Representing and Evolving Smooth Manifolds of Arbitrary Dimension Embedded in Rn as the Intersection of n Hypersurfaces : The Vector Distance Functions

José Gomes, Olivier Faugeras

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Thème 3 — Interaction homme-machine,
images, données, connaissances
Projet Robotvis

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Abstract: We present a novel method for representing and evolving objects of arbitrary dimension. The method, called the Vector Distance Function (VDF) method, uses the vector that connects any point in space to its closest point on the object. It can deal with smooth manifolds with and without boundaries and with shapes of different dimensions. It can be used to evolve such objects according to a variety of motions, including mean curvature. If discontinuous velocity fields are allowed the dimension of the objects can change. The evolution method that we propose guarantees that we stay in the class of VDF's and therefore that the intrinsic properties of the underlying shapes such as their dimension, curvatures can be read off easily from the VDF and its spatial derivatives at each time instant. The main disadvantage of the method is its redundancy: the size of the representation is always that of the ambient space even though the object we are representing may be of a much lower dimension. This disadvantage is also one of its strengths since it buys us flexibility.

Key-words: *Implicit representations of shape, Mean curvature motion in arbitrary codimension, Change of dimension, Manifolds with a boundary.*

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Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : 04 92 38 77 77 - International : +33 4 92 38 77 77 — Fax : 04 92 38 77 65 - International : +33 4 92 38 77 65

Représenter et faire évoluer des variétés différentiables de co-dimension arbitraire plongées dans $\mathbb{R}^n$ comme l'intersection de n hypersurfaces: les fonctions distance vectorielles.

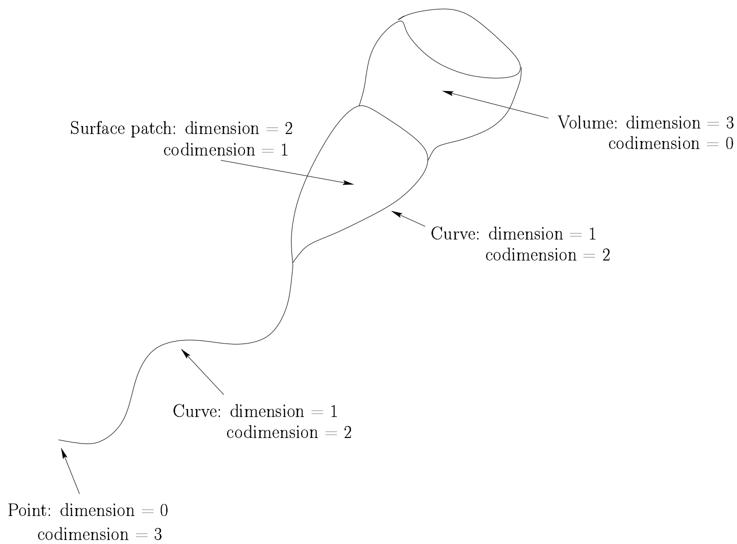
Résumé : Nous présentons une méthode pour représenter et faire évoluer des objets de dimension arbitraire. La méthode, dite de la "Fonction Distance Vectorielle" (VDF), utilise le vecteur qui joint tout point de l'espace à son point le plus proche sur l'objet et est donc applicable à des objets de toute dimension, avec ou sans bord. Elle peut être utilisée pour simuler, entre autres, le mouvement par courbure moyenne. L'utilisation de champs de vitesse discontinus permet à l'objet de changer de dimension en cours d'évolution, la fonction vectorielle associéee restant toujours dans la classe des VDF. Par conséquent, les propriétés intrinsèques de ladite variété, comme sa dimension et sa courbure, sont accessibles à chaque instant *via* le calcul des dérivées spatiales d'ordre 2 de sa VDF. La principale faiblesse de cette méthode réside dans sa redondance: la taille de la représentation est toujours celle de l'espace ambiant alors même que la variété décrite peut être de dimension faible. Dans le même temps, ceci est une force de la méthode puisque que cela procure une plus grande flexibilité.

Mots-clés : *Représentations implicites des formes, Mouvement de courbure moyenne en co-dimension arbitraire, Changement de dimension, Variétés à bords.*

Contents

1 Introduction and history

In this paper we present a general method for representing objects of arbitrary dimension embedded in spaces of arbitrary dimension. The representation method is also the basis for evolving such objects according to a variety of motions, including mean-curvature. We are not limited to objects of constant dimension, for example we can cope with open curves or surfaces, or even with objects such as the one shown in figure 1 which is the union of an open curve, an open surface, and a volume. The history of this program is quite long and can be



traced back to the early work on snakes by Kass, Witkins and Terzopoulos [33]. But the motivation is of course much older, see [35]. Their idea was to find a practical method for evolving planar curves, considered as embedded in an image, when submitted to force fields created by elements in the image, e.g. edges, corners, or interactively, by the user. Mathematically, these fields were forcing the curve to evolve according to a second order ordinary differential equation (ODE). Evolving the curve was then equivalent to solving this ODE starting from an initial curve provided by the user. Computationally, the curve was represented by a set of splines and its evolution reduced to the evolution of their control points. The method was quite efficient and could cope with open and closed curves. Improvements of the computational side allowed the Oxford and Cambridge groups around Andrew Blake and Roberto Cipolla to come up with real-time implementations on sequences of images for tracking silhouettes of objects ([10, 19]).

It was not too difficult to generalize the idea to 3-D in order to cope with surfaces and this was done by a number of people after the original paper [50]. See [38] for a nice survey of these methods. One of the problems that was met in both the 3-D and the 2-D cases is the problem that the control points of the approximating splines, even though they were chosen somewhat uniformly distributed along the initial curve or surface, had a tendency not to remain so during the course of the evolution, creating numerical problems. These numerical problems were not well understood in particular because the general problem of curve and surface evolution was not well understood to start with.

As it is often the case in science and technology, it turned out that this problem was receiving at the same time a great deal of attention in the community of mathematicians. They were looking at a special case, namely that of the evolution of a closed simple smooth planar curve when submitted to some velocity field. In a series of papers, Epstein, Gage, Grayson, Hamilton, and others proved three important facts. First they proved that under some mild hypotheses it is only the normal part of the velocity field, *i.e.* its component along the normal to the curve that is relevant for the evolution of the curve [22]. Second, in the case where the velocity is the curvature of the curve, a type of motion known as mean curvature motion, they showed that even if the initial curve is not convex, it becomes so in finite time [31]; and third they showed that after becoming convex, the curve goes through a singularity in finite time, disappearing as a circle with zero radius [26, 27]. The importance of these results cannot be underestimated because in many applications where one evolves planar curves, one of the terms in the velocity is very often a function of curvature. It is in effect always the case when the velocity of the curve is derived from a variational principle written intrinsically, *i.e.* as a function of its arc-length.

The extension to higher dimensions is both straightforward and difficult. It is in effect straightforward to see that the mean curvature motion equation

$$C_t = \kappa \mathbf{N}$$

of a planar closed smooth curve C becomes

$$\mathcal{M}_t = \mathcal{H}\mathbf{N} \tag{1}$$

in the case of a smooth closed hypersurface $\mathcal{M}$ of $\mathbb{R}^n$ $(n > 2)$ (its dimension is $n-1$, therefore its codimension is 1, meaning that it can be described locally by one equation in the space coordinates, or that its tangent space $T_{\mathbf{x}}\mathcal{M}$ is of dimension $n-1$ at every point $\mathbf{x}$, or that its normal space $N_{\mathbf{x}}\mathcal{M}$ is of dimension 1 at every point), where $\mathcal{H}$ is the mean curvature of the manifold, *i.e.* the mean of its $n-1$ principal curvatures, and $\mathbf{N}$ is one of its unit normal. In the case of a smooth closed manifold of dimension strictly less than $n-1$, *i.e.* of codimension strictly greater than 1, the evolution equation simply becomes

$$\mathcal{M}_t = \boldsymbol{\mathcal{H}},$$

where $\boldsymbol{\mathcal{H}}$ is the mean curvature vector of the manifold, a vector that belongs to the normal space of the manifold (of dimension $k > 1$) and whose definition is given in appendix 9.1.

But the simplicity of these equations hides the complexity of the corresponding motion. There are no theorems analogous to the ones by Epstein, Gage, Grayson and Hamilton for the case of hypersurfaces, and practically nothing is known in the case of arbitrary codimension. In fact there exist counterexamples to the straightforward generalization of their theorems:

the case of the dumbbell which breaks into two pieces before becoming convex is one such counterexample [32, 45].

Beside this work in differential geometry, another community was busy at the same time with inventing numerical methods and algorithms for evolving interfaces between different materials. Since these interfaces were often closed planar curves or surfaces that could develop discontinuities (shocks) and change topology, the method of level sets was introduced by Osher and Sethian [39] to deal exactly with these issues with which the standard Lagrangian techniques could not deal. Their Eulerian approach consists in replacing the evolution of the manifold of interest (a planar curve or a 3D surface in their case) with that of a function defined in a space of one dimension higher than the manifold and whose zero-crossings are precisely the evolving manifold. This function satisfies a Partial Differential Equation (PDE) which is very closely related to that satisfied by the evolving manifold:

$$u_t = \frac{1}{n-1} div(\frac{\nabla u}{\|\nabla u\|})\|\nabla u\|. \tag{2}$$

While the Computational Physics community went ahead to develop efficient algorithms for implementing this idea such as the narrow-banding [18, 37, 1] and the fast marching methods [51, 44, 2], other mathematicians, such as Evans and Spruck [24, 17] characterized the set of functions for which the method was indeed valid. One such function, and an obvious candidate, is the signed distance function to the manifold: by definition, its zero-crossings are the manifold under consideration and it is smooth in its vicinity. This function, being easy to compute, is often used in the applications of the level set method. One major problem it suffers from is the fact that the signed distance function is *not* a solution of the above PDE, as shown for example in [8].

On the other hand, as shown by the authors [28], the solution of the above PDE causes quickly numerical problems because its level sets tend to get very close to each other. The solution which has been adopted by the vast majority of authors is to reinitialize the function u after a fixed number of iterations (sometimes at each iteration!) so that it remains the distance function to its 0 level set. This is of course incorrect unless one proves that one is solving the initial PDE. As shown by Zhao, Chan, Meriman and Osher [52] and by the authors [30], it is possible to preserve the distance function, but the usual level-set PDE has to be modified in a non-trivial manner.

As an interesting application of these ideas Steve Zucker and his collaborators developed a nice theory of two-dimensional shape representation [34] based on the deformation of the boundary C_0 of the shape by an equation of the type

$$C_t = (1 + \alpha\kappa)\mathbf{N}, \quad C(.,0) = C_0(.).$$

They use the singularities of the evolving curve to represent the original shape; the resulting representation can be used for matching [46].

Another interesting contribution to the world of snakes was made by Caselles and his collaborators who observed that since the mean curvature motion is the gradient flow that minimizes the $n-1$-dimensional area of a hypersurface (a curve in the plane, a surface in space, etc...), if one changed the metric of the hypersurface by introducing e.g. a function of the intensity of the n-dimensional image in which it evolves, one obtained a new evolution equation that is similar to (1) and can be implemented by the level set technique using an equation similar to (2) [13, 14, 15, 12]. This was the starting point of a large body of work using these ideas in such areas as image segmentation [49, 16], motion analysis [40], and stereo [25].

All this was done in the framework of the evolution of manifolds of codimension 1 and the case of higher codimension has been less investigated. Some recent contributions in this direction are the following.

In 1996, Ambrosio and Soner, inspired by ideas of De Giorgi, published a paper [4] in which they showed that the level-set method could be extended to the case of arbitrary codimension. Their idea is to replace the evolution of the smooth manifold under mean curvature motion by that of a tubular neighborhood of the manifold, in effect a hypersurface. They show that the evolution of this tube is related to that of the manifold in a simple way and that it is not the mean curvature motion of the hypersurface. This theory has been applied by Lorigo et al. [36] to the problem of detecting blood vessels in volumetric medical images, the blood vessels being considered as the tubular neighborhoods of 3D curves. A different approach, closer to what we propose in the present paper, has been proposed by Sapiro and collaborators in a preliminary paper on tracking curves on a surface [9].

There are a number of problems with Ambrosio and Soner's approach, the main one being that it sweeps in some sense the dust under the rug: even though it does evolve correctly the manifold of interest, it turns out that recovering this manifold is in itself a major problem since it is not explicitly represented.

It is therefore natural to turn to a different approach and to attempt to represent an arbitrary smooth manifold $\mathcal{M}$ of dimension k as the intersection of $n-k$ hypersurfaces; the evolution of the hypersurfaces is computed in order to guarantee that their intersection evolves as required for $\mathcal{M}$ and that they remain transverse. This approach is natural since it is based on the definition of the dimension (or the codimension). It was suggested by Ambrosio and Soner in [4] but not pursued because it was thought to be too difficult. The corresponding program can nonetheless be studied, as described in [11, 29]. We shall not pursue it here because it is not easy to deal with such objects as the one represented in figure 1 and to deal with changes in the dimension of the manifold during the evolution.

In 1998, Ruuth, Merriman, Xin and Osher [41], considering the particular features of the complex Ginzburg-Landau equation, introduce another method for evolving space curves according to the mean curvature motion. The curve is represented by means of a (two-dimensional) complex function of unit magnitude defined on $\mathbb{R}^3$ whose phase angle "winds" around the curve. The time evolution is "diffusion-generated", *i.e.* it is the consequence of a diffusion-renormalization loop. Very convincing results are shown demonstrating in particular the possibility for the curve to have its topology altered during the evolution. Nevertheless, this function is not defined at points of the curve of interest and it is a serious disadvantage in the context of sampled functions.

Alternatively, we shall follow a slightly counterintuitive idea that was proposed by Ruuth, Merriman and Osher in a discrete setting [42] and whose roots are in a work by Steinhoff, Fan and Wang [48]. We were inspired by the last section of this technical report and our paper elaborates on some of the suggestions of these authors and generalizes them in a variety of directions. The idea is to introduce redundancy in the representation of the manifold $\mathcal{M}$: instead of representing it as the intersection of k hypersurfaces, we propose to represent it as the intersection of n hypersurfaces. These hypersurfaces are related in a natural manner to the distance of the points of $\mathbb{R}^n$ to $\mathcal{M}$ and evolve in such a way as to guarantee that their intersection evolves according to the desired evolution for $\mathcal{M}$. Introducing this redundancy allows for more flexibility in the representation: manifolds with non constant dimensions (in space) and boundaries such as the one in figure 1 can now be represented and evolved. Their dimension can even change in time, *i.e.* increase or decrease.

The plan of the paper is as follows. In section 2 we introduce our redundant representation, called the Vector Distance Function (VDF), for arbitrary smooth manifolds of dimension k. In section 3 we study some of its differential properties. In section 4 we start looking at the problem of evolving a manifold by evolving its VDF instead: we show that this problem has a very simple solution that guarantees that the VDF remains a VDF at all times. In section 5 we work out an example. In section 6 we tackle the problem of manifolds with a boundary and we show that it is closely related to changes in the dimension. In section 7 we illustrate some of these ideas on a simple example. We conclude in section 8.

2 The vector distance function (VDF) to a smooth manifold

Let $\mathcal{M}$ be a closed subset of $\mathbb{R}^n$. For every point x we note $\delta(x)$ the distance $dist(\mathbf{x}, \mathcal{M})$ of $\mathbf{x}$ to $\mathcal{M}$. This function is Lipschitz continuous and therefore almost everywhere differentiable [23]. The same holds for the function $\eta(\mathbf{x}) = \frac{1}{2}\delta^2(\mathbf{x})$. We note $\mathbf{u}(\mathbf{x})$ its derivative, defined almost everywhere:

$$\mathbf{u}(\mathbf{x}) = D\eta(\mathbf{x}) = \delta(\mathbf{x})D\delta(\mathbf{x}).$$

This equation shows that, since $\delta(\mathbf{x})$ satisfies *a.e.* the eikonal equation,

$$\|D\delta\| = 1, \tag{3}$$

$\mathbf{u}(\mathbf{x})$ is a vector of length $\delta(\mathbf{x})$. Moreover, at a point $\mathbf{x}$ where δ is differentiable, let $\mathbf{y} = P_{\mathcal{M}}(\mathbf{x})$ be the unique projection of $\mathbf{x}$ onto $\mathcal{M}$. This point is such that

$$\delta(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\|.$$

Besides, if $\mathcal{M}$ is smooth at $\mathbf{y}$, the vector $\mathbf{x} - \mathbf{y}$ is normal to $\mathcal{M}$ at $\mathbf{y}$ and parallel to $D\delta(\mathbf{x})$, see figure 1:

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} - \mathbf{y} \equiv \mathbf{x} - P_{\mathcal{M}}(\mathbf{x}).$$

The vectors such as $\mathbf{x}-\mathbf{y}$, normal at $\mathbf{y}$ to $\mathcal{M}$, define the *characteristics* of the distance

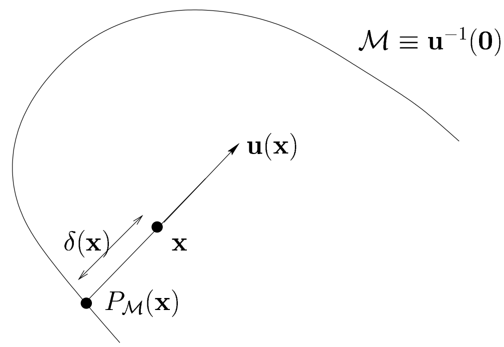


Figure 1: The projection of the point $\mathbf{x}$ on the smooth manifold $\mathcal{M}$ is noted $P_{\mathcal{M}}(\mathbf{x})$. The VDF to $\mathcal{M}$ at $\mathbf{x}$ is equal to $\mathbf{x} - P_{\mathcal{M}}(\mathbf{x})$.

function δ. Starting from a point $\mathbf{y}$ of $\mathcal{M}$ and following a characteristic, *i.e.* a direction in the normal space $N_{\mathbf{y}}\mathcal{M}$, we either go to infinity or reach a point $\mathbf{z}$ at finite distance where δ is not differentiable and therefore $\mathbf{u}$ is not defined. Such a point belongs to the skeleton of $\mathcal{M}$. Because of the previous properties of the function $\mathbf{u}$, we have the following proposition

Proposition 1 *Let* $\mathbf{x}$ *be a point of* $\mathbb{R}^n$ *where* $\mathbf{u}$ *is defined. The following relation*

$$\mathbf{u}(\mathbf{x}+\alpha\mathbf{u}(\mathbf{x})) = (1+\alpha)\mathbf{u}(\mathbf{x}) \tag{4}$$

holds true for all values of $+\infty \geq \alpha_m > \alpha \geq -1$ *such that* $\mathbf{x}+\alpha_m\mathbf{u}(\mathbf{x})$ *is the first point on the characteristic where* $\mathbf{u}$ *is not defined.*

Proof : Use the equation $\mathbf{u}(\mathbf{x}) = \mathbf{x} - P_{\mathcal{M}}(\mathbf{x})$ and the fact that $P_{\mathcal{M}}(\mathbf{x}+\alpha\mathbf{u}(\mathbf{x})) = P_{\mathcal{M}}(\mathbf{x})$ for all α's such that $\alpha_m > \alpha \geq -1$, see figure 2.

