# Shape Representation as the Intersection of n-k Hypersurfaces

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**Abstract:** We investigate the feasibility of representing implicitly a $k$-dimensional manifold embedded in the Euclidean space $\mathbb{R}^n$ as the intersection of $n-k$ transverse hypersurfaces. From the analytical point of view, the embedded manifold is defined as the inverse image of a *regular* value of a *vector* function. This approach is *a priori* appealing since the corresponding function is differentiable at any point of the embedded manifold. We focus on time-dependent manifolds and establish the link between the velocity field of the evolving manifold and a Partial Differential Equation (PDE) satisfied by its describing function.

**Key-words:** *Implicit representations of shape, Mean curvature motion in arbitrary codimension.*

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# Représentation des formes par l'intersection de $n-k$ hypersurfaces.

**Résumé :** Nous étudions la possibilité de représenter implicitement une variété de dimension $k$ plongée dans l'espace Euclidien $\mathbb{R}^n$ comme l'intersection de $k$ hypersurfaces. Du point de vue analytique, cela revient à définir la variété plongée comme l'image réciproque d'une valeur *régulière* d'une fonction vectorielle. Cette approche est *a priori* séduisante car, dans ce cas, la fonction vectorielle en question est différentiable en tout point de la variété d'intérêt. Nous nous intéressons plus particulièrement aux variétés se déformant au cours du temps et mettons en évidence une Équation aux Dérivées Partielles vérifiée par ladite fonction vectorielle.

**Mots-clés :** *Représentations implicites des formes, Mouvement de courbure moyenne en co-dimension arbitraire.*

# Contents

# 1 Introduction

The important question of designing practical implicit representations of evolving manifolds for computer simulation purposes has been widely investigated by the Computational Physics [20, 24, 25, 28] and Computer Vision [4, 19, 27, 6, 3, 5, 7, 12, 18, 21] communities but also by the mathematicians [13, 14, 11, 16, 17]. Very satisfactory solutions have been proposed for the case of planar curves and even for general hypersurfaces (nevertheless, the strongest theoretical results apply only to planar curves). The case of manifolds of higher codimension has been less explored. Two recent contributions in this direction are the following.

Ambrosio and Soner [1] propose to evolve an $\varepsilon$-neighbourhood of the manifold (*i.e.* an hypersurface). For instance, in the case of a curve in $\mathbb{R}^3$, it consists in evolving a tubular neighbourhood of the initial curve which has a small radius $\varepsilon$. The relation between the motion of this hypersurface and the one of the curve is established. This scheme was successfully adopted by Lorigo *et al.* [18] to the problem of detecting blood vessels in volumetric medical images. A drawback of this approach is that the manifold of interest (*i.e.* the curve) is somewhat "lost" in the middle of the evolving hypersurface (*i.e.* the tubular neighbourhood) and is not as accurately positioned as it would be, for instance, by the zero-crossing of a smooth function.

Alternatively, Ruuth *et al.* [22] propose to represent a curve in $\mathbb{R}^3$ by means of a (two-dimensional) complex function of unit magnitude defined on $\mathbb{R}^3$ whose phase angle "winds" around the curve. This function is not defined at points of the curve of interest. The time evolution is "diffusion-generated", *i.e.* it is the consequence of a diffusion-renormalisation loop. Very convincing results are shown demonstrating in particular the possibility for the curve to have its topology altered during the evolution. Nevertheless, in this approach, the curve of interest is defined as the set of singularities of a function and it is a drawback when dealing with sampled functions.

In the context of computer simulations where the sampling side-effects cannot be neglected, we believe it is an important advantage to define manifolds as the zero-crossings of smooth functions, and it is not the case of these two propositions. This naturally suggests to consider a manifold of dimension $k$ as the intersection of $n-k$ hypersurfaces $\mathbf{S}_i$, each one being described implicitly by a smooth scalar function, *i.e.* $\mathbf{S}_i = u_i^{-1}(0)$. The intersection of the hypersurfaces $\mathcal{M} = \mathbf{S}_1 \cap \cdots \cap \mathbf{S}_{n-k}$ is then equal to $\mathbf{u}^{-1}(\mathbf{0})$, where $\mathbf{u} = (u_1, \cdots, u_{n-k})$. We present some ideas for realising this program and start by pointing out difficulties that must be faced:

*i)* According to the Theory of Submanifolds, the intersection of hypersurfaces is not necessarily a submanifold. The adequate notion is that of the *transversality condition* [9]: the tangent spaces $\mathbf{T}_{\boldsymbol{x}}\mathbf{S}_i$ should be linearly independent at each point $\boldsymbol{x} \in \mathcal{M}$. Geometrically speaking, it means the hypersurfaces $\mathbf{S}_i$ represent $n-k$ non-degenerated constrains and we shall only consider the ideal case where the $\mathbf{S}_i$'s intersect orthogonally at each time instant.

*ii)* *A priori*, infinitely many sets of $n-k$ hypersurfaces can share the same intersection.

*iii)* A single hypersurface can be described implicitly by infinitely many different scalar functions.

The remainder of the paper is organised as follows. Section 2 is a preliminary discussion of the equivalence between the geometrical point of view (*i.e.* an evolving curve is seen as the intersection of two evolving transverse surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$) and the analytical point of view (*i.e.* the same evolving curve is seen as the inverse image of a regular value of a time-dependant vector function, $\mathcal{M} = \boldsymbol{u}^{-1}(\mathbf{0})$). In section 3, we apply some Differential Geometry results exposed in [26] in order to build, in an *intrinsic way*, a set of $n-k$ evolving hypersurfaces intersecting orthogonally at the manifold of interest. Then, in section 4, we consider the interesting case of *arbitrary transverse hypersurfaces* intersecting at the manifold of interest $\mathcal{M}$. Given a certain desired motion of this manifold, we establish a PDE that must be satisfied by the describing function $\mathbf{u}$. The motions of the corresponding hypersurfaces are discussed at the same time.

# 2 Notations and preliminaries

In this paper, the terms "curve" and "surface" have their classical sense, *i.e.* submanifolds of $\mathbb{R}^3$ with dimensions 1 and 2. A curve is embedded in $\mathbb{R}^3$ by $p \to \mathbf{C}(p)$, $p \in \mathbb{R}$ and a surface by $p \to \mathbf{S}(p)$, $p \in \mathbb{R}$. For the sake of simplicity, the ideas shall be presented for differentiable curves of $\mathbb{R}^3$ evolving according to the mean curvature motion. Nevertheless, it is understood that the objective of the authors is to design a scheme which can be generalised to any dimension and codimension and other motions as well. Guidelines are given each time this generalisation is not trivial.

We begin by making the notion of a manifold *deforming* precise. Then we recall the notion of *the inverse image of a regular value* and discuss its geometrical interpretation.

