Lyapunov Equation for the Stability of 2-D Systems

Pierre-Alexandre Bliman

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Lyapunov equation for the stability of 2-D systems

Pierre-Alexandre Bliman

N° 4014

Septembre 2000

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4014--FR+ENG

Lyapunov equation for the stability of 2-D systems

Pierre-Alexandre Bliman*

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Sosso

Rapport de recherche n° 4014 — Septembre 2000 — 13 pages

Abstract: In this report, an analogue of the characterization of asymptotic stability of the rational systems by the solvability of associated Lyapunov equation is proposed for 2-D systems. It is shown that internal stability of Roesser model is equivalent to the feasibility of some linear matrix inequality (LMI), related to quadratic Lyapunov functions.

Key-words: 2-D systems, internal stability, quadratic Lyapunov functions, linear matrix inequalities.

* Email : `pierre-alexandre.bliman@inria.fr`. Part of this work was done when the author was on leave from I.N.R.I.A. in National Technical University of Athens, Greece, with the support of European Commission's Training and Mobility of Researchers Programme ERB FMRXCT-970137 "Breakthrough in the control of nonlinear systems". The author is indebted to the network coordinator Françoise Lamnabhi-Lagarrigue, and the coordinator of the greek team Ioannis Tsinias, for their help and support.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Equation de Lyapunov pour la stabilité des systèmes 2-D

Résumé : Dans ce rapport, on propose pour les systèmes 2-D un résultat analogue à la caractérisation de la stabilité asymptotique des systèmes rationnels par la solvabilité d'une équation de Lyapunov associée. On montre que la propriété de stabilité interne du modèle de Roesser est équivalente à la solvabilité d'une inégalité linéaire matricielle, obtenue à partir de fonctions de Lyapunov quadratiques.

Mots-clés : Systèmes 2-D, stabilité interne, fonctions de Lyapunov quadratiques, inégalités linéaires matricielles.

1 Introduction

It is a well-known property that the asymptotic stability of the discrete system

$$x_{i+1} = Ax_i \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, may be equivalently characterized by the spectral property:

$$\forall z \in \mathbb{C}, |z| \leq 1 \Rightarrow \det(I_n - zA) \neq 0 \ , \tag{2}$$

or by the solvability of the Lyapunov inequation

$$\exists Q \in \mathbb{R}^{n\times n}, \ Q = Q^T > 0, \ A^T QA - Q < 0 \ . \tag{3}$$

The link between the two properties is made by using the quadratic function

$$V(x) \stackrel{\text{def}}{=} x^T Qx \ , \tag{4}$$

which is a Lyapunov function for system (1) when Q is solution of the linear matrix inequality (3). Stated otherwise, this means that system (1) is asymptotically stable *if and only if* it possesses a quadratic Lyapunov function of the form (4).

The goal of the present contribution is to present an analogue of this property for the characterization of the internal stability of 2-D systems.

On the one hand, internal stability of 2-D systems [6] has been expressed under a spectral form, with two frequency variables instead of one in (2).

On the other hand, simple class of quadratic Lyapunov functions have been proposed for 2-D systems [5]. The computation of their evolution along the trajectories of the system under study leads to some stability conditions (usually called *2-D Lyapunov equations*) expressed as linear matrix inequalities, analogue to (3). A major benefit of this approach lies in the fact that there exists fast and powerful numerical algorithms of resolution for linear matrix inequalities [4]. However, solvability of the 2-D Lyapunov equation is only *sufficient* for stability, but in no way necessary [3]. The involved quadratic forms appear as the sum of two terms, and are parametrized by two symmetric definite positive matrices, instead of one in (4).

Further study reveals that the stability property is indeed equivalent to the solvability of some Lyapunov equation dependent upon the frequency, a so-called *frequency dependent 1-D Lyapunov equation* [9, 3]. The use of such a characterization is difficult in practice, as it involves the verification of the solvability of infinitely many linear matrix inequalities. All these aspects will be detailed further.

Here, we provide a characterization of the internal stability property by linear matrix inequalities. Roughly speaking, the main idea is the following. Instead of considering the usual state variables of the considered systems, say $x(i,j)$, we use a non minimal state, allowing the use of ancient values of the state, more precisely $\begin{pmatrix} x(i,j)^T & x(i,j-1)^T & \dots & x(i,j-k+1)^T \end{pmatrix}^T$ for k an integer. The supplementary available information which has been temporarily ignored when augmenting the state space (as any choice of $\begin{pmatrix} x(i,j)^T & x(i,j-1)^T & \dots & x(i,j-k+1)^T \end{pmatrix}^T$ cannot be part of a trajectory when $k > 1$), is then reintroduced when estimating the variation of the candidate Lyapunov function.

It turns out that the linear matrix inequalities found by this approach constitute *necessary and sufficient* stability conditions, provided that enough supplementary information is considered (when k is large enough). A precise statement and a complete proof of this fact constitute the core of the present paper. In particular, Lyapunov theory is not used here to prove the stability, but rather as a guideline, in order to find adequate algebraic conditions of stability.

The problem is presented in more details and the background is recalled in Section 2. The main result is enunciated in Section 3. Its proof is given in Section 4.

The notations are quite standard. I_n, $0_{m\times n}$ stand respectively for the identity matrix of size n and the null matrix of size $m \times n$ (simply abbreviated 0_n when $m = n$). The Kronecker product is denoted $\otimes$. The spectral radius and maximal singular value of a matrix M are respectively denoted $\rho(M)$ and $\|M\|$. The conjugate and transconjugate of M, are denoted M^T and M^*.