□

It may be interesting to pause here and make the remark that the VDF to a smooth manifold $\mathcal{M}$ is an implicit representation of this manifold:

Proposition 2 *Let* $\mathcal{M}$ *be a smooth closed manifold and* $\mathbf{u}$ *its VDF, defined a.e.. Then*

$$\mathcal{M} = \mathbf{u}^{-1}(\mathbf{0}). \tag{5}$$

In effect, $\mathcal{M}$ is the intersection of the n hypersurfaces of equations $u_i(\mathbf{x}) = 0$, $i = 1, \cdots, n$. Since $\mathbf{u}$ represents implicitly $\mathcal{M}$, the rank of the differential $D\mathbf{u}$ of $\mathbf{u}$ which is defined *a.e.* if $\mathcal{M}$ is smooth provides some interesting information about the dimension of $\mathcal{M}$. Indeed, we have

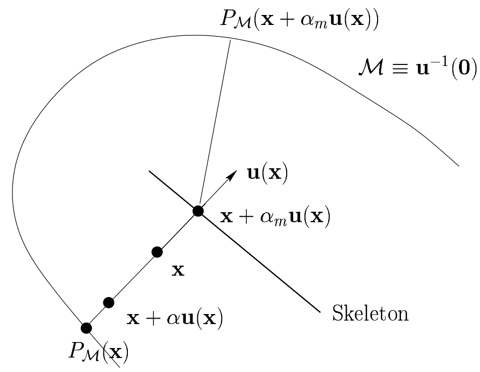


Figure 2: The VDF is defined *a.e.*. In particular it is not defined on the skeleton of the manifold $\mathcal{M}$: the point $\mathbf{x}+\alpha_m\mathbf{u}(\mathbf{x})$ is at equal distances from the two points $P_{\mathcal{M}}(\mathbf{x})$ and $P_{\mathcal{M}}(\mathbf{x}+\alpha_m\mathbf{u}(\mathbf{x})$.

Lemma 1 *The codimension of $\mathcal{M}=\mathbf{u}^{-1}(\mathbf{0})$ is equal to the rank of the differential $D\mathbf{u}(\mathbf{x})$ at points of $\mathcal{M}$.*

Proof : This is a particular case of the implicit function theorem. Let $\mathbf{x}^0\in\mathbb{R}^n$ such that $\mathbf{u}(\mathbf{x}^0)=\mathbf{0}$ and $\text{rank}(D\mathbf{u}(\mathbf{x}^0))=r$. Then we can, without loss of generality, give a privileged role to the r^{th} first coordinates of $\mathbf{x}^0$ and suppose

$$\frac{\partial(u_1(\mathbf{x}^0),\cdots,u_r(\mathbf{x}^0))}{\partial(x_1,\cdots,x_r)}\neq 0. \tag{6}$$

Now, we define the function

$$\begin{cases} f:\mathbb{R}^n\to\mathbb{R}^n \\ (x_1,\cdots,x_n)\to(u_1(\mathbf{x}),\cdots,u_r(\mathbf{x}),x_{r+1},\cdots,x_n). \end{cases} \tag{7}$$

It follows easily from (6) that the Jacobian matrix of f verifies $det(\frac{\partial f}{\partial\mathbf{x}}(\mathbf{x}^0))\neq 0$. By the inverse function theorem, f is a diffeomorphism between certain neighborhoods $V_{\mathbf{x}^0}\subset\mathbb{R}^n$ and $W_{\tilde{\mathbf{0}}}\subset\mathbb{R}^n$ of $\mathbf{x}_0$ and $\tilde{\mathbf{0}}=[0,\cdots,0,x^0_{r+1},\cdots,x^0_n]^T$.
Hence, we can consider the function $M(x_{r+1},\cdots,x_n)=f^{-1}(0,\cdots,0,x_{r+1},\cdots,x_n)$ defined from a $(n-r)$-dimensional subset of $W_{\tilde{\mathbf{0}}}$ onto $V_{\mathbf{x}^0}$.
The function M is the embedding of a manifold $\mathcal{M}$ of dimension $n-r$ in $\mathbb{R}^n$, *i.e.* a $(n-r)$-dimensional submanifold of $\mathbb{R}^n$ such that $\mathbf{u}(\mathcal{M})=\mathbf{0}$. This result is true for all points in $\mathbf{u}^{-1}(\mathbf{0})$ which concludes the proof.

□

This latter fact is not particular to VDF's but, as we shall see in the next section, the relation between $D\mathbf{u}$ and the codimension of $\mathcal{M}$ is even more remarkable in the case of VDF's since the codimension of $\mathcal{M}$ can be determined by the value of $D\mathbf{u}$ at points *off* $\mathcal{M}$.

Because we are interested in evolving the manifold $\mathcal{M}$ through the evolution of $\mathbf{u}$, while keeping $\mathbf{u}$ a VDF, we are interested in finding a characterization of the VDF's analogous to the one for distance functions, (3). This will be our first step in the exploration of the differential properties of the function $\mathbf{u}$ that will be pursued in the next section.

Proposition 3 *Let* $\mathbf{u} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be such that*

$$(D\mathbf{u})^T \mathbf{u} = \mathbf{u} \quad a.e. \tag{8}$$

and $\mathbf{u}$ *is continuous at all points of the set* $\mathcal{M} = \mathbf{u}^{-1}(\mathbf{0})$. *Then* $\mathbf{u}(\mathbf{x}) = D\eta(\mathbf{x})$ *a.e., where* $\eta(\mathbf{x})$ *is the function* $\frac{1}{2}dist^2(\mathbf{x}, \mathcal{M})$ *to the set* $\mathcal{M}$.

Proof : Define the function $\phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\forall \mathbf{x} \in \mathbb{R}^n, \quad \phi(\mathbf{x}) = \|\mathbf{u}(\mathbf{x})\| \tag{9}$$

and compute its first order derivative with respect to $\mathbf{x}$

$$\nabla\phi = \frac{D\mathbf{u}^T\,\mathbf{u}}{\|\mathbf{u}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Hence

$$\|\nabla\phi\| = 1 \quad a.e.,$$

which means that ϕ is equal to the distance function to the set $\mathcal{M}$ plus a constant

$$\phi = \delta + C.$$

In addition, the combination of $\phi = \delta + C$ and (9) shows that

$$\mathbf{u} = (\delta + C)\nabla\delta.$$

The continuity of $\mathbf{u}$ on $\mathbf{u}^{-1}(0)$ implies that $C = 0$. Indeed, let $\mathbf{x}_0$ be a point of $\mathcal{M}$ and $\mathbf{n}$ be a unit vector of $N_{\mathbf{x}_0}\mathcal{M}$, the normal space of $\mathcal{M}$ at $\mathbf{x}_0$. We consider the line $\lambda : \mathbf{x}_0 + \lambda\mathbf{n}$ and the variations of δ and $\nabla\delta$ along this line. Figure 3 shows that the product $\delta\nabla\delta$ is continuous on $\mathcal{M}$ but $\nabla\delta$ is not.

Finally

$$\mathbf{u} = \phi\nabla\phi = \nabla\left(\frac{\delta^2}{2}\right).$$

□

Equation (8) is the characteristic equation of the class of Vector Distance Functions. Unfortunately, since the notion of viscosity solutions is not readily available for systems of PDE's we cannot state and prove a result analogous to the one characterizing the distance function to $\mathcal{M}$ as the unique viscosity solution of the eikonal equation (see for example [7, 23]).

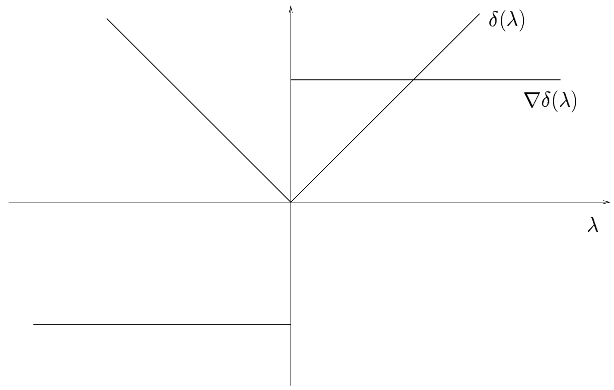


Figure 3: The function $\nabla\delta$ is discontinuous at $\lambda = 0$ along the line $\mathbf{x}_0 + \lambda \boldsymbol{n}$. On the other hand, the product $\delta\nabla\delta$ is continuous at $\lambda = 0$.

The previous proposition says nothing about the regularity of the set $\mathcal{M}$. In fact, the proof of proposition 3 assumes that $\mathcal{M}$ is smooth enough to have a normal space at every point. This is explained later in the paper. If we want that set to be a smooth manifold, then it is likely that $\mathbf{u}$ must satisfy some extra regularity conditions. We have not pursued this direction but a clue can be found in the paper [3] where it is shown that, given a smooth manifold $\mathcal{M}$, the first and second fundamental forms of $\mathcal{M}$ can be recovered from the *third* order derivatives of the distance function δ to $\mathcal{M}$ at all points where it is differentiable.

We can also make good use of some standard results in Differential Geometry but we need first some properties of VDF's that will be proved in section 3.

Let us make two additional remarks that are useful in the application of the theory of VDF's.

Corollary 1 *A VDF* $\mathbf{u}$ *satisfies a.e. the following two properties:*

$$\mathbf{u} = \frac{1}{2} D\|\mathbf{u}\|^2$$

and

$$D\mathbf{u} = (D\mathbf{u})^T.$$

Proof : Both properties follow from the fact that we have proved in proposition 3 that the magnitude of $\mathbf{u}$ is a distance function: the first one is the last equation in the proof, the second one is a consequence of the fact that $D\mathbf{u}$ is the second order derivative of $\frac{\delta^2}{2}$, hence a Hessian, hence symmetric.

□

To provide the reader with some intuition, we show in figure 4 the VDF of a smooth manifold of dimension 0, a point, embedded in $\mathbb{R}^2$. Similarly, figure 5 shows the VDF of a smooth

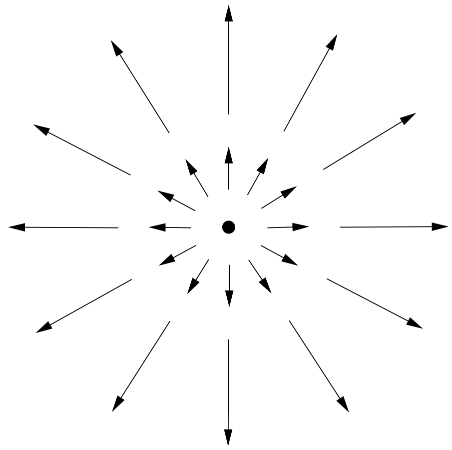


Figure 4: The VDF of a point: a radial field equal to $\mathbf{0}$ at the point.

manifold of dimension 1, an infinite line, embedded in $\mathbb{R}^2$, and figure 6 shows the VDF of another smooth manifold of dimension 1, a circle, also embedded in $\mathbb{R}^2$.

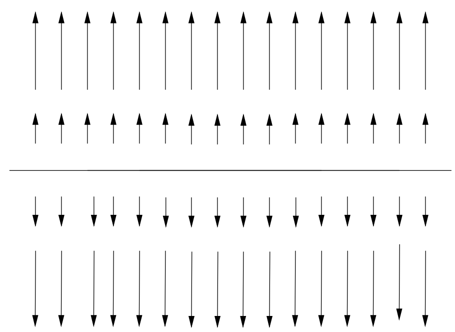


Figure 5: The VDF of a line: a translationally invariant field equal to $\mathbf{0}$ on the line.

Remark 1 *Figure 7 proposes an alternate visualization of the VDF's.*

3 Properties of VDF's

We now study some differential properties of the VDF $\mathbf{u}$. Most of them can be found in [4]. In some cases we give alternate proofs of these.

Equation (4) yields, for $\alpha = -1$ the (almost obvious) equation

$$\mathbf{u}(\mathbf{x} - \mathbf{u}(\mathbf{x})) = \mathbf{0}. \tag{10}$$

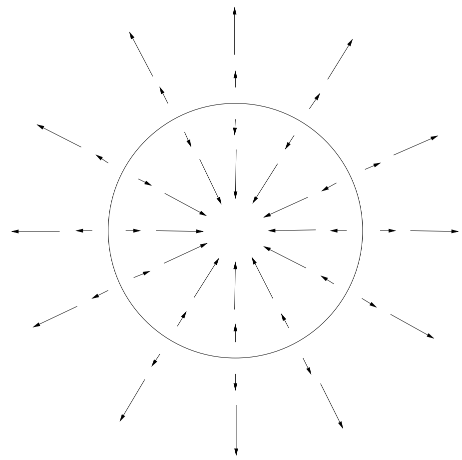


Figure 6: The VDF of a circle: a radial field equal to $\mathbf{0}$ on the circle and undefined at the center.

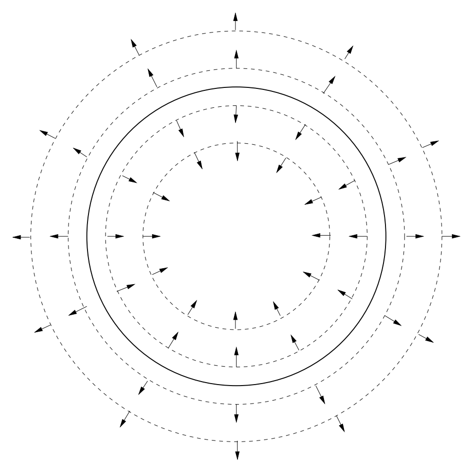


Figure 7: Since VDF's are vector fields, it might be useful to visualize separately the direction and the magnitude fields. In this figure, the direction field of the VDF of a circle is represented by unit vectors and its magnitude is visualized by some of its level-sets (dashed lines). Any vector function can be represented in that way but in the case of VDF's, the dashed lines are parallel to the represented curve and the unit vectors are normal to the dashed lines.

We use this equation to prove the following proposition:

Proposition 4 *The derivative of a VDF satisfies the following relation at each point of $\mathcal{M}$:*

$$D\mathbf{u} = (D\mathbf{u})^2. \tag{11}$$

Therefore it is a projector on a vector subspace.

Proof : Take the derivative of (10) with respect to $\mathbf{x}$ to obtain

$$D\mathbf{u}(\mathbf{x} - \mathbf{u}(\mathbf{x}))(\mathbf{I} - D\mathbf{u}(\mathbf{x})) = \mathbf{0}, \tag{12}$$

which is true *a.e.*. Let us specialize this relation to the case where $\mathbf{u}(\mathbf{x}) = \mathbf{0}$, *i.e.* for $\mathbf{x}$ on $\mathcal{M}$:

$$D\mathbf{u} = (D\mathbf{u})^2.$$

This equation says that $D\mathbf{u}(\mathbf{x})$ is a projector for all points of $\mathcal{M}$.

□

It is not difficult to find out which vector subspace it is a projector on by using some other properties of the distance vector functions (see below).

In the sequel we denote by u_i, $i = 1, \cdots, n$ the coordinates of $\mathbf{u}$. The equation $(D\mathbf{u})^T\mathbf{u} = \mathbf{u}$ can be rewritten in terms of the coordinates u_i of $\mathbf{u}$ and their gradients:

$$\mathbf{u} = \sum_{i=1}^{n} u_i \nabla u_i. \tag{13}$$

We group in the next proposition a number of useful results on $D\mathbf{u}$ and $\mathbf{D}^2\mathbf{u}$.

Proposition 5 *The square $(D\mathbf{u})^2$ of the derivative $D\mathbf{u}$ of $\mathbf{u}$ is given by*

$$(D\mathbf{u})^2 = \sum_{i=1}^{n} \nabla u_i \nabla u_i^T.$$

The second order derivative $\mathbf{D}^2\mathbf{u}$ of $\mathbf{u}$ is related to $D\mathbf{u}$ by the following matrix equation

$$\mathbf{D}^2\mathbf{u}\ \mathbf{u} + (D\mathbf{u})^2 = D\mathbf{u}, \tag{14}$$

where the matrix $\mathbf{D}^2\mathbf{u}\ \mathbf{u}$ is given by

$$\mathbf{D}^2\mathbf{u}\ \mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{D}^2 u_i$$

and $\mathbf{D}^2 u_i$ is the Hessian matrix of the function $u_i : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Proof : Take the derivative of both sides of equation (13) to obtain an expression for $D\mathbf{u}$:

$$D\mathbf{u} = \sum_{i=1}^{n} \left(\nabla u_i \nabla u_i^T + u_i \mathbf{D}^2 u_i \right). \tag{15}$$

We compare this equation with the one obtained by deriving $D\mathbf{u}\mathbf{u} = \mathbf{u}$, *i.e.*

$$\mathbf{D}^2\mathbf{u}\ \mathbf{u} + (D\mathbf{u})^2 = D\mathbf{u}.$$

Indeed, we show that

$$(D\mathbf{u})^2 = \sum_{i=1}^{n} \nabla u_i \nabla u_i^T$$

and hence that the matrix $\mathbf{D}^2\mathbf{u}\ \mathbf{u}$ is equal to $\sum_{i=1}^{n} u_i \mathbf{D}^2 u_i$. In general we have

$$\mathbf{D}^2\mathbf{u}\ \mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{D}^2 u_i\ . \tag{16}$$

In order to see this, write $\left(\sum_{i=1}^{n} \nabla u_i \nabla u_i^T\right)\mathbf{x} = \sum_{i=1}^{n} (\nabla u_i \cdot \mathbf{x}) \nabla u_i$ and note that this is equal to $D\mathbf{u}\,\mathbf{y}$ where the ith coordinate of $\mathbf{y}$ is $\nabla u_i \cdot \mathbf{x}$. Hence $\mathbf{y} = D\mathbf{u}\,\mathbf{x}$ and the result follows.

□

The previous proof allows to show the following important proposition.

Proposition 6 *When evaluated at $\mathbf{x}$ on $\mathcal{M}$, $D\mathbf{u}(\mathbf{x})$ is the projector on the normal space $N_{\mathbf{x}}\mathcal{M}$. This implies that $D\mathbf{u}(\mathbf{x})$ has $n-k$ (the dimension of $N_{\mathbf{x}}\mathcal{M}$) eigenvalues equal to 1 and k (the dimension of $T_{\mathbf{x}}\mathcal{M}$) eigenvalues equal to 0. In particular, the rank of $D\mathbf{u}$ is equal to $n-k$ on $\mathcal{M}$.*

Proof : In the special case where $\mathbf{u} = \mathbf{0}$, *i.e.* on $\mathcal{M}$, equation (15) simplifies to

$$D\mathbf{u} = \sum_{i=1}^{n} \nabla u_i \nabla u_i^T, \tag{17}$$

which shows that the image of the linear operator $D\mathbf{u}(\mathbf{x})$, *i.e.* $D\mathbf{u}(\mathbf{x})(\mathbb{R}^n)$, at every point $\mathbf{x}$ of $\mathcal{M}$, is the vector space of dimension $n-k$ generated by the vectors ∇u_i, $i = 1, \cdots, n$, *i.e.* the normal space $N_{\mathbf{x}}\mathcal{M}$ to $\mathcal{M}$ at point $\mathbf{x}$. Combining this with our observation that $D\mathbf{u}$ is a projector at every point of $\mathcal{M}$ allows us to state that $D\mathbf{u}(\mathbf{x})$ is the projector on the normal space to $\mathcal{M}$ at $\mathbf{x}$. In particular we have

$$D\mathbf{u} \nabla u_i = \nabla u_i \quad i = 1, \cdots, n \tag{18}$$

on $\mathcal{M}$, which is not obvious from (17).

□

This property allows us to say something about the regularity of $\mathcal{M}$. Let $\mathbf{u}$ be a solution of the problem described in proposition 3. If $\mathbf{u}$ is such that the rank of $D\mathbf{u}$ is constant and equal to k on $\mathcal{M} = \mathbf{u}^{-1}(\mathbf{0})$, then because the set where the rank of $D\mathbf{u}$ is equal to k is open (the rank is a lower semi-continuous function), this open set contains $\mathcal{M}$ and theorem 12 of chapter 2 in volume I of [47] says that $\mathcal{M}$ is a smooth manifold of dimension $n-k$ (we do not make more precise what we mean by smooth here).