## 2.1 Evolution of a manifold

An "evolution of a curve" will denote a smooth one-parameter family of curves $t \to \mathbf{C}(p,t)$ where $t \in \mathbb{R}^+$ is the *time* parameter. This can be modelled as the solution to the following PDE with Dirichlet boundary condition:

$$\left\{ \begin{array}{lr} \mathbf{C}_{|t=0} \quad \text{is given} & \text{(a)} \\ \dfrac{\partial \mathbf{C}}{\partial t} = \beta \mathbf{N}, & \text{(b)} \end{array} \right. \tag{1}$$

where $\beta : \mathbf{C} \to \mathbb{R}$ is a smooth scalar velocity function and $\mathbf{N}$ is a smooth normal vector field of $\mathbf{C}$. An "evolution of a surface" can be defined similarly and a geometrical interpretation of (1) is then given in Fig. 1. The fundamental curve evolution we are most interested in is the curve shortening flow for which $\beta = \kappa$, the Euclidean curvature and $\mathbf{N}$ is *the* unit

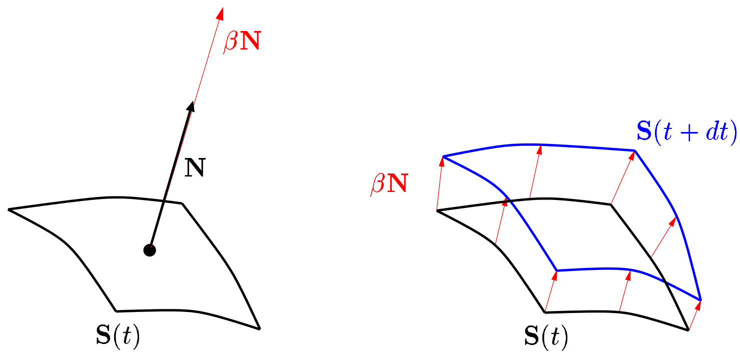



Figure 1: If a family of surfaces $t \to \mathbf{S}(p,t)$ satisfies $\frac{\partial \mathbf{S}}{\partial t} = \beta\mathbf{N}$, then at time $t$, the velocity of a point over the surface $\mathbf{S}(t)$ is $\beta$ times the normal vector $\mathbf{N}$ to $\mathbf{S}(t)$ at this point.

normal vector field of the curve. This particular motion has been studied in great detail for planar curves by the mathematicians mentioned in the introduction.

## 2.2 Inverse image of a regular value

This notion, introduced by the Theory of Isometric Immersions (*cf* [8, 10] for example) motivated a lot this work. We shall only need a definition and a lemma.

**Definition 1** *Let* $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^k$*, with* $k \leq n$*, be a differentiable function and* $\mathbf{y}_0 \in \mathbb{R}^k$*.* $\mathbf{y}_0$ *is said to be a* regular value *of* $\mathbf{u}$ *if* $D\mathbf{u} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$ *is surjective at all points* $\mathbf{x}^0 \in \mathbf{u}^{-1}(\mathbf{y}_0)$*.*

**Lemma 1** *Let* $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^k$ *a differentiable function and* $\mathbf{y}_0 \in \mathbb{R}^k$ *a regular value of* $\mathbf{u}$*. Then the set* $\mathbf{u}^{-1}(\mathbf{y}^0)$ *is a* $(n-k)$*-dimensional submanifold of* $\mathbb{R}^n$*.*

**Proof :** This is a particular case of the implicit function theorem. Let $(\mathbf{x}^0, \mathbf{y}^0) \in \mathbb{R}^n \times \mathbb{R}^k$ such that $\mathbf{u}(\mathbf{x}^0) = \mathbf{y}^0$ and $\mathbf{y}^0$ is a regular value of $\mathbf{u}$. Since $D_{x^0}\mathbf{u}$ is surjective, we can without loss of generality give a privileged role to the $k^{th}$ first coordinates of $\mathbf{x}^0$ and suppose

$$\frac{\partial(u_1(\mathbf{x}), \cdots, u_k(\mathbf{x}))}{\partial(x_1^0, \cdots, x_k^0)} \neq 0. \tag{2}$$

Now, we define the function

$$\left\{ \begin{array}{l} f : \mathbb{R}^n \to \mathbb{R}^n \\ (x_1, \cdots, x_n) \to (u_1(\mathbf{x}), \cdots, u_k(\mathbf{x}), x_{k+1}, \cdots, x_n). \end{array} \right. \tag{3}$$

It follows easily from (2) that the Jacobian matrix of $f$ verifies $det(\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}^0)) \neq 0$. By the inverse function theorem, $f$ is a diffeomorphism between certain neighbourhoods $V_{\mathbf{x}^0} \subset \mathbb{R}^n$

and $W_{\tilde{\mathbf{y}}_0} \subset \mathbb{R}^n$ of $\mathbf{x}_0$ and $\tilde{\mathbf{y}}_0 = [y_1^0, \cdots, y_k^0, y_{k+1}^0, x_{k+1}^0, \cdots, x_n^0]^T$.
Hence, we can consider the function $\mathbf{C}(x_{k+1}, \cdots, x_n) = f^{-1}(y_1^0, \cdots, y_k^0, x_{k+1}, \cdots, x_n)$ defined from a $(n-k)$-dimensional subset of $W_{\tilde{\mathbf{y}}_0}$ onto $V_{\mathbf{x}^0}$.
The function $\mathbf{C}$ is the embedding of a manifold of dimension $n-k$ in $\mathbb{R}^n$, *i.e.* a $(n-k)$-dimensional submanifold of $\mathbb{R}^n$ such that $\mathbf{u}(\mathbf{C}) = \mathbf{y}^0$. This result is true for all points in $\mathbf{u}^{-1}(\mathbf{y}^0)$ which concludes the proof.

□

## 2.3 Link between the inverse image of a regular value and the intersection of hypersurfaces

The *transversality* of the hypersurfaces is obviously related to the surjectivity condition of the lemma. More precisely, let $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\mathbf{0} \in \mathbb{R}^2$ is a regular value of $\mathbf{u}$. Then, according to the lemma, $\mathbf{u}^{-1}(\mathbf{0})$ is a curve. Indeed, denoting by $(u_1, u_2)$ the two components of $\mathbf{u}$, the sets $\mathbf{S}_1 = u_1^{-1}(0)$ and $\mathbf{S}_2 = u_2^{-1}(0)$ are oriented isosurfaces of the two functions $\mathbf{x} \to u_1(\mathbf{x})$ and $\mathbf{x} \to u_2(\mathbf{x})$ if and only if 0 is a regular value of $u_1$ and $u_2$ (*i.e.* if $\nabla u_1$ and $\nabla u_2$ are not null on $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively). If , in addition, $\nabla u_1 \times \nabla u_2 \neq \mathbf{0}$ on $\mathcal{M} = \mathbf{S}_1 \cap \mathbf{S}_2$, then the two surfaces are transverse (indeed $\nabla u_i$ is normal to $\mathbf{S}_i$) and their intersection is a curve. The condition $\nabla u_1 \times \nabla u_2 \neq \mathbf{0}$ means exactly that $\mathrm{D}\mathbf{u}$ is surjective at $\mathcal{M}$ or equivalently that $\mathbf{0}$ is a regular value of $\mathbf{u}$.

Although these considerations show the $u_i$'s are not independent from one another, the way they should be constructed is still unclear. This is in particular a consequence of iii), namely of the non-uniqueness of the implicit representation of an hypersurface. In order to simplify considerably our study, it is desirable to be allowed to *identify* an hypersurface and its implicit representation. This uniqueness can elegantly be achieved by the following choice: each coordinate of $\mathbf{u}$ is chosen to be the oriented distance function to the corresponding surface, namely:

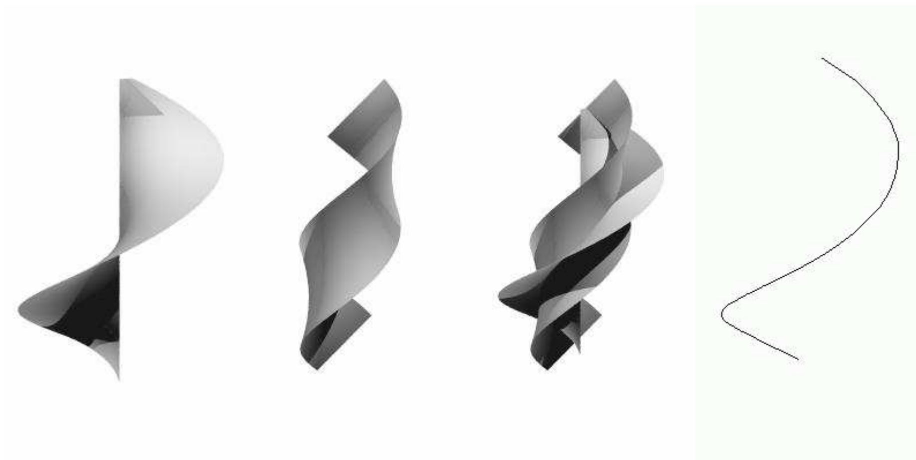
$$u_i(\boldsymbol{x}, t) = \pm \inf_{\boldsymbol{y} \in \mathbf{S}_i(t)} ||\boldsymbol{x} - \boldsymbol{y}||,$$

where the sign $\pm$ is chosen according to the orientation of $\mathbf{S}_i$.