2 Presentation and background

Consider the following Roesser state-space model (input and output do not appear, as one is interested in internal stability only) defined by [12]

$$\begin{aligned} x_1(i+1,j) &= Ax_1(i,j) + Bx_2(i,j),\\ x_2(i,j+1) &= Cx_1(i,j) + Dx_2(i,j)\ . \end{aligned} \tag{5}$$

This model is more suitable for our goal, but the results to be enunciated apply also to the other, equivalent, models of 2-D systems, especially Fornasini-Marchesini's model [6]. Here and in the sequel,

$$A \in \mathbb{R}^{n_1\times n_1},\ B \in \mathbb{R}^{n_1\times n_2},\ C \in \mathbb{R}^{n_2\times n_1},\ D \in \mathbb{R}^{n_2\times n_2},$$

for certain $n_1, n_2 \in \mathbb{N}\setminus\{0\}$. System (5) is asymptotically stable if and only if [6]

$$\forall (z_1,z_2)\in\mathbb{C}^2,\ |z_1|,|z_2|\leq 1 \Rightarrow \det\begin{pmatrix} I_{n_1}-z_1A & -z_1B\\ -z_2C & I_{n_2}-z_2D\end{pmatrix} \neq 0\ ,$$

or equivalently[1]

$$\rho(A) < 1 \text{ and } \forall z\in\mathbb{C}, |z|\leq 1 \Rightarrow \rho(zC(I_{n_1}-zA)^{-1}B + D) < 1\ . \tag{6}$$

In [5] the following nonnegative state function is considered

$$V(i,j) \stackrel{\text{def}}{=} x_1(i,j)^TQ_1x_1(i,j) + x_2(i,j)^TQ_2x_2(i,j)\ .$$

Defining $V_{11}(i,j) \stackrel{\text{def}}{=} x_1(i+1,j)^TQ_1x_1(i+1,j) + x_2(i,j+1)^TQ_2x_2(i,j+1)$, it is shown there that, if the quantity $V(i,j) - V_{11}(i,j)$ is negative, or null if $x_1(i,j) = 0$, $x_2(i,j) = 0$, then system (5) is asymptotically stable. In the present case,

$$V_{11}(i,j) - V(i,j) = \begin{pmatrix} x_1(i,j)\\ x_2(i,j)\end{pmatrix}^T R \begin{pmatrix} x_1(i,j)\\ x_2(i,j)\end{pmatrix}\ , \tag{7}$$

where

$$R \stackrel{\text{def}}{=} \begin{pmatrix} A^TQ_1A-Q_1 & A^TQ_1B\\ B^TQ_1A & B^TQ_1B\end{pmatrix} + \begin{pmatrix} C^TQ_2C & C^TQ_2D\\ D^TQ_2C & D^TQ_2D-Q_2\end{pmatrix}\ .$$

This suggests that the following LMI, called *2-D Lyapunov equation*,

$$Q_1 = Q_1^T > 0, Q_2 = Q_2^T > 0, R < 0 \tag{8}$$

is sufficient for asymptotic stability of (5), and this is indeed the case [5].

[1] Using the maximum modulus principle, one may restrict the set on which the conditions have to be verified, see e.g. [8].

However, contrary to the 1-D case presented in the Introduction, this property is not necessary [3]. A simple way to see that is as follows [3]. By Kalman-Yakubovich-Popov lemma (see Appendix), condition (8) is equivalent to

$$\exists Q_2 = Q_2^T > 0,\ \rho(A) < 1 \text{ and } \forall z \in \mathbb{C},$$
$$|z| \leq 1 \Rightarrow \begin{pmatrix} z(I_{n_1} - zA)^{-1}B \\ I_{n_2} \end{pmatrix}^* \begin{pmatrix} C^TQ_2C & C^TQ_2D \\ D^TQ_2C & D^TQ_2D - Q_2 \end{pmatrix} \begin{pmatrix} z(I_{n_1} - zA)^{-1}B \\ I_{n_2} \end{pmatrix} < 0 ,$$

that is:

$$\rho(A) < 1 \text{ and } \exists Q = Q^T > 0, \forall z \in \mathbb{C}, |z| \leq 1 \Rightarrow (zC(I_{n_1} - zA)^{-1}B + D)^*Q(zC(I_{n_1} - zA)^{-1}B + D) < Q .$$

On the other hand, (6) is clearly equivalent to

$$\rho(A) < 1 \text{ and } \forall z \in \mathbb{C},$$
$$|z| \leq 1 \Rightarrow \exists Q(z) = Q(z)^T > 0, (zC(I_{n_1} - zA)^{-1}B + D)^*Q(z)(zC(I_{n_1} - zA)^{-1}B + D) < Q(z) . \quad (9)$$

It hence appears that the LMI (8) is a conservative criterion for the asymptotic stability of (5). Condition (9) is refered in the literature as a *frequency dependent 1-D Lyapunov equation* [9].

3 Main result

Our method is based on an adaptation of the previous one. For $k \in \mathbb{N} \setminus \{0\}$, denote

$$\mathcal{X}_{1,k}(i,j) \stackrel{\text{def}}{=} \begin{pmatrix} x_1(i,j) \\ x_1(i,j-1) \\ \vdots \\ x_1(i,j-k+1) \end{pmatrix}, \quad \mathcal{X}_{2,k}(i,j) \stackrel{\text{def}}{=} \begin{pmatrix} x_2(i,j) \\ x_2(i,j-1) \\ \vdots \\ x_2(i,j-k+1) \end{pmatrix} . \quad (10)$$

The vectors $\mathcal{X}_{1,k}$, $\mathcal{X}_{2,k}$ are elements of $\mathbb{R}^{kn_1}$, $\mathbb{R}^{kn_2}$ respectively.

