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Abstract: Natural neighbour coordinates and natural neighbour interpolation have been introduced by Sibson for interpolating multivariate scattered data. In this paper, we consider the case where the data points belong to a smooth surface $\mathcal{S}$, i.e. a $(d-1)$-manifold of $\mathbb{R}^d$. We show that the natural neighbour coordinates of a point X belonging to $\mathcal{S}$ tends to behave as a local system of coordinates on the surface when the density of points increases. Our result does not assume any knowledge about the ordering, connectivity or topology of the data points or of the surface. An important ingredient in our proof is the fact that a subset of the vertices of the Voronoi diagram of the data points converges towards the medial axis of $\mathcal{S}$ when the sampling density increases.

Key-words: Computational Geometry, Voronoi diagrams, Medial Axis, Natural Neighbour Interpolation, Surface Reconstruction.

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Coordonnées naturelles de points sur une surface

Résumé : Les coordonnées naturelles ont été introduites par Sibson pour interpoler des fonctions de plusieurs variables à partir de données non structurées. On considère dans cet article le cas où les points de données appartiennent à une surface $\mathcal{S}$, c'est-à-dire une variété de dimension $d-1$ de $\mathbb{R}^d$, lisse et sans bord. On montre que lorsque la densité de l'échantillonnage augmente, les coordonnées naturelles d'un point de la surface tendent à se comporter comme un système de coordonnées locales sur la surface. Ce résultat ne suppose aucune connaissance sur l'ordre ou la connectivité des points, ni d'information sur la topologie de la surface. Il est basé sur le fait qu'un sous-ensemble des sommets du diagramme de Voronoï des points de données tend vers le squelette de $\mathcal{S}$ lorsque la densité d'échantillonnage augmente.

Mots-clés : Géométrie algorithmique, Diagrammes de Voronoï, Squelette, Interpolation, Voisins Naturels, Reconstruction de Surfaces.

1 Introduction

Natural neighbour coordinates and natural neighbour interpolation have been introduced by Sibson [19] for interpolating multivariate scattered data. Given a set of points $\mathcal{A} = \{A_1, \ldots, A_n\}$, the associated system of natural neighbour coordinates is a set of continuous functions $\lambda_i : \mathbb{R}^d \longrightarrow \mathbb{R}$, $i = 1, \ldots, n$, defined from the Voronoi diagram of $\mathcal{A}$.

In this paper, we consider the case where the data points are scattered over a surface $\mathcal{S}$, i.e. a $(d-1)$-manifold of $\mathbb{R}^d$. We show that the set of natural neighbour coordinates of a point X belonging to $\mathcal{S}$ tends to behave as a local system of coordinates on the surface when the density of points increases. Our result does not assume any knowledge about the ordering, connectivity or topology of the data points or of $\mathcal{S}$.

This work is motivated by the many application domains where surfaces are to be reconstructed from a sample of unorganized data. Such data may be provided by various sensors or may result from a mathematical analysis. In a companion paper [6], we derive a new method for surface reconstruction based on the natural neighbour interpolation and the results of this paper.

The paper is organized as follows. In section 2, we recall the definition and the main properties of natural neighbour coordinates in $\mathbb{R}^d$. In section 3, we consider the case where the points belong to a surface. We recall such definitions as medial axis, local feature size and Voronoi diagram of points on a surface and derive some basic results. In section 4, we prove that some vertices of the Voronoi diagram of $\mathcal{A}$ converge towards the medial axis of $\mathcal{S}$ when the sampling density increases. Finally, in section 5, we analyze the behaviour of the natural neighbour coordinates of a point X of $\mathcal{S}$ when the sampling density increases.

2 Natural neighbour coordinates

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of points. The Voronoi cell of A_i is

$$V(A_i) = \{X \in \mathbb{R}^d : \|X - A_i\| \leq \|X - A_j\| \;\; \forall j = 1, \ldots, n\}$$

where $\|X - Y\|$ denotes the Euclidean distance between points $X, Y \in \mathbb{R}^d$. The collection of Voronoi cells is called the *Voronoi diagram* of $\mathcal{A}$. Let $\mathcal{A}'$ be a subset of points of $\mathcal{A}$ whose Voronoi cells have a non empty intersection. The convex hull $conv(\mathcal{A}')$ is called a Delaunay face and the collection of all Delaunay faces is a geometric complex called the Delaunay triangulation of $\mathcal{A}$, denoted $Del(\mathcal{A})$. It is well known that if there is no sphere passing through $d+2$ points of $\mathcal{A}$, $Del(\mathcal{A})$ is a simplicial complex, and that the balls circumscribing the d-simplices in $Del(\mathcal{A})$ cannot contain a point of $\mathcal{A}$ in their interior.

Given a point X, we define $Vor^+ = Vor(\mathcal{A} \cup \{X\})$ and $Del^+ = Del(\mathcal{A} \cup \{X\})$. In addition $V^+(X)$ denotes the Voronoi cell of X in Vor^+.

Definition 1 *A ball is said to be* empty *if its interior does not contain any point of $\mathcal{A}$.*

Definition 2 *The natural neighbours of a point X with respect to $\mathcal{A}$ are the vertices other than X of the simplices of Del^+ incident to X.*

Let $V(X, A_i) = V^+(X) \cap V(A_i)$ (see Figure 1). If $V(X, A_i) \neq \emptyset$, A_i is a natural neighbour of X. Let $w_i(X)$ be the volume of $V(X, A_i)$ and let $w(X)$ be the sum, over all natural neighbours, of the $w_i(X)$.

Observe that $w_i(X)$ is bounded unless X lies outside the convex hull $CH(\mathcal{A})$ of $\mathcal{A}$ and A_i is a vertex of the convex hull. In the rest of this section, we restrict our attention to points X that lie in $CH(\mathcal{A})$.

Definition 3 (Sibson) *The natural neighbour coordinates of X with respect to $\mathcal{A}$ are the $\lambda_i(X) = \frac{w_i(X)}{w(X)}$, $i = 1, \ldots, n$.*

The natural neighbour coordinates have several interesting properties.

Property 1 *For any $i \leq n$, $\lambda_i(A_j) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol.*

Property 2 *For $i = 1, \ldots, n$, $\lambda_i(X) \geq 0$ and $\sum_{i=1}^{n} \lambda_i(X) = 1$.*

Sibson has proved the following important property that justifies the term coordinates [19]. Besides the initial proof of Sibson, several alternative proofs are known [5, 8, 13].

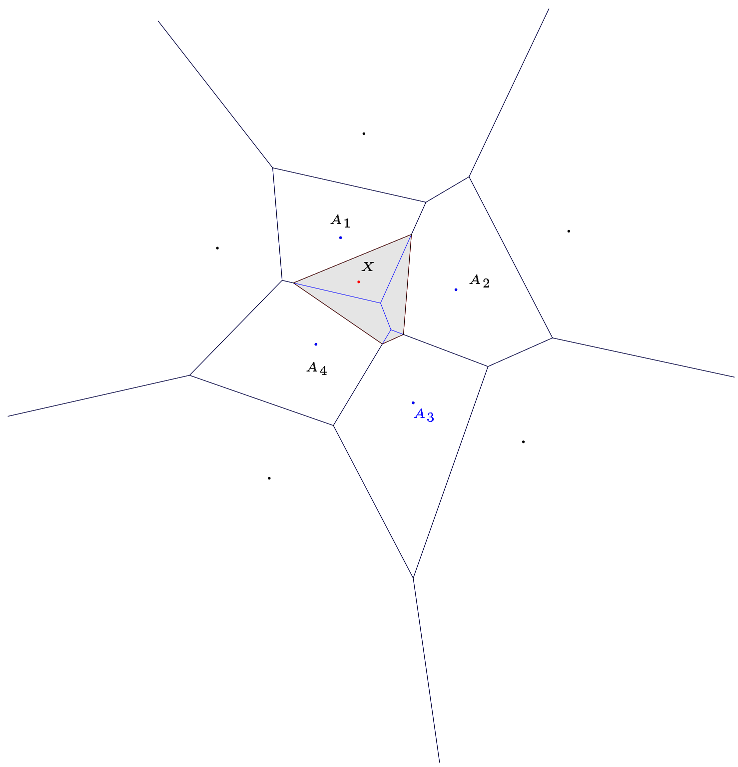

Figure 1: X has four natural neighbours $A_1, \ldots, A_4$.