**Remark 1** *This can be generalised to higher dimensions since an hypersurface is always orientable [10].*

A good understanding of the properties of the distance functions shows that this choice is appropriate. Indeed, it is important to have at our disposal the relation between a desired motion of the surface $\mathbf{S}_i$ and an associated parabolic PDE satisfied by $u_i$. Thanks to this relation, not only $\mathbf{S}_i$ and $u_i$ can be identified but also their time evolution. This relation is known for the oriented distance functions [28, 15] and will be used in the section 4.

It remains to design a pair of proper evolving transverse hypersurfaces.



$\mathbf{S}_1$ $\mathbf{S}_2$ $\mathbf{S}_1 \cup \mathbf{S}2$ $\mathbf{C} = \mathbf{S}_1 \cap \mathbf{S}_2$

Figure 2: In this figure, a segment of an circular helix $\mathbf{C}$ is shown to be the intersection of two transverse surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ which are ruled surfaces respectively spanned by the normal and the binormal vectors of $\mathbf{C}$.

# 3 Intrinsic transverse hypersurfaces

This section is an attempt to design, in an intrinsic manner, a set of $n-k$ hypersurfaces of $\mathbb{R}^n$ intersecting properly at a given $(n-k)$-dimensional submanifold of $\mathbb{R}^n$. The natural idea that is developed is the following: in the case of a space curve, one can construct two ruled surfaces whose generatrix is the curve and rulings are respectively the normal and the binormal vector field of the Frenet frame (*cf* Fig.2). The two surfaces intersect orthogonally, which is the ideal situation. Then, if the curve initiates a motion, one can calculate the evolutions of the two corresponding ruled surfaces by studying how the Frenet frame is affected by the motion of the curve.

The question we address here is the *pros* and *cons* of this intrinsic approach in arbitrary dimension. As a first step in the direction of general submanifolds of $\mathbb{R}^n$, we begin with space curves, then we give guidelines for generalising.

Given a curve $\mathbf{C}(p,t)$ evolving through the equation

$$\frac{\partial \mathbf{C}}{\partial t} = \kappa \mathbf{N},$$

we are looking for a pair of surfaces $\mathbf{S}_i(p, \lambda_i, t)$, $i = 1, 2$ such that $\mathbf{C}(p,t)$ is on $\mathbf{S}_i$ at each time instant and the normal vectors to the two surfaces are orthogonal along $\mathbf{C}$. We call such a pair a *transverse pair of surfaces for the curve* $\mathbf{C}$.

We recall the Frenet formula

$$\begin{aligned}\mathbf{T}_s &= \kappa\mathbf{N}\\ \mathbf{N}_s &= -\kappa\mathbf{T} + \rho\mathbf{B}\\ \mathbf{B}_s &= -\rho\mathbf{N},\end{aligned}$$

where $\kappa$ is the curvature, $\rho$ is the torsion of the curve and $s$ is the arc-length.

## 3.1 Finding transverse pairs

We write

$$\mathbf{S}(p,\lambda,t) = \mathbf{C}(p,t) + \lambda(u\mathbf{T} + v\mathbf{N} + w\mathbf{B}),$$

where $u, v$ and $w$ are unknown functions of $p$, hence of $s$, to be determined.

In general, such ruled surfaces are only regular in a neighbourhood of their generatrix but it is enough for our purpose. Indeed, the differentiability of the describing functions $u_i$'s in the vicinity of the embedded manifold is sufficient for positioning it with a good degree of accuracy and for computing its differential properties.

Let us compute two tangent vectors of $\mathbf{S}$:

$$\begin{aligned}\mathbf{S}_s = \mathbf{C}_s + \lambda(u'\mathbf{T} + u\mathbf{T}_s + v'\mathbf{N} + v\mathbf{N}_s + w'\mathbf{B} + w\mathbf{B}_s) = \\ (1+\lambda(u'-\kappa v))\mathbf{T} + \lambda(v' + u\kappa - w\rho)\mathbf{N} + \lambda(w' + v\rho)\mathbf{B},\end{aligned}$$

$$\mathbf{S}_\lambda = u\mathbf{T} + v\mathbf{N} + w\mathbf{B}.$$

Let us compute the cross-product of these two vectors, yielding the normal to the surface

$$\begin{aligned}\mathbf{S}_\lambda \times \mathbf{S}_s = \lambda(v(w' + v\rho) - w(v' + \kappa - w\rho)\mathbf{T} + \\ (w(1+\lambda(u'-\kappa v)) - \lambda u(w' + v\rho))\mathbf{N} + \\ (\lambda u(v' + u\kappa - w\rho) - v(1 + \lambda(u' - \kappa v))\mathbf{B}.\end{aligned}$$

Additional constrains on the ruled surfaces have to be chosen in order to find some proper $u, v$ and $w$ functions. We propose two solutions, the first one being the more natural.

#### First solution

We begin by dropping the tangential component (*i.e.* $u = 0$) arguing that the rulings should not be parallel to the tangent vector of the generatrix otherwise the surfaces would be singular. Next, we observe that $v$ and $w$ are only defined up to a scale factor so that one can add the constrain $\sqrt{v^2 + w^2} = 1$. Finally, we impose that the two surfaces intersect orthogonally (*i.e.* we expand $(\mathbf{S}_{1\lambda} \times \mathbf{S}_{1s}) \cdot (\mathbf{S}_{2\lambda} \times \mathbf{S}_{2s}) = 0$ for $\lambda = 0$) and obtain the solution

$$\begin{cases}\mathbf{S}_1(p,\lambda,t) = \mathbf{C}(p,t) + \lambda\left(\cos\theta\mathbf{N} + \sin\theta\mathbf{B}\right)\\ \mathbf{S}_2(p,\lambda,t) = \mathbf{C}(p,t) - \lambda\left(\sin\theta\mathbf{N} + \cos\theta\mathbf{B}\right),\end{cases}$$

where $\theta$ is an arbitrary angle.

This solution is weakly constrained in the sense that the ruled surfaces are not orthogonal to the curve along the whole rulings but only at the intersection, *i.e.* for $\lambda = 0$.

**Second solution**

We also propose a more constrained solution by imposing that the cross-product $\mathbf{S}_\lambda \times \mathbf{S}_s$ of the two normal vectors is in the normal plane of the curve $\mathbf{C}$ for all $\lambda$. This yields one equation in the three unknown functions $u, v$ and $w$:

$$w'v - wv' + \rho(v^2 + w^2) - uw\kappa = 0.$$

We can conveniently express this as an expression in the ratios $V = \frac{v}{w}$ and $U = \frac{u}{w}$, if $w \neq 0$:

$$V' - \rho(1 + V^2) + \kappa U = 0. \tag{4}$$

We want this to be true for the two surfaces:

$$V_i' - \rho(1 + V_i^2) + \kappa U_i = 0 \quad w_i \neq 0, \quad i = 1, 2. \tag{5}$$