From (5) one deduces that

$$\begin{aligned} \mathcal{X}_{1,k}(i+1,j) &= (I_k \otimes A)\mathcal{X}_{1,k}(i,j) + (I_k \otimes B)\mathcal{X}_{2,k}(i,j), \\ \mathcal{X}_{2,k}(i,j+1) &= (I_k \otimes C)\mathcal{X}_{1,k}(i,j) + (I_k \otimes D)\mathcal{X}_{2,k}(i,j) . \end{aligned} \quad (11)$$

Defining, for definite positive matrices $Q_{1,k} \in \mathbb{R}^{kn_1 \times kn_1}$, $Q_{2,k} \in \mathbb{R}^{kn_2 \times kn_2}$, the scalar, nonnegative, state function

$$V_k(i,j) \stackrel{\text{def}}{=} \mathcal{X}_{1,k}(i,j)^TQ_{1,k}\mathcal{X}_{1,k}(i,j) + \mathcal{X}_{2,k}(i,j)^TQ_{2,k}\mathcal{X}_{2,k}(i,j) ,$$

one has (compare with (7))

$$\begin{aligned} V_{k,11}(i,j) - V_k(i,j) = \begin{pmatrix} \mathcal{X}_{1,k}(i,j) \\ \mathcal{X}_{2,k}(i,j) \end{pmatrix}^T & \left[\begin{pmatrix} (I_k \otimes A)^TQ_{1,k}(I_k \otimes A) - Q_{1,k} & (I_k \otimes A)^TQ_{1,k}(I_k \otimes B) \\ (I_k \otimes B)^TQ_{1,k}(I_k \otimes A) & (I_k \otimes B)^TQ_{1,k}(I_k \otimes B) \end{pmatrix} \right. \\ & \left. + \begin{pmatrix} (I_k \otimes C)^TQ_{2,k}(I_k \otimes C) & (I_k \otimes C)^TQ_{2,k}(I_k \otimes D) \\ (I_k \otimes D)^TQ_{2,k}(I_k \otimes C) & (I_k \otimes D)^TQ_{2,k}(I_k \otimes D) - Q_{2,k} \end{pmatrix} \right] \begin{pmatrix} \mathcal{X}_{1,k}(i,j) \\ \mathcal{X}_{2,k}(i,j) \end{pmatrix} . \end{aligned}$$

But we are only interested in some special trajectories of system (11) in the space $\mathbb{R}^{kn_1} \times \mathbb{R}^{kn_2}$: those which obey the equations (5) when x_1, x_2 are defined by (10). The key point now, is the fact that $\mathcal{X}_{1,k}$ and $\mathcal{X}_{2,k}$ are constrained by the following identity

$$\mathcal{X}_{2,k}(i,j) = F_k\mathcal{X}_{1,k}(i,j) + f_kx_2(i,j-k+1) ,$$

where $F_k \in \mathbb{R}^{kn_2\times kn_1}$, $f_k \in \mathbb{R}^{kn_2\times n_2}$ are defined by induction by

$$f_1 \stackrel{\text{def}}{=} I_{n_2},\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}D \\ I_{n_2}\end{pmatrix}, \tag{12}$$

$$F_1 \stackrel{\text{def}}{=} 0_{n_2\times n_1},\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1}C \\ 0_{n_2\times(k-1)n_1} & 0_{n_2\times n_1}\end{pmatrix}. \tag{13}$$

As an example,

$$f_2 = \begin{pmatrix} D \\ I_{n_2}\end{pmatrix},\ f_3 = \begin{pmatrix} D^2 \\ D \\ I_{n_2}\end{pmatrix},\ F_2 = \begin{pmatrix} 0_{n_2\times n_1} & C \\ 0_{n_2\times n_1} & 0_{n_2\times n_1}\end{pmatrix},\ F_3 = \begin{pmatrix} 0_{n_2\times n_1} & C & DC \\ 0_{n_2\times n_1} & 0_{n_2\times n_1} & C \\ 0_{n_2\times n_1} & 0_{n_2\times n_1} & 0_{n_2\times n_1}\end{pmatrix}, \dots$$

One deduces that

$$V_{k,11}(i,j) - V_k(i,j) = \begin{pmatrix} \mathcal{X}_{1,k}(i,j) \\ x_2(i,j-k+1)\end{pmatrix}^T R_k \begin{pmatrix} \mathcal{X}_{1,k}(i,j) \\ x_2(i,j-k+1)\end{pmatrix},$$

where

$$\begin{aligned} R_k \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k\end{pmatrix}^T \Bigg[&\begin{pmatrix} (I_k\otimes A)^T Q_{1,k}(I_k\otimes A) - Q_{1,k} & (I_k\otimes A)^T Q_{1,k}(I_k\otimes B) \\ (I_k\otimes B)^T Q_{1,k}(I_k\otimes A) & (I_k\otimes B)^T Q_{1,k}(I_k\otimes B)\end{pmatrix} \\ &+ \begin{pmatrix} (I_k\otimes C)^T Q_{2,k}(I_k\otimes C) & (I_k\otimes C)^T Q_{2,k}(I_k\otimes D) \\ (I_k\otimes D)^T Q_{2,k}(I_k\otimes C) & (I_k\otimes D)^T Q_{2,k}(I_k\otimes D) - Q_{2,k}\end{pmatrix}\Bigg] \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k\end{pmatrix}. \end{aligned} \tag{14}$$

One is hence naturally led to study the solvability of the following LMI, defined, for any $k \in \mathbb{N}\backslash\{0\}$, by:

$$\begin{gathered} Q_{1,k}\in\mathbb{R}^{kn_1\times kn_1}, Q_{2,k}\in\mathbb{R}^{kn_2\times kn_2}, Q_{1,k} = Q_{1,k}^T > 0, Q_{2,k} = Q_{2,k}^T > 0,\ R_k < 0, \\ \textit{where } R_k \in \mathbb{R}^{(kn_1+n_2)\times(kn_1+n_2)} \textit{is defined by } (12), (13), (14). \end{gathered} \tag{15_k}$$

Of course, the case $k = 1$ reduces to (8).