Property 3 (Sibson) $X = \sum_i \lambda_i(X)\ A_i$.

Let us consider the Delaunay balls circumscribing the d-simplices of the Delaunay triangulation of $\mathcal{A}$, called the *Delaunay balls* of $\mathcal{A}$ for short. We say that a Delaunay ball *conflicts* with a point X if X belongs to the interior of the ball. The natural neighbours of X are the vertices of the Delaunay d-simplices whose circumscribing balls conflict with X.

Let $\mathcal{D}$ denote the arrangement of the spheres bounding the Delaunay balls of $\mathcal{A}$. Any point X in the interior of a cell of $\mathcal{D}$ has the same natural neighbours. We associate to each cell Γ of $\mathcal{D}$ a set of indexes $I(\Gamma)$ as follows. If a ball circumscribing a d-simplex S of $Del(\mathcal{A})$ covers Γ, we add the indexes of the vertices of S to $I(\Gamma)$.

All points in the interior of Γ have the same natural neighbours which are the points of $\mathcal{A}$ with indexes in $I(\Gamma)$.

When X reaches a $(d-1)$-face of the boundary of a cell Γ of $\mathcal{D}$, X gains or looses a natural neighbour, depending whether X enters a Delaunay ball that does not cover Γ or goes out of a Delaunay ball that covers Γ. More generally, if X reaches a k-face of Γ, $0 < k < d$, $d-k$ natural neighbours of X change, some of them being gained, others being lost.

This discussion leads to the following property.

Property 4 (Combinatorial structure) *Let $\mathcal{N}$ be the equivalence relation that relates two points if they have the same natural neighbours with respect to $\mathcal{A}$. The equivalence classes of $\mathcal{N}$ are the faces of the arrangement of the spheres circumscribing the d-simplices of $Del(\mathcal{A})$.*

Let Λ_i denote the *support* of λ_i, i.e. the subset of X such that $\lambda_i(X) \neq 0$.

Property 5 (Support) *Λ_i is included in the union of the balls circumscribing the d-simplices of $Del(\mathcal{A})$ that are incident to A_i.*

Property 6 (Locality) *Let $\mathcal{A}$ be a finite set of points in $\mathbb{R}^d$ and $\mathcal{R} \subset CH(\mathcal{A})$ a region such that, for all $X \in \mathcal{R}$, there exists $A_i \in \mathcal{A}$, $\|X - A_i\| \leq \varepsilon$. Then for all $X \in \mathcal{R}$, Λ_i is included in the ball $B(A_i, 2\varepsilon)$ with center A_i and radius 2ε, $i = 1, \ldots, n$.*

Proof. Let $X \in \mathcal{R}$ be such that $\lambda_i(X) \neq 0$. A_i is a natural neighbour of X and therefore X belongs to the circumscribing ball of a d-simplex S of $Del(\mathcal{A})$ incident to A_i. Let V be the circumcenter of S. We have $\|V - X\| \leq \|V - A_i\|$ and $\|V - A_i\| \leq \varepsilon$ by hypothesis. Hence, $\|X - A_i\| \leq \|X - V\| + \|V - A_i\| \leq 2\varepsilon$. □

As a consequence of this last property, when the sampling density increases (i.e. when ε decreases), the support Λ_i of λ_i becomes a small neighborhood of A_i ($i = 1, \ldots, n$).

The following property is stated without proof in [19] and discussed in more detail in [11]. The formula for the gradient is due to Piper [16].

Property 7 (Differentiability) *$\lambda_i(X)$ is a function that is continuous at all X and continuously differentiable at all $X \notin \mathcal{A}$. We have*

$$\nabla w_i(X) = \frac{\mu_i(X)}{\|X - A_i\|} \overrightarrow{XC_i},$$

where $\mu_i(X)$ and C_i are respectively the $(d-1)$-volume and the centroid of the Voronoi facet common to $V^+(X)$ and $V(X, A_i)$.

The next property is a direct consequence of the definition of the natural neighbours. Implementation details and experimental results can be found in the companion paper [6].

Property 8 (Time complexity) *The time complexity of computing the natural neighbour coordinates of a point X is the same as the time complexity of inserting a point in the Delaunay triangulation of $\mathcal{A}$.*

In view of Properties 1, 2, 3 and 6, and of the continuity of the λ_i, we say that the set of λ_i is a *local coordinate system* associated to $\mathcal{A}$.

Since the pionneering work of Sibson, other Voronoi-based systems of coordinates have been proposed [8, 13].

3 Sampled surfaces : definitions and preliminary results

Let $\mathcal{O}$ be a compact region of $\mathbb{R}^d$ whose boundary $\mathcal{S}$ is a smooth surface, i.e. a twice-differentiable $(d-1)$-manifold. $B(X, r)$ denotes the ball centered at X with radius r and $\Sigma(X, r)$ its bounding sphere. $\vec{n}_{\mathrm{X}}$ denotes the unit normal to $\mathcal{S}$ at X directed outwards from $\mathcal{O}$ (see Figure 2).

Let $\mathcal{A}$ denote a set of n points $A_1, \ldots, A_n$ on $\mathcal{S}$. A *local system of coordinate on $\mathcal{S}$ associated to $\mathcal{A}$* is a set of continuous functions $\sigma_i : \mathcal{S} \longrightarrow \mathbb{R}$, $i = 1, \ldots, n$ such that, for all $X \in \mathcal{S}$:

1. $X = \sum_i \sigma_i(X)\, A_i$,

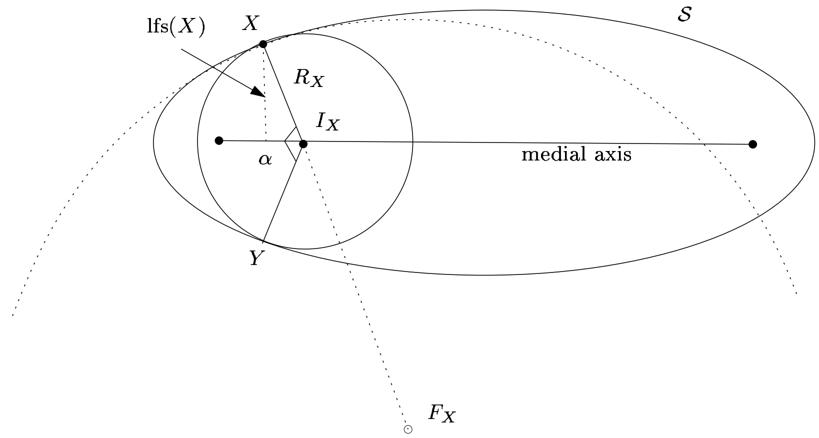

Figure 2: For the definitions.

2. For any $i \leq n$, $\sigma_i(A_j) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol,
3. $\sum_i \sigma_i(X) = 1$,
4. If the surface is well sampled, for any $i \leq n$, the support of σ_i is a small neighborhood of A_i.

One way to determine such σ_i could be to resort to the Voronoi diagram of $\mathcal{A}$ on the surface $\mathcal{S}$ where the Euclidean distance is replaced by a Riemannian metric on $\mathcal{S}$. However such diagrams are much more complicated than Euclidean diagrams and difficult to compute [14]. Moreover, in some applications, such as surface reconstruction, the surface itself is unknown and computing Voronoi diagrams on the surface is impossible. We prefer to follow a different approach that only uses Euclidean Voronoi diagrams and natural neighbours in $\mathbb{R}^d$.

This section introduces some basic definitions and results, and make precise what we mean by a good sample. These results will be used in the next two sections and, in particular, in section 5, where local systems of coordinates will be considered again.