We also want the two normals to be orthogonal along $\mathbf{C}$:

$$w_1 w_2 + v_1 v_2 = 0,$$

or

$$V_1 V_2 + 1 = 0.$$

This last equation implies

$$V_2' = \frac{V_1'}{V_1^2}.$$

By comparing the two equations (5) we obtain, if $\kappa \neq 0$,

$$U_1 = U_2 V_1^2. \tag{6}$$

A possible choice is

$$U_1 = U_2 = 0.$$

In that case one can integrate (5) and obtain

$$V_1 = \tan \int \rho \equiv \varphi, \tag{7}$$

and therefore $V_2 = -\frac{1}{\varphi}$. We have found the equations of two surfaces containing the curve $\mathbf{C}$, intersecting at a right angle along $\mathbf{C}$ and such that their normals along $\mathbf{C}$ are in the normal plane:

$$\mathbf{S}_1(p, \lambda_1, t) = \mathbf{C}(p, t) + \lambda_1 w_1 (\mathbf{N} - \varphi \mathbf{B})$$
$$\mathbf{S}_2(p, \lambda_2, t) = \mathbf{C}(p, t) + \lambda_2 w_2 (\mathbf{N} + \frac{\mathbf{B}}{\varphi})$$

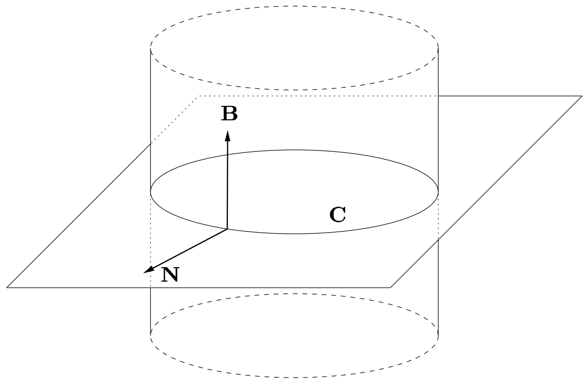



Figure 3: A circle is shown to be the intersection of two ruled surfaces: the plane containing the binormal vectors (and the circle itself) and the cylinder generated by the circle and ruled by the binormal vectors.

A natural choice is to take $w_1 = 1$ and $w_2 = \varphi$:

$$\mathbf{S}_1(p, \lambda_1, t) = \mathbf{C}(p, t) + \lambda_1(\mathbf{N} - \varphi\mathbf{B}) \tag{8}$$
$$\mathbf{S}_2(p, \lambda_2, t) = \mathbf{C}(p, t) + \lambda_2(\varphi\mathbf{N} + \mathbf{B}) \tag{9}$$

Note that in the case where the curve is planar, $\varphi = 0$ and one obtains the "natural" equations (*cf* Fig.3)

$$\mathbf{S}_1(p, \lambda_1, t) = \mathbf{C}(p, t) + \lambda_1\mathbf{N}$$
$$\mathbf{S}_2(p, \lambda_2, t) = \mathbf{C}(p, t) + \lambda_2\mathbf{B}$$

## 3.2 Evolution of the transverse pair

We now want to compute the time evolution of the transverse pair of surfaces. In order to do this, we need to compute the time evolution $\varphi_t$ of $\varphi$, *i.e.* the time evolution of the torsion $\rho$, and the time evolution of the normal $\mathbf{N}$ and the binormal $\mathbf{B}$. It turns out that we will also need the time evolution of the tangent $\mathbf{T}$ and the curvature. We do it for the surface $\mathbf{S}_1$ and drop for convenience the index 1:

$$\mathbf{S}_t = \mathbf{C}_t + \lambda(\mathbf{N}_t - \varphi_t\mathbf{B} - \varphi\mathbf{B}_t).$$

We can do this systematically using a combination of the following well-known formula [23]

$$\frac{\partial^2}{\partial t \partial s} = \alpha\kappa\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial t} \tag{10}$$

and the Frenet formulae. We assume a more general time evolution for $\mathbf{C}$ than mean curve motion, *i.e.*

$$\frac{\partial \mathbf{C}}{\partial t} = \alpha \mathbf{N} + \beta \mathbf{B}.$$

**Evolution of the tangent**

Equation (10) yields

$$\frac{\partial^2}{\partial t \partial s}\mathbf{C} = \mathbf{T}_t = \alpha\kappa\mathbf{T} + \frac{\partial}{\partial s}(\alpha\mathbf{N} + \beta\mathbf{B}) = (\alpha' - \rho\beta)\mathbf{N} + (\beta' + \rho\alpha)\mathbf{B} \equiv a\mathbf{N} + b\mathbf{B}, \tag{11}$$

where $a = \alpha' - \rho\beta$ and $b = \beta' + \rho\alpha$.

**Evolution of the normal**

Let us compute the evolution of the normal. The first Frenet equation and (10) yield

$$\frac{\partial^2}{\partial t \partial s}\mathbf{T} = \kappa_t\mathbf{N} + \kappa\mathbf{N}_t = \alpha\kappa(-\kappa\mathbf{T} + \rho\mathbf{B}) + \frac{\partial}{\partial s}\mathbf{T}_t.$$

We combine this result with equation (11):

$$\begin{aligned}\kappa_t\mathbf{N} + \kappa\mathbf{N}_t = \alpha\kappa(-\kappa\mathbf{T} + \rho\mathbf{B}) + a'\mathbf{N} + b'\mathbf{B} + a(-\kappa\mathbf{T} + \rho\mathbf{B}) - b\rho\mathbf{N} = \\ -\kappa(a + \alpha\kappa)\mathbf{T} + (a' - b\rho)\mathbf{N} + (b' + \rho(\alpha\kappa + a))\mathbf{B}.\end{aligned}$$

From which we obtain the evolution of the curvature and of the normal:

$$\kappa_t = a' - b\rho \tag{12}$$

$$\kappa\mathbf{N}_t = -\kappa(a + \alpha\kappa)\mathbf{T} + (b' + \rho(\alpha\kappa + a))\mathbf{B} \equiv c\mathbf{N} + d\mathbf{B}. \tag{13}$$

**Evolution of the binormal**

Once again we use equation (10) and combine it with the second Frenet equation

$$\frac{\partial^2}{\partial t \partial s}\mathbf{N} = -\kappa_t\mathbf{T} - \kappa\mathbf{T}_t + \rho_t\mathbf{B} + \rho\mathbf{B}_t = \alpha\kappa(-\kappa\mathbf{T} + \rho\mathbf{B}) + \frac{\partial}{\partial s}\mathbf{N}_t.$$

We reorganise this equation as

$$\rho_t\mathbf{B} + \rho\mathbf{B}_t = (\kappa_t - \alpha\kappa^2)\mathbf{T} + \kappa a\mathbf{N} + \kappa(b + \rho\alpha)\mathbf{B} + \frac{\partial}{\partial s}\mathbf{N}_t. \tag{14}$$

Equation (13) yields

$$\kappa'\mathbf{N}_t + \kappa\frac{\partial}{\partial s}\mathbf{N}_t = c'\mathbf{N} + c(-\kappa\mathbf{T} + \rho\mathbf{B}) + d'\mathbf{B} - d\rho\mathbf{N} = -c\kappa\mathbf{T} + (c' - \rho d)\mathbf{N} + (d' + \rho c)\mathbf{B}.$$

We will not pursue the calculation. It is clear that this equation combined with (13) yields the term $\frac{\partial}{\partial s}\mathbf{N}_t$ that is needed in (14) from which we will immediately obtain the time derivative of the torsion, $\rho_t$, and the time derivative of the binormal $\mathbf{B}_t$. Equation (7) implies that $\varphi_t = \tan \int \rho_t$.

In the end, using also (13), we have a complete definition of $\mathbf{S}_t(p, \lambda, t)$ in the coordinate system $(\mathbf{T}, \mathbf{N}, \mathbf{B})(p, t)$. But the previous calculations show that the orders of the derivatives of the curve $\mathbf{C}$ that have to be evaluated are too high for an implementation of this method. This is confirmed by the analysis in the next section.