Our main result tells that the solvability of (15_k) becomes indeed *equivalent* to the asymptotic stability of (5) "at the limit", when k goes to infinity.

Theorem 1. *The following properties are equivalent.*

1. *System* (5) *is asymptotically stable.*
2. *For any* $(z_1, z_2) \in \mathbb{C}^2$,

$$|z_1|, |z_2| \leq 1 \Rightarrow \det\begin{pmatrix} I_{n_1} - z_1A & -z_1B \\ -z_2C & I_{n_2} - z_2D\end{pmatrix} \neq 0.$$

3. $\rho(D) < 1$ *and for any* $z \in \mathbb{C}$,

$$|z| \leq 1 \Rightarrow \rho(A + zB(I_{n_2} - zD)^{-1}C) < 1.$$

4. $\rho(A) < 1$ *and for any* $z \in \mathbb{C}$,

$$|z| \leq 1 \Rightarrow \rho(zC(I_{n_1} - zA)^{-1}B + D) < 1.$$

5. *There exists* $k \in \mathbb{N} \setminus \{0\}$ *such that* (15_k) *is feasible.*

6. *There exists* $k^* \in \mathbb{N} \setminus \{0\}$ *such that, for any* $k \geq k^*$, (15_k) *is feasible.*

■

Theorem 1 furnishes a sequence of LMI criteria, of arbitrary precision. The precision may be estimated as follows. One may check from the proof, that a sufficient condition for solvability of (15_k) is

$$\rho(A) < 1 \text{ and } \sup_{\substack{z \in \mathbb{C}, \\ |z| \leq 1}} \|[zC(I_{n_1} - zA)^{-1}B + D]^k\| < 1 \ .$$

On the other hand, stability is equivalent (condition 4. of Theorem 1) to

$$\rho(A) < 1 \text{ and } \lim_{k \to +\infty} \sup_{\substack{z \in \mathbb{C}, \\ |z| \leq 1}} \|[zC(I_{n_1} - zA)^{-1}B + D]^k\| = 0 \ .$$

Theorem 1 also provides more information on the form of a solution of the frequency dependent 1-D Lyapunov equations, see the Remark at the end of the proof.

Remark that the nature of the results presented here is close from a result given by Fornasini *et al.* [7], stating that the 2-D system

$$x(i+1, j+1) = A_1 x(i, j+1) + A_2 x(i+1, j)$$

is stable *iff* for k large enough, the sum on all i, j such that $i + j = k$, of the norms of all products including i times the matrix A_1 and j times the matrix A_2 (*shuffle product*) is less than 1. As a matter of fact, both conditions reduce the determination of the stability to checking if a certain condition involving the matrices of the representation is fulfilled for a certain integer k. The LMI condition is simpler to check.

From Theorem 1 may be derived the following three types of results. First, one may deduce similar LMI characterizations for stability of continuous or mixed (discrete/continuous) 2-D systems, see [1, 2] where frequency dependent 1-D Lyapunov equations and 2-D Lyapunov equations are presented for these systems. As an example, using the property that

$$\forall z \in \mathbb{C}, \mathrm{Re}\, \frac{1-z}{1+z} \geq 0 \Leftrightarrow |z| \leq 1 \ ,$$

one shows that

$$\begin{aligned}
&\forall (s,z) \in \mathbb{C}^2,\ \mathrm{Re}\, s \geq 0, |z| \leq 1 \Rightarrow \det \begin{pmatrix} sI_{n_1} - A & -B \\ -zC & I_{n_2} - zD \end{pmatrix} \neq 0 \\
&\Leftrightarrow\ \forall (z_1, z_2) \in \mathbb{C}^2,\ |z_1|, |z_2| \leq 1 \Rightarrow \det \begin{pmatrix} I_{n_1} - A - z_1(I_{n_1} + A) & -(1+z_1)B \\ -z_2 C & I_{n_2} - z_2 D \end{pmatrix} \neq 0 \\
&\Leftrightarrow\ \forall (z_1, z_2) \in \mathbb{C}^2,\ |z_1|, |z_2| \leq 1 \\
&\qquad \Rightarrow \det \begin{pmatrix} I_{n_1} - A - z_1(I_{n_1} + A) & -z_1(I_{n_1} + (I_{n_1} + A)(I_{n_1} - A)^{-1})B \\ -z_2 C & I_{n_2} - z_2(C(I_{n_1} - A)^{-1}B + D) \end{pmatrix} \neq 0 \\
&\Leftrightarrow\ \forall (z_1, z_2) \in \mathbb{C}^2,\ |z_1|, |z_2| \leq 1 \\
&\qquad \Rightarrow \det \begin{pmatrix} I_{n_1} - z_1(I_{n_1} + A)(I_{n_1} - A)^{-1} & -2z_1(I_{n_1} - A)^{-1}B \\ -z_2 C(I_{n_1} - A)^{-1} & I_{n_2} - z_2(C(I_{n_1} - A)^{-1}B + D) \end{pmatrix} \neq 0 \ .
\end{aligned}$$

The last condition has the same structure than (15_k), and Theorem 1 applies. Also, results based directly on quadratic Lyapunov functions are under progress.

