0.1000000 \end{cases}$	$0.1 < C$	$A_6 = \begin{cases} 1.0010101 \\ 0.1000000 \end{cases}$

We then get  $y = 0.1010101$ , whereas the exact value of  $x \bmod \pi/4$  is  $0.10101010001\dots$

## 6 Conclusion

The redundant algorithm presented in Section 4 allows fast, on-the-fly, range reduction. The accuracy of this method is the same as that of the Conventional Modular range reduction method (see [3, 8]).

## References

- [1] W. Cody and W. Waite. *Software Manual for the Elementary Functions*. Prentice-Hall, Englewood Cliffs, NJ, 1980.
- [2] W. J. Cody. Implementation and testing of function software. In P. C. Messina and A. Murli, editors, *Problems and Methodologies in Mathematical Software Production*, Lecture Notes in Computer Science 142. Springer-Verlag, Berlin, 1982.
- [3] M. Dumas, C. Mazenc, X. Merrheim, and J. M. Muller. Modular range reduction: A new algorithm for fast and accurate computation of the elementary functions. *Journal of Universal Computer Science*, 1(3):162–175, March 1995.

- 
- [4] M. D. Ercegovac and T. Lang. On-the-fly conversion of redundant into conventional representations. *IEEE Transactions on Computers*, C-36(7), July 1987. Reprinted in E. E. Swartzlander, *Computer Arithmetic*, Vol. 2, IEEE Computer Society Press Tutorial, Los Alamitos, CA, 1990.
  - [5] M. D. Ercegovac and T. lang. On-line arithmetic: a design methodology and applications in digital signal processing. In *VLSI Signal Processing III*, pages 252–263, 1988. Reprinted in E. E. Swartzlander, *Computer Arithmetic*, Vol. 2, IEEE Computer Society Press Tutorial, Los Alamitos, CA, 1990.
  - [6] M. D. Ercegovac and T. Lang. On-the-fly rounding. *IEEE Transactions on Computers*, 41(12):1497–1503, December 1992.
  - [7] M. D. Ercegovac and T. Lang. *Division and Square Root: Digit-Recurrence Algorithms and Implementations*. Kluwer Academic Publishers, Boston, 1994.
  - [8] J.M. Muller. *Elementary Functions, Algorithms and Implementation*. Birkhauser, Boston, 1997.
  - [9] M. Payne and R. Hanek. Radian reduction for trigonometric functions. *SIGNUM Newsletter*, 18:19–24, 1983.
  - [10] J. E. Robertson. A new class of digital division methods. *IRE Transactions on Electronic Computers*, EC-7:218–222, 1958. Reprinted in E. E. Swartzlander, *Computer Arithmetic*, Vol. 1, IEEE Computer Society Press Tutorial, Los Alamitos, CA, 1990.
  - [11] R. A. Smith. A continued-fraction analysis of trigonometric argument reduction. *IEEE Transactions on Computers*, 44(11):1348–1351, November 1995.
  - [12] K. S. Trivedi and M. D. Ercegovac. On-line algorithms for division and multiplication. In *3rd IEEE Symposium on Computer Arithmetic*, pages 161–167, Dallas, Texas, USA, 1975. IEEE Computer Society Press, Los Alamitos, CA.



---

Unit e de recherche INRIA Lorraine, Technop le de Nancy-Brabois, Campus scientifique,  
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY  
Unit e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex  
Unit e de recherche INRIA Rh ne-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN  
Unit e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex  
Unit e de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

---

 diteur  
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)  
<http://www.inria.fr>  
ISSN 0249-6399