3.1 Medial axis and local feature size

Definition 4 *A ball is said to be* maximal *if (1) its interior does not intersect $\mathcal{S}$, (2) it cannot be included in a larger ball satisfying (1).*

There are two maximal balls passing through a point $X \in \mathcal{S}$. We denote by B_{X} the one that is contained in $\mathcal{O}$, Σ_{X} its bounding sphere, I_{X} its center and R_{X} its radius (see Figure 2). We use the superscript e for the other ball B^e_{X}, its bounding sphere Σ^e_{X}, its center I^e_{X} and its radius R^e_{X}.

Definition 5 *The medial axis of $\mathcal{O}$ consists of the centers of the maximal balls.*

Definition 6 *Let $X \in \mathcal{S}$. A point on the line $\{X + t\vec{n}_X, t \in \mathbb{R}\}$ is called a focal point if $t = \kappa_i^{-1}(X)$ where $\kappa_i(X)$ is one of the principal curvatures of $\mathcal{S}$ at X.*

For any $X \in \mathcal{S}$, let F_{X} be the focal point on the line $\{X + t\vec{n}_{\mathrm{X}}, t < 0\}$ that is closer to X. I_{X} belongs to the line segment $[XF_{\mathrm{X}}]$ and $R_{\mathrm{X}} \leq \min_i(|\kappa_i^{-1}(X)|)$.

If I_{X} is distinct from F_{X} and therefore belongs to the relative interior of $[XF_{\mathrm{X}}]$, the maximal sphere Σ_{X}, which is centered at point I_{X} and tangent to $\mathcal{S}$ at X, is tangent to $\mathcal{S}$ in at least another point $Y_{\mathrm{X}} \neq X$. We denote by η_{X} the minimum over all such points Y_{X} of $\frac{\|X-Y_{\mathrm{X}}\|}{2R_{\mathrm{X}}}$. Observe that η_{X} belongs to the interval $[0,1]$ and is 0 when $I_{\mathrm{X}} = F_{\mathrm{X}}$. Moreover,

$$2\,\eta_{\mathrm{X}}^2 = 2\sin^2\frac{\alpha_{\mathrm{X}}}{2} = 1 - \cos\alpha_{\mathrm{X}},$$

where α_{X} denotes the *bisector angle* $\angle XI_{\mathrm{X}}Y_{\mathrm{X}}$ of $\mathcal{S}$ at X.

Similarly, for the maximal ball B^e_{X}, we denote by Y^e_{X} one of its contact point other than X, and by η^e_{X} the minimum over all such points Y^e_{X} of $\frac{\|X-Y^e_{\mathrm{X}}\|}{2R^e_{\mathrm{X}}}$.

We borrow from Amenta and Bern [1] the notion of local feature size. A related notion is the r-regularity introduced by Serra [18] (see also [4, 7]).

Definition 7 (Amenta & Bern) *The* local feature size $\mathrm{lfs}(X)$ *at a point $X \in \mathcal{S}$ is the Euclidean distance from X to the medial axis of $\mathcal{S}$.*

Lemma 9 *For any* $X, Y \in \mathcal{S}$, $\mathrm{lfs}(X) \leq \mathrm{lfs}(Y) + \|X - Y\|$.

Proof. $B(X, \mathrm{lfs}(Y) + \|X - Y\|)$ contains $B(Y, \mathrm{lfs}(Y))$. Since, by definition of the local feature size, the latter intersects the medial axis of $\mathcal{S}$, the same is true for the former, which proves the lemma. □

3.2 Voronoi diagram on a surface

We first define in this section the Voronoi diagram of a set of points restricted to a surface, following previous work by Chew [9] and Edelsbrunner and Shah [10].

Definition 8 (Chew) *The Voronoi diagram of* $\mathcal{A}$ *restricted to* $\mathcal{S}$ *is the (curved) cell complex obtained by intersecting each face of* $Vor(\mathcal{A})$ *with* $\mathcal{S}$. *We denote it by* $Vor_{\mathcal{S}}(\mathcal{A})$.

Similarly, we can define the Voronoi diagram of $\mathcal{A}$ restricted to $\mathcal{O}$, denoted $Vor_{\mathcal{O}}(\mathcal{A})$, as the cell complex obtained by intersecting each face of $Vor(\mathcal{A})$ with $\mathcal{O}$.

We denote by $V_{\mathcal{S}}(A_i)$ the cell of $Vor_{\mathcal{S}}(\mathcal{A})$ consisting of the points of $\mathcal{S}$ that are closer to A_i than to any A_j, $j \neq i$. A vertex of $V_{\mathcal{S}}(\mathcal{A})$ is the intersection of an edge of the Voronoi diagram of $\mathcal{A}$ with $\mathcal{S}$. Hence, it is the center of an empty ball passing through d points of $\mathcal{A}$.

Definition 9 *The Delaunay triangulation of* $\mathcal{A}$ *restricted to* $\mathcal{S}$ *is the subcomplex of* $Del(\mathcal{A})$ *consisting of the facets of* $Del(\mathcal{A})$ *whose dual Voronoi edges intersect* $\mathcal{S}$. *We denote it by* $Del_{\mathcal{S}}(\mathcal{A})$.

Observe that the facets of $Del_{\mathcal{S}}(\mathcal{A})$ are the facets of $Del(\mathcal{A})$ that can be circumscribed by an empty ball centered on $\mathcal{S}$. Such a ball is called a *$\mathcal{S}$-Delaunay ball.*

Let us now look at the natural neighbours of a point X of $\mathcal{S}$. Typically X has natural neighbours that are close to X on the surface and others that are far away, usually on both sides of the tangent plane to $\mathcal{S}$ at X.

Definition 10 *The* $\mathcal{S}$*-natural neighbours of a point* X *of* $\mathcal{S}$ *are the vertices of the facets of* $Del_{\mathcal{S}}(\mathcal{A} \cup \{X\})$ *that are incident to* X.

3.3 ε-samples

Definition 11 (Amenta & Bern) *$\mathcal{A}$ is called a ε-*sample of *$\mathcal{S}$ if $\mathcal{A} \subset \mathcal{S}$ and if, for all $X \in \mathcal{S}$, there exists a point A_i such that $\|X - A_i\| \leq \varepsilon \mathrm{lfs}(X)$. When $\varepsilon < 1$, the sample is said to be a* good *sample.*

Lemma 10 *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a good ε-sample of $\mathcal{S}$, and X a point of $\mathcal{S}$. 1. If A_i is the point of $\mathcal{A}$ closest to X, $\|X - A_i\| \leq \frac{\varepsilon}{1-\varepsilon} \mathrm{lfs}(A_i)$. 2. For any $\mathcal{S}$-natural neighbour of X, we have $\|X - A_i\| \leq \frac{2\varepsilon}{1-\varepsilon} \mathrm{lfs}(X)$ and $\|X - A_i\| \leq \frac{2\varepsilon}{1-\varepsilon} \mathrm{lfs}(A_i)$.*

Proof. 1. Since $\mathcal{A}$ is a good ε-sample and thanks to Lemma 9, we have $\|X - A_i\| \leq \varepsilon \mathrm{lfs}(X) \leq \varepsilon(\mathrm{lfs}(A_i) + \|X - A_i\|)$, which implies the first part of the lemma.

2. Let A_i be a $\mathcal{S}$-natural neighbour of X. A_i is a vertex of some facet F of $Del_{\mathcal{S}}(\mathcal{A})$ whose circumscribing $\mathcal{S}$-Delaunay ball B_F contains X. Denoting by V the center of B_F, we have $\|V - X\| \leq \|V - A_i\|$ and $\|V - A_i\| \leq \varepsilon \mathrm{lfs}(V)$ since $\mathcal{A}$ is a ε-sample.

Moreover, $\|X - A_i\| \leq \|X - V\| + \|V - A_i\| \leq 2\varepsilon \mathrm{lfs}(V)$. By Lemma 9, $\mathrm{lfs}(V) \leq \mathrm{lfs}(X) + \|X - V\| \leq \mathrm{lfs}(X) + \varepsilon\, \mathrm{lfs}(V)$. Hence $\mathrm{lfs}(V) \leq \frac{1}{1-\varepsilon} \mathrm{lfs}(X)$.