Second, as demonstrated by Pandolfi [10], any asymptotically stable 2-D system is also exponentially stable. It may be interesting to estimate the rate of convergence to zero of the solutions. System (5) is exponentially stable with rate $\alpha \in (0,1)$ if and only if the system governing the evolution of $\alpha^{-(i+j)}\begin{pmatrix} x_1(i,j)^T & x_2(i,j)^T \end{pmatrix}^T$ is still asymptotically stable. It is hence clear that this property may be checked numerically as in Theorem 1, replacing A, B, C, D respectively by $\alpha^{-1}A, \alpha^{-1}B, \alpha^{-1}C, \alpha^{-1}D$.

Last, the result given here permits straightforward deduction of necessary and sufficient conditions of stabilizability of 2-D systems by (state or output) static feedback, or by dynamic feedback of given order.

4 Proof of Theorem 1

The equivalence between 1., 2., 3. and 4. is known [6], and the implication 6. $\Rightarrow$ 5. is straightforward. One will show that 5. implies 3., and then that 4. implies 6.

4.1 Proof of the implication 5. $\Rightarrow$ 3.

Consider first that the feasibility of (15_k) implies that

$$\begin{aligned} 0 &> \begin{pmatrix} 0_{kn_1\times n_2} \\ I_{n_2} \end{pmatrix}^* R_k \begin{pmatrix} 0_{kn_1\times n_2} \\ I_{n_2} \end{pmatrix} = f_k^T[(I_k\otimes B)^T Q_{1,k}(I_k\otimes B) + (I_k\otimes D)^T Q_{2,k}(I_k\otimes D) - Q_{2,k}]f_k \\ &\geq f_k^T[(I_k\otimes D)^T Q_{2,k}(I_k\otimes D) - Q_{2,k}]f_k = D^T f_k^T Q_{2,k} f_k D - f_k^T Q_{2,k} f_k \ . \end{aligned}$$

The previous formula appears as a Lyapunov equation, and permits to show that $\rho(D) < 1$.

Define now, for any $z \in \mathbb{C}$ and for $k \in \mathbb{N}\setminus\{0\}$, the matrices $v_{1,k}(z) \in \mathbb{R}^{kn_1\times n_1}$, $v_{2,k}(z) \in \mathbb{R}^{kn_2\times n_2}$ and $w_k(z) \in \mathbb{R}^{(kn_1+n_2)\times n_1}$ by

$$v_{1,k}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I_{n_1} \\ zI_{n_1} \\ \vdots \\ z^{k-1}I_{n_1} \end{pmatrix},\ v_{2,k}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I_{n_2} \\ zI_{n_2} \\ \vdots \\ z^{k-1}I_{n_2} \end{pmatrix},\ w_k(z) \stackrel{\text{def}}{=} \begin{pmatrix} v_{1,k}(z) \\ z^k(I_{n_2}-zD)^{-1}C \end{pmatrix} \ .$$

Then,

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} w_k(z) = \begin{pmatrix} v_{1,k}(z) \\ F_k v_{1,k}(z) + z^k f_k(I-zD)^{-1}C \end{pmatrix} \ ,$$

and, using (12), (13),

$$F_k v_{1,k}(z) + z^k f_k(I-zD)^{-1}C = \begin{pmatrix} F_{k-1}v_{1,k-1}(z) + z^{k-1}f_{k-1}(I-zD)^{-1}C \\ z^k(I-zD)^{-1}C \end{pmatrix} \ .$$

One then proves by induction that

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} w_k(z) = \begin{pmatrix} I_{kn_1} \\ zI_k\otimes(I_{n_2}-zD)^{-1}C \end{pmatrix} v_{1,k}(z) \ .$$

From the solvability of (15_k) one hence deduces that, for any $z \in \mathbb{C}$ such that $I - zD$ is invertible,

$$\begin{aligned}
0 >& w_k(z)^* R_k w_k(z) \\
=& \left[\begin{pmatrix} I_{kn_1} \\ zI_k \otimes (I-zD)^{-1}C \end{pmatrix} v_{1,k}(z)\right]^* \left[\begin{pmatrix} (I_k\otimes A)^T Q_{1,k}(I_k\otimes A) - Q_{1,k} & (I_k\otimes A)^T Q_{1,k}(I_k\otimes B) \\ (I_k\otimes B)^T Q_{1,k}(I_k\otimes A) & (I_k\otimes B)^T Q_{1,k}(I_k\otimes B)\end{pmatrix}\right. \\
&+ \left.\begin{pmatrix} (I_k\otimes C)^T Q_{2,k}(I_k\otimes C) & (I_k\otimes C)^T Q_{2,k}(I_k\otimes D) \\ (I_k\otimes D)^T Q_{2,k}(I_k\otimes C) & (I_k\otimes D)^T Q_{2,k}(I_k\otimes D) - Q_{2,k}\end{pmatrix}\right] \begin{pmatrix} I_{kn_1} \\ zI_k \otimes (I-zD)^{-1}C \end{pmatrix} v_{1,k}(z) \\
=& \left[(I_k \otimes (A+zB(I-zD)^{-1}C))v_{1,k}(z)\right]^* Q_{1,k}(I_k\otimes(A+zB(I-zD)^{-1}C))v_{1,k}(z) \\
&- v_{1,k}(z)^* Q_{1,k} v_{1,k}(z) + (1-|z|^2)\left[(I_k\otimes(I-zD)^{-1}C)v_{1,k}(z)\right]^* Q_{2,k}(I_k\otimes(I-zD)^{-1}C)v_{1,k}(z) \\
=&(A+zB(I-zD)^{-1}C)^* v_{1,k}(z)^* Q_{1,k} v_{1,k}(z)(A+zB(I-zD)^{-1}C) \\
&- v_{1,k}(z)^* Q_{1,k} v_{1,k}(z) + (1-|z|^2)[(I-zD)^{-1}C]^* v_{2,k}(z)^* Q_{2,k} v_{2,k}(z)^* (I-zD)^{-1}C\ .
\end{aligned}$$