We now come to the study of the eigenvalues and eigendirections of $D\mathbf{u}$ outside $\mathcal{M}$. The results are quite simple if we consider the line defined by the two points $\mathbf{x}$ and $P_{\mathcal{M}}(\mathbf{x})$: let $\mathbf{n}$ be the unit vector parallel to $\mathbf{x} - P_{\mathcal{M}}(\mathbf{x})$ and consider the line $s \longrightarrow P_{\mathcal{M}}(\mathbf{x}) + s\mathbf{n} \equiv \mathbf{x}(s)$; such a line is called a *characteristic* line. Consider further the values $s_{min} < 0$ and $s_{max} > 0$ (possibly infinite) such that $D\mathbf{u}(\mathbf{x}(s))$ is defined on the open interval $I =]s_{min}, s_{max}[$; the eigendirections of $D\mathbf{u}(\mathbf{x}(s))$ are constant and $n-k$ eigenvalues are equal to 1 for all s in I. Such an open segment is called a *characteristic* segment. More precisely, we have the following proposition

Proposition 7 *The eigendirections of the symmetric matrix $D\mathbf{u}$ are constant along each characteristic segment.*
Moreover, if a ray is parameterized by its arc-length s, starting at $P_{\mathcal{M}}(\mathbf{x})$, then each one of the eigenvalues of $D\mathbf{u}$ has one of the three following forms:

$$\begin{aligned} \lambda(s) &= 1, \quad \textit{or} \\ \lambda(s) &= 0, \quad \textit{or} \\ \lambda(s) &= \frac{s}{s \pm c}, \; c > 0, \\ \forall s \in I, & \end{aligned} \tag{19}$$

where c depends on the particular eigendirection. The first form corresponds to the eigenvectors of $D\mathbf{u}$ which are elements of the normal space to $\mathcal{M}$ at $P_{\mathcal{M}}(\mathbf{x})$; there are $n-k$ such eigenvalues. The second and third forms (the second form is obtained from the third form by taking $c = \infty$) correspond to eigenvectors of $D\mathbf{u}$ which are elements of the tangent space to $\mathcal{M}$ at $P_{\mathcal{M}}(\mathbf{x})$. There are k such eigenvalues.

Since this result is basically proved in [4], our proof appears only in the appendix.
As a final property of the VDF's, we state without proof (the proof is in the appendix, section 9.3) the following proposition

Proposition 8 *The mean-curvature vector of $\mathcal{M}$, of dimension k, at $\mathbf{x}$ is equal, up to a scale factor, to the Laplacian $\Delta\mathbf{u}(\mathbf{x})$ of $\mathbf{u}$ at the same point:*

$$\boldsymbol{\mathcal{H}}(\mathbf{x}) = -\frac{1}{k}(\Delta\mathbf{u}(\mathbf{x})), \quad \forall \mathbf{x} \in \mathcal{M}.$$

4 How to evolve a smooth manifold by evolving its VDF

Let us consider a family $\mathcal{M}(\mathbf{p}, t)$ of smooth manifolds of dimension k, where $\mathbf{p}$ is a k-dimensional vector parameterizing $\mathcal{M}$ at each time instant t. We assume the initial conditions

$$\mathcal{M}(\cdot, 0) = \mathcal{M}_0(\cdot),$$

where $\mathcal{M}_0$ is a smooth manifold of dimension k. Furthermore the evolution of the family $\mathcal{M}$ is governed by the following PDE

$$\mathcal{M}_t(\mathbf{p}, t) = \boldsymbol{\mathcal{H}}(\mathcal{M}(\mathbf{p}, t), t) + \Pi^N_{\mathcal{M}(\mathbf{p},t)}(\boldsymbol{\mathcal{D}}(\mathcal{M}(\mathbf{p}, t), t)) \stackrel{def}{=} \mathbf{V}(\mathcal{M}(\mathbf{p}, t), t), \tag{20}$$

where $\boldsymbol{\mathcal{H}}(\mathcal{M}(\mathbf{p}, t), t)$ is the mean curvature vector at the point $\mathcal{M}(\mathbf{p}, t)$ of the manifold $\mathcal{M}$ (see section 9.1) and $\boldsymbol{\mathcal{D}}(\mathbf{x}, t)$ is a vector field defined on $\mathbb{R}^n \times \mathbb{R}^+$ representing a velocity induced on $\mathcal{M}$ by the data. $\Pi^N_{\mathcal{M}(\mathbf{p},t)}$ is the projection operator on the normal space $N\mathcal{M}_{\mathcal{M}(\mathbf{p},t)}$ to $\mathcal{M}$ at the point $\mathcal{M}(\mathbf{p}, t)$, see figure 8. The goal of this section is to explore ways of evolving

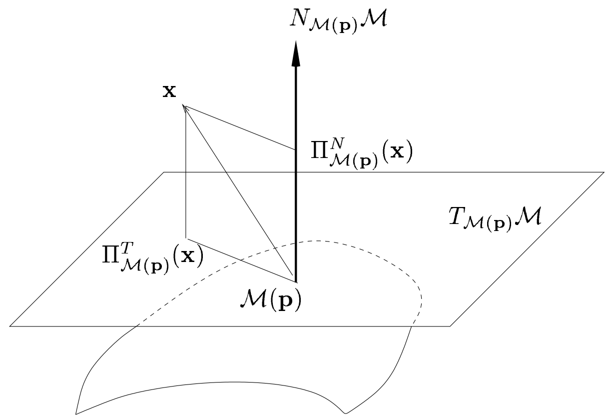


Figure 8: Any point $\mathbf{x}$ of $\mathbb{R}^n$ can be projected orthogonally onto the normal, noted $N_{\mathcal{M}(\mathbf{p})}\mathcal{M}$, and tangent, noted $T_{\mathcal{M}(\mathbf{p})}\mathcal{M}$, spaces of the smooth manifold $\mathcal{M}$ at the point $\mathcal{M}(\mathbf{p})$ by the linear operators $\Pi^N_{\mathcal{M}(\mathbf{p})}(\mathbf{x})$ and $\Pi^T_{\mathcal{M}(\mathbf{p})}(\mathbf{x})$, respectively.

$\mathbf{u}$, the VDF to $\mathcal{M}$, instead of $\mathcal{M}$ while guaranteeing three conditions:

1. That $\mathbf{u}$ remains a VDF at all time instants.
2. That $\mathbf{u}^{-1}(\mathbf{0}) = \mathcal{M}$ at all times where $\mathcal{M}$ is defined.
3. That the manifold $\mathcal{M}$ evolves according to (20).

In other words, we look for a vector field $\mathbf{b}(\mathbf{x}, t)$ defined *a.e.* on $\mathbb{R}^n$ and for some interval $[0, T[$ of $\mathbb{R}^+$ such that

$$\begin{cases} (D\mathbf{u})^T\mathbf{u} = \mathbf{u} & \text{(a)} \\ \mathbf{u}(\mathcal{M}(\mathbf{p}, t), t) = \mathbf{0} & \text{(b)} \\ \mathbf{u}_t(\mathbf{x}, t) = \mathbf{b}(\mathbf{x}, t) & \text{(c)} \\ (\mathbf{u}^{-1}(\mathbf{0}))_t = \boldsymbol{\mathcal{H}} + \Pi^N_{\mathcal{M}}(\boldsymbol{\mathcal{D}}(\mathcal{M}, t)) & \text{(d)} \end{cases} \tag{21}$$

4.1 Some intuition about $\mathbf{u}_t$

The following simple considerations should provide to the reader some intuition about the results we must expect to obtain with the more formal and general methodology presented further in the section.

Consider an evolving curve embedded in the plane by $(s, t) \to \mathcal{C}(s, t)$, where s is the arc-length, and satisfying

$$\mathcal{C}_t = \beta\mathbf{N},$$

where $\mathbf{N}$ is a unit normal vector and β is a smooth velocity field defined on $\mathcal{C}$. Let us fix a point $\boldsymbol{x} \notin \mathcal{C}$ and let $\boldsymbol{y}$ be its closest point on $\mathcal{C}$. Since the VDF of this curve at $\boldsymbol{x}$ is equal to $\boldsymbol{x} - \boldsymbol{y}$, its time derivative, $\mathbf{u}_t(\boldsymbol{x}, t)$, is equal to $-\dot{\boldsymbol{y}} \equiv -\boldsymbol{y}_t$. Thus, the evolution of the VDF is due to the motion of the closest point to $\boldsymbol{x}$ on the curve. Furthermore, let us note $t \to s_{\boldsymbol{x}}(t)$ the curve parameter corresponding to $\boldsymbol{y}$, *i.e.* such that $\boldsymbol{y} = \mathcal{C}(s_{\boldsymbol{x}}(t), t)$. Then, since

$$\mathbf{u}(\boldsymbol{x}, t) = \boldsymbol{x} - \mathcal{C}(s_{\boldsymbol{x}}, t),$$

we have

$$\mathbf{u}_t(\boldsymbol{x}, t) = -(\dot{s}_{\boldsymbol{x}}\mathbf{T} + \beta\mathbf{N}),$$

where $\mathbf{T} = \mathcal{C}_s$. This shows that the motion of the closest point $\boldsymbol{y}$ has two contributions: a normal one, of magnitude β, corresponding to the fact that the curve is moving further or closer to the point $\boldsymbol{x}$ and a tangential one, of magnitude $\dot{s}_{\boldsymbol{x}}$, corresponding to the fact that the point $\boldsymbol{y}$ moves on the curve (*cf* figure 9).

In order to learn more about the tangential component, we calculate $\dot{s}_{\boldsymbol{x}}$ by differentiating the equation

$$\mathcal{C}_s(s_{\boldsymbol{x}}, t) \cdot (\boldsymbol{x} - \mathcal{C}(s_{\boldsymbol{x}}, t)) = 0$$

with respect to t. This yields

$$d\,(\mathcal{C}_{ss}\dot{s}_{\boldsymbol{x}} + \mathcal{C}_{st}) \cdot \mathbf{N} - \dot{s}_{\boldsymbol{x}} = 0,$$

where $d = \|\boldsymbol{x} - \boldsymbol{y}\|$. Now, we use the formula [43]

$$\frac{\partial^2}{\partial t \partial s} = \beta\kappa\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial t} \tag{22}$$

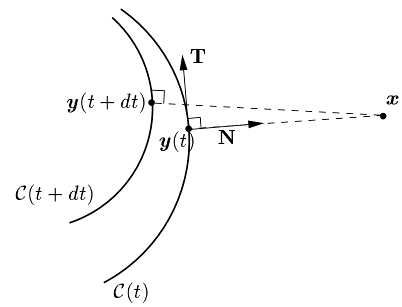


Figure 9: This figure shows the fixed point $\boldsymbol{x}$ and its projections, $\boldsymbol{y}(t)$ and $\boldsymbol{y}(t+dt)$, on the curves $\mathcal{C}(t)$ and $\mathcal{C}(t+dt)$. The motion of $\boldsymbol{y}$ has a normal component due to the fact that $\mathcal{C}$ is evolving further from $\boldsymbol{x}$ and a tangential component due to the fact that the "parameter s" of the closest point is changing.

and obtain

$$\dot{s}_{\boldsymbol{x}} = \frac{d\beta_s}{1 - d\kappa}, \tag{23}$$

where κ is the curvature of $\mathcal{C}$ at $\boldsymbol{y}$. A geometric interpretation of this formula is discussed in figure 10.

Remark 2 *In higher dimension, i.e. with the manifold $\mathcal{M}$ presented at the beginning of the section, this writes*

$$\dot{\boldsymbol{p}}_{\boldsymbol{x}} = d\left(\boldsymbol{I} - \boldsymbol{a}^{\mathbf{N}} d\right)^{-1} (D\boldsymbol{V})^T \mathbf{N}, \tag{24}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix, $\boldsymbol{a}^{\mathbf{N}}$ is the matrix of the endomorphism of Weingarten of $\mathcal{M}$ in the direction $\mathbf{N}$ (see Appendix, section 9.1), $D\boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{p}}$, $d = \|\boldsymbol{x} - \boldsymbol{y}\|$ and $\mathbf{N} = (\boldsymbol{x} - \boldsymbol{y})/d$.

From the practical point of view, this formula may not be very useful because the computation of the term β_s requires in general high order spatial differentiation of the VDF (for example, if $\beta = \kappa$, *i.e.* the curve shortens its length, then β_s depends upon the *third* order spatial derivative of the VDF) but it shades some light on the evolution of VDF's.

4.2 Calculating $\mathbf{u}_t$ using the method of characteristics

Since one of the goals of this paper is to find a tractable way of evolving VDF's, we shall develop now the idea that the vector field $\mathbf{b} = \mathbf{u}_t$ is itself the solution to a certain PDE which can be solved numerically.
It is not too difficult, from (21), to find a characterization of $\mathbf{b}$.

Proposition 9 *The velocity field $\mathbf{b}$ of the VDF $\mathbf{u}$ is characterized by the first order, quasi-linear Partial Differential Equation*

$$D\mathbf{b}\mathbf{u} = (\mathbf{I} - D\mathbf{u})\mathbf{b}, \tag{25}$$

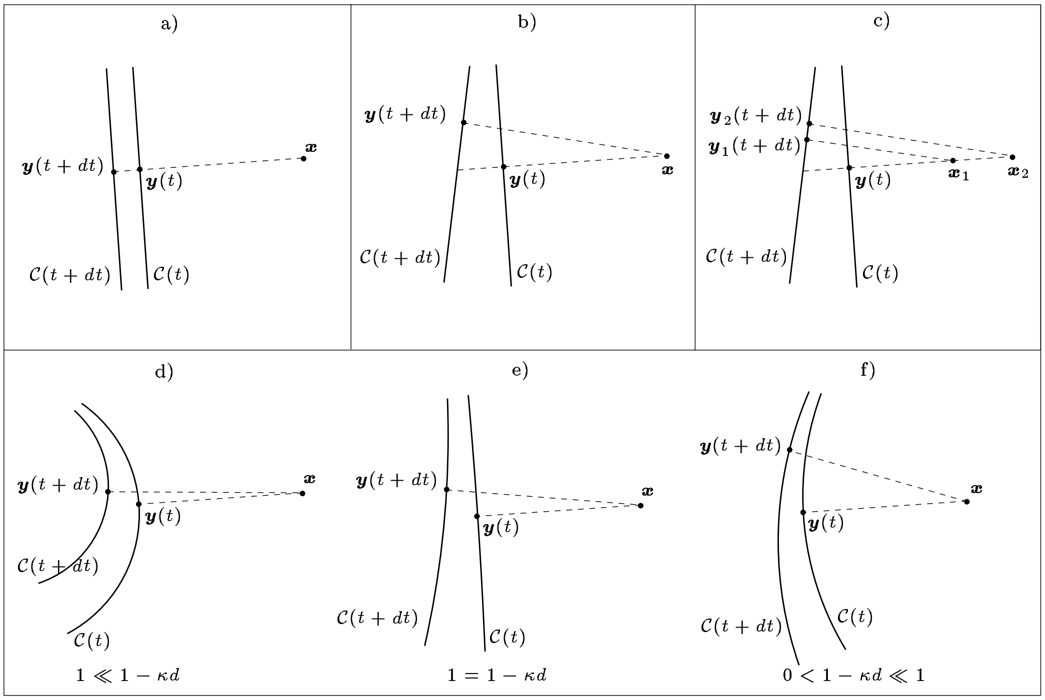


Figure 10: If a planar curve evolves according to $\mathcal{C}_t = \beta\mathbf{N}$, see text, then the curve parameter, $s_{\boldsymbol{x}}$, of the closest point on $\mathcal{C}$, $\boldsymbol{y}$, to a fixed point, $\boldsymbol{x}$, satisfies $\dot{s}_{\boldsymbol{x}} = \frac{d\beta_s}{1-d\kappa}$. We depict some particular cases in order to interpret this formula. **a)** The curve is a line evolving with a constant velocity (*i.e.* $\beta_s = 0$) so that it remains parallel to the initial line. In this case, we have $\dot{s}_{\boldsymbol{x}} = 0$ since the parameter of the closest point to $\boldsymbol{x}$ on the curve does not change. **b)** The initial curve is a line and the velocity field is such that $\beta_s = 1$. The curve remains a straight line but its direction changes. In this case, the point $\boldsymbol{y}$ slides on the curve and the amount of sliding, $\dot{s}_{\boldsymbol{x}}$, is proportional to β_s. **c)** The amount of sliding is also proportional to $d = \|\boldsymbol{x} - \boldsymbol{y}\|$, as shown by taking two different fixed points, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, both with projection $\boldsymbol{y}(t)$ on $\mathcal{C}(t)$ but with different projections on $\mathcal{C}(t+dt)$. Indeed, one has $\frac{\dot{s}_{\boldsymbol{x}_1}}{\dot{s}_{\boldsymbol{x}_2}} = \frac{d_1}{d_2}$. **def)** The last three figures help interpreting the term $1-\kappa d$. If $1 \ll 1-\kappa d$, *i.e.* if the curve is convex, seen from the fixed point $\boldsymbol{x}$, then the amount of sliding is small. If $1-\kappa d = 1$, *i.e.* the curve is a straight line, then the amount of sliding is as in abc). If $0 < 1-\kappa d \ll 1$, *i.e.* if the curve is concave, seen from the fixed point $\boldsymbol{x}$, then the amount of sliding is even more important. It is useful to see the term $1-\kappa d$ as a measure of the "degree of uniqueness" of the closest point to $\boldsymbol{x}$ on $\mathcal{C}$. Indeed, if this quantity is null at a point $\boldsymbol{x}$, it means that point lies on the skeleton of the curve (in particular, $\boldsymbol{x}$ is a center of curvature). If $\boldsymbol{x}$ is close to a center of curvature, then this quantity is close to zero and the position of $\boldsymbol{y}$ is very sensitive to perturbations of the positions either of $\mathcal{C}$ or of $\boldsymbol{x}$. On the contrary, if this quantity is high, case d), the curve is convex, seen from $\boldsymbol{x}$, and the point $\boldsymbol{y}$ is "better defined". The quantity $1-\kappa d$ is a positive measure of the uniqueness of the closest point and the amount by which $\boldsymbol{y}$ slides is inversely proportional to it.

with initial conditions

$$\mathbf{b}(\mathcal{M}(\mathbf{p},t),t) = -\mathbf{V}(\mathcal{M}(\mathbf{p},t),t), \tag{26}$$

wherever $\mathbf{u}$ *is defined and differentiable.*

Proof : We start with the equation (8) that characterizes the VDF's *a.e.* and take its time derivative

$$(D\mathbf{u}_t)^T\mathbf{u} + (D\mathbf{u})^T\mathbf{u}_t = \mathbf{u}_t.$$

We then replace $\mathbf{u}_t$ with its value $\mathbf{b}$ and obtain

$$(D\mathbf{b})^T\mathbf{u} = (\mathbf{I} - (D\mathbf{u})^T)\mathbf{b}.$$

Since $D\mathbf{u}$ is a symmetric operator (corollary 1) we can force $D\mathbf{b}$ to be symmetric and we obtain equation (25).

Conversely, if the field $\mathbf{b}$ satisfies (25) wherever $\mathbf{u}$ is defined and differentiable, and its derivative $D\mathbf{b}$ is symmetric as is the derivative $D\mathbf{u}$, we have:

$$(D\mathbf{b})^T\mathbf{u} + (D\mathbf{u})^T\mathbf{b} = \mathbf{b}.$$

Using the fact that $\mathbf{b} = \mathbf{u}_t$, we rewrite this equation as

$$(D\mathbf{u})_t^T\mathbf{u} + (D\mathbf{u})^T\mathbf{u}_t = \mathbf{u}_t.$$

The left-hand side appears as the time derivative of $(D\mathbf{u})^T\mathbf{u}$ hence

$$(D\mathbf{u})^T\mathbf{u} = \mathbf{u} + \mathbf{c},$$

where $\mathbf{c}(\mathbf{x},t)$ is a vector field such that $\mathbf{c}_t = \mathbf{0}$. If we choose $\mathbf{c}(.,0) = \mathbf{0}$ we then have

$$(D\mathbf{u})^T\mathbf{u} = \mathbf{u} \quad a.e. \quad \forall t \geq 0,$$

which, according to proposition 3, implies that $\mathbf{u}$ is a VDF for all times $t \geq 0$ where $\mathbf{u}$ is defined *a.e.*.