Similarly, $\mathrm{lfs}(V) \leq \mathrm{lfs}(A_i) + \|V - A_i\| \leq \mathrm{lfs}(A_i) + \varepsilon\, \mathrm{lfs}(V)$. Hence $\mathrm{lfs}(V) \leq \frac{1}{1-\varepsilon} \mathrm{lfs}(A_i)$.

□

The following lemma has been proved by Amenta and Bern [1, Lemma 5]. Although the lemma was originally stated for $d = 3$, the proof holds for any d.

Lemma 11 (Amenta & Bern) *Assume that $\mathcal{A}$ is a good ε-sample of $\mathcal{S}$. Let $X \in \mathcal{S}$ and let V be a vertex of $V^+(X)$ such that $\|V - X\| \geq \eta\, \mathrm{lfs}(X)$ for $\eta > 0$. The angle at X between the normal to $\mathcal{S}$ at X and the vector to V (oriented so that the angle is acute) is at most* $\arcsin(\frac{\varepsilon}{\eta(1-\varepsilon)}) + \arcsin(\frac{\varepsilon}{1-\varepsilon})$.

The following lemma will be useful later.

Lemma 12 *If θ is as in Lemma 11 and $\eta \geq 1$, then* $\cos\theta \geq 1 - \frac{2\,\varepsilon^2}{(1-\varepsilon)^2}$.

Proof. $\cos\theta \geq \cos(2 \arcsin \frac{\varepsilon}{1-\varepsilon}) = 1 - 2 \sin^2(\arcsin \frac{\varepsilon}{1-\varepsilon}) = 1 - 2\, \frac{\varepsilon^2}{(1-\varepsilon)^2}$ □

3.4 Topological balls on the surface

The following lemma extends to higher dimensions an analogous result proved by Amenta, Bern and Eppstein [2] for $d = 2$.

Proposition 13 *Let $B = B(X, R)$ be a ball that intersects $\mathcal{S}$. If $B \cap \mathcal{S}$ is not a topological ball, then B contains a point of the medial axis of $\mathcal{S}$.*

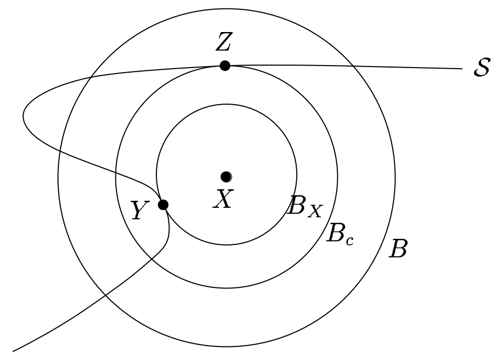

Figure 3: For the proof of Proposition 14.

Proof. The result is clearly true if X lies on the medial axis. Consider the other case and assume that $B \cap \mathcal{S}$ is not empty nor a topological ball. Let X be the center of B and R be its radius. We denote by B_{X} the maximal ball centered at X and by R_{X} its radius. B_{X} is tangent to $\mathcal{S}$ in a unique (since X does not belong to the medial axis) point Y. $B_{\mathrm{X}} \cap \mathcal{S} = \{Y\}$ and is therefore a topological ball. It follows from Lemma 22 and Theorem 23 that the distance function $\delta_{\mathrm{X}}(Y) = \|X - Y\|^2$ has a critical point Z, with $R_{\mathrm{X}} < \|X - Z\| \leq R$ (see Figure 3). By Lemma 22 again, $B_c = B(X, \|X - Z\|)$ is tangent to $\mathcal{S}$ at Z. One of the two maximal balls $B_{\mathrm{X}}(Z)$ or $B_{\mathrm{X}}^{e}(Z)$ tangent to $\mathcal{S}$ at Z is contained in B_c (since it cannot contain Y) and therefore in B, and its center belongs to the medial axis of $\mathcal{S}$. This proves the lemma. $\square$

Proposition 14 *For any $X \in \mathcal{S}$ and any $r < \operatorname{lfs}(X)$, $\mathcal{S} \bigcap B(X, r)$ is a topological $(d-1)$-ball.*

Proof. By definition of $\text{lfs}(X)$, $B(X, r)$ cannot intersect the medial axis of $\mathcal{S}$. Proposition 13 then implies that $\mathcal{S} \bigcap B(X, r)$ is a topological $(d-1)$-ball. $\square$

Corollary 15 *$\mathcal{S}$ cuts $B(X, \text{lfs}(X))$ into two regions, one entirely inside $\mathcal{O}$ and one entirely outside $\mathcal{O}$.*

4 Approximation of the medial axis

In this section, we show that the centers of a subset of the Delaunay balls converge towards the medial axis when the sampling density increases. This is an extension to higher dimensions of a result proved in the plane by Schmitt [17] (see also [7]). More precisely, Schmitt proved that, when ε tends to 0, the centers of all the Delaunay circles converge towards the medial axis of $\mathcal{S}$. This result does not extend in higher dimensions. Indeed, Amenta, Bern and Eppstein have shown that, even in three dimensions, the centers of some Delaunay balls may be far away from the medial axis [2]. Propositions 17, 18 and 19 below provide convergence results that hold in any dimension. We first prove the following technical lemma.

Given are two spheres Σ and Σ_{X} of the same radius R_{X} and passing through a point X, and a point A (see Figure 4). Let I and I_{X} denote the centers of Σ and Σ_{X}, θ the angle[1] $\angle IXI_{\mathrm{X}}$, Σ_A the sphere tangent to Σ at X and passing through A, C its center and α be the angle $\angle XI_{\mathrm{X}}A$.

Lemma 16 *Assume that A lies at distance $R_X(1+\rho)$ from I_X, for some small positive ρ, $\theta = c\,\rho(1+O(\rho))$ for some (positive) constant c, and $\cos\alpha \leq \frac{\cos\theta}{1+\rho}$. Then $\overrightarrow{XI} \cdot \overrightarrow{XA} > 0$ and we have for any point P on the line segment $[IC]$,*

$$\|I_X - P\| \leq \sqrt{c^2 + \frac{(1+c)^2}{(1-\cos\alpha)^2}}\, R_X\,\rho\ (1+O(\rho)).$$

Proof. For the proof that $\overrightarrow{XI} \cdot \overrightarrow{XA} > 0$, refer to Figure 5. H denotes the hyperplane passing through X and tangent to Σ at X, and J the projection of I_{X} onto H. The

[1] In the paper, angles are taken in the interval $[0, 2\pi)$.

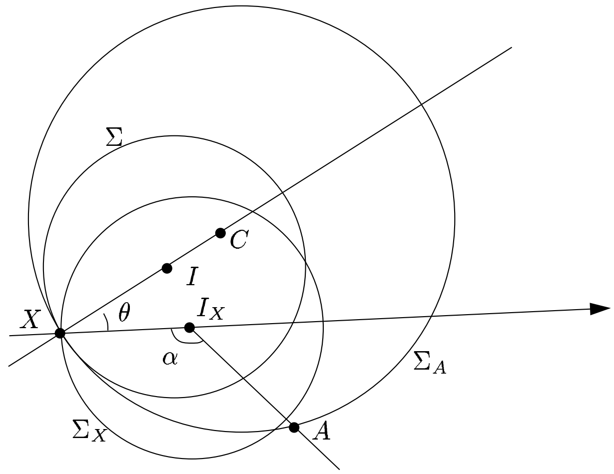

Figure 4: For Lemma 16.

portion of the ball centered at I_{X} of radius $R_{\mathrm{X}}(1+\rho)$ that is in the halfspace H^- limited by H not containing I is contained in the cone of revolution with apex at I_{X} and a 2ϕ apex angle. We have $R_{\mathrm{X}}\cos\theta = R_{\mathrm{X}}(1+\rho)\cos\phi$. When $\cos\alpha \leq \cos\phi = \frac{\cos\theta}{1+\rho}$, A cannot belong to H^-, which implies the first part of the lemma.