In particular, if $|z| \leq 1$ (and then $I - zD$ invertible, as $\rho(D) < 1$)), this yields

$$v_{1,k}(z)^* Q_{1,k} v_{1,k}(z) > [A+zB(I-zD)^{-1}C]^* v_{1,k}(z)^* Q_{1,k} v_{1,k}(z)[A+zB(I-zD)^{-1}C]\ .$$

As the matrix $v_{1,k}(z)^* Q_{1,k} v_{1,k}(z)$ is positive definite, one deduces that for any z such that $|z| \leq 1$, the matrix $A + zB(I-zD)^{-1}C$ fulfills a Lyapunov equation, so $\rho(A + zB(I-zD)^{-1}C) < 1$. This achieves the proof of the implication 5. $\Rightarrow$ 3.

4.2 Proof of the implication 4. $\Rightarrow$ 6.

- One first transforms condition 4. It is well-known that, for any square matrix M,

$$\rho(M) < 1 \Leftrightarrow \lim_{k\to+\infty} \|M^k\| = 0 \Leftrightarrow \exists k^*, \sup_{k\geq k^*} \|M^k\| < 1$$

(where the second equivalence is obtained using the fact that the matrix norm induced by the euclidian norm is submultiplicative). One hence deduces that condition 4. is indeed *equivalent* to

$$\forall z\in\mathbb{C}, |z|\leq 1 \Rightarrow \det(I_{n_1} - zA) \neq 0 \text{ and } \exists k^* \in \mathbb{N}\setminus\{0\}, \sup_{k\geq k^*} \|[zC(I_{n_1}-zA)^{-1}B + D]^k\| < 1\ . \quad (16)$$

- One now transforms (15_k). Developing the first term in R_k leads to the identity

$$\begin{aligned}
&\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix}^T \begin{pmatrix} (I_k\otimes A)^T Q_{1,k}(I_k\otimes A) - Q_{1,k} & (I_k\otimes A)^T Q_{1,k}(I_k\otimes B) \\ (I_k\otimes B)^T Q_{1,k}(I_k\otimes A) & (I_k\otimes B)^T Q_{1,k}(I_k\otimes B)\end{pmatrix} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} \\
=&\ \left((I_k\otimes A) + (I_k\otimes B)F_k \quad (I_k\otimes B)f_k\right)^T Q_{1,k} \left((I_k\otimes A) + (I_k\otimes B)F_k \quad (I_k\otimes B)f_k\right) \\
&- \left(I_{kn_1} \quad 0_{kn_1\times n_2}\right)^T Q_{1,k} \left(I_{kn_1} \quad 0_{kn_1\times n_2}\right)^T\ .
\end{aligned}$$

Written under this form, one may apply Kalman-Yakubovich-Popov lemma (see Appendix) to (15_k). Denoting

$$\mathcal{S}_k = \mathcal{S}_k(z) \stackrel{\text{def}}{=} z(I_{kn_1} - z(I_k\otimes A) - z(I_k\otimes B)F_k)^{-1}\ ,$$

a sufficient condition for solvability of (15_k) appears to be

$$\begin{aligned}
&\exists Q_{2,k} = Q_{2,k}^T > 0, \forall z\in\mathbb{C}, |z|\leq 1 \Rightarrow \det(I_{kn_1} - z(I_k\otimes A) - z(I_k\otimes B)F_k) \neq 0 \text{ and} \\
&[\cdot]^* \begin{pmatrix} (I_k\otimes C)^T Q_{2,k}(I_k\otimes C) & (I_k\otimes C)^T Q_{2,k}(I_k\otimes D) \\ (I_k\otimes D)^T Q_{2,k}(I_k\otimes C) & (I_k\otimes D)^T Q_{2,k}(I_k\otimes D) - Q_{2,k}\end{pmatrix} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} \begin{pmatrix} \mathcal{S}_k(z)(I_k\otimes B)f_k \\ I_{n_2} \end{pmatrix} < 0\ ,
\end{aligned}$$

where the dot in the brackets has to be replaced by the last two matrices, that is: $\exists Q_{2,k} = Q_{2,k}^T > 0, \forall z \in \mathbb{C}, |z| \leq 1 \Rightarrow \det(I_{n_1} - zA) \neq 0$ and

$$[(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k+(I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k)]^* Q_{2,k}[(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k+(I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k)] \\ < [F_k\mathcal{S}_k(I_k \otimes B)f_k + f_k]^* \mathcal{S}_k^* Q_{2,k} \mathcal{S}_k [F_k\mathcal{S}_k(I_k \otimes B)f_k + f_k]\ . \tag{17}$$

Let us simplify these expressions. From (13), (12), one gets that

$$\begin{aligned} \mathcal{S}_k(z) &= z \begin{pmatrix} I_{(k-1)n_1} - z(I_{k-1}\otimes A) - z(I_{k-1}\otimes B)F_{k-1} & -z(I_{k-1}\otimes B)f_{k-1}C \\ 0_{n_1\times(k-1)n_1} & I_{n_1} - zA \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathcal{S}_{k-1} & z\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}C(I_{n_1}-zA)^{-1} \\ 0_{n_1\times(k-1)n_1} & z(I_{n_1}-zA)^{-1} \end{pmatrix}, \\ (I_k\otimes B)f_k &= \begin{pmatrix} (I_{k-1}\otimes B)f_{k-1}D \\ B \end{pmatrix}\ . \end{aligned}$$