□

The next problem we want to address is that of relating in a simple way the field $\mathbf{b}$ to the velocity field of the manifold $\mathcal{M} = \mathbf{u}^{-1}(\mathbf{0})$. We have the following simple result, see figure 12:

Proposition 10 *At all points* $\mathbf{x}$ *where* $\mathbf{b}$ *is defined, its component in the normal space* $N_{P_\mathcal{M}(\mathbf{x})}\mathcal{M}$ *to* $\mathcal{M}$ *at the point* $P_\mathcal{M}(\mathbf{x})$ *is equal to minus the normal velocity* $\mathbf{V}(P_\mathcal{M}(\mathbf{x}),t)$ *of the point* $P_\mathcal{M}(\mathbf{x})$*:*

$$\mathbf{b}(\mathbf{x},t)_{|N_{P_\mathcal{M}(\mathbf{x})}\mathcal{M}} = \mathbf{b}(\mathbf{x}-\mathbf{u}(\mathbf{x},t))_{|N_{P_\mathcal{M}(\mathbf{x})}\mathcal{M}} \equiv \mathbf{b}(P_\mathcal{M}(\mathbf{x}))_{|N_{P_\mathcal{M}(\mathbf{x})}\mathcal{M}} \equiv -\mathbf{V}(P_\mathcal{M}(\mathbf{x}),t). \tag{27}$$

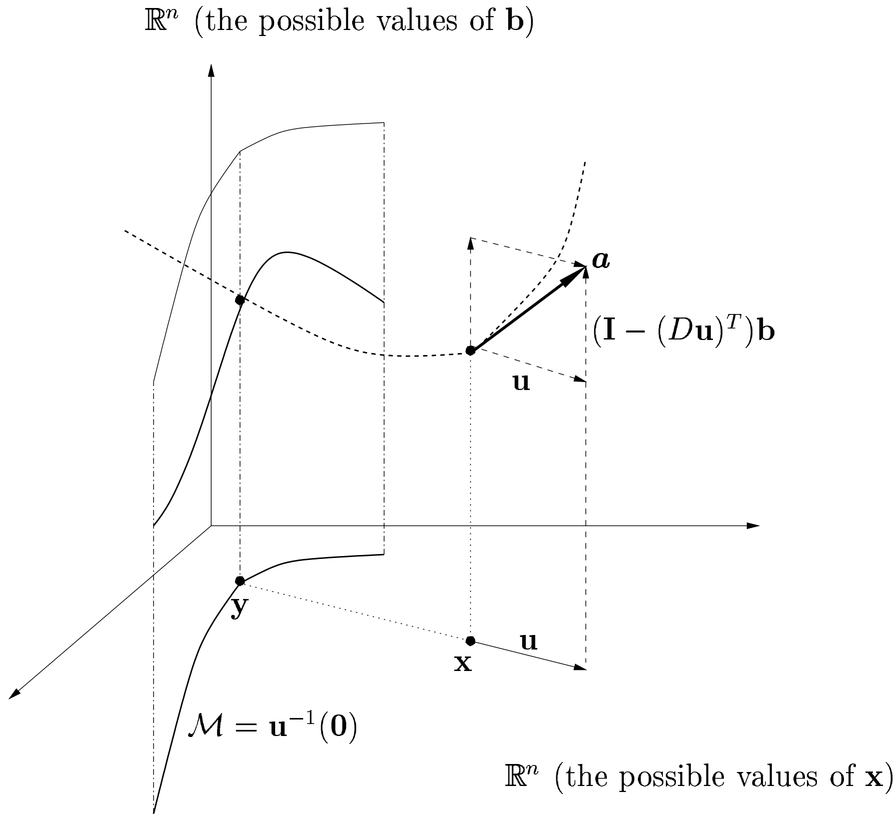


Figure 11: The horizontal space represents the Euclidean space $\mathbb{R}^n$ where $\mathcal{M}$ lives: it is the set of possible values of $\boldsymbol{x}$. The vertical space represents $\mathbb{R}^n$, the set of possible values for $\boldsymbol{b}$. A "surface" over the horizontal plane whose tangent plane is never vertical represents the graph of a function $\mathbb{R}^n \to \mathbb{R}^n$.

Proof : We use the method of characteristics (see e.g. [5]) and rewrite this equation along the characteristics of $\mathbf{b}$ so that a set of ODE's are obtained.

In detail, we build the $2n$-dimensional characteristic vectors (see figure 11)

$$\mathbf{a} = \begin{pmatrix} \mathbf{u} \\ (\mathbf{I} - (D\mathbf{u})^T)\mathbf{b} \end{pmatrix}. \tag{28}$$

We fix a point $\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{b}_0 \end{pmatrix} \in \mathbb{R}^{2n}$ such that $\mathbf{x}_0$ is not on $\mathcal{M}$, *i.e.* $\mathbf{u}(\mathbf{x}_0) \neq \mathbf{0}$. The characteristic curve of the vector field $\mathbf{b}$, parameterized by q, passing through this point is tangent to $\mathbf{a}$ so that it can be described by the following ODE's

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{b}} \end{pmatrix} = \mathbf{a}, \tag{29}$$

where the dot $\dot{}$ indicates the derivative with respect to q. The first three equations of this set of six yield

$$\dot{\mathbf{x}}(q) = \mathbf{u}(\mathbf{x}) = \mathbf{x}(q) - P_{\mathcal{M}}(\mathbf{x}_0),$$

since the characteristic going through $\mathbf{x}_0$ is the line defined by $\mathbf{x}_0$ and $P_{\mathcal{M}}(\mathbf{x}_0)$. This ODE is readily integrated:

$$\mathbf{x}(q) - P_{\mathcal{M}}(\mathbf{x}_0) = (\mathbf{x}_0 - P_{\mathcal{M}}(\mathbf{x}_0))e^q.$$

The parameter q is such that for $q = 0$ the point $\mathbf{x}(q)$ is at $\mathbf{x}_0$, and when q goes to $-\infty$ it tends to $P_{\mathcal{M}}(\mathbf{x}_0)$. Let us rewrite the last equation as

$$\mathbf{x}(q) = P_{\mathcal{M}}(\mathbf{x}_0) + \mathbf{u}(\mathbf{x}_0)e^q,$$

and let us do the change of variable $s = d_0 e^q$ ($d_0 = \|\mathbf{u}(\mathbf{x}_0)\|$ is the distance of x_0 to $P_{\mathcal{M}}(\mathbf{x}_0)$) which corresponds to a change of origin on the characteristic (the origin being now $P_{\mathcal{M}}(\mathbf{x}_0)$ instead of $\mathbf{x}_0$), and a change of velocity.

Let us now look at the last three equations of (28) and (29):

$$\dot{\mathbf{b}}(\mathbf{x}(q)) = \left(\mathbf{I} - D\mathbf{u}^T(\mathbf{x}(q))\right) \mathbf{b}(\mathbf{x}(q)).$$

This equation is readily integrated as

$$\mathbf{b}(p) = \exp\left(\int_0^p (\mathbf{I} - (D\mathbf{u})^T(q))dq\right) \mathbf{b}(0).$$

We compute an analytical form of the right-hand side which, by means of the previous change of variable $s = d_0 e^q$ and an abuse of notations, namely $\mathbf{b}(s) \stackrel{def}{=} \mathbf{b}(\mathbf{x}(s))$ and similarly $D\mathbf{u}(s) \stackrel{def}{=} D\mathbf{u}(\mathbf{x}(s)))$ is also integrated by quadrature with the help of proposition 7 which tells us that along a characteristic ray, all symmetric matrices $D\mathbf{u}$ can be diagonalized in the same orthonormal basis, their eigenvalues being given by equations (19) in the same proposition. In this basis, the matrix $D\mathbf{u}$ can be written

$$D\mathbf{u}(q) = diag(1, \cdots, 1, \frac{d_0 e^q}{d_0 e^q \pm c_1}, \cdots, \frac{d_0 e^q}{d_0 e^q \pm c_k}),$$

where there are $n-k$ 1's and k positive values $c_l, l = 1, \cdots, k$, some of them possibly infinite to take into account the fact that some of the eigenvalues not equal to 1 can be equal to 0.

Hence the integrand $\mathbf{I} - (D\mathbf{u})^T(q)$ is equal to

$$diag(0, \cdots, 0, 1 - \frac{d_0 e^q}{d_0 e^q \pm c_1}, \cdots, 1 - \frac{d_0 e^q}{d_0 e^q \pm c_k}).$$

We then have to compute the values of the integrals

$$\int_0^p (1 - \frac{d_0 e^q}{d_0 e^q \pm c_l}) dq, \quad l = 1, \cdots, k.$$

Applying the previous change of variable $s = d_0 e^q$, these integrals can be rewritten as

$$\int_{d_0}^{d_0 e^p} (\frac{1}{s} - \frac{1}{s \pm c_l}) ds,$$

which is found to be equal to

$$\log \frac{(d_0 \pm c_l) e^p}{d_0 e^p \pm c_l}.$$

Taking the exponential of these values we obtain the relation between the $(n-k+l)$th components of the velocity field at the point $P_{\mathcal{M}}(\mathbf{x}_0) + \mathbf{u}(\mathbf{x}_0) e^p$, noted $b_l(p)$ and at the point $\mathbf{x}_0$, noted $b_l(0)$:

$$b_l(p) = \frac{(d_0 \pm c_l) e^p}{d_0 e^p \pm c_l} b_l(0), \quad l = 1, \cdots, k.$$

Let us rewrite this expression as

$$b_l(0) = \frac{d_0 e^p \pm c_l}{(d_0 \pm c_l) e^p} b_l(p).$$

When $p \longrightarrow -\infty$, $b_l(p)$ goes to 0 since it is a component of the velocity field of $\mathcal{M}$ in its tangent space $T\mathcal{M}$. Since $b_l(0)$ is well defined the product $b_l(p) e^{-p}$ must have a limit, noted α_l when $p \longrightarrow -\infty$:

$$b_l(0) = \pm \frac{c_l}{d_0 \pm c_l} \alpha_l. \tag{30}$$

This shows that the components of $\mathbf{b}(\mathbf{x}_0)$ in the tangent space to $\mathcal{M}$ at $P_{\mathcal{M}}(\mathbf{x}_0)$ are in general non zero. The constants α_l depend upon the geometry of $\mathcal{M}$ and of the various derivatives of the velocity field on $\mathcal{M}$. The exact values for these parameters are not important in practice, see section 8. To summarize, in an orthonormal basis where $D\mathbf{u}(\mathbf{x}_0)$ and $D\mathbf{u}(P_{\mathcal{M}}(x_0))$ are diagonal, the first $n-k$ coordinates of $\mathbf{b}(\mathbf{x}_0)$ and $\mathbf{b}(P_{\mathcal{M}}(x_0))$ are equal (they correspond to components of the velocity in the normal space to $\mathcal{M}$). The remaining k components, corresponding to the tangent space to $\mathcal{M}$ are in general non zero and are given by (30) or (24).

□

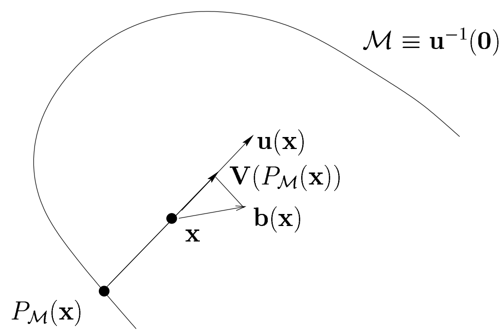


Figure 12: The normal component of the velocity field $\mathbf{b}(\mathbf{x})$ of the VDF $\mathbf{u}$ at point $\mathbf{x}$ where $\mathbf{u}$ is defined is equal to the velocity of the projection $P_{\mathcal{M}}(\mathbf{x})$ of $\mathbf{x}$ on the smooth manifold $\mathcal{M}$.

5 Example I: the helix

We consider a circular helix $\mathcal{M}$ embedded in $\mathbb{R}^3$ by

$$\sigma \to \mathbf{x}(\sigma) = \begin{pmatrix} re^{i\sigma} \\ \sigma \end{pmatrix}, \quad \sigma \in \mathbb{R}.$$

It is a smooth manifold of dimension 1, codimension 2, without a boundary. Its Frenet frame is made of the tangent $\boldsymbol{\tau}$, the normal $\boldsymbol{\nu}$ and the binormal $\boldsymbol{\beta}$, respectively

$$\boldsymbol{\tau}(\sigma) = \frac{1}{\sqrt{r^2+1}} \begin{pmatrix} re^{i(\sigma+\frac{\pi}{2})} \\ 1 \end{pmatrix}, \quad \boldsymbol{\nu}(\sigma) = \begin{pmatrix} -e^{i\sigma} \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta}(\sigma) = \frac{1}{\sqrt{r^2+1}} \begin{pmatrix} -e^{i(\sigma+\frac{\pi}{2})} \\ r \end{pmatrix}.$$

This helix is invariant by the one-parameter family F_ϕ of rigid motions (screw motions) defined by

$$\forall \begin{pmatrix} \rho e^{i\theta} \\ z \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \quad F_\phi \begin{pmatrix} \rho e^{i\theta} \\ z \end{pmatrix} = \begin{pmatrix} \rho e^{i(\theta+\phi)} \\ z+\phi \end{pmatrix}.$$

Indeed, if $\mathbf{x}(\sigma)$ is a point of the helix, then $F_\phi(\mathbf{x}(\sigma)) = \mathbf{x}(\sigma+\phi)$ is also a point of the helix. This proves that any function defined over $\mathbb{R}^3$ which depends only upon the geometry of the helix (like the VDF) is also invariant by these rigid motions. As a corollary, once the restriction $\boldsymbol{A} = \mathbf{u}_{|\Pi}$ of $\mathbf{u}$ to the half plane $\Pi = \{(\rho e^{i\theta}, z) = (x, y, z) \in \mathbb{R}^3, \quad \theta = 0\}$ is known, this invariance property yields the expression of $\mathbf{u}$ in $\mathbb{R}^3$. More formally, if

$$\boldsymbol{A} : \Pi \to \mathbb{R}^3 \tag{31}$$

$$(\rho, z) \to \mathbf{u}(\rho, 0, z), \tag{32}$$

then

$$\mathbf{u}(\rho, \theta, z) = F_\theta(\boldsymbol{A}(\rho, z - \theta)). \tag{33}$$

Furthermore it is clear, from the symmetry of the problem, that $\boldsymbol{A}$ is periodic in z with smallest period 2π. So, we consider only the restriction $\boldsymbol{a}_{|I}$ of $\boldsymbol{A}$ to the half band $I = \mathbb{R}^+ \times]-\pi, \pi[$, so that

$$\boldsymbol{A}(\rho, z) = \boldsymbol{a}\left(\rho, z - 2\pi E\left(\frac{z+\pi}{2\pi}\right)\right),$$

where $E(x)$ denotes the integer part of the positive real number x. It is also possible to use the symmetry of $\boldsymbol{a}$ with respect to the $z = 0$ line but it is useless as we soon shall obtain it as a function which is odd in z.

Next we compute an expression of $\boldsymbol{a}(\rho, z)$ for $z \in [0, \pi[$ by seeking the closest point to $(\rho\boldsymbol{i}, z) \in I$ which belongs to the helix. The squared distance from $\mathbf{x}(\sigma)$ to the point $(\rho\boldsymbol{i}, z)$ is given by

$$\left\| \begin{pmatrix} \rho\boldsymbol{i} \\ z \end{pmatrix} - \begin{pmatrix} re^{i\sigma} \\ \sigma \end{pmatrix} \right\|^2, \quad \sigma \in [0, \pi[.$$

Writing the first order minimum necessary condition of this expression with respect to σ yields the following relationship between the variables ρ, z and σ

$$\rho r \sin\sigma + \sigma - z = 0. \tag{34}$$

This implicitly expresses σ as a function of ρ and z but it cannot be explicitly inverted. Nevertheless, the implicit function theorem gives

$$\frac{\partial\sigma(\rho, z)}{\partial\rho} = -\frac{r\sin\sigma}{r\cos\sigma + 1} \quad \text{and} \quad \frac{\partial\sigma(\rho, z)}{\partial z} = -\frac{1}{r\cos\sigma + 1},$$

which will be enough for our purpose.

Finally, we have

$$\boldsymbol{a}(\rho, z) = \begin{pmatrix} \rho\boldsymbol{i} \\ z \end{pmatrix} - \begin{pmatrix} re^{i\sigma(\rho,z)} \\ \sigma \end{pmatrix}, \tag{35}$$

where $\sigma(\rho, z)$ is given by (34).

Next, we compute $D\mathbf{u}$ in I (*i.e.* for $\theta = 0$ and $0 \le z < \pi$) and verify the results of the previous section. We express it in the Frenet's frame $(\boldsymbol{\tau}(\sigma), \boldsymbol{\nu}(\sigma), \boldsymbol{\beta}(\sigma))$ where its expression is (hopefully) diagonal. This is a straightforward but fastidious calculation and we shall only give the guidelines of the calculus.

Starting from (33), calculate $\frac{\mathbf{u}(\rho,\theta,z)}{(\rho,\theta,z)}$ using the chain rule and the expressions of $\frac{\partial\sigma(\rho,z)}{\partial\rho}$, $\frac{\partial\sigma(\rho,z)}{\partial z}$, $\boldsymbol{a}$, $\frac{\partial\boldsymbol{a}}{\partial(\rho,z)}$, $\boldsymbol{A}$ and $\frac{\partial\boldsymbol{A}}{\partial(\rho,z)}$. Then calculate

$$D\mathbf{u} = \frac{\partial\mathbf{u}}{\partial\mathbf{x}} = \frac{\partial\mathbf{u}}{\partial(\rho,\theta,z)}\frac{\partial(\rho,\theta,z)}{\partial\mathbf{x}},$$

where

$$\frac{\partial(\rho,\theta,z)}{\partial\mathbf{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now pass to the intrinsic Frenet frame, *i.e.*

$$\frac{\partial\mathbf{u}}{\partial\mathbf{x}'} = \frac{\partial\mathbf{x}}{\partial\mathbf{x}'}^T D\mathbf{u}\,\frac{\partial\mathbf{x}}{\partial\mathbf{x}'},$$

where

$$\frac{\partial\mathbf{x}}{\partial\mathbf{x}'} = \begin{pmatrix}\boldsymbol{\tau} & \boldsymbol{\nu} & \boldsymbol{\beta}\end{pmatrix}.$$

The resulting differential operator is

$$\frac{\partial\mathbf{u}}{\partial\mathbf{x}'} = \begin{pmatrix} \frac{r(\rho\cos\sigma - r)}{r\rho\cos\sigma + 1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is the expression of $D\mathbf{u}$ in the Frenet frame of the helix.

As expected, the Frenet frame makes this matrix diagonal and the eigenvalue corresponding to the directions $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ (*i.e.* the normal directions) is equal to 1. The first eigenvalue (corresponding to the tangential vector $\boldsymbol{\tau}$) is better understood by introducing explicitly the distance to the helix, *i.e.* $s = ||\mathbf{u}||$. To do so, we remark that $\mathbf{u} = s\mathbf{n}$ where $\mathbf{n}$ is a unit normal vector, thus of the form $\mathbf{n} = \boldsymbol{\nu}\cos\alpha + \boldsymbol{\beta}\sin\alpha$. Now, noticing the following

$$\boldsymbol{a}\cdot\boldsymbol{\nu} = r - \rho\cos\sigma,$$

we see that the first eigenvalue is

$$\frac{r(\rho\cos\sigma - r)}{r\rho\cos\sigma + 1} = \frac{s}{\frac{1}{\kappa\cos\alpha} + s},$$

where $\kappa = \frac{r}{r^2+1}$ is the curvature of the helix, which agrees with the result of proposition 7.