Let $R = R_{\mathrm{X}}(1+\rho)$. For simplicity, we take X as the origin of the reference frame and XI_{X} as the first axis (see Figure 4). Moreover, we choose the second axis so that I lies in the plane defined by the first two axis. If C and r denote respectively the center and the radius of Σ_A, we have

$$I_{\mathrm{X}} = \begin{pmatrix} R_{\mathrm{X}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad I = \begin{pmatrix} R_{\mathrm{X}}\cos\theta \\ R_{\mathrm{X}}\sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad A = \begin{pmatrix} R_{\mathrm{X}} - R\cos\alpha \\ \phi_2 \, R\sin\alpha \\ \phi_3 \, R\sin\alpha \\ \vdots \\ \phi_d \, R\sin\alpha \end{pmatrix}$$

with $r > 0$ and $\sum_{i=2}^{d} \phi_i^2 = 1$. From $r^2 = \|C - A\|^2$, simple computations lead to

$$r = R_{\mathrm{X}} \frac{(1-\cos\alpha) + (1-\cos\alpha)\rho + \rho^2/2}{(1-\cos\alpha)\cos\theta - \cos\alpha\cos\theta\,\rho + \phi_2\sin\alpha\,\sin\theta\,(1+\rho)}.$$

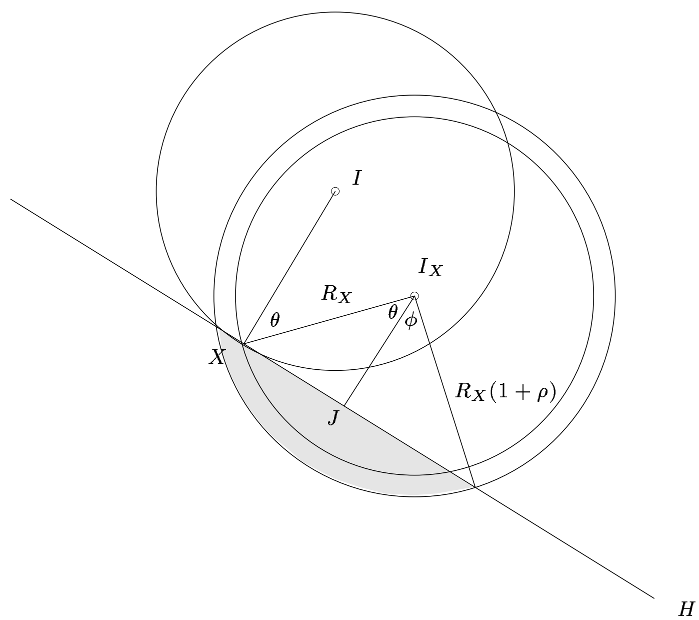

Figure 5: For the first part of Lemma 16.

With $\theta = c\,\rho(1+O(\rho))$, we get

$$r \leq R_{\mathrm{X}} \left(1 + \frac{\rho - \phi_2 \sin\alpha \sin\theta + O(\rho^2)}{1-\cos\alpha}\right) \leq R_{\mathrm{X}} \left(1 + \frac{1+c}{1-\cos\alpha}\,\rho\,(1+O(\rho)\right)$$

Finally

$$\begin{aligned} \|I_{\mathrm{X}} - C\| &= \left((r\cos\theta - R_{\mathrm{X}})^2 + r^2 \sin^2\theta\right)^{\frac{1}{2}} \\ &\leq \left((R_{\mathrm{X}} - r)^2 + r R_{\mathrm{X}} \theta^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{c^2 + \frac{(1+c)^2}{(1-\cos\alpha)^2}}\, R_{\mathrm{X}} \rho\ (1 + O(\rho)) \\ &\stackrel{\text{def}}{=} M \end{aligned}$$

The claim is therefore proved for $P = C$. Since we also have

$$\|I_{\mathrm{X}} - I\| = 2R_{\mathrm{X}} \sin\frac{\theta}{2} \leq R_{\mathrm{X}}\theta = cR_{\mathrm{X}}\ \rho\ (1+O(\rho)) \leq M,$$

the claim is also proved for any point P on the line segment $[IC]$. $\square$

The following lemma shows that, if the radius of a Delaunay d-simplex S incident to a given point $X \in \mathcal{S}$ is greater than the radius R_{X} of the maximal sphere Σ_{X}, then the circumcenter of S is close to the medial axis of $\mathcal{S}$.

Proposition 17 *Let $\mathcal{A}$ be a good ε-sample of $\mathcal{S}$. Let X be a point of $\mathcal{S}$ such that $\eta_X^2 \geq \varepsilon - \frac{\varepsilon^2}{(1-\varepsilon)^2}$, and let S be a d-simplex of Del^+ incident to X. If the circumcenter V of S satisfies $\overrightarrow{XV} \cdot \overrightarrow{XI_X} > 0$ and is at distance ηR_X from X, for $\eta \geq 1$, we have*

$$\|V - I_X\| \leq \omega(\eta)\, R_X\, \varepsilon\, (1 + O(\sqrt{\varepsilon})), \quad \text{where } \omega(\eta) = \sqrt{\left(1 + \frac{1}{\eta}\right)^2 + \frac{1}{\eta_X^4}\left(1 + \frac{1}{2\eta}\right)^2}.$$

Remark. Observe that, since $V_{\mathcal{O}}(X)$ contains I_{X}, a d-simplex as in the lemma exists for any X. A similar result has been independently obtained by Amenta and Kolluri [3] in the special case where S is the d-simplex incident to X whose circumcenter (called a pole) is the farthest from X.

Proof. Since $\eta_X \neq 0$, I_{X} is not a focal point of $\mathcal{S}$ and therefore Σ_{X} is tangent to $\mathcal{S}$ at two distinct points, X and at some other point $Y_{\mathrm{X}} \neq X$ (see Figure 6). Let Σ be a moving and deformable sphere that initially coincides with Σ_{X}, I its center, and $Y \in \Sigma$ the point that initially coincides with Y_{X}. We first rotate Σ around X until its center lies on the ray going from X towards V. Let I' denote the new position of I and Σ' the corresponding sphere.

By Lemma 11, the angle θ between the vectors $\overrightarrow{XV}$ and $\overrightarrow{XI_{\mathrm{X}}}$ is at most $\arcsin(\frac{\varepsilon}{\eta(1-\varepsilon)}) + \arcsin(\frac{\varepsilon}{1-\varepsilon}) = (1 + \frac{1}{\eta})\, \varepsilon\, (1 + O(\varepsilon))$.

Since V is farther from X than I_{X}, I' lies between X and V and Σ' does not contain any point of $\mathcal{A}$ in its interior. We then grow Σ until it passes through $A_1, \ldots, A_d$ and X. More precisely, we move the center I of Σ along the line XV towards V while keeping Σ passing through X. We stop when $I = V$, i.e. when Σ coincides with the sphere circumscribing S. During this second motion, Σ cannot grow much. Indeed, since $\mathcal{A}$ is a ε-sample, there exists a sample point A_i at distance at most $\varepsilon\, \mathrm{lfs}(Y_{\mathrm{X}}) \leq \varepsilon\, R_{Y_{\mathrm{X}}} = \varepsilon\, R_{\mathrm{X}}$ from Y_{X}. The sample point A_i is therefore at distance at most $R_{\mathrm{X}}(1 + \varepsilon)$ from I_{X}. Since, at the end of the motion, Σ is a Delaunay sphere, the interior of the ball bounded by Σ cannot contain A_i. We now apply Lemma 16 to Σ_{X}, Σ' and A_i, with $\rho = \varepsilon$ and $c = 1 + \frac{1}{\eta}$.