This permits to establish that

$$\mathcal{S}_k(I_k\otimes B)f_k = \begin{pmatrix} \mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}(zC(I_{n_1}-zA)^{-1}B+D) \\ z(I_{n_1}-zA)^{-1}B \end{pmatrix}\ ,$$

and then

$$F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k = \begin{pmatrix} [F_{k-1}\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1} + f_{k-1}](zC(I_{n_1}-zA)^{-1}B+D) \\ I_{n_2} \end{pmatrix}$$

and

$$\begin{aligned} &(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k + (I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k) \\ &= \begin{pmatrix} [(I_{k-1}\otimes C)\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1} + (I_{k-1}\otimes D)(F_{k-1}\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}+f_{k-1})](zC(I_{n_1}-zA)^{-1}B+D) \\ zC(I_{n_1}-zA)^{-1}B+D \end{pmatrix}\ . \end{aligned}$$

So, one proves recursively that

$$(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k + (I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k) = \begin{pmatrix} [zC(I_{n_1}-zA)^{-1}B+D]^k \\ \vdots \\ zC(I_{n_1}-zA)^{-1}B+D \end{pmatrix}, \tag{18}$$

$$F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k = \begin{pmatrix} [zC(I_{n_1}-zA)^{-1}B+D]^{k-1} \\ \vdots \\ I_{n_2} \end{pmatrix}\ . \tag{19}$$

Taking $Q_{2,k} = I_{kn_2}$, one finally gets from (17) that a *sufficient* condition for realization of 5. is

$$\exists k^* \in \mathbb{N}\setminus\{0\}, \forall k \in \mathbb{N}, k \geq k^*, \forall z \in \mathbb{C}, |z| \leq 1 \Rightarrow \det(I_{n_1} - zA) \neq 0 \text{ and } \|[zC(I_{n_1}-zA)^{-1}B+D]^k\| < 1\ . \tag{20}$$

• In view of (16) and (20), it remains, in order to prove that 4. implies 6., to show that one may choose in (16) the index k *uniformly* wrt $z \in \mathbb{C}$, $|z| \leq 1$. The final argument, based on compacity, is itself decomposed into two parts. It will be shown first that (16) implies

$$\exists k \in \mathbb{N}\setminus\{0\}, \forall z \in \mathbb{C}, |z| \leq 1 \Rightarrow \det(I_{n_1} - zA) \neq 0 \text{ and } \|[zC(I_{n_1}-zA)^{-1}B+D]^k\| < 1\ . \tag{21}$$

and then that (21) implies (20).

• For $k \in \mathbb{N} \setminus \{0\}$, let

$$K_k \stackrel{\text{def}}{=} \{z \in \mathbb{C} \ : \ |z| \leq 1, \ \|[zC(I_n - zA)^{-1}B + D]^k\| \geq 1\} \ .$$

By continuity, the sets K_k are closed, and hence compact. Moreover,

$$z \in K_{2k} \Rightarrow 1 \leq \|[zC(I_n - zA)^{-1}B + D]^{2k}\| \leq \|[zC(I_{n_1} - zA)^{-1}B + D]^k\|^2 \Rightarrow z \in K_k \ .$$

Hence $K_{2k} \subset K_k$, for any $k \in \mathbb{N} \setminus \{0\}$. The sequence K_{2^k} is thus a sequence of nested compact sets.

Assume now that (21) does not hold. If $\rho(A) \not< 1$, then (16) does not hold. Otherwise, for any $k \in \mathbb{N} \setminus \{0\}$, the sets K_k are nonempty. In particular,

$$\exists z_0 \in \bigcap_{k \in \mathbb{N}} K_{2^k} \ ,$$

that is

$$\exists z_0 \in \mathbb{C}, |z_0| \leq 1, \forall k \in \mathbb{N}, \ \|[z_0 C(I - z_0 A)^{-1}B + D]^{2^k}\| \geq 1 \ .$$

Hence,

$$\forall k \in \mathbb{N} \setminus \{0\}, \sup_{k \geq k^*} \|[z_0 C(I - z_0 A)^{-1}B + D]^k\| \geq 1 \ ,$$

and (16) does not hold either. One has hence proved by contradiction that (16) implies (21).

• Let us prove now that (21) implies (20). Suppose that (21) holds, and let $k^* \in \mathbb{N} \setminus \{0\}$ and $c_1 > 0$ be such that

$$\forall z \in \mathbb{C}, |z| \leq 1 \Rightarrow \det(I_{n_1} - zA) \neq 0 \text{ and } \|[zC(I_{n_1} - zA)^{-1}B + D]^{k^*}\| \leq c_1 < 1 \ .$$

Define also

$$c_2 \stackrel{\text{def}}{=} \sup \left\{ \sup_{z \in \mathbb{C}, |z| \leq 1} \|[zC(I_{n_1} - zA)^{-1}B + D]^k\| \ : \ k \in \{0, \dots, k^* - 1\} \right\} \ .$$

Then c_2 is finite.