We shall now make this helix evolve in time according to mean curvature motion and verify that its (time dependent) distance vector satisfies the equation (27). First of all, we show the helix remains an helix with decreasing radius until it becomes (in infinite time) the y axis. The fact, the mean-curvature vector of the helix is $\kappa\boldsymbol{\nu}$ suggests that the trace of the evolving helix in the $(\boldsymbol{i}, \boldsymbol{j})$ plane (a circle of initial radius r) evolves with a radius satisfying the following ODE

$$\dot{R}(t) = -\frac{R(t)}{R^2(t)+1} = -\kappa(t),$$

with initial condition $R(0) = r$. This integrates as

$$\frac{R^2(t) - r^2}{2} + \ln\frac{R(t)}{r} = t,$$

which expresses implicitly the function $t \to R(t)$.

We then verify that the time dependent helix embedded by

$$\sigma \to \begin{pmatrix} R(t)e^{i\sigma} \\ \sigma \end{pmatrix}, \quad \forall\sigma \in \mathbb{R},$$

satisfies the following ODE

$$\mathcal{M}_t = \begin{pmatrix} \dot{R}(t)e^{i\sigma} \\ 0 \end{pmatrix} = -\kappa(t)\boldsymbol{\nu},$$

i.e. it is *the* solution to the mean curvature motion and has the correct initial value. As a corollary, the distance vector to this evolving helix is

$$\mathbf{u}(\rho, 0, z, t) = \begin{pmatrix} \rho\boldsymbol{i} \\ z \end{pmatrix} - \begin{pmatrix} R(t)e^{i\sigma(t)} \\ \sigma(t) \end{pmatrix},$$

where $\sigma(t)$ is defined by

$$\rho R(t)\sin\sigma(t) + \sigma(t) - z = 0.$$

.

We compute

$$\mathbf{u}_t(\rho, 0, z, t) = -\begin{pmatrix} (\dot{R} + iR\dot{\sigma})e^{i\sigma} \\ \dot{\sigma} \end{pmatrix} = -\kappa(t)\boldsymbol{\nu}(\sigma(t)) - \dot{\sigma}\sqrt{R^2+1}\boldsymbol{\tau}(\sigma(t)).$$

We find, as proved in proposition 10, that the normal component of $\mathbf{b}(\rho, 0, z, t)$ is equal to minus the velocity of the point $P_{\mathcal{M}}(\rho, 0, z)$.

6 Smooth manifolds with a boundary and changes in dimension

A simple example will introduce the new issues of this section. In the plane $\mathbb{R}^2$, we consider the time dependent segment $[A(t), B(t)]$ whose endpoints have respectively the velocities $\boldsymbol{v}_A(t) = \alpha \boldsymbol{i}$ and $\boldsymbol{v}_B(t) = \beta \boldsymbol{i}$ with $\alpha < 0 < \beta$ and with initial condition $A(0) = B(0) = O$. This situation describes a point transforming into a segment with increasing length: it is a prototype of a change of dimension followed by the evolution of a smooth manifold with boundary. We note $\mathbf{x} = [x, y]^T$ the coordinates of a point of $\mathbb{R}^2$. The VDF to this object is easily shown to satisfy

$$\mathbf{u}_t(x,y,t) + D\mathbf{u}\mathbf{V} = \mathbf{0} \quad \text{with} \quad \mathbf{V}(x,y,t) = \begin{cases} \boldsymbol{v}_A & \text{if } x < \alpha t, \\ \mathbf{0} & \text{if } \alpha t < x < \beta t, \\ \boldsymbol{v}_B & \text{if } x > \beta t. \end{cases} \tag{36}$$

This equation is a variant of a well-known class of PDE's called the *transport equations* [23]. One of its solution is readily shown to be

$$\begin{cases} \mathbf{u}_0(\mathbf{x} - \mathbf{A}) & \text{if} \quad x < \alpha t, \\ \mathbf{u}_0(x, 0) & \text{if} \quad \alpha t < x < \beta t, \\ \mathbf{u}_0(\mathbf{x} - \mathbf{B}) & \text{if} \quad x > \beta t. \end{cases}$$

At first glance (see figure 13), the vector field $\mathbf{b} = -D\mathbf{u}\mathbf{V}$ is not of the form presented in the previous section because it is only piecewise-smooth, $\mathbf{b}$ being discontinuous on the lines of equations $x = \alpha t$ and $x = \beta t$. But, as shown in the figure, the evolution of the point O is quite remarkable: it is a smooth manifold of dimension 0 which is turned into a smooth manifold with boundary (the segment AB). The velocity field $\mathbf{b}$ is discontinuous on the vertical axis at time $t = 0$ which has the effect of allowing the point to "spread" to a line segment. At time $t > 0$ the velocity field $\mathbf{b}(., t)$ is of the form $\mathbf{V}(P_{AB}(.))$ everywhere except on the previous two lines. We note that the *normal* component of $\mathbf{V}$ is continuous across these two lines, being equal to 0 everywhere, while its *tangential* component is discontinuous over them. This last point is the reason why the segment $[AB]$ can grow in time. We shall see more on this in section 6.1.

The situation we have just described is archetypal of all cases in higher dimensions and codimensions and reveals an undesirable lack of generality in the analysis of the previous section and suggests that *non-continuous* $\mathbf{b}$'s may also be interesting since they can account for changes of dimension and tangent velocities at the boundary of a manifold (both are intimately related as it can be learned from the segment example).

In order to deal with these new issues, it is necessary in the first place to provide ourselves with a model for the intuitive but vague notion of "changing dimension". The model we shall present is valid regardless of the dimensions but for the sake of simplicity, it will be presented in the case of manifolds embedded in $\mathbb{R}^3$. In this model, the target manifold $\mathcal{M}$, of dimension

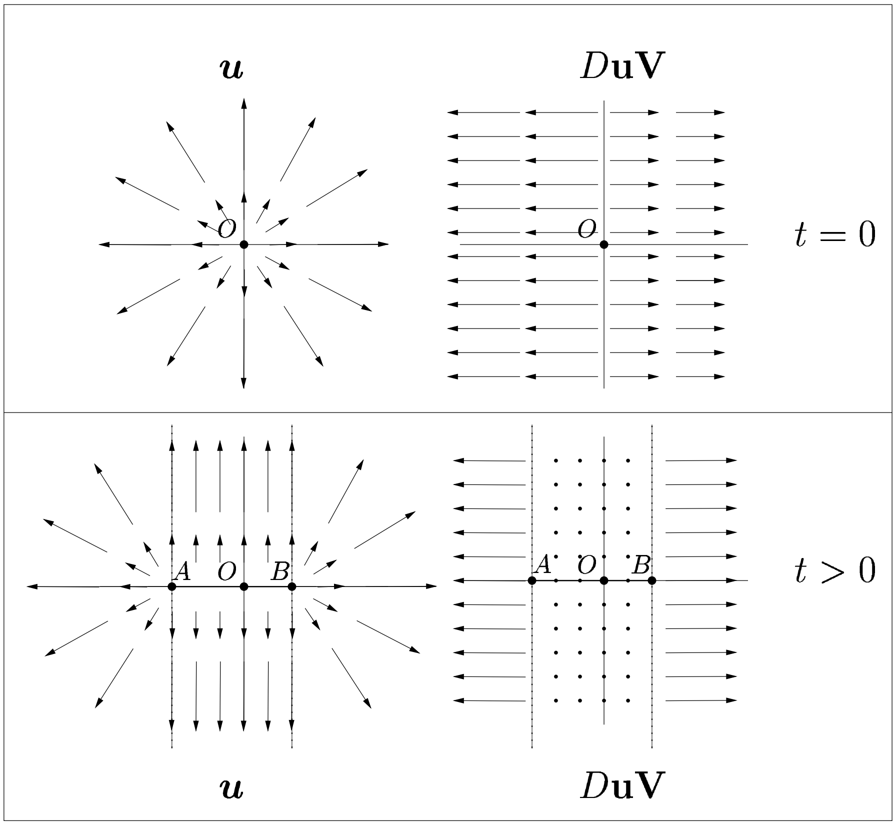


Figure 13: Changing dimension: If we allow the velocity field $\mathbf{b}$ to be discontinuous, we can induce changes in the dimension of the manifold. The example shows the simplest of all manifolds, a single point O in the plane and its VDF (upper left-hand corner), the velocity field $\mathbf{b}(., 0) = -D\mathbf{uV}$ is discontinuous on the vertical axis (upper right-hand corner). At a later time $t > 0$ the point O has become the line segment $[AB]$ (its VDF is shown in the lower left-hand corner); the new velocity $\mathbf{V}(., t)$ is shown in the lower right-hand corner. The initial point, a smooth manifold without boundary of dimension 0 is turned into a closed (in the topological sense) open (in the usual sense) curve, a smooth manifold with boundary (the two endpoints A and B) of dimension 1.

k, has a smooth boundary $\partial\mathcal{M}$ of dimension $k-1$. Thus, the study of the VDF's to manifolds with boundary cannot be dissociated from the content of this section.

6.1 VDF's of manifolds with a boundary and their singularities

The previous section has motivated the introduction of manifolds with boundaries. Well-known examples of such manifolds are open curves in the plane or in 3D, surface patches in 3D, regions of the plane or volumes in 3D. Such manifolds are easily modeled by noticing that, if their dimension is k each of their points is contained in a neighborhood which is either homeomorphic to $\mathbb{R}^k$ or to the closed half-space $H^k \equiv \{(x_1, \cdots, x_k) \in \mathbb{R}^k : x^k \geq 0\}$. We note $\mathcal{M}$ the manifold and $\partial\mathcal{M} \subset \mathcal{M}$ its boundary which we assume to be a smooth manifold of dimension $k-1$. $\mathcal{M}$ is also a function of the time t.

We show that the boundary $\partial\mathcal{M}$ introduces new singularities for the VDF to $\mathcal{M}$, in effect singularities of order 1: both $D\mathbf{u}$ and $\mathbf{u}_t$ are discontinuous on some hypersurfaces defined by $\partial\mathcal{M}$ and the normal space of $\mathcal{M}$ at points of $\partial\mathcal{M}$.

This is shown in the next proposition

Proposition 11 *The spatial derivative $D\mathbf{u}$, of the VDF to the smooth manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ is discontinuous on a special ruled hypersurface $R_{\partial\mathcal{M}}$, defined in the proof, generated by $\partial\mathcal{M}$ and the normal space to $\mathcal{M}$ at points of $\partial\mathcal{M}$.*

Proof : To see this we consider the set, noted $\mathcal{M}_a$ (a for above), of points $\mathbf{x}$ such that $P_{\mathcal{M}}(\mathbf{x}) \in \mathcal{M}\backslash\{\partial\mathcal{M}\}$, and the set, noted $\mathcal{M}_b$ (b for beside), of points $\mathbf{x}$ such that $P_{\mathcal{M}}(\mathbf{x}) \in \partial\mathcal{M}$. The boundary between these two sets is the ruled hypersurface, noted $R_{\partial\mathcal{M}}$, of $\mathbb{R}^n$ generated by $\partial\mathcal{M}$ and ruled by all the vectors normal to $\mathcal{M}$ at a point of $\partial\mathcal{M}$. For example, in the case of an open curve in the plane, this set is equal to the two lines orthogonal to the curve at its two endpoints, see figure 14. As further examples, in the case of an open 3D curve, $R_{\partial\mathcal{M}}$ is made of the two planes orthogonal to the curve at its two endpoints, in the case of an open surface patch $R_{\partial\mathcal{M}}$ is the ruled surface generated by its boundary and the normals to the surface, see figure 15.

In region $\mathcal{M}_a$, the VDF is that of a manifold of dimension k whereas in region $\mathcal{M}_b$, the VDF "sees" only $\partial\mathcal{M}$, a manifold of dimension $k-1$.

Because of proposition 7, in region $\mathcal{M}_b$, $D\mathbf{u}$ has $k-1$ eigenvalues equal to 1 whereas it has k such eigenvalues in region $\mathcal{M}_a$, therefore $D\mathbf{u}$ is discontinuous on $R_{\partial\mathcal{M}}$.

□

Let us now look at the discontinuities of $\mathbf{u}_t$. In each of the two regions $\mathcal{M}_a$ and $\mathcal{M}_b$ we are actually evolving the VDF's of $\mathcal{M}\backslash\{\partial\mathcal{M}\}$ and $\partial\mathcal{M}$, respectively, since this is what the VDF "sees". Therefore, in each of these two regions, $\mathbf{u}_t$ satisfies the properties determined in section 4. But since the normal spaces at a point of $\partial\mathcal{M}$ considered as a point of $\partial\mathcal{M}$ and as a point of $\mathcal{M}$ differ (the first one contains the second and is of dimension higher by one)

$$N_{\mathbf{x}}\mathcal{M} \subset N_{\mathbf{x}}\partial\mathcal{M} \quad \forall \mathbf{x} \in \partial\mathcal{M} \quad \text{and} \quad dim(N_{\mathbf{x}}\partial\mathcal{M}) = dim(N_{\mathbf{x}}\mathcal{M}) + 1,$$

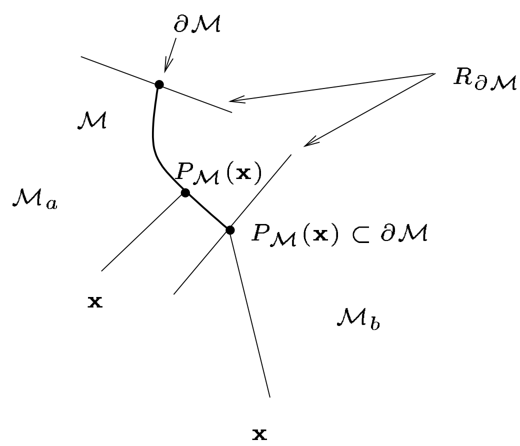


Figure 14: The manifold $\mathcal{M}$ is an open curve of $\mathbb{R}^2$; the two endpoints are the boundary $\partial\mathcal{M}$.

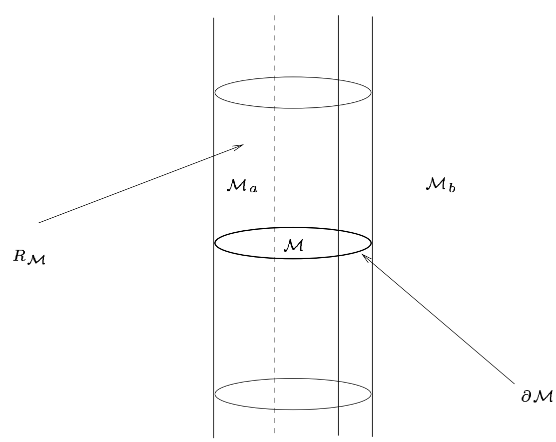


Figure 15: The manifold $\mathcal{M}$ is a surface patch of $\mathbb{R}^3$; its boundary $\partial\mathcal{M}$ is a smooth closed curve which generates a ruled surface: the rulings are the normals to $\mathcal{M}$.

we may expect some problems. We have the following proposition for the discontinuities of $\mathbf{u}_t$

Proposition 12 *The tangential component of the time derivative* $\mathbf{u}_t$*, of the VDF to the smooth manifold* $\mathcal{M}$ *with boundary* $\partial\mathcal{M}$ *can be discontinuous on the hypersurface* $R_{\partial\mathcal{M}}$ *generated by* $\partial\mathcal{M}$ *and the normal space to* $\mathcal{M}$ *at points of* $\partial\mathcal{M}$*. The normal component is continuous.*

Proof : Let us consider a point $\mathbf{x}$ of $\mathcal{M}_b$. We note $\mathbf{y} = P_{\partial\mathcal{M}}(\mathbf{x})$. $\mathbf{u}_t(\mathbf{x}, t)$ has two components, one in $N_{\mathbf{y}}\partial\mathcal{M}$ and one in $T_{\mathbf{y}}\partial\mathcal{M}$

$$\mathbf{u}_t(\mathbf{x}, t) = \mathbf{u}_t(\mathbf{x}, t)_{|\, N_{\mathbf{y}}\partial\mathcal{M}} + \mathbf{u}_t(\mathbf{x}, t)_{|\, T_{\mathbf{y}}\partial\mathcal{M}}.$$

The first component, noted $\mathbf{N}(\mathbf{y}, t)$ depends only on $\mathbf{y}$ and t as long as $P_{\mathcal{M}}(\mathbf{x}) = \mathbf{y}$. It can be uniquely decomposed as the sum of a vector $\mathbf{N}_1(\mathbf{y}, t)$ in $N_{\mathbf{y}}\mathcal{M}$ and a vector $\mathbf{N}_2(\mathbf{y}, t)$ in $N_{\mathbf{y}}\partial\mathcal{M} \cap T_{\mathbf{y}}\mathcal{M}$. This comes from the fact that $N_{\mathbf{y}}\mathcal{M} + T_{\mathbf{y}}\mathcal{M} = \mathbb{R}^n$ and $N_{\mathbf{y}}\mathcal{M} \subset N_{\mathbf{y}}\partial\mathcal{M}$ which imply that

$$N_{\mathbf{y}}\mathcal{M} + (N_{\mathbf{y}}\partial\mathcal{M} \cap T_{\mathbf{y}}\mathcal{M}) = N_{\mathbf{y}}\partial\mathcal{M}.$$

Let us now consider a point $\mathbf{x}'$ of $R_{\partial\mathcal{M}}$ such that $P_{\mathcal{M}}(\mathbf{x}') = \mathbf{y}$. Since $\mathbf{x}'$ is in $\overline{\mathcal{M}}_a$, the velocity $\mathbf{u}_t(\mathbf{x}', t)$ has two components, one, noted $\mathbf{N}'(\mathbf{y}, t)$ is in the normal space $N_{\mathbf{y}}\mathcal{M}$ at $\mathbf{y}$ and the other one is in $T_{\mathbf{y}}\mathcal{M}$:

$$\mathbf{u}_t(\mathbf{x}', t) = \mathbf{N}'(\mathbf{y}, t) + \mathbf{u}_t(\mathbf{x}', t)_{|\, T_{\mathbf{y}}\mathcal{M}}.$$

Therefore we must have $\mathbf{N}_1(\mathbf{y}, t) = \mathbf{N}'(\mathbf{y}, t)$ since otherwise the manifold $\mathcal{M}$ would spread out in its normal space and create a discontinuity of $\mathbf{u}$ at $\mathbf{y}$, a contradiction with the continuity hypothesis of proposition 3).

The tangential components are equal to $\mathbf{N}_2(\mathbf{y}) + \mathbf{u}_t(\mathbf{x}, t)_{|\, T_{\mathbf{y}}\partial\mathcal{M}}$ for $\mathbf{u}(\mathbf{x}, t)$ using the fact that $T_{\mathbf{y}}\mathcal{M} = T_{\mathbf{y}}\partial\mathcal{M} + (N_{\mathbf{y}}\partial\mathcal{M} \cap T_{\mathbf{y}}\mathcal{M})$, $\mathbf{u}_t(\mathbf{x}', t)_{|\, N_{\mathbf{y}}\partial\mathcal{M} \cap T_{\mathbf{y}}\mathcal{M}} + \mathbf{u}_t(\mathbf{x}', t)_{|\, T_{\mathbf{y}}\partial\mathcal{M}}$. If we let $\mathbf{x}$ tend to $\mathbf{x}'$, the tangential components may be discontinuous without introducing a discontinuity on $\mathcal{M}$.

□

An example of this is shown in figure 16 in the case of an open curve of $\mathbb{R}^2$. Another example is shown in in figure 17 in the case of a surface patch of $\mathbb{R}^3$.

6.2 Modeling a change of dimension

The previous section has characterized the singularities of the VDF to a manifold which has a boundary. Since the next step in this investigation should naturally be to do it for the manifolds which have their dimension altered as well, we introduce now a modelization of this process.