## 3.3 One-dimensional manifolds embedded in $\mathbb{R}^n$

The previous results can be extended by means of a generalisation of the Frenet frame ([26], IV, p.29). Unfortunately, the determination of an intrinsic basis of the normal space requires high order differentiation of the involved functions. Indeed, let $s \to \mathbf{C}(s)$ be an arc-length parameterisation of a one-dimensional submanifold of $\mathbb{R}^n$ and $\boldsymbol{v}_1 = \mathbf{C}_s$ its unit tangent vector. Then

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = 1$$

implies

$$\frac{d}{ds}(\boldsymbol{v}_1 \cdot \boldsymbol{v}_1) = 0 = 2\boldsymbol{v}_1 \cdot \boldsymbol{v}_{1s}. \tag{15}$$

The function $\kappa_1 = |\boldsymbol{v}_{1s}|$ is the "first" curvature of $\mathbf{C}$ (it is the same definition as for space curves) and, if it does not vanish, it allows to define

$$\boldsymbol{v}_2 = \frac{1}{\kappa_1}\boldsymbol{v}_{1s},$$

which is orthogonal to $\boldsymbol{v}_1$ and of unit norm

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0,$$

$$\boldsymbol{v}_2 \cdot \boldsymbol{v}_2 = 1.$$

By differentiation of these last two equations with respect to $s$, we obtain

$$\boldsymbol{v}_2 \cdot \boldsymbol{v}_{2s} = 0$$

and

$$\kappa_1 + \boldsymbol{v}_1 \cdot \boldsymbol{v}_{2s} = 0,$$

which prove the vector $\boldsymbol{v}_{2s}$ has a component along $\boldsymbol{v}_1$ equal to $-\kappa_1$ and another component orthogonal to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. In consequence, it is of the form

$$\boldsymbol{v}_{2s} = -\kappa_1\boldsymbol{v}_1 + \kappa_2\boldsymbol{v}_3,$$

where

$$\kappa_2 = |\boldsymbol{v}_{2s} + \kappa_1\boldsymbol{v}_1|$$

and

$$\boldsymbol{v}_3 = \frac{1}{\kappa_2}(\boldsymbol{v}_{2s} + \kappa_1 \boldsymbol{v}_1),$$

provided that the "second" curvature function $\kappa_2$ of $\mathbf{C}$ does not vanish.

By induction, it is thus possible to define the $i_{\text{th}}$ curvature

$$\kappa_i = |\boldsymbol{v}_{is} + \kappa_{i-1}\boldsymbol{v}_{i-1}|$$

and the $(i+1)_{\text{th}}$ vector of the generalised Frenet frame

$$\boldsymbol{v}_{i+1} = \frac{1}{\kappa_i}(\boldsymbol{v}_{is} + \kappa_{i-1}\boldsymbol{v}_{i-1}).$$

The corresponding Frenet formula, needed in our computation would be

$$\boldsymbol{v}_{is} = -\kappa_{i-1}\boldsymbol{v}_{i-1} + \kappa_i \boldsymbol{v}_i.$$

These last results discouraged us to pursue in this interesting direction for two reasons. First, the computation of an intrinsic basis of the normal space requires high order differentiation ($n-1$ order) of the describing functions which, from the numerical standpoint, is reasonable for space curves but is not for greater dimensions. Second, this basis is defined only for "nicely" curved manifolds, *i.e.* when none of the curvatures vanishes. For example. it is simply undefined for a straight line in $\mathbb{R}^3$ and this is too limiting for real applications.

Nevertheless, using an intrinsic set of transverse hypersurfaces is not the only possible solution to the problem we are considering. An alternative approach is to consider we are given "out of the blue" a set of initial transverse hypersurfaces whose intersections define a manifold of lower dimension (in applications, initial manifolds are often very simple analytical manifolds for which it is possible to guess such tranverse hypersurfaces). The problem is then to study how to evolve properly these initial hypersurfaces.

# 4 Arbitrary transverse hypersurfaces

## 4.1 The case of space curves

Alternatively then, we suppose (any) two orthogonal surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ are given implicitly in the form of their oriented distance functions $u_1$ and $u_2$. Their intersection is a curve. These two surfaces are not intrinsic to this curve since another pair of surfaces $(\mathbf{S}'_1, \mathbf{S}'_2)$ possibly intersect at the same curve. The question is then to find *any* evolution of these two surfaces which meets the following two requirements: (i) The surfaces stay orthogonal (ii) Their intersection is a curve evolving under mean curvature flow.

As explained in section 2, by "finding an evolution of $\mathbf{S}_i$", we mean the design of a velocity function $\beta_i : \mathbf{S}_i \to \mathbb{R}$. In [28, 15], it is shown that if $\mathbf{S}_i(t)$ is modelled implicitly by its oriented distance function $u_i(.,t)$, then the relation

$$\frac{\partial u_i}{\partial t} = \beta_i(x - u_i \nabla u_i), \quad i = 1, 2\,, \tag{16}$$

holds. It is the relation between the evolutions of $\mathbf{S}_i$ and $u_i$. The two scalar velocity functions $\beta_i,\ i = 1, 2$ are the unknowns of our problem since, once known, the two equations (16) define what we called the $\mathbf{u}$ evolution. The scalar function $\mathbf{x} \to \beta_i(\mathbf{x} - u_i \nabla u_i)$ will be denoted by $b_i$. The functions $\beta_i$ and $b_i$ are of different nature ($\beta_i$ is defined over $\mathbf{S}_i$ whether $b_i$ is defined in $\mathbb{R}^3$) but are closely related since $b_i$ is a continuation of $\beta_i$ in $\mathbb{R}^3$.

We formalise these ideas by writing the following set of equations

$$\left\{\begin{array}{ll} \dfrac{\partial \mathbf{C}}{\partial t} = \kappa \mathbf{N} & \text{(i)} \\[2ex] \mathbf{u}\big(\mathbf{C}(p,t), t\big) = 0,\ \forall t & \text{(ii)} \\[2ex] \big(\nabla u_i \cdot \nabla u_j\big)_{|\mathbf{C}} = 0,\ \forall i \neq j & \text{(iii)} \\ \dfrac{\partial \mathbf{S}_i}{\partial t} = \beta_i \mathbf{N}_i, & \text{(iv)} \end{array}\right. \tag{17}$$

where $\mathbf{S}_i = u_i^{-1}(0)$, $\mathbf{C} = \mathbf{S}_1 \cap \mathbf{S}_2$ and initial conditions are given (*i.e.* $\mathbf{u}(., t = 0)$ is known). The interpretation of these equations follows easily from the previous discussion and the unknowns are the velocity functions $\beta_i : \mathbf{S}_i \to \mathbb{R}$.

We find necessary conditions by calculating

- the first order derivative of (17ii) with respect to $t$: it provides a relation between $\beta_{i|\mathbf{C}}$ and the curvature of $\mathbf{C}$.
- the second order derivative of (17ii) with respect to $p$: it provides relations between the curvature of $\mathbf{C}$ and the spatial derivatives of $\mathbf{u}$ up to the second order.
- the first order derivative of (17iii) with respect to $t$: it provides a relation between $\nabla\beta_1$ and $\nabla\beta_2$.