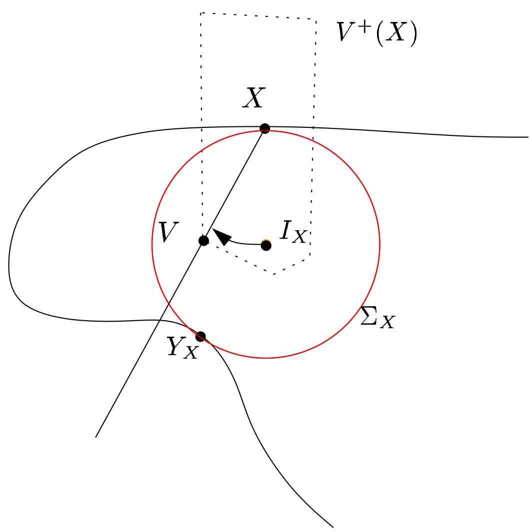

Figure 6: For the proof of Proposition 17.

If we note $\alpha_i = \angle X I_X A_i$, $\alpha = \angle X I_X Y_X$ and $\phi = \angle Y_X I_X A_i$, we have $\sin\phi \leq \frac{\varepsilon\,\mathrm{lfs}(Y_X)}{R_X} \leq \varepsilon$, $\alpha_i \geq \alpha - \phi$ and, since $1-\cos\alpha = 2\,\eta_X^2$, $1-\cos\alpha_i \geq 2\eta_X^2 - \varepsilon\,\sin\alpha \geq 2\eta_X^2 - \varepsilon$. We now show that $\cos\alpha_i \leq \frac{\cos\theta}{1+\varepsilon}$. By Lemma 12, $\frac{\cos\theta}{1+\varepsilon} \geq (1 - 2\,\frac{\varepsilon^2}{(1-\varepsilon)^2})\,\frac{1}{1+\varepsilon} \geq 1-\varepsilon-2\,\frac{\varepsilon^2}{(1-\varepsilon)^2}$. Therefore, if $\eta_X^2 \geq \varepsilon - \frac{\varepsilon^2}{(1-\varepsilon)^2}$, $\cos\alpha_i \leq \frac{\cos\theta}{1+\varepsilon}$ and we can apply Lemma 16. Observing that

$$\frac{1}{1-\cos\alpha_i} \leq \frac{1}{1-\cos\alpha-\varepsilon\,\sin\alpha} = \frac{1}{1-\cos\alpha}\,(1+O(\frac{\varepsilon}{\eta_X})) = \frac{1}{\eta_X^4}\,(1+O(\sqrt{\varepsilon})),$$

we finally obtain

$$\begin{aligned}\|I - I_X\| &\leq \sqrt{c^2 + \frac{(1+c)^2}{(1-\cos\alpha_i)^2}}\,R_X\varepsilon\;(1+O(\varepsilon))\\ &\leq \sqrt{\left(1+\frac{1}{\eta}\right)^2 + \frac{1}{\eta_X^4}\left(1+\frac{1}{2\eta}\right)^2}\,R_X\,\varepsilon\,(1+\,O(\sqrt{\varepsilon}))\end{aligned}$$

□

The following lemma shows that the circumcenters of the Delaunay d-simplices that have a long edge are close to the medial axis.

Proposition 18 *Let $\mathcal{A}$ be a good ε-sample of $\mathcal{S}$. Let X be a point of $\mathcal{S}$ such that $\eta_X \neq 0$, and let A_r be a natural neighbour of X lying at distance $\geq 2\eta R_X$ from X for $\varepsilon^{\frac{1}{3}} \leq \eta \leq \eta_X$. The circumcenter V of any d-simplex incident to $[XA_r]$ and such that $\overrightarrow{XV} \cdot \overrightarrow{XI_X} > 0$ satisfies*

$$\|V - I_X\| \leq \omega(\eta)\, R_X\, \varepsilon\, (1 + O(\varepsilon))) \text{ where } \omega(\eta) = \left(1 + \frac{1}{\eta}\right) \sqrt{1 + \frac{1}{\eta^4}}.$$

Proof. For convenience, let $r = 1$ and $S = [A_1 \ldots A_d X]$ be a d-simplex of Del^+ incident to $[XA_1]$. Let V be the circumcenter of S.

The proof is similar to the proof of lemma 17. Let Σ be a moving and deformable sphere that initially coincides with Σ_{X}, I its center, and $Y \in \Sigma$ the point that initially coincides with Y_{X}. We first rotate Σ around X until its center lies on the ray going from X towards V. Let I' denote the new position of I and Σ' the corresponding sphere.

By Lemma 11, the angle θ between the vector $\overrightarrow{VX}$ and the normal to $\mathcal{S}$ at X is at most $\arcsin(\frac{\varepsilon}{\eta(1-\varepsilon)}) + \arcsin(\frac{\varepsilon}{1-\varepsilon}) = (1 + \frac{1}{\eta})\, \varepsilon\, (1 + O(\varepsilon))$.

If I' lies between X and V, the previous lemma allows to conclude. Otherwise, Σ contains $A_1, \ldots, A_d$. We shrink Σ until it passes through $A_1, \ldots, A_d$ and X. More precisely, we move the center I of Σ as above along the line XV towards V while keeping Σ passing through X. We stop when $I = V$. During this second motion, Σ cannot be shrunk much. Indeed, since A_1 lies in the interior of the ball bounded by Σ after the first motion, we have

$$\|I_{\mathrm{X}} - A_1\| \leq \|I_{\mathrm{X}} - I'\| + \|I' - A_1\| \leq \|I_{\mathrm{X}} - I'\| + R_{\mathrm{X}} \leq R_{\mathrm{X}}(1 + 2\sin\frac{\theta}{2}) \leq R_{\mathrm{X}}(1 + \theta).$$

We wish to apply Lemma 16 to Σ_{X}, Σ' and A_1 (with $c = 1$ and $\rho = \theta$).

Let $\alpha_1 = \angle X I_{\mathrm{X}} A_1$. We bound α_1 using the fact that edge $[XA_1]$ is long. Since A_1 does not lie inside Σ_{X}, we have $\sin(\frac{\alpha_1}{2}) = \frac{\|X - A_1'\|}{2R_{\mathrm{X}}} \geq \frac{\|X - A_1\| - \|A_1 - A_1'\|}{2R_{\mathrm{X}}}$ where A_1' is the intersection of Σ_{X} and the line segment $[A_1 I_{\mathrm{X}}]$. Since $\|A_1 - A_1'\| \leq R_{\mathrm{X}}\,\theta$, we have $\sin(\frac{\alpha_1}{2}) \geq \eta - \frac{\theta}{2}$. In order to apply Lemma 16, we need to show that

$\cos\alpha_1 \le \frac{\cos\theta}{1+\theta}$. Since $\frac{\cos\theta}{1+\theta} > 1-\theta-\frac{\theta^2}{2}$ and $\cos\alpha_1 \le 1-2(\eta-\frac{\theta}{2})^2$, the inequality holds if $2\eta^2 - 2\,\theta\,\eta - \theta \ge 0$. It can then be verified that this last inequality holds when $\eta \ge \varepsilon^{\frac{1}{3}}$.

We can now apply Lemma 16 and obtain

$$\begin{aligned}\|I - I_{\mathrm{X}}\| &\le \sqrt{1+\frac{4}{(1-\cos\alpha_1)^2}}\, R_{\mathrm{X}}\,\theta\,(1+O(\theta)) \\ &\le \left(1+\frac{1}{\eta}\right)\sqrt{1+\frac{1}{\eta^4}}\, R_{\mathrm{X}}\,\varepsilon\,(1+O(\varepsilon)).\end{aligned}$$

This achieves the proof of the lemma. □

Propositions 17 and 18 state that certain vertices of $V^+(\mathcal{A})$ converge towards the medial axis of $\mathcal{S}$. The next lemma states a similar result for the vertices of $V(X, A_r)$ provided that A_r is sufficiently far from X.