We shall say that an initial smooth manifold $\mathcal{W}$ of dimension $m < n$ at $t = 0$ increases its dimension if it "spreads" out in the direction of some privileged normal directions and

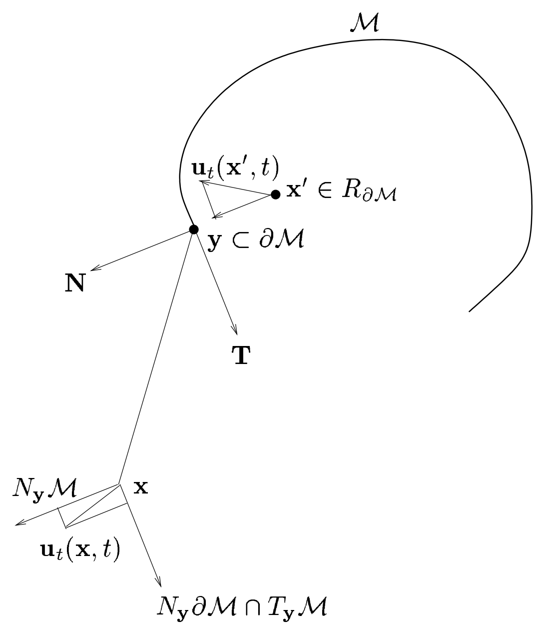


Figure 16: The time derivative $\mathbf{u}_t(\mathbf{x}, t)$ of the VDF of the open planar curve $\mathcal{M}$ at point $\mathbf{x} \in \mathcal{M}_b$ can be uniquely decomposed as the sum of a component in $N_{\mathbf{y}}\mathcal{M}$ and a component in $N_{\mathbf{y}}\partial\mathcal{M} \cap T_{\mathbf{y}}\mathcal{M} = T_{\mathbf{y}}\mathcal{M}$. The component in $N_{\mathbf{y}}\mathcal{M}$ must be equal to that of $\mathbf{u}_t(\mathbf{x}', t)$ at any point $\mathbf{x}' \in R_{\partial\mathcal{M}}$ such that $P_{\mathcal{M}}(\mathbf{x}') = P_{\mathcal{M}}(\mathbf{x})$. See proof of proposition 12.

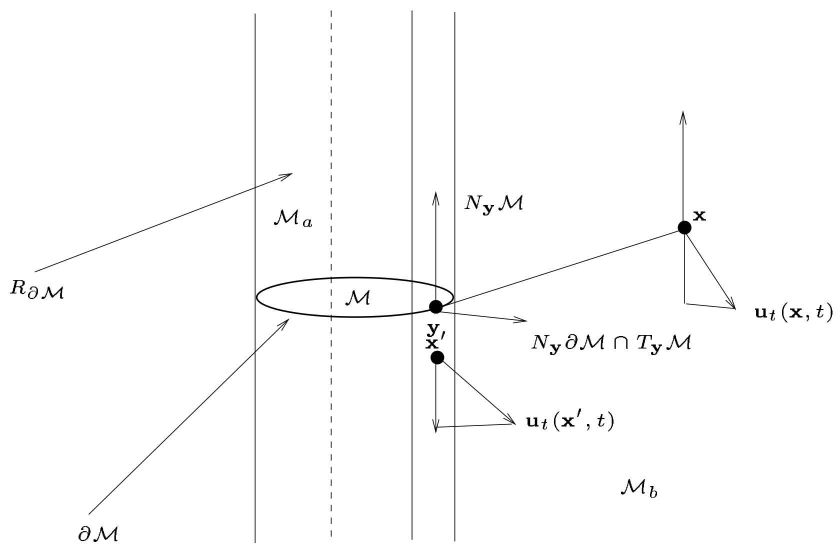


Figure 17: The time derivative $\mathbf{u}_t$ of the VDF of the surface patch $\mathcal{M}$ at point $\mathbf{x} \in \mathcal{M}_b$ can be uniquely decomposed in the sum of a component in $N_{\mathbf{y}}\mathcal{M}$ and a component in $N_{\mathbf{y}}\partial\mathcal{M} \cap T_{\mathbf{y}}\mathcal{M} \subset T_{\mathbf{y}}\mathcal{M}$. The component in $N\mathbf{y}\mathcal{M}$ must be equal to that of $\mathbf{u}_t(\mathbf{x}', t)$ at any point $\mathbf{x}' \in R_{\partial\mathcal{M}}$ such that $P_{\mathcal{M}}(\mathbf{x}') = P_{\mathcal{M}}(\mathbf{x})$. See proof of proposition 12.

becomes another manifold $\mathcal{M}(t)$, $t > 0$, of higher dimension $k \in \{m+1, \cdots, n\}$. The increment in dimension is equal to the number of linearly independent orthogonal normal directions where this filling occurs. The modeling of a decrease in dimension is obtained by reversing the direction of time.

Let us make this clearer in some simple cases in $\mathbb{R}^3$. For instance (see figure 18), the normal space to a point (a smooth manifold of dimension 0) is $\mathbb{R}^3$ itself, *i.e.* it is three-dimensional. In order to change its dimension, the point may "choose" a certain normal direction and spread out in that direction to become an arc of a curve (case a). Alternatively, it may "choose" to spread out over a two-dimensional subspace of its normal space and transform smoothly into a small surface patch (case b). It can even fill the entire embedding space by spreading out in all of its normal directions (case c). In all the directions of spread, the velocity field is like the one described in the previous section, equation (36) with the values of α and β varying with the normal. Similarly, an embedded curve in $\mathbb{R}^3$ (a smooth

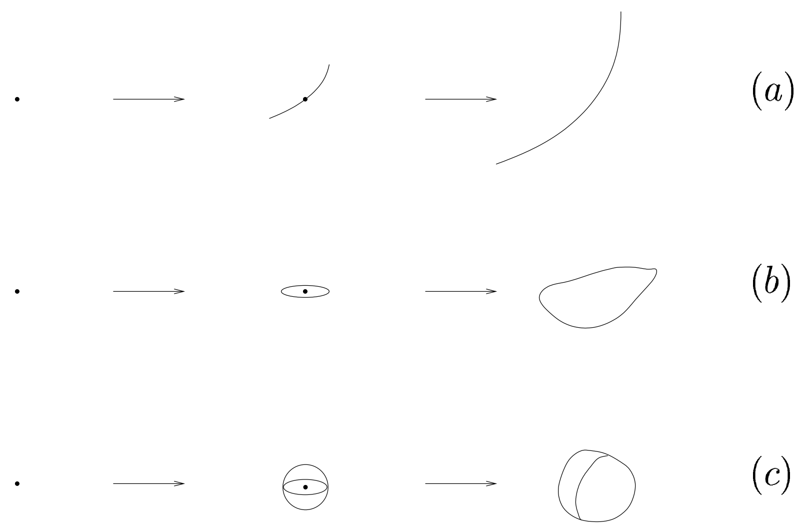


Figure 18: A point ($m = 0$) spreads out to become a curve ($k = 1$): (a), a patch of a surface ($k = 2$): (b) and a volume ($k = 3$): (c). See text.

manifold of dimension 1) has a two-dimensional normal space at each of its points and thus, its dimension can increase by 1 or 2 according to the number of orthogonal normal directions that are given a privileged role. By privileging a particular smooth normal vector field, the curve can spread in these normal directions to become a ribbon (case d) which is a smooth manifold with boundary of dimension 2. Alternatively, the curve may spread in all the directions of its normal spaces, thereby becoming a smooth manifold with boundary of dimension 3, a volume (see figure 19). To be complete, a surface patch (a smooth manifold of dimension 2 with or without boundary) has a one-dimensional normal space. By allowing the velocity field $\mathbf{b}$ to be discontinuous on the patch with jumps that may depend on the normal

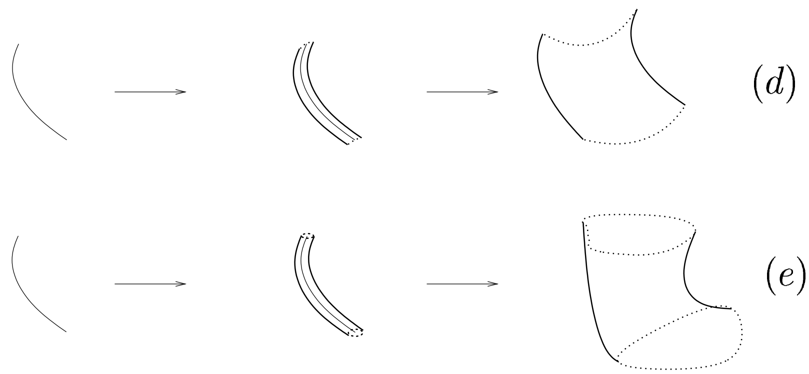


Figure 19: A curve ($m = 1$) spreads out to become a surface patch ($k = 2$): (d), and a volume ($k = 3$): (e). See text.

at each point, the patch can become a smooth manifold of dimension 3 with boundary, a volume (see figure 20).

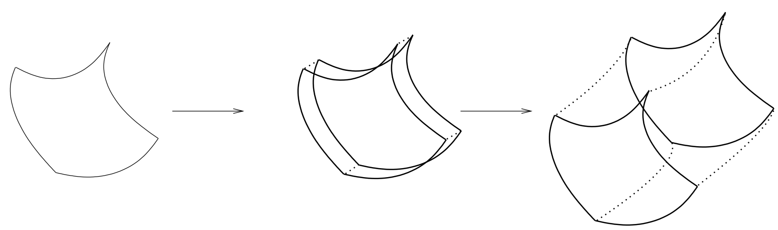


Figure 20: A surface patch ($m = 2$) spreads out to become a volume ($k = 3$). See text.

Palpably, the important missing ingredient for the formalization of this "spreading" is that of a k-dimensional neighborhood of a smooth manifold of dimension $m < k \leq n$, $\mathcal{W}$. Now, the key tool in Riemannian Geometry for discussing neighborhoods of embedded manifolds is the *exponential map*: a parameterization by arc-length of the geodesic curves.

Definition 1 *Let g be a Riemannian metric*[1] *of $\mathbb{R}^n$ and*

$$\mathbb{R} \to \mathbb{R}^n \tag{37}$$

$$\lambda \to \exp_g(\boldsymbol{x}, \boldsymbol{v}, \lambda) \tag{38}$$

be an embedding of the corresponding geodesic curve passing through $\boldsymbol{x}$, tangent to the unit vector $\boldsymbol{v}$ such that λ is the arc length.

[1] $g(\mathbf{x})$ is at every point $\mathbf{x}$ of $\mathbb{R}^n$ a positive quadratic form of $\mathbb{R}^n$; the function $\mathbf{x} \longrightarrow g(\mathbf{x})$ is assumed to be smooth.

Remark 3 *We insist on the fact that we are considering geodesic curves of the embedding space $\mathbb{R}^n$ and not of an embedded manifold. Notice also that we choose $\boldsymbol{v}$ to be unitary: this may slightly differ from other definitions but is more suitable in our case.*

See [47, 21] for important results concerning the exponential map and its use in the study of neighborhoods. As a simple example, if $g(\mathbf{x}) = \mathbf{I}_3\ \forall \mathbf{x}$, $\exp_g(\boldsymbol{x}, \boldsymbol{v}, \lambda) = \mathbf{x} + \lambda \boldsymbol{v}$ is the line going through $\mathbf{x}$ and parallel to $\boldsymbol{v}$. By choosing different metrics g, in particular non-constant ones, one obtains curved geodesics which can be used to model smooth manifolds $\mathcal{M}$ containing $\mathcal{W}$. The appeal of this notion is due to the fact it allows to define what is a neighborhood of radius λ of a manifold. We are going to use it right now for this purpose with $\lambda = t$, the time, in order to clarify the way a manifold can smoothly spread out over one of its neighborhoods. To begin with, let us revisit some of the examples of figure 19.

First, we return to the example of a curve transforming into a ribbon (19d). Let $\mathcal{W}$ be this curve, $\boldsymbol{n}$ a differentiable unit normal vector field along $\mathcal{W}$ and g a Riemannian metric of $\mathbb{R}^3$ (see figure 19d). Consider a point $\boldsymbol{x} \in \mathcal{W}$ and the geodesic curve passing through $\mathbf{x}$ and tangent to $\boldsymbol{n}(\mathbf{x})$ *i.e.* the curve $\mathcal{G}_\mathbf{x}(s)$ embedded by $s \to \exp_g(\mathbf{x}, \pm\boldsymbol{n}, s)$, $0 \le s \le t$. The open curve $\mathcal{G}_\mathbf{x}$ is normal to $\mathcal{W}$ at $\mathbf{x}$ and models the change of dimension of the point $\mathbf{x}$ into a curve whose boundary is the made of the two points $\{A_\mathbf{x} \equiv \exp_g(\boldsymbol{x}, \boldsymbol{n}, t), B_\mathbf{x} \equiv \exp_g(\boldsymbol{x}, -\boldsymbol{n}, t)\}$.

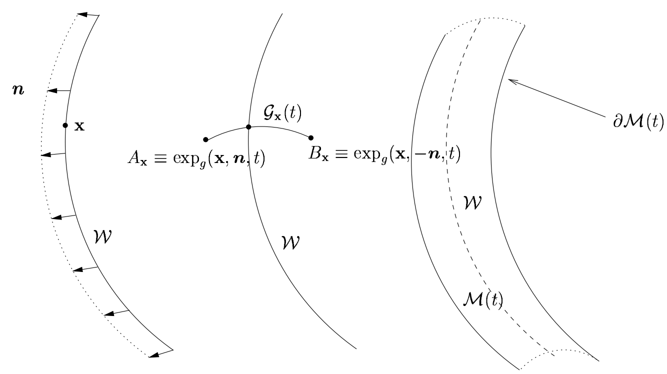


Figure 21: Given a smooth field of normals $\boldsymbol{n}(\mathbf{x})$ along the curve $\mathcal{W}$, and a Riemannian metric g on $\mathbb{R}^3$, the set $\mathcal{M}(t)$ for t small enough, union of all the geodesic curves $\mathcal{G}_\mathbf{x}$ is a smooth surface patch whose boundary $\partial\mathcal{M}(t)$ is the union of all the points $A_\mathbf{x}$ and $B_\mathbf{x}$, see text.

For $t > 0$ small enough, consider the union of all these geodesic curves

$$\mathcal{M}(t) = \cup_{\mathbf{x} \in \mathcal{W}} \mathcal{G}_\mathbf{x}.$$

It is a smooth surface of $\mathbb{R}^3$ which has for boundary a smooth curve. Indeed, $\mathcal{M}(t)$ can be embedded by

$$\begin{aligned} \mathcal{W} \times [0, t[&\to \mathbb{R}^3 \\ (\mathbf{x}, s) &\to \exp_g(\mathbf{x}, \pm\boldsymbol{n}(\mathbf{x}), s) \end{aligned}$$

and its border $\partial\mathcal{M}(t)$ by

$$\begin{aligned} \mathcal{W} &\to \mathbb{R}^3 \\ \mathbf{x} &\to \exp_g(\mathbf{x}, \pm\boldsymbol{n}(\mathbf{x}), t). \end{aligned}$$

The proof that these two maps are embeddings is a straightforward consequence of the important known result that, for short geodesics, the exponential map is a diffeomorphism.

Next, we show how these ideas apply in the case of a curve transforming into a tubular volume (see figure 19e). The increase of dimension should be two and since the normal space at each point $N_{\mathbf{x}}\mathcal{W}$ of the curve is two-dimensional, it means that the spread must occur in all the normal directions. We note $P\mathcal{W}$ the submanifold of the normal bundle $N\mathcal{W}$ corresponding to unit normal vectors (we have dimension($P\mathcal{W}$) = dimension($N\mathcal{W}$) − 1). Hence, the target manifold $\mathcal{M}(t)$ can be embedded by

$$P\mathcal{W} \times [0, t[\to \mathbb{R}^3 \tag{39}$$
$$((\mathbf{x}, \boldsymbol{n}), s) \to \exp_g(\mathbf{x}, \boldsymbol{n}, s) \tag{40}$$

and its border $\partial\mathcal{M}(t)$ by

$$P\mathcal{W} \to \mathbb{R}^3 \tag{41}$$
$$(\mathbf{x}, \boldsymbol{n}) \to \exp_g(\mathbf{x}, \boldsymbol{n}, t), \tag{42}$$

which shows that the dimension of $\mathcal{M}(t)$ is 3 and that of $\partial\mathcal{M}(t)$ is 2.

To conclude the presentation of the model with a general note, the change of dimension is entirely defined by the choice of a submanifold of $P\mathcal{W}$, noted $S\mathcal{W}$, and a metric g of the embedding space. The submanifold $S\mathcal{W} \subset P\mathcal{W}$ describes the set of directions where $\mathcal{W}$ is supposed to spread out and its dimension is equal to the increment of dimension minus one. As far as the metric g is concerned, its purpose is to describe the shape of $\mathcal{M}(t)$. The resulting manifold $\mathcal{M}$, of dimension k, is embedded by

$$S\mathcal{W} \times [0, t[\to \mathbb{R}^n \tag{43}$$
$$((\mathbf{x}, \boldsymbol{n}), s) \to \exp_g(\mathbf{x}, \boldsymbol{n}, s), \tag{44}$$

its border $\partial\mathcal{M}$, of dimension $k-1$, is embedded by

$$S\mathcal{W} \to \mathbb{R}^n \tag{45}$$
$$(\mathbf{x}, \boldsymbol{n}) \to \exp_g(\mathbf{x}, \boldsymbol{n}, t), \tag{46}$$

and the ruled hypersurface $R_{\partial\mathcal{M}}$, of dimension $n-1$, is embedded by

$$\partial\mathcal{M} \times (N\partial\mathcal{M} \cap N\mathcal{M}) \rightarrow \mathbb{R}^n \quad (47)$$

$$(\mathbf{x}, \boldsymbol{n}) \rightarrow \mathbf{x} + \boldsymbol{n}. \quad (48)$$

Indeed, the generatrix of $R_{\partial\mathcal{M}}$ is $\partial\mathcal{M}$ and its ruled directions are $(N\partial\mathcal{M} \cap N\mathcal{M})$.

6.3 Singularities during a change of dimension

The section 6.1 describes the singularities of the VDF that are involved when representing manifolds which have a boundary. Here, we extend this study to the case of manifolds which have their dimension altered. The methodology is to use the model introduced in the previous section and to observe how $R_{\partial\mathcal{M}}$ is affected when $t \rightarrow 0^+$. Again, we consider the manifold $\mathcal{W}$, of dimension m, transforming, at time $t = 0$, into a manifold $\mathcal{M}$ of dimension $k > m$ and with boundary $\partial\mathcal{M}$.

Proposition 13 *At $t = 0$, the vector field $\mathbf{b} = \mathbf{u}_t$ is discontinuous on the ruled manifold, noted $\mathcal{M}^\perp$, generated by $\mathcal{W}$ and with ruled directions $(S\mathcal{W})^\perp = N\mathcal{W} \backslash S\mathcal{W}$. The dimension of this ruled surface is m+n-k. Nevertheless, the components of the VDF parallel to these rules are continuous.*

Proof : At $t > 0$, $\mathbf{u}$ is the VDF to $\mathcal{M}$ which has a boundary and according to proposition 12 $\mathbf{u}_t$ is not continuous across the ruled hypersurface $R_{\partial\mathcal{M}}$. To understand the discontinuities of the VDF at the very moment when the change of dimension is initiated, we consider the set $\lim_{t\rightarrow 0^+} R_{\partial\mathcal{M}}$. This set is a ruled surface whose generatrix is

$$\lim_{t\rightarrow 0^+} \partial\mathcal{M} = \mathcal{W}$$

and ruled directions are

$$\lim_{t\rightarrow 0^+} (N\partial\mathcal{M} \cap N\mathcal{M}) = N\mathcal{W} \backslash S\mathcal{W}.$$

The continuity of the components of the VDF in the ruled directions still holds. We also observe that both $R_{\partial\mathcal{M}}$ and $\mathcal{M}^\perp$ are ruled surfaces: they share the same rules directions but their generatrix has different dimensions. Figure 22 depicts this situation.