In detail, we obtain for points of $\mathbf{C}$

$$\frac{\partial u_i}{\partial t} = \kappa \mathbf{N} \cdot \nabla u_i \tag{18}$$

and

$$\frac{\partial \mathbf{C}}{\partial p} \cdot \mathbf{D^2} u_i \, \frac{\partial \mathbf{C}}{\partial p} - \kappa \mathbf{N} \cdot \nabla u_i = 0, \tag{19}$$

which yields the coordinates of the mean curvature vector $\kappa\mathbf{N}$ in the orthonormal basis $(\nabla u_1,\, \nabla u_2)$. This, along with (17iv), gives $\beta_i$ over $\mathbf{C} \subset \mathbf{S}_i$ (*cf.* Fig. 4) by the formula

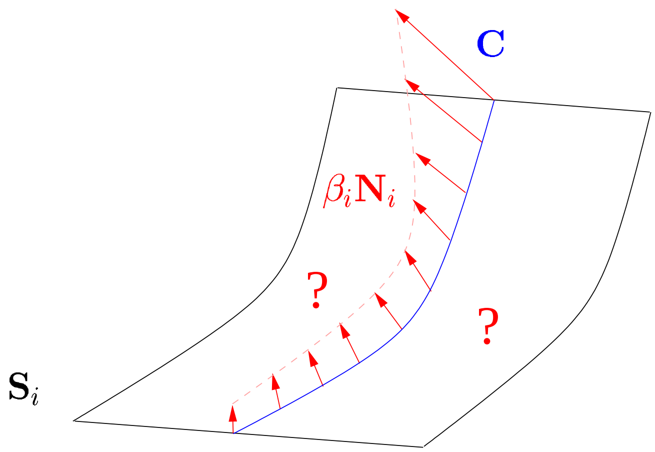



Figure 4: The equation (22) gives the value of $\beta_i$ over the curve $\mathbf{C}$ which is traced on $\mathbf{S}_i$. The questions marks remind us that $\beta_i$ is not only defined over $\mathbf{C}$ but over $\mathbf{S}_i$.

$$\beta_i(\mathbf{C}) = \left( \frac{\partial \mathbf{C}}{\partial p} \cdot \mathbf{D^2} u_i \, \frac{\partial \mathbf{C}}{\partial p} \right)_{|\mathbf{C}} \tag{20}$$

$$= \left( \Delta u_i - \nabla u_1 \cdot \mathbf{D^2} u_i \, \nabla u_1 - \nabla u_2 \cdot \mathbf{D^2} u_i \, \nabla u_2 \right)_{|\mathbf{C}} \tag{21}$$

$$= \left( \Delta u_i - \nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j \right)_{|\mathbf{C}}, j \neq i \tag{22}$$

since $u_i$ is a distance function and consequently $\nabla u_i \cdot \mathbf{D^2} u_i \, \nabla u_i = 0$.

Next, the third differentiation yields the linear equation

$$\nabla \beta_i \cdot \nabla u_j + \nabla \beta_j \cdot \nabla u_i + \beta_i \nabla u_i \cdot \mathbf{D^2} u_j \, \nabla u_i + \beta_j \nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j = 0, \; i \neq j, \tag{23}$$

which is a relation between $\nabla \beta_i$ and $\nabla \beta_j$ at points of $\mathbf{C}$, and therefore informs us on the way $\beta_i$ and $\beta_j$ should be extended from $\mathbf{C}$ to the whole surfaces $\mathbf{S}_i$ and $\mathbf{S}_j$. The existence of such a relation is a consequence of the fact that the two surfaces need coupled motions in order to remain tranverse.

We propose an interpretation of the terms of this equation.

Since, at points of $\mathbf{C}$, the normal vector $\nabla u_j$ is orthogonal to $\mathbf{C}$ and tangent to $\mathbf{S}_i$, the quantity $\nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j$ is the normal curvature of $\mathbf{S}_i$ in the direction orthogonal to $\mathbf{C}$. In particular, if the surface $\mathbf{S}_i$ is ruled orthogonally to $\mathbf{C}$, then this quantity is null.

More over, notice that, at points of $\mathbf{C}$, $\nabla \beta_i$ is a tangent vector of $\mathbf{S}_i$ which can be decomposed in a component tangent to the curve and another one normal to the curve. Besides, since the value of $\beta_i$ is known along $\mathbf{C}$ by (22), so is the component $\nabla \beta_i \cdot \mathbf{T}$ of $\nabla \beta_i$ tangent to $\mathbf{C}$. So, only the component of $\nabla \beta_i$ normal to the curve is unknown and the true unknowns of the system are the two scalars $\nabla \beta_i \cdot \nabla u_j$ and $\nabla \beta_j \cdot \nabla u_i$. We interpret these

two quantities as the velocities of the "rotations" of $\mathbf{S}_i$ and $\mathbf{S}_j$ around $\mathbf{C}$. Indeed, consider a point $\boldsymbol{x} \in \mathbf{C}$ and a curve $\boldsymbol{\gamma}_i$ of $\mathbf{S}_i$ tangent to $\nabla u_j$ and passing through $\boldsymbol{x}$ (any such curve will do but the most natural choice is the corresponding geodesic curve of $\mathbf{S}_i$ (*cf* Fig.5)). We parameterise $\boldsymbol{\gamma}_i$ by arc length, starting at $\boldsymbol{x}$.
The motion of this curve is given by

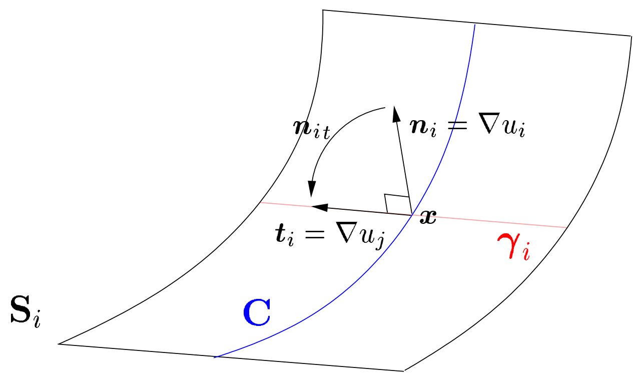



Figure 5: This figure depicts the rotation of the curve $\boldsymbol{\gamma}_i$ around $\mathbf{C}$ (*i.e.* around $\nabla u_1 \times \nabla u_2$), see text.

$$\boldsymbol{\gamma}_{it} = \alpha_i \boldsymbol{n}_i,$$

where $\alpha_i(0) = \beta_i(\boldsymbol{x})$ and $\boldsymbol{n}_i = \nabla u_i$.

Next, we differentiate this last relation with respect to $s$ and use once again (10) in order to obtain

$$\boldsymbol{n}_{it} = \alpha_i' \boldsymbol{t}_i,$$

where $\boldsymbol{t}_i = \nabla u_j$ and $\alpha_i'(0) = \nabla \beta_i(\boldsymbol{x}) \cdot \nabla u_j(\mathbf{x})$, which quantifies the rotation velocity of $\boldsymbol{\gamma}_i$ around the tangent to $\mathbf{C}$, $\boldsymbol{n}_i \times \boldsymbol{t}_i$.

Hence, if we consider also the curve $\boldsymbol{\gamma}_j$ of $\mathbf{S}_j$, the equation (23) can be written

$$\alpha_i' + \alpha_i \kappa_j + \alpha_j' + \alpha_j \kappa_i = 0,$$

where $\alpha_i$ is the velocity of the curve $\boldsymbol{\gamma}_i$, $\kappa_i$ is its curvature (and similar definitions for the index $j$). The two scalar unknowns are $\alpha_i'$ and $\alpha_j'$. This equation means that the two surfaces should turn around $\mathbf{C}$ with rotation velocities which are related to one another but also depend upon the curvatures of $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_j$: this will maintain the transversality (even well, the orthogonality) of the surfaces. Notice that if $\kappa_i = \kappa_j = 0$, *i.e.* if $\mathbf{S}_i$ and $\mathbf{S}_j$ are ruled orthogonally to $\mathbf{C}$, the equation just says that the surfaces should turn with the same rotation velocity. There is a one-parameter family of solutions to this affine equation

$$\alpha_i' = r - c$$
$$\alpha_j' = -r - c,$$

where $c = \frac{\alpha_i \kappa_j + \alpha_j \kappa_i}{2}$ and $r \in \mathbb{R}$.

We choose the trivial and symmetric solution $r = 0$ because it is simple (this choice corresponds to the "laziest" possible motion of the two surfaces).

Setting

$$\nabla \beta_i \cdot \nabla u_j = \nabla \beta_j \cdot \nabla u_i = -c, \quad i \neq j, \tag{24}$$

we extend $\beta_i(\boldsymbol{x})$ along $\boldsymbol{\gamma}_i$ with slope $c$, *cf* Fig.6.