Now, fix $k^{**} \in \mathbb{N} \setminus \{0\}$ such that

$$k^{**} > \left(-\frac{\log c_2}{\log c_1} + 2 \right) k^* \ ,$$

and let $k \in \mathbb{N}$, $z \in \mathbb{C}$ be such that $k \geq k^{**}$ and $|z| \leq 1$. Denote q and r the quotient and the rest of the euclidian division of k by k^*, that is:

$$q \in \mathbb{N}, \ r \in \{0, 1, \dots, k^* - 1\}, \ k = qk^* + r \ .$$

Remark that $k \geq k^{**}$ implies

$$q \geq -\frac{\log c_2}{\log c_1} + 1 > -\frac{\log c_2}{\log c_1} \ . \tag{22}$$

Then,

$$\|[zC(I_{n_1} - zA)^{-1}B + D]^k\| \leq \|[zC(I_{n_1} - zA)^{-1}B + D]^{k^*}\|^q \|[zC(I_{n_1} - zA)^{-1}B + D]^r\| \leq c_1^q c_2 < 1 \ ,$$

due to (22) and the fact that $c_1 < 1$. From (21), one has hence deduced the existence of k^{**} such that

$$\forall k \in \mathbb{N}, k \geq k^{**}, \forall z \in \mathbb{C}, |z| \leq 1 \Rightarrow \det(I_{n_1} - zA) \neq 0 \text{ and } \|[zC(I_{n_1} - zA)^{-1}B + D]^k\| < 1 \ ,$$

that is (20).

To summarize, it has been successively shown that: Condition 4. $\Leftrightarrow$ (16), (20) $\Rightarrow$ Condition 6., (16) $\Rightarrow$ (21), (21) $\Rightarrow$ (20). This shows finally that condition 4. implies condition 6., and concludes the proof of Theorem 1.

Remark. If $Q_{2,k}$ fulfills (17) (that is, there exists $Q_{1,k}$ such that $(Q_{1,k}, Q_{2,k})$ is solution of (15_k)), then (18), (19) show that (9) is fulfilled with $Q(z)$ chosen as

$$Q(z) \stackrel{\text{def}}{=} \begin{pmatrix} [zC(I_{n_1} - zA)^{-1}B + D]^{k-1} \\ \vdots \\ I_{n_2} \end{pmatrix}^* Q_{2,k} \begin{pmatrix} [zC(I_{n_1} - zA)^{-1}B + D]^{k-1} \\ \vdots \\ I_{n_2} \end{pmatrix} .$$

A Appendix – Kalman-Yakubovich-Popov lemma

The following form of Kalman-Yakubovich-Popov lemma is used [11]. Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, M = M^T \in \mathbb{R}^{(n+p)\times(n+p)}$ with

$$\begin{pmatrix} I_n & 0_{n\times p} \end{pmatrix} M \begin{pmatrix} I_n \\ 0_{p\times n} \end{pmatrix} \geq 0. \tag{23}$$

Lemma. *For any matrix M fulfilling (23), the following two statements are equivalent:*

(i) There exists $Q = Q^T \in \mathbb{R}^{n\times n}$, positive definite, such that

$$M + \begin{pmatrix} A^TQA - Q & A^TQB \\ B^TQA & B^TQB \end{pmatrix} < 0 .$$

(ii) $\rho(A) < 1$ and, for any $z \in \mathbb{C}$ with $|z| = 1$,

$$\begin{pmatrix} (zI_n - A)^{-1}B \\ I_p \end{pmatrix}^* M \begin{pmatrix} (zI_n - A)^{-1}B \\ I_p \end{pmatrix} < 0 .$$

■

Maximum modulus principle permits to replace equivalently (ii) by:

$\rho(A) < 1$ *and, for any $z \in \mathbb{C}$ with $|z| \leq 1$,*

$$\begin{pmatrix} z(I_n - zA)^{-1}B \\ I_p \end{pmatrix}^* M \begin{pmatrix} z(I_n - zA)^{-1}B \\ I_p \end{pmatrix} < 0 .$$

References

[1] P. Agathoklis, Conditions for the 2-D characteristic polynomial of a matrix to be very strict Hurwitz, *IEEE Trans. Acoust. Speech Signal Proc.* **37** (1989) 1284–1286

[2] P. Agathoklis, S. Foda, Stability and the matrix Lyapunov equation for delay differential systems, *Int. J. Control* **49** no 2 (1989) 417–432

[3] B.D.O. Anderson, P. Agathoklis, E.I. Jury, M. Mansour, Stability and the matrix Lyapunov equation for discrete 2-dimensional systems, *IEEE Trans. Circ. Syst.* vol. **CAS-33** no 3 (1986) 261–267

[4] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear matrix inequalities in system and control theory*, SIAM Studies in Applied Mathematics vol. 15, 1994

[5] N.G. El-Agizi, M.M. Fahmy, Two-dimensional digital filters with no overflow oscillations, *IEEE Trans. on Acoust. Speech and Sign. Process.* vol. **ASSP-27** no 5 (1979) 465–469

[6] E. Fornasini, G. Marchesini, Doubly-indexed dynamical systems: state-space models and structural properties, *Math. Systems Theory* **12** (1978) 59–72

[7] E. Fornasini, G. Marchesini, Stability analysis of 2-D systems, *IEEE Trans. Circ. Syst.* vol. **CAS-27** no 12 (1980) 1210–1217

[8] T.S. Huang, Stability of two-dimensional recursive filters, *IEEE Trans. Audio. Elec.* vol. **AU-20** (1972) 158–163

[9] W.-S. Lu, E.B. Lee, Stability analysis for two-dimensional systems via a Lyapunov approach, *IEEE Trans. Circ. Syst.* vol. **CAS-32** no 1 (1985) 61–68

[10] L. Pandolfi, Exponential stability of 2-D systems, *Systems Control Letters* 4 (1984) 381–385

[11] A. Rantzer, On the Kalman-Yakubovich-Popov lemma, *Syst. Contr. Lett.* **28** no 1 (1996) 7–10

[12] R.P. Roesser, A discrete state-space model for linear image processing, *IEEE Trans. Automat. Control* vol. **AC-20** no 1 (1975) 1–10

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399