□

Remark 4 *Notice that the situation $k = n$, i.e. the manifold $\mathcal{W}$ transforms into a region of $\mathbb{R}^n$ (which is the interior of the hypersurface $\partial\mathcal{M}$), is a limit case since we have $S\mathcal{W} = N\mathcal{W}$ and consequently $(S\mathcal{W})^\perp = \emptyset$. Thus, the VDF is discontinuous on the "generatrix" $\mathcal{W}$ itself and there are no rules.*

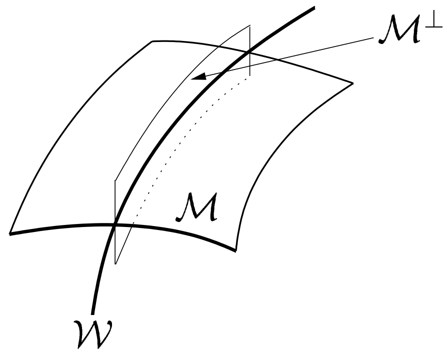


Figure 22: The VDF of a manifold $\mathcal{W}$ initiating a change of dimension at $t = 0$ to transform into $\mathcal{M}$ are on $\mathcal{M}^\perp$ at time $t = 0$. The components of the VDF parallel to the rules of $\mathcal{M}^\perp$ are continuous. At $t > 0$, $\mathcal{M}^\perp$ has transformed into the ruled hypersurface $R_{\partial\mathcal{M}}$

7 Example II: the sphere

Consider the spherical sector, part of the sphere of radius 1, centered at the origin O from which we take away the part above the plane of equation $x_3 = \cos\alpha$, see figure 23. This is a smooth manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$ equal to the circle of radius $\sin\alpha$ centered at the point C of coordinates $[0, 0, \cos\alpha]^T$. The surface $R_{\partial\mathcal{M}}$ is the half cone of vertex O containing $\partial\mathcal{M}$. For a point $\mathbf{x}$ in $\mathcal{M}_a$, the VDF is equal to

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

For a point $\mathbf{x}'$ in $\mathcal{M}_b$, the VDF is equal to

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{z}}{\|\mathbf{z}\|}\sin\alpha - \mathbf{C},$$

where $\mathbf{z} = [x_1, x_2, 0]^T$ is the projection of $\mathbf{x}$ on the plane of equation $x_3 = 0$. It is therefore easy to compute $D\mathbf{u}$

$$D\mathbf{u}(\mathbf{x}) = \begin{cases} (1 - \frac{1}{\|\mathbf{x}\|})\mathbf{I}_3 + \frac{\mathbf{x}\mathbf{x}^T}{\|\mathbf{x}\|^3} & \text{if} \quad \mathbf{x} \in \mathcal{M}_a, \\ \mathbf{I}_3 - \frac{\sin\alpha}{\|\mathbf{z}\|}\mathbf{I}_3^2 + \frac{\mathbf{z}\mathbf{z}^T}{\|\mathbf{z}\|^3}\sin\alpha & \text{if} \quad \mathbf{x} \in \mathcal{M}_b, \end{cases}$$

where $\mathbf{I}_3^2 = diag(1, 1, 0)$. We verify that in an orthonormal basis built from the two vectors $(\mathbf{x} - \mathbf{u}(\mathbf{x}),\ \mathbf{z}^\dagger)$ $(\mathbf{z}^\dagger = [-x_2, x_1, 0]^T)$ and their cross-product, the matrix $D\mathbf{u}$ becomes diagonal

$$D\mathbf{u}(\mathbf{x}) = \begin{cases} diag(1, 1 - \frac{1}{\|\mathbf{x}\|}, 1 - \frac{1}{\|\mathbf{x}\|}) & \text{if} \quad \mathbf{x} \in \mathcal{M}_a, \\ diag(1, 1, 1 - \frac{\sin\alpha}{\|\mathbf{z}\|}) & \text{if} \quad \mathbf{x} \in \mathcal{M}_b. \end{cases}$$

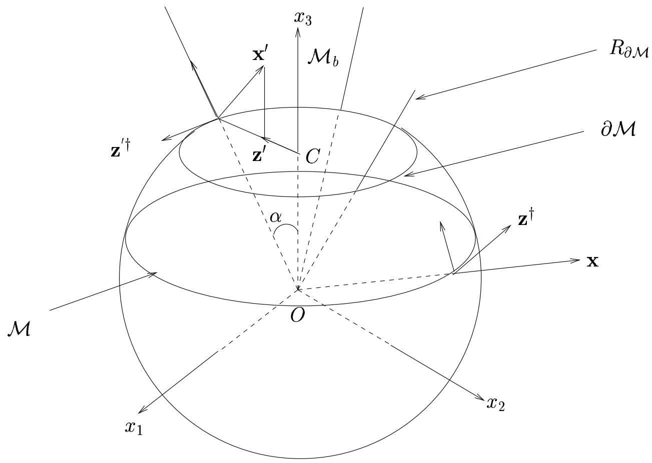


Figure 23: A spherical sector.

In region $\mathcal{M}_a$, the VDF "sees" a smooth manifold of dimension 2 (a sphere) and, according to proposition 7, there is one eigenvalue equal to 1 and the other two are equal to $\frac{\|\mathbf{x}\|-1}{\|\mathbf{x}\|} \equiv \frac{s}{s+1}$ if we let $s = \|\mathbf{x}\| - 1$. As expected, these two eigenvalues tend to 0 when s does, *i.e.* when the point $\mathbf{x}$ approaches the sphere. They blow up at the center ($s = -1$) which is both a center of curvature and on the skeleton.

In region $\mathcal{M}_b$, the VDF "sees" a smooth manifold of dimension 1 (a circle) and, according to the same proposition, there are two eigenvalues equal to 1 and one equal to $\frac{\|\mathbf{z}\|-\sin\alpha}{\|\mathbf{z}\|}$. This eigenvalue tends to zero when $\|\mathbf{z}\|$ tends to $\sin\alpha$, *i.e.* when $\mathbf{x}$ approaches the circle.It blows up when $\|\mathbf{z}\| = 0$, *i.e.* on the x_3-axis, the skeleton of the circle.

Moreover, when $\mathbf{x}$ is on the half cone $R_{\partial\mathcal{M}}$ of equation $\|\mathbf{z}\|^2 - x_3^2 \tan^2\alpha = 0$ one verifies that $\|\mathbf{x}\|^2 \sin^2\alpha = \|\mathbf{z}\|^2$. Hence when $\mathbf{x}$ tends to $R_{\partial\mathcal{M}}$ in $\mathcal{M}_b$ $D\mathbf{u}(\mathbf{x})$ tends to $diag(1, 1, 1 - \frac{1}{\|\mathbf{x}\|})$ which is different from $diag(1, 1 - \frac{1}{\|\mathbf{x}\|}, 1 - \frac{1}{\|\mathbf{x}\|})$ outside the sphere. Hence we have an example of the result in proposition 11.

8 Some remarks and conclusion

The method of the VDF's for representing arbitrary smooth manifolds with or without a boundary finds its roots in the work of Ambrosio and Soner [4] where we found inspiration and some of the technical results that we needed, and in the work of Ruuth, Merriman, Xin and Osher [42] that inspired us the idea of the VDF representation. Our contributions are the development of a) a method for evolving a VDF instead of a smooth closed manifold while guaranteeing that it stays a VDF over time and that the manifold evolves correctly, b) a theory that describes changes of dimension by a generalized transport equation and

c) a theory that extends a) to deal seamlessly with smooth manifolds with boundaries. Moreover, in our approach, the function we evolve is regular on the manifold of interest unlike for example the method presented in [41].

These theoretical developments are being implemented. Let us make two remarks concerning this implementation. The first remark is related to the computation of the velocity field $\mathbf{b}$. Even though proposition 10 provides a characterization of $\mathbf{b}(\mathbf{x})$ in terms of $\mathbf{V}(\mathbf{x}-\mathbf{u}(\mathbf{x}))$, and the tangential terms given in the proof, our favorite method is to compute $\mathbf{b}$ by applying proposition 9 and solving the quasi-linear PDE (25) with initial conditions (26).

The second remark is that because of the accumulation of numerical errors, the function $\mathbf{u}$ may drift away from the class of VDF's. In order to correct for this drift, we suggest that it may be a good idea to combine the solution of (21c) with that of

$$\mathbf{u}_t = ((D\mathbf{u})^T + (\alpha - 1)\mathbf{I})\mathbf{u}, \tag{49}$$

where

$$\alpha = Trace(D\mathbf{u}(D\mathbf{u})^T) + \mathbf{u}\cdot\Delta\mathbf{u} - div\mathbf{u}.$$

Equation (49) is the Euler-Lagrange equation of the following functional

$$\frac{1}{2}\int_\Omega \|(D\mathbf{u})^T\mathbf{u} - \mathbf{u}\|^2 \quad d\mathbf{x},$$

where Ω is some neighborhood of $\mathcal{M}$. This functional arises naturally from proposition 3.

Similarly, since the VDF's are gradient vector fields, it might be useful to consider also the functional

$$\frac{1}{2}\int_\Omega \|D\mathbf{u} - (D\mathbf{u})^T\|^2 \quad d\mathbf{x}.$$

To conclude, we think that the VDF method for representing and evolving shapes has the following advantages: it can deal with smooth manifolds with and without boundaries, with shapes of different dimensions; if discontinuous velocity fields are allowed dimension can change; the evolution method that we propose guarantees that we stay in the class of VDF's and therefore that the intrinsic properties of the underlying shapes such as their dimension, curvatures can be read off easily from the VDF and its spatial derivatives. The main disadvantage is its redundancy: the size of the representation is always that of the ambient space even though the object we are representing is of a much lower dimension. This disadvantage is also one of its strengths since it buys us flexibility.

9 Appendix

9.1 Mean curvature vector of a smooth manifold

Dimension 2

This case is classical, see e.g. [20].

Let $\mathcal{M} \subset \mathbb{R}^3$ be an oriented hypersurface, $(u, v) \to \mathbf{x}(u, v)$ a chart in a neighborhood of $\mathbf{x}_0 \in \mathcal{M}$ and $\mathbf{N}$ the inward unit normal vector. Let $t \to \alpha(t)$ be a curve on $\mathcal{M}$ such that $\alpha(0) = \mathbf{x}_0$ and $\alpha'(0) = \mathbf{y}$, where $\mathbf{y}$ is a tangent vector.

Definition 2 **The shape operator** *is the linear map* $S(\mathbf{y}) = -\mathbf{N}_t = -\left(\mathbf{N}(\alpha(t))\right)'$.

This map describes how the normal vector varies when following a curve in the direction $\mathbf{y}$. The shape operator is self-adjoint *i.e.* $\mathbf{y}_1 \cdot S(\mathbf{y}_2) = S(\mathbf{y}_1) \cdot \mathbf{y}_2$ and $\mathbf{N} \cdot S(\mathbf{y}) = 0$ (since $|\mathbf{N}| = 1$).

So, the shape operator can be expressed in the basis $(\mathbf{x}_u, \mathbf{x}_v)$ of the tangent plane by

$$\begin{cases} -S(\mathbf{x}_u) = \mathbf{N}_u = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v \\ \\ -S(\mathbf{x}_v) = \mathbf{N}_u = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v, \end{cases} \tag{50}$$

where the (Weingarten) endomorphism[2], $\boldsymbol{a} = [a_{ij}]$, is the shape operator expressed in this particular basis of the tangent plane (Notice that this matrix is not always symmetric but it is the case if the first fundamental form is the identity matrix).

So, let us compute this matrix. We suppose we are given the following numbers $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = \mathbf{x}_v \cdot \mathbf{x}_u$, $e = -\mathbf{N} \cdot \mathbf{x}_{uu} = \mathbf{N}_u \cdot \mathbf{x}_u$, $f = -\mathbf{N} \cdot \mathbf{x}_{uv} = -\mathbf{N} \cdot \mathbf{x}_{vu} = \mathbf{N}_u \cdot \mathbf{x}_v = \mathbf{N}_v \cdot \mathbf{x}_u$ and $g = -\mathbf{N} \cdot \mathbf{x}_{vv} = \mathbf{N}_v \cdot \mathbf{x}_v$. (One can indeed consider these numbers as given since they derive directly from the given parameterization). Notice that identities like $\mathbf{N} \cdot \mathbf{x}_{uu} = -\mathbf{N}_u \cdot \mathbf{x}_u$ can be proved by differentiation of $\mathbf{N} \cdot \mathbf{x}_u = 0$ with respect to u. In order to set the a_{ij}'s, one computes the scalar products of the two Weingarten equations with $\mathbf{x}_u$ and $\mathbf{x}_v$, hence

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{51}$$

and finally

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}. \tag{52}$$

The numbers $H = \frac{1}{2}tr(S)$ and $K = det(S)$ are invariants of the shape operator (independent of the choice of the basis). H is the mean curvature and K the Gaussian curvature of $\mathcal{M}$ at $\mathbf{x}_0$.

The general case

Let $\mathcal{M} \subset \mathbb{R}^n$ be an oriented smooth k-dimensional manifold and $\mathbf{p} \to \mathcal{M}(\mathbf{p})$ a local parameterization of $\mathcal{M}$ in a neighborhood of a point $\mathbf{x}_0 \in \mathcal{M}$. Let $t \to \alpha(t)$ be a curve on $\mathcal{M}$ such that $\alpha(0) = \mathbf{x}_0$ and $\alpha'(0) = \mathbf{y}$ where $\mathbf{y}$ is a vector tangent to Mat $\mathbf{x}_0$.

[2] $S(T\mathcal{M}_{\mathbf{x}_0})$ and $T\mathcal{M}_{\mathbf{x}_0}$ are identified since $\|\mathbf{N}\| = 1$.

Definition 3 *If* $\mathbf{N}$ *is a normal vector,* **the shape operator in the normal direction N** *is the linear map* $S_{\mathbf{N}}(\mathbf{y}) = -\mathbf{N}_t = -(\mathbf{N}(\alpha(t)))'$.

This shape operator is also self-adjoint, see e.g. [47].

The tangential part of the shape operator can be expressed in the basis $(\mathcal{M}_{p_1}, \cdots, \mathcal{M}_{p_k})$ of the tangent plane by

$$\begin{cases} -S(\mathcal{M}_{p_1})^T = \Pi^T_{\mathcal{M}(\mathbf{p})}(\mathbf{N}_{p_1}) = a^N_{11}\mathcal{M}_{p_1} + \cdots + a^N_{1k}\mathcal{M}_{p_k} \\ \vdots \\ -S(\mathcal{M}_{p_k})^T = \Pi^T_{\mathcal{M}(\mathbf{p})}(N_{p_k}) = a^N_{k1}\mathcal{M}_{p_1} + \cdots + a^N_{kk}\mathcal{M}_{p_k}, \end{cases} \tag{53}$$

where $\Pi^T_{\mathcal{M}(\mathbf{p})}$ is the projection operator on the tangent plane $T\mathcal{M}_{\mathcal{M}(\mathbf{p})}$ to $\mathcal{M}$ at the point $\mathcal{M}(\mathbf{p})$, see figure 8.

Now, consider the two $k \times k$ matrices ϵ^N and Φ defined by $\epsilon^N_{ij} = -\mathbf{N}_{p_i} \cdot \mathcal{M}_{p_j} = \mathbf{N} \cdot \mathcal{M}_{p_i p_j}$ and $\Phi_{ij} = \mathcal{M}_{p_i} \cdot \mathcal{M}_{p_j}$. Then define

$$a^N = -\epsilon^N \Phi^{-1} \tag{54}$$

and $\mathcal{H}^N = \frac{1}{k} tr(a^N)$ and $K^N = det(a^N)$.

Given an orthonormal basis $(\mathbf{N}_1, \cdots, \mathbf{N}_{n-k})$ of the normal space, the mean curvature vector, $\boldsymbol{\mathcal{H}}$ to $\mathcal{M}^k$ is given in that basis by

$$\boldsymbol{\mathcal{H}} = -\sum_{i=1}^{n-k} \mathcal{H}^{N_i} \mathbf{N}_i. \tag{55}$$

Note that this works in the case of 3D curves. Assume that the curve is parameterized by arc-length s. The normal space is spanned by the two orthonormal vectors $\mathbf{N}$, the normal, and $\mathbf{B}$, the binormal. We have two 1×1 matrices ϵ^N and ϵ^B. The 1×1 matrix Φ is equal to $\|\mathbf{T}\|^2 = 1$, where $\mathbf{T}$ is the unit tangent vector to the curve. Because of the Frenet formula (see e.g. [20]), we find $\epsilon^N = -\mathbf{N}_s \cdot \mathbf{T} = \kappa$ and $\epsilon^B = -\mathbf{B_s} \cdot \mathbf{T} = \mathbf{0}$. Hence $a^N = -\kappa$ and $a^B = 0$, therefore $\boldsymbol{\mathcal{H}} = \kappa\mathbf{N}$, as expected.

9.2 Proof of proposition 7

We give a proof of proposition 7.

Proof : Let $\mathbf{x}_0 = P_{\mathcal{M}}(\mathbf{x}) \in \mathbf{u}^{-1}(0)$ and $\mathbf{n} \in N_{\mathbf{x}_0}\mathcal{M}$ so that the corresponding characteristic ray's equation is $s \to \mathbf{x}_s = \mathbf{x}_0 + s\mathbf{n}$. We consider the variation of $D\mathbf{u}$ on the corresponding characteristic segment:

$$\frac{d}{ds}(D\mathbf{u}(\mathbf{x}_s)) = \mathbf{D}^2\mathbf{u}\ (\mathbf{x}_s)\frac{d}{ds}(\mathbf{x}_s) = \mathbf{D}^2\mathbf{u}\ (\mathbf{x}_s)\mathbf{n}.$$

Using the two identities $\mathbf{u}(\mathbf{x}_s) = s\mathbf{n}$ and equation (14) of proposition 5 we obtain

$$\frac{d}{ds}D\mathbf{u}(\mathbf{x}_s) = \frac{D\mathbf{u}(\mathbf{x}_s) - (D\mathbf{u}(\mathbf{x}_s))^2}{s}. \tag{56}$$

Writing this equation in the orthonormal basis where the symmetric matrix $D\mathbf{u}(\mathbf{x}_s)$ is diagonal yields an ODE for each eigenvalue of $D\mathbf{u}(\mathbf{x}_s)$:

$$\dot{\lambda} = \frac{d\lambda}{ds} = \frac{\lambda(1-\lambda)}{s}.$$

One then writes

$$\frac{d\lambda}{\lambda(1-\lambda)} = \frac{ds}{s} \tag{57}$$

and sees that this is a non-autonomous differential equation with separable variables s and λ. A well-known theorem, see for example [6], says that for each pair (s_0, λ_0) such that $s_0 \neq 0$ and $\lambda_0 \neq 0$ or 1, there exists a unique solution λ of our ODE subject to the condition $\lambda(s_0) = \lambda_0$ in a neighborhood of s_0. Moreover, this solution is given by

$$\int_{s_0}^{s} \frac{d\,s}{s} = \int_{\lambda_0}^{\lambda(s)} \frac{d\,\lambda}{\lambda(1-\lambda)}.$$

This can be immediately integrated and yields

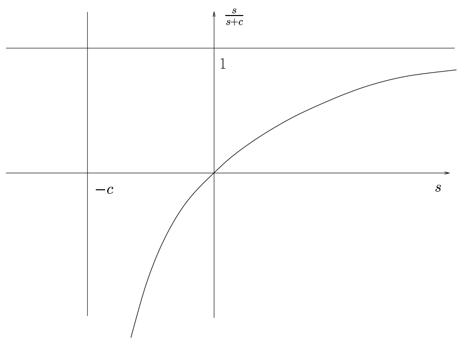
$$\frac{\lambda(s)}{1-\lambda(s)} = \frac{s\lambda_0}{s_0(1-\lambda_0)}.$$

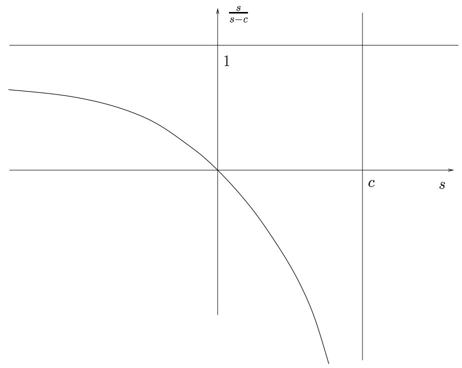
If we let $c = \frac{s_0(1-\lambda_0)}{\lambda_0}$ we obtain

$$\lambda(s) = \frac{s}{s \pm c}, \quad c > 0. \tag{58}$$

Note that this solution is defined in a neighborhood of $s_0 \neq 0$. The fact that the values $s = 0$, $\lambda = 0, 1$ pose problem in the application of the theorem is seen by the fact that $\lambda(s) = 0$ is clearly another solution of (57) which also satisfies, as (58) does, the initial condition $\lambda(0) = 0$. It may be seen as the limit of (58) when $c \longrightarrow \infty$. Another solution is obtained by choosing $\lambda(s) = 1$ which can be considered as the limit of (58) when $c \longrightarrow 0$. Overall we find that $\lambda(0) = 0$ or 1, in agreement with the fact that, according to proposition 6, $D\mathbf{u}(\mathbf{x}_0)$ is the projector on $N_{x_0}M$. The shapes of the curves $s \longrightarrow \lambda(s)$ are shown in figure 9.2. Coming back to equation (56), we see that $\frac{d}{ds}(D\mathbf{u}(\mathbf{x}_s))$ is a polynomial in $D\mathbf{u}(\mathbf{x}_s)$ with coefficients that are rational functions of s. By further differentiation, we see that this is true for all orders of derivatives of $D\mathbf{u}(\mathbf{x}_s)$. Considering the Taylor expansion of $D\mathbf{u}(\mathbf{x}_{s+\Delta s})$ as a function of the various derivatives of $D\mathbf{u}(\mathbf{x}_s)$, we see that both symmetric matrices are diagonal in the same basis of eigenvectors which are those of $D\mathbf{u}(\mathbf{x}_0)$ and fall into two categories: the vectors in $T_{x_0}M$, a vector subspace of dimension k, to which are attached the eigenvalues of $D\mathbf{u}(\mathbf{x}_s)$ equal to 0 or given by (58) and the vectors in $N_{x_0}M$, a vector subspace of dimension $n-k$, to which are attached the eigenvalues of $D\mathbf{u}(\mathbf{x}_s)$ equal to 1.