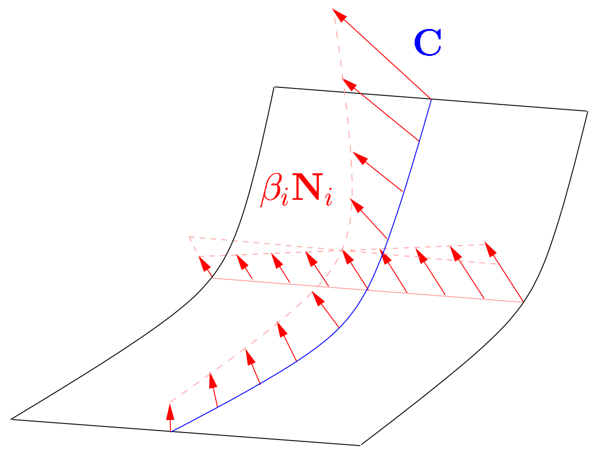



Figure 6: The equation (24) expresses how to extend the value of $\beta_i$ from $\mathbf{C}$ to the whole surface.

Putting it all together, $\beta_i$ is first estimated over $\mathbf{C}$ by (22), then over the whole surface $\mathbf{S}_i$ by (24), and finally it is propagated from $\mathbf{S}_i$ to the whole embedding space in order to obtain $b_i$ by

$$b_i(\boldsymbol{x}) = \beta_i(\boldsymbol{x} - \boldsymbol{u}(\boldsymbol{x})).$$

A practical way to construct the scalar field $b_i$ is to solve the following first order system (at each time instant $t$ and for $i = 1, 2$) until it reaches a stationary solution

$$\left\{ \begin{array}{llr} b_i = \Delta \mathbf{u} - \nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j & \text{, if } u_i = 0 \text{ and } u_j = 0, & \text{(a)} \\ \dfrac{\partial b_i}{\partial \tau} = sgn(u_j) \left( -c - \nabla b_i \cdot \nabla u_j \right) & \text{, if } u_i = 0 \text{ and } u_j \neq 0, & \text{(b)} \\ \dfrac{\partial b_i}{\partial \tau} = -\nabla b_i \cdot \nabla u_i & \text{, if } u_i \neq 0 \text{ and } u_j \neq 0, & \text{(c)} \end{array} \right. \tag{25}$$

where $\tau$ is an auxiliary "time" variable and

$$c = \frac{1}{2} \left( \beta_i \nabla u_i \cdot \mathbf{D^2} u_j \, \nabla u_i + \beta_j \nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j \right).$$

The interpretation of this system is as follows. The purpose of equation (a) is to compute $b_i$ over the curve $\mathbf{C}$ (*i.e.* $u_i = 0$ and $u_j = 0$). The equation[1] (b) propagates the values of $b_i$ from $\mathbf{C}$ to the whole surface $\mathbf{S}_i$ (*i.e.* $u_i = 0$ and $u_j \neq 0$) by varying these values with a slope equal to $-c$. The transport equation (c) propagates the values of $b_i$ to the whole space (*i.e.* $u_i \neq 0$ and $u_j \neq 0$). Remark (a) is the initial condition for (b), which is itself the initial condition for (c).

## 4.2 Generalisation to any dimension

We shall finish this section by a discussion on some aspects of the generalisation of the previous equations to the case of a manifold $\mathcal{M}$ of dimension $k$ embedded in $\mathbb{R}^n$. We focus on a particular hypersurface among the $n-k$ ones, for instance the $i^{\text{th}}$.

The equation (22) is generalised trivially by subtracting from $\Delta u_i$ all the normal components $\nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j, \forall j \neq i$. Indeed, the component of the mean curvature vector $\boldsymbol{\mathcal{H}}$ of $\mathcal{M}$ along $\nabla u_i$ is related to the Hessian of $u_i$ by

$$\boldsymbol{\mathcal{H}} \cdot \nabla u_i = \sum_{\ell=1,\cdots,k} \boldsymbol{T}_\ell \cdot \mathbf{D^2} u_i \, \boldsymbol{T}_\ell,$$

where the $\boldsymbol{T}_\ell$'s form an orthonormal basis of the tangent plane of $\mathcal{M}$.
Besides, since the matrix $\mathbf{D^2} u_i$ is symmetric we have

$$\Delta u_i = \sum_{\ell=1,\cdots,k} \boldsymbol{T}_\ell \cdot \mathbf{D^2} u_i \, \boldsymbol{T}_\ell + \sum_{m=1,\cdots,n-k} \boldsymbol{N}_m \cdot \mathbf{D^2} u_i \, \boldsymbol{N}_m,$$

where the $\boldsymbol{N}_m$'s form *any* orthonormal basis of the orthogonal complement of the tangent plane of $\mathcal{M}$. By choosing the $n-k$ vectors $\nabla u_m$'s for this purpose and remembering that $\nabla u_i \cdot \mathbf{D^2} u_i \, \nabla u_i = 0$, one proves that

$$b_i(\mathbf{C}) = \Delta u_i - \sum_{j \neq i} \nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j.$$

As depicted in Fig.4, this last equation defines the value of $\beta_i$ only over a $k$-dimensional submanifold of the hypersurface $\mathbf{S}_i$ and it remains to extended this values in $n-k-1$ directions to "fill" $\mathbf{S}_i$. As far as equation (23) is concerned, it is replaced by the system of $\frac{(n-k)(n-k-1)}{2}$ equations (it is the number of pairs of hypersurfaces) and $(n-k)(n-k-1)$ unknowns

$$\nabla \beta_i \cdot \nabla u_j + \nabla \beta_j \cdot \nabla u_i + \beta_i \nabla u_i \cdot \mathbf{D^2} u_j \, \nabla u_i + \beta_j \nabla u_j \cdot \mathbf{D^2} u_i \, \nabla u_j = 0, \; \forall i \neq j, \qquad (26)$$

[1] This equation plays the same role as the classic equation $\frac{\partial v}{\partial t} = sgn(v)\,(1 - |\nabla v|)$ in the reinitialisation of the oriented distance functions. Here it is not used for this purpose but for building a function with slope $-c$ instead of 1.

which correspond to the fact that each one of the $\beta_i$'s has to be extended in its $n-k-1$ tangential directions $\nabla u_j, \forall j \neq i$ so that each pair of the hypersurfaces remains orthogonal. There are $\frac{(n-k)(n-k-1)}{2}$ degrees of freedom (one can choose the more symmetric solution as in the case of the curve) and it can be given a geometric interpretation by means of a generalisation of the "rotation" introduced in the previous study of curves. In the first place we have to understand what it means for a $(n-1)$-dimensional vector space to turn around a $k$-dimensional subspace of it. Such a notion is developed in [2] where general rotations in Euclidean spaces are considered.

In order to interpret it, we first define what is a *simple* rotation in $\mathbb{R}^n$ by considering two orthogonal unit vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$. The *simple* rotation corresponding to this *bivector* leaves invariant the $(n-2)$-dimensional space which is orthonormal to the bivector and performs a "regular" rotation in the two-dimensional plane $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$. In an orthonormal basis of $\mathbb{R}^n$ whose first two vectors are $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, this simple rotation can be represented by the $n \times n$ matrix

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & & & \\ \sin\alpha & \cos\alpha & & (0) & \\ & & 1 & & \\ (0) & & & \ddots & \\ & & & & 1 \end{pmatrix}, \tag{27}$$

where $\alpha$ is the angle of rotation.