Proposition 19 *Let $\mathcal{A}$, X and A_r be as in Lemma 18. Any vertex of $V(X, A_r)$ such that $\overrightarrow{XV}\cdot\overrightarrow{XI_X} > 0$ satisfies*

$$\|V - I_X\| \le \varpi(\eta)\, R_X\,\varepsilon(1+O(\varepsilon)) \;\text{ where }\; \varpi(\eta) = \frac{1}{2\eta^2} + \left(1+\frac{1}{\eta}\right)\sqrt{1+\frac{1}{\eta^4}}.$$

Proof. Consider first the face F of Vor^+ that is common to $V^+(X)$ and $V(X, A_r)$. The vertices of F are the circumcenters of the d-simplices of Del^+ incident to $[XA_r]$. By Proposition 18, the vertices of F all lie in a ball B centered at I_{X} whose radius is $R \le \omega(\eta)\, R_{\mathrm{X}}\,\varepsilon\,(1+O(\varepsilon))$. This proves the claim for the vertices of $V(X, A_r)$ that are also vertices of $V^+(X)$. Moreover, by convexity of F and B, the claim holds for any point of F.

Let us consider now a vertex V of $V(X, A_r)$ that is not a vertex of F. Therefore, V lies strictly inside $V^+(X)$ and we have $\|V - X\| < \|V - A_r\|$. Let Σ_V be the sphere centered at V and passing through A_r. We have $\|V - A_r\| \le \|V - X\| + \varepsilon\,\mathrm{lfs}(X)$ since the interior of the ball bounded by Σ_V contains X but not the sample point closest to X. Hence

$$\|V - A_r\| - \varepsilon\,\mathrm{lfs}(X) \le \|V - X\|. \tag{1}$$

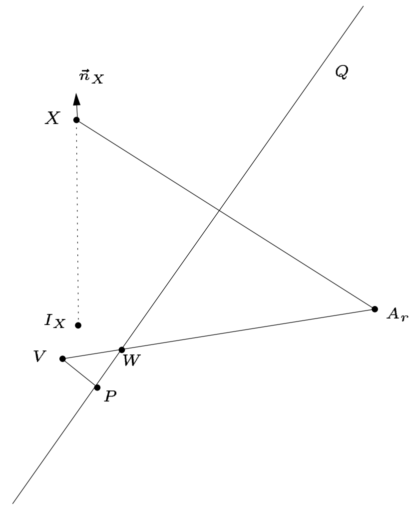

Figure 7: For the proof of Proposition 19.

Refer to Figure 7. Let Q be the affine hull of F, i.e. the bisector hyperplane of X and A_r, let P be the orthogonal projection of V onto Q, and let W be the point of intersection of Q and the line segment $[A_rV]$. We first observe that $W \in F$. Indeed, $[A_rV]$ is contained in $V(A_r)$ and $F = V(A_r) \cap Q$. Moreover, we have $\|V - A_r\|^2 - \|V - X\|^2 = 2(A_r - X) \cdot (\frac{A_r + X}{2} - V) = 2\|X - A_r\| \, \|V - P\|$. Moreover,

$$\frac{\|V - P\|}{\|V - W\|} = \frac{\frac{1}{2}\,\|X - A_r\|}{\|W - A_r\|}.$$

We therefore get

$$\|V - X\|^2 = \|V - A_r\|^2 - \|X - A_r\|^2 \, \frac{\|V - W\|}{\|W - A_r\|}.$$

By squaring the two sides of Equation (1), we obtain

$$-2\,\|V - A_r\|\,\varepsilon\,\mathrm{lfs}(X) + \varepsilon^2\,\mathrm{lfs}^2(X) \leq -\|X - A_r\|^2 \, \frac{\|V - W\|}{\|W - A_r\|},$$

and, since $\|V - A_r\| = \|V - W\| + \|W - A_r\|$,

$$\|V - W\| \leq 2\,\frac{\|W - A_r\|^2}{\|X - A_r\|^2}\,\mathrm{lfs}(X)\,\varepsilon\,(1 + O(\varepsilon)). \tag{2}$$

Since $W \in F$, as observed above, W is at distance $R \leq \omega(\eta)\, R_{\mathrm{X}}\; \varepsilon(1+O(\varepsilon))$ from I_{X}. Hence, we have

$$\|W - A_r\| = \|X - W\| \leq \|X - I_{\mathrm{X}}\| + \|I_{\mathrm{X}} - W\| \leq R_{\mathrm{X}} + R,$$

This last inequality together with Equation (2) and $\|X - A_r\| \geq 2\,\eta\, R_{\mathrm{X}}$, leads to

$$\|V - W\| \leq \frac{1}{2\eta^2}\, \mathrm{lfs}(X)\, \varepsilon\, (1 + O(\varepsilon)) \leq \frac{1}{2\eta^2}\, R_{\mathrm{X}}\, \varepsilon(1 + O(\varepsilon)).$$

Finally,

$$\|I_{\mathrm{X}} - V\| \leq \|I_{\mathrm{X}} - W\| + \|V - W\| \leq \left(\omega(\eta) + \frac{1}{2\eta^2}\right)\, R_{\mathrm{X}}\, \varepsilon\, (1 + O(\varepsilon)),$$

which proves the lemma. □

Remark 1 Propositions 17, 18 and 19 are still valid if one considers $\mathbb{R}^d \setminus \mathcal{O}$ instead of $\mathcal{O}$. More precisely, if one replaces $\overrightarrow{XV} \cdot \overrightarrow{XI_{\mathrm{X}}} > 0$ by $\overrightarrow{XV} \cdot \overrightarrow{XI_{\mathrm{X}}} < 0$ in the propositions, the same bounds hold provided that R_{X}, I_{X} and η_{X} are replaced by R^e_{X}, I^e_{X} and η^e_{X}. However since R^e_{X} can be arbitrarily large, the results are only meaningful when only a bounded region of the plane (containing $\mathcal{S}$) is considered. For the lemmas to hold, the boundary $\mathcal{B}$ of that region must be smooth and $\mathcal{A}$ must be a ε-sample of $\mathcal{S} \cup \mathcal{B}$.

5 Natural neighbour coordinates of points on $\mathcal{S}$

The set of natural neighbour coordinates of a point X of $\mathcal{S}$ (computed in $\mathbb{R}^d$) do not constitute a local system of coordinates on $\mathcal{S}$. Indeed, the support of the λ_i is *not* a small neighborhood of A_i even if the sampling is very dense. This is illustrated in Figure 8.

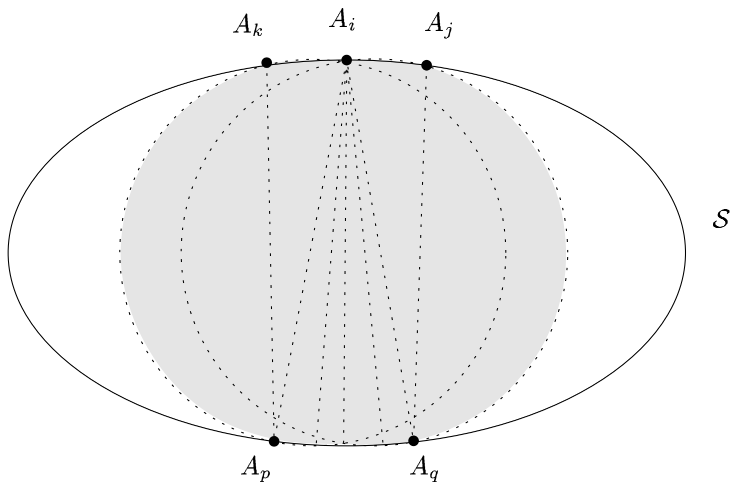

Figure 8: The grey region is the union of the bounded Delaunay balls passing through A_i. Its intersection with $\mathcal{S}$ is the support of A_i. It consists of two arcs, A_jA_k and A_pA_q.

In this section, we show however that the set of natural neighbour coordinates of $X \in \mathcal{S}$ tends to behave as a local coordinate system on the surface $\mathcal{S}$ when the density of points increases, i.e. when ε tends to 0.