□





9.3 Proof of proposition 8

In this section, we assume that $(p_1, \cdots, p_k) \in \mathbb{R}^k \longrightarrow M$ is a parameterization of the smooth manifold $M = \mathbf{u}^{-1}(\mathbf{0})$. We start with the following lemma.

Lemma 2 *The projection of the mean-curvature vector* $\boldsymbol{\mathcal{H}}$ *along the gradient* ∇u_i *of the ith component of* $\mathbf{u}$ *is given by*

$$\boldsymbol{\mathcal{H}} \cdot \nabla u_i = -\frac{1}{k} \sum_{l=1}^{k} M_{p_l} \cdot \mathbf{D}^2 u_i \, M_{p_l}.$$

Proof : Since

$$\nabla u_i \cdot M_{p_l} = 0 \quad i = 1, \cdots, n-k \quad l = 1, \cdots, k,$$

if we take the partial derivative of this expression with respect to p_m, $m = 1, \cdots, k$, we obtain

$$M_{p_m} \cdot \mathbf{D}^2 u_i \, M_{p_l} + \nabla u_i \cdot M_{p_l p_m} = 0 \quad i = 1, \cdots, n-k \quad l, m = 1, \cdots, k. \tag{59}$$

The $k \times k$ matrix $\epsilon^i = [\epsilon^i_{lm}]_{l,m=1,\cdots,k} = [\nabla u_i \cdot M_{p_l p_m}]_{l,m=1,\cdots,k}$ yields the scalar product of the mean curvature vector $\boldsymbol{\mathcal{H}}$ of M with ∇u_i:

$$\boldsymbol{\mathcal{H}} \cdot \nabla u_i = -\frac{1}{k} tr(a^i),$$

where

$$a^i = -\epsilon^i \Phi^{-1}.$$

Equation (59) shows that

$$-\epsilon^i = \phi^i = [M_{p_m} \cdot \mathbf{D}^2 u_i \, M_{p_l}]_{l,m=1,\cdots,k}.$$

Hence we have

$$\boldsymbol{\mathcal{H}} \cdot \nabla u_i = -\frac{1}{k} tr(a^i) = -\frac{1}{k} tr(\phi^i \Phi^{-1}).$$

Let us assume now that the parameters $p_l, l = 1, \cdots, k$ are such that the vectors $M_{p_l}, l = 1, \cdots, k$ form an orthonormal basis of $T_x M$. Hence Φ is the identity matrix and the result follows.

□

We need another ingredient related to the properties of the matrixes $\mathbf{D}^2 u_i$:

Lemma 3 *At each point* $\mathbf{x}$ *of* M*, the linear operators* $\mathbf{D}^2 u_i\ (\mathbf{x})$*,* $i = 1, \cdots, n$ *map the normal space* NM_x *to the tangent space* TM_x*.*

Proof : In order to prove this, we consider equation (12) again and write it in coordinates. We note $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ the canonical basis of $\mathbb{R}^n$. Consider the ith column of both sides of equation (12):

$$\nabla u_i(\mathbf{x} - \mathbf{u}(\mathbf{x})) = D\mathbf{u}(\mathbf{x} - \mathbf{u}(\mathbf{x}))\nabla u_i(\mathbf{x}).$$

Since $D\mathbf{u}$ is symmetric *a.e.*, the right-hand side can be rewritten as $\sum_{j=1}^n \nabla u_j(\mathbf{x} - \mathbf{u}(\mathbf{x})) \cdot \nabla u_i(\mathbf{x})\mathbf{e}_j$, hence:

$$\nabla u_i(\mathbf{x} - \mathbf{u}(\mathbf{x})) = \sum_{j=1}^n \nabla u_j(\mathbf{x} - \mathbf{u}(\mathbf{x})) \cdot \nabla u_i(\mathbf{x})\mathbf{e}_j.$$

We now take the derivative of both sides of this equality and obtain

$$\mathbf{D}^2 u_i\ (\mathbf{x} - \mathbf{u}(\mathbf{x}))(I - D\mathbf{u}(\mathbf{x})) = \\ \sum_{j=1}^n \mathbf{e}_j \left[\mathbf{D}^2 u_j\ (\mathbf{x} - \mathbf{u}(\mathbf{x}))(I - D\mathbf{u}(\mathbf{x}))\nabla u_i(\mathbf{x}) + \mathbf{D}^2 u_i\ (\mathbf{x})\nabla u_j(\mathbf{x} - D\mathbf{u}(\mathbf{x}))\right]^T.$$

If we evaluate these expressions on M we obtain (without indicating the dependency with respect to $\mathbf{x}$):

$$\mathbf{D}^2 u_i\ (I - D\mathbf{u}) = \sum_{j=1}^n \mathbf{e}_j \left[\mathbf{D}^2 u_j\ (I - D\mathbf{u})\nabla u_i + \mathbf{D}^2 u_i\ \nabla u_j\right]^T.$$

We saw in the previous discussion that, on M, $D\mathbf{u}$ is the projector on $N_x M$; This implies that $(I - D\mathbf{u})\nabla u_i = \mathbf{0}$, $i = 1, \cdots, n$. Hence we can write

$$\mathbf{D}^2 u_i\ (I - D\mathbf{u}) = \sum_{j=1}^n \mathbf{e}_j \left[\mathbf{D}^2 u_i\ \nabla u_j\right]^T.$$

We now multiply both sides of this equality on the right with ∇u_k and take once again advantage of the fact that $D\mathbf{u}$ is the projector on $N_x M$:

$$\sum_{j=1}^n \left(\nabla u_k \cdot \mathbf{D}^2 u_i\ \nabla u_j\right) \mathbf{e}_j = \mathbf{0},$$

and since $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ is a vector basis, we obtain

$$\nabla u_k \cdot \mathbf{D}^2 u_i \; \nabla u_j = 0 \quad i, j, k = 1, \cdots, n,$$

which implies that $\mathbf{D}^2 u_i \; \nabla u_j$ is in TM_x for all i's and j's.

□

We are now ready to prove proposition 8. **Proof :** Because of lemma 2 we know that $\boldsymbol{\mathcal{H}} \cdot \nabla u_i = -\frac{1}{k} \sum_{l=1}^{k} M_{p_l} \cdot \mathbf{D}^2 u_i \; M_{p_l}$ carries over directly to this case. Now consider an orthonormal basis $(\mathbf{N}_1, \cdots, \mathbf{N}_{n-k})$ of $N_x M$. We certainly have

$$-\sum_{l=1}^{k} M_{p_l} \cdot \mathbf{D}^2 u_i \; M_{p_l} = -\Delta u_i + \sum_{l=1}^{n-k} \mathbf{N}_l \cdot \mathbf{D}^2 u_i \; \mathbf{N}_l.$$

Since we proved before that $\mathbf{D}^2 u_i \; \mathbf{N}_l$ is in TM_x for all $i = 1, \cdots, n$ and $l = 1, \cdots, n-k$ (proposition 3), we conclude that

$$\boldsymbol{\mathcal{H}} \cdot \nabla u_i = -\frac{1}{k} \Delta u_i.$$

Let us now write $\boldsymbol{\mathcal{H}} = \sum_{i=1}^{n} \boldsymbol{\mathcal{H}}^i \mathbf{e}_i$. Taking the inner product with ∇u_j, we obtain

$$\boldsymbol{\mathcal{H}} \cdot \nabla u_j = \sum_{i=1}^{n} (\nabla u_j \cdot \mathbf{e}_i) \boldsymbol{\mathcal{H}}^i,$$

which we rewrite in vector form as

$$H = D\mathbf{u}\boldsymbol{\mathcal{H}},$$

where H is the $n \times 1$ vector of coordinates $H^i = \boldsymbol{\mathcal{H}} \cdot \nabla u_i$, $i = 1, \cdots, n$, in the basis $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$.

This equation shows , since $D\mathbf{u}$ is the projector on $N_x M$ (proposition 5), and $\boldsymbol{\mathcal{H}}$ is in $N_x M$, that

$$H = \boldsymbol{\mathcal{H}}.$$

This can be rewritten as

$$\boldsymbol{\mathcal{H}} = \sum_{i=1}^{n} H^i \mathbf{e}_i = -\frac{1}{k} \sum_{i=1}^{n} \Delta u_i \mathbf{e}_i = -\frac{1}{k} \Delta \mathbf{u},$$

as announced.

□

References

[1] D. Adalsteinsson and J. A. Sethian. A Fast Level Set Method for Propagating Interfaces. *Journal of Computational Physics*, 118(2):269–277, 1995.

[2] D. Adalsteinsson and J. A. Sethian. The fast construction of extension velocities in level set methods. *Journal of Computational Physics*, 1(148):2–22, 1999.

[3] L. Ambrosio and C. Mantegazza. Curvature and distance function from a manifold. *J. Geom. Anal.*, 1996. To appear.

[4] Luigi Ambrosio and Halil M. Soner. Level set approach to mean curvature flow in arbitrary codimension. *J. of Diff. Geom.*, 43:693–737, 1996.

[5] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer-Verlag New York Inc., 1983.

[6] V.I. Arnold. *Ordinary Differential Equations.* MIT Press, 1973.

[7] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi.* Springer-Verlag, 1994.

[8] G. Barles, H.M. Soner, and P.E. Souganidis. Front propagation and phase field theory. *SIAM J. Control and Optimization*, 31(2):439–469, March 1993.

[9] M. Bertalmio, G. Sapiro, and G. Randall. Region Tacking on Surfaces Deforming via Level-Sets Methods. In Mads Nielsen, P. Johansen, O.F. Olsen, and J. Weickert, editors, *Scale-Space Theories in Computer Vision*, volume 1682 of *Lecture Notes in Computer Science*, pages 58–69. Springer, September 1999.

[10] A. Blake and M. Isard. *Active Contours.* Springer-Verlag, 1998.

[11] Paul Burchard, Li-Tien Cheng, Barry Merriman, and Stanley Osher. Motion of curves in three spatial dimensions using a level set approach. Technical report, Department of Mathematics, University of California Los Angeles, September 2000.

[12] V. Caselles and B. Coll. Snakes in Movement. *SIAM Journal on Numerical Analysis*, 33:2445–2456, December 1996.

[13] V. Caselles, R. Kimmel, and G. Sapiro. Geodesic active contours. In *Proceedings of the 5th International Conference on Computer Vision*, pages 694–699, Boston, MA, June 1995. IEEE Computer Society Press.

[14] V. Caselles, R. Kimmel, and G. Sapiro. Geodesic active contours. *IJCV*, 22(1):61–79, 1997.

[15] V. Caselles, R. Kimmel, G. Sapiro, and C. Sbert. 3d active contours. In M-O. Berger, R. Deriche, I. Herlin, J. Jaffre, and J-M. Morel, editors, *Images, Wavelets and PDEs*, volume 219 of *Lecture Notes in Control and Information Sciences*, pages 43–49. Springer, June 1996.

[16] V. Caselles, R. Kimmel, G. Sapiro, and C. Sbert. Minimal surfaces based object segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 9(4):394–398, 1997.

[17] Y.G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geometry*, 33:749–786, 1991.

[18] David L. Chopp. Computing minimal surfaces via level set curvature flow. *Journal of Computational Physics*, 106:77–91, 1993.

[19] R. Cipolla and P. Giblin. *Visual Motion of Curves and Surfaces*. Cambridge University Press, 2000.

[20] M. P. DoCarmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.

[21] M. P. DoCarmo. *Riemannian Geometry*. Birkhäuser, 1992.

[22] C.L. Epstein and Michael Gage. The curve shortening flow. In A.J. Chorin, A.J. Majda, and P.D. Lax, editors, *Wave motion: theory, modelling and computation*. Springer-Verlag, 1987.

[23] L.C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 1998.

[24] L.C. Evans and J. Spruck. Motion of level sets by mean curvature: I. *Journal of Differential Geometry*, 33:635–681, 1991.

[25] Olivier Faugeras and Renaud Keriven. Variational principles, surface evolution, pde's, level set methods and the stereo problem. *IEEE Trans. on Image Processing*, 7(3):336–344, March 1998.

[26] M. Gage. Curve shortening makes convex curves circular. *Invent. Math.*, 76:357–364, 1984.

[27] M. Gage and R.S. Hamilton. The heat equation shrinking convex plane curves. *J. of Differential Geometry*, 23:69–96, 1986.

[28] J. Gomes and O. Faugeras. Reconciling Distance Functions and Level Sets. In Mads Nielsen, P. Johansen, O.F. Olsen, and J. Weickert, editors, *Scale-Space Theories in Computer Vision*, volume 1682 of *Lecture Notes in Computer Science*, pages 70–81. Springer Verlag, 1999.

[29] J. Gomes and O. Faugeras. Shape representation as the intersection of k hypersurfaces. Technical report, INRIA, 2000.

[30] J. Gomes and O.D. Faugeras. Reconciling Distance Functions and Level Sets. *Journal of Visual Communication and Image Representation*, 11:209–223, 2000.

[31] M. Grayson. The heat equation shrinks embedded plane curves to round points. *J. of Differential Geometry*, 26:285–314, 1987.

[32] M. Grayson. A short note on the evolution of surfaces via mean curvature. *Duke Math J.*, pages 555–558, 1989. Proof that the dumbell can split in 2 under mean curvature motion.

[33] M. Kass, A. Witkin, and D. Terzopoulos. SNAKES: Active contour models. *The International Journal of Computer Vision*, 1:321–332, January 1988.

[34] B. Kimia, A. R. Tannenbaum, and S. W. Zucker. Shapes, schoks and deformations i: The components of two-dimensional shape and the reaction-diffusion space. *ijcv*, 15:189–224, 1995.

[35] Jan J. Koenderink. *Solid Shape*. MIT Press, 1990.

[36] L. Lorigo, O. Faugeras, W.E.L. Grimson, R. Keriven, R. Kikinis, and C-F. Westin. Co-dimension 2 geodesic active contours for mra segmentation. In *Proc. Int'l Conf. Information Processing in Medical Imaging*, pages 126–139, June 1999.

[37] R. Malladi, J. A. Sethian, and B.C. Vemuri. Evolutionary fronts for topology-independent shape modeling and recovery. In J-O. Eklundh, editor, *Proceedings of the 3rd European Conference on Computer Vision*, volume 800 of *Lecture Notes in Computer Science*, Stockholm, Sweden, May 1994. Springer-Verlag.

[38] J. Montagnat, H. Delingette, N. Scapel, and N. Ayache. Representation, shape, topology and evolution of deformable surfaces. Application to 3D medical image segmentation. Technical Report 3954, INRIA, 2000.

[39] S. Osher and J. Sethian. Fronts propagating with curvature dependent speed : algorithms based on the Hamilton-Jacobi formulation. *Journal of Computational Physics*, 79:12–49, 1988.

[40] Nikos Paragios and Rachid Deriche. Geodesic active contours and level sets for the detection and tracking of moving objects. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22:266–280, March 2000.

[41] S. J. Ruuth, B. Merriman, J. Xin, and S. Osher. Diffusion-Generated Motion by Mean Curvature for filaments. Technical Report 98-47, UCLA Computational and Applied Mathematics Reports, November 1998.

[42] S.J. Ruuth, B. Merriman, and S. Osher. A fixed grid method for capturing the motion of self-intersecting interfaces and related pdes. Technical Report 99-22, UCLA Computational and Applied Mathematics Reports, July 1999.

[43] G. Sapiro and A. Tannenbaum. On affine plane curve evolution. *Journal of Functional Analysis*, 119(1):79–120, January 1994.

[44] J. A. Sethian. A Fast Marching Level Set Method for Monotonically Advancing Fronts. In *Proc. Nat. Ac. Science*, volume 93, pages 1591–1694, 1996.

[45] J.A. Sethian. Recent numerical algorithms for hypersurfaces moving with curvature-dependent speed:Hamilton-Jacobi equations and conservation laws. *J. Differential Geometry*, 31:131–136, 1990.

[46] K. Siddiqi, A. Shokoufandeh, S. J. Dickinson, and S. W. Zucker. Shock Graphs and Shape Matching. *The International Journal of Computer Vision*, 35(1):13–32, November 1999.

[47] Michael Spivak. *A Comprehensive Introduction to Differential Geometry*, volume I–V. Publish or Perish, Berkeley, CA, 1979. Second edition.

[48] J. Steinhoff, M. Fan, and L. Wang. A new eulerian method for the computation of propagating short acoustic and electromagnetic pulses. *Journal of Computational Physics*, 157:683–706, 2000.

[49] H. Tek and B. B. Kimia. Image Segmentation by Reaction-Diffusion Bubbles. In *Proceedings of the 5th International Conference on Computer Vision*, Boston, MA, June 1995. IEEE Computer Society Press.

[50] D. Terzopoulos, A. Witkin, and M. Kass. Constraints on Deformable Models: Recovering 3D shape and Nonrigid Motion. *Artificial Intelligence*, 36(1):91–123, 1988.

[51] J. N. Tsitsiklis. Efficient algorithms for globally optimal trajectories. *IEEE Transactions on Automatic Control*, 40(9):1528–1538, September 1995.

[52] Hong-Kai Zhao, T. Chan, B. Merriman, and S. Osher. A variational level set approach to multiphase motion. *Journal of Computational Physics*, 127(0167):179–195, 1996.

Unité de recherche INRIA Sophia Antipolis
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Unité de recherche INRIA Lorraine : Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 Villers lès Nancy Cedex (France)
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