Theorem 20 *Let $\mathcal{A}$ be a ε-sample of $\mathcal{S}$. Let X be a point of $\mathcal{S}$ such that $\eta_X \neq 0$ and let $N_\eta(X)$ denote the set of indexes of the natural neighbours of X lying at distance $< 2\,\eta\,R_X$ from X, for $\varepsilon^{\frac{1}{3}} \leq \eta \leq \eta_X$. We have*

$$\sum_{i \notin N_\eta(X)} \lambda_i(X) \leq d\,\varpi(\eta)\,\varepsilon\,(1 + O(\varepsilon)), \tag{3}$$

where $\varpi(\eta)$ is defined as in Proposition 19.

Proof. We first consider the case where the $\mathcal{S}$-natural neighbours of X all lie in the hyperplane $H(X)$ passing through X and normal to $\vec{n}_\mathrm{X}$. Let H^i denote the half space limited by $H(X)$ that contains I_X (i.e. opposite to $\vec{n}_\mathrm{X}$) and H^e the other halfspace.

Under this assumption, $V_{\mathcal{S}}^{+}(X)$ is a convex $(d-1)$-polytope contained in $H(X)$. Let $v(X)$ denote its area, i.e. $(d-1)$-dimensional volume. Since $I_{\mathrm{X}} \in V_{\mathcal{O}}^{+}(X)$, the volume of $V_{\mathcal{O}}^{+}(X)$ is at least $\frac{1}{d} R_{\mathrm{X}} v(X)$.

When points are appropriately added on a bounding box (see Remark 1), a similar inequality holds for the portion of $V^{+}(X)$ that is outside $\mathcal{O}$: we simply need to replace R_{X} by the radius R_{X}^{e} of the other maximal sphere tangent to $\mathcal{S}$ at X. We therefore have

$$w(X) \geq \frac{1}{d}(R_{\mathrm{X}} + R_{\mathrm{X}}^{e})\, v(X). \tag{4}$$

Let A_r be a natural neighbour of X lying at distance $\geq 2\eta \max(R_{\mathrm{X}}, R_{\mathrm{X}}^{e})$ from X.

We denote by C the cylinder intersecting $\mathcal{S}$ along $V_{\mathcal{S}}^{+}(X)$ and whose axis is parallel to $\vec{n}(X)$. It follows from Propositions 18 and 19 that all the vertices of $V(X, A_r)$ that lie in H^i belong to a portion $\mathcal{Z}$ of C. $\mathcal{Z}$ contains I_{X} and has height $\leq \varpi(\eta)\, R_{\mathrm{X}}\, \varepsilon\, (1 + O(\varepsilon))$. Hence, $V_{\mathcal{O}}(X, A_r) = V(X, A_r)$ and we have

$$\sum_{r \notin N_\eta(X)} \mathrm{vol}(V_{\mathcal{O}}(X, A_r)) \leq \mathrm{vol}(\mathcal{Z}) \leq v(X)\ \varpi(\eta)\, R_{\mathrm{X}}\, \varepsilon\, (1 + O(\varepsilon)).$$

The same inequality holds for the vertices of $V(X, A_r)$ that lie in H^e provided that R_{X} is replaced by R_{X}^{e}. Therefore, we have

$$\sum_{r \notin N_\eta(X)} \mathrm{vol}(V(X, A_r)) \leq v(X)\, \varpi(\eta)\, (R_{\mathrm{X}} + R_{\mathrm{X}}^{e})\ \varepsilon\, (1 + O(\varepsilon)). \tag{5}$$

From Equations (4) and (5), we get

$$\sum_{r \notin N_\eta(X)} \lambda_r(X) = \frac{\sum_{r \notin N_\eta(X)} \mathrm{vol}(V(X, A_r))}{w(X)} \leq d\varpi(\eta)\, \varepsilon\, (1 + O(\varepsilon)). \tag{6}$$

We now show that the same result holds when the $\mathcal{S}$-natural neighbours of X do not belong to the hyperplane $H(X)$. Let $A_1, \ldots, A_m$ be the vertices of these facets other than X and $A'_1, \ldots, A'_m$ their projection onto $H(X)$.

It is easy to see that $\|A_i - A'_i\| \leq \frac{2\varepsilon^2}{(1-\varepsilon)^2}\, \mathrm{lfs}(X)$. Indeed, $\|A_i - A'_i\| \leq \|A_i - X\|\ \sin\theta$, where $\theta = \angle A_i X A'_i$. Since A_i does not belong to the two balls of radius $\mathrm{lfs}(X)$ that are tangent to $\mathcal{S}$ at X,

$$\sin\theta \leq \frac{\|A_i - X\|}{2\, \mathrm{lfs}(X)}.$$

With $\|A_i - X\| \leq \frac{2\varepsilon}{1-\varepsilon}\,\mathrm{lfs}(X)$ (Lemma 10), we finally get $\|A_i - A_i'\| \leq \frac{2\varepsilon^2}{(1-\varepsilon)^2}\,\mathrm{lfs}(X)$.

Since the element of area dv on a surface is the element of area in the tangent plane, Equation (4) holds up to second order terms in ε. The rest of the proof is unchanged. We conclude that Equation (6) still holds when the $\mathcal{S}$-natural neighbours of X do not belong to the hyperplane $H(X)$.

$\square$

Corollary 21 *Let X and $N_\eta(X)$ be as in Theorem 20. We have*

$$X = \sum_{i \in N_\eta(X)} \lambda_i(X)\, A_i \; + \; Y \;\text{ with }\; \|Y\| = d\, \varpi(\eta)\, \varepsilon\, (1 + O(\varepsilon)).$$

Proof. We have $X = \sum_i \lambda_i(X)\ A_i = \sum_{i \in N_\eta(X)} \lambda_i(X)\ A_i \ + \ \sum_{i \notin N_\eta(X)} \lambda_i(X)\ A_i$. By the previous result, $\lambda = \sum_{i \notin N_\eta(X)} \lambda_i(X) = d\, \varpi(\eta)\, \varepsilon\, (1 + O(\varepsilon))$. Hence, $X = \sum_{i \in N_\eta(X)} \lambda_i(X)\ A_i + Y$ where Y belongs to the convex hull $CH(\{\lambda A_i, i \notin N_\eta(X)\}) = d\, \varpi(\eta)\, \varepsilon\, (1 + O(\varepsilon))\ CH(\{A_i, i \notin N_\eta(X)\})$. Since, the diameter of $CH(\{A_i, i \notin N_\eta(X)\})$ remains bounded, the result follows. $\square$

Remark 2 Observe that $\|Y\|$ tends to zero with ε when $\eta = \Omega(\varepsilon^{\frac{1}{3}})$.

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6 Appendix : distance to a manifold and Morse theory

Let f be a smooth function on a smooth manifold $\mathcal{S}$. A point $Y \in \mathcal{S}$ is called *critical* if all the partial derivatives of f are zero at Y. A critical point is called *degenerate* if the determinant of the Hessian matrix is zero at Y. A smooth function on a smooth manifold is called a *Morse function* if all its critical points are nondegenerate.

Let X be a fixed point of $\mathbb{R}^d$ and consider the distance function δ_{X} that associates to any point $Y \in \mathcal{S}$ its squared distance to X. The following lemma is well-known (see, for instance, [12], section 9.4) :

Lemma 22 *A point $Y \in \mathcal{S}$ is a critical point of δ_X iff the vector XY is normal to $\mathcal{S}$ at Y. Moreover, δ_X has a degenerate critical point exactly when X is a focal point of $\mathcal{S}$. Hence, δ_X is a Morse function when X is not a focal point of $\mathcal{S}$. In particular, δ_X is a Morse function for any $X \in \mathcal{S}$.*

The following theorem is a basic theorem in Morse theory (see [15] for a proof).

Theorem 23 *Let ϕ be a smooth function on a compact smooth surface $\mathcal{S}$ and assume that, for two reals $a < b$, $\phi^{-1}([a,b]) = \{x : a \leq \phi(x) \leq b\}$ contains no critical point. Then $\phi_a = \{x : \phi(x) \leq a\}$ and $\phi_b = \{x : \phi(x) \leq b\}$ are diffeomorphic.*

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