It is demonstrated in [2] that a general rotation in $\mathbb{R}^n$ is the product of a certain number $\leq \frac{n}{2}$ of simple rotations whose bivectors are orthogonal to one another. Returning to the case of a $(n-k)$-dimensional manifold turning around a $k$-dimensional subspace of it, this fact implies the existence of an orthonormal basis of $\mathbb{R}^n$ whose $k$ first vectors are in $T_{\boldsymbol{x}}\mathcal{M}$ and remaining ones are in $N_{\boldsymbol{x}}\mathcal{M}$ such that the local rotation of $T_{\boldsymbol{x}}\mathbf{S}_i$ around $T_{\boldsymbol{x}}\mathcal{M}$ can be expressed by a $n \times n$ matrix of the form

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & (0) & & \\ & & & \cos\alpha_1 & -\sin\alpha_1 & & & & \\ & & & \sin\alpha_1 & \cos\alpha_1 & & & & \\ & & & & & \ddots & \ddots & & \\ & & (0) & & & & & \cos\alpha_{E(\frac{n-k}{2})} & -\sin\alpha_{E(\frac{n-k}{2})} \\ & & & & & & & \sin\alpha_{E(\frac{n-k}{2})} & \cos\alpha_{E(\frac{n-k}{2})} \end{pmatrix}. \tag{28}$$

Thus, this rotation is a product of $E(\frac{n-k}{2})$ simple rotations leaving invariant $T_{\boldsymbol{x}}\mathcal{M}$ (*cf* Fig.7). Besides, simple rotations corresponding to bivectors belonging entirely to $T_{\boldsymbol{x}}\mathbf{S}_i$ do not count since they describe a motion of $\mathbf{S}_i$ within itself (*i.e.* an alteration of the parameterisation) and do not change its geometry. Consequently, we consider only simple rotations which involve the normal vector $\nabla u_j$ in their bivector. A rotation of $\mathbf{S}_i$ which does not leave $T_{\boldsymbol{x}}\mathbf{S}_i$ invariant is thus exactly equal to one simple rotation whose bivector is formed by $\nabla u_j$ and another unit vector orthogonal to $\mathbf{C}$ and $T_{\boldsymbol{x}}\mathcal{M}$. Indeed, if this rotation were the composition of several simple rotations then all of them would need to have $\nabla u_j$ involved in their bivector which contradicts the orthogonal requirement of the bivectors. The description of such a rotation requires the choice of the second vector of the bivector ($n-k-2$ parameters are necessary to choose a unit vector in a $n-k-1$ dimensional space) and the choice of the angle (1 parameter) which makes $n-k-1$ parameters. To complete the geometric interpretation, each hypersurface has to remain orthogonal to the other $n-k-1$ hypersurfaces (we have $n-k-1$ constrains and $n-k-1$ degrees of freedom by hypersurface) and each constrain on a pair of hypersurfaces adds 1 parameter like in the case of the curve which makes a total of $\frac{(n-k)(n-k-1)}{2}$ degrees of freedom.

Returning to our natural choice (*i.e.* the more symmetric solution which we note $\nabla b_i \cdot u_j = \nabla b_j \cdot u_i = -c_{ij}$), it remains to generalise the equation (25b). The principle is to extend the values of $\beta_i$ from $\mathbf{C}$ to the whole hypersurface $\mathbf{S}_i$ by changing these values with the slopes $c_{ij}$. A straightforward generalisation is to solve the equation

$$\frac{\partial b_i}{\partial \tau} = \sum_{j \neq i} sgn(u_j)\left(-c_{ij} - \nabla b_i \cdot \nabla u_j\right). \tag{29}$$

To complete, the equation (25c) is already valid regardless of the dimension but $n-k$ such equations (one per hypersurface) have to be solved.

# 5 Conclusion

This paper has investigated a scheme to represent and evolve a $k$-dimensional manifold embedded in $\mathbb{R}^n$ as the intersection of $n-k$ transverse hypersurfaces. We have enlightened the difficulties of this approach and proposed a methodology to deal with them.

Nevertheless, *a posteriori*, this method appears to be unappropriated to deal in practice with general situations. Indeed, the construction and evolution of hypersurfaces in an *intrinsic* way requires high order differentiation of the involved functions which is, numerically speaking, not reasonable. A natural fix to this problem was to consider *arbitrary* transverse hypersurfaces (involving only second order differentiations). But, the presented results are local and would have to be implemented in a *narrow band* of the evolving manifold. This very last consideration discouraged us to pursue in this interesting direction because of the issue of topological changes. Indeed, if two initially disconnected manifolds are to be merged (which occurs quite often in practice) then the corresponding implicit descriptions have no reason to be smooth across the two narrow bands at the time they touch one another (since in each narrow band, the hypersurfaces are initially arbitrary!).

In addition, we believe it is also an important feature for a practical implicit representation to deal, in an homogeneous way, with embedded manifold of any dimension. It is not the case of the presented method since, for instance, the intersection of two tranverse

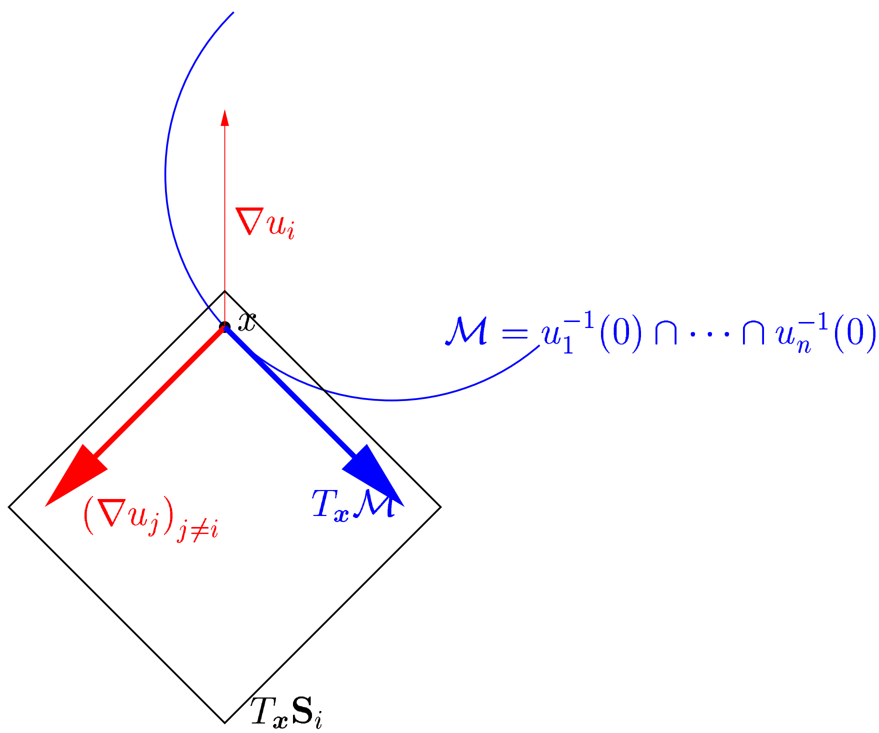



Figure 7: This figure depicts the general situation of a manifold $\mathcal{M}$ of dimension $k$ defined as the intersection of $n-k$ hypersurfaces $\mathbf{S}_i$ of $\mathbb{R}^n$. We consider a point $\boldsymbol{x}$ of $\mathcal{M}$. A privileged role is given in the figure to the $i^{\text{th}}$ hypersurface whose tangent linear space $T_{\boldsymbol{x}}\mathbf{S}_i$ is represented as an "horizontal" hyperplane. The manifold $\mathcal{M}$ is traced over $\mathbf{S}_i$ and is consequently tangent to $T_{\boldsymbol{x}}\mathbf{S}_i$: $\mathcal{M}$ is represented as a curve tangent to the "horizontal" plane. The vector $\nabla u_i$ is the unit normal vector to $\mathbf{S}_i$ and also to $T_{\boldsymbol{x}}\mathbf{S}_i$. The remaining $\nabla u_j$'s, $j \neq i$ are both normal to $\mathcal{M}$ and to $\nabla u_i$: the linear space spanned by these $\nabla u_j$'s is represented as a $(n-k-1)$-dimensional *multivector* of $T_{\boldsymbol{x}}\mathbf{S}_i$ orthogonal to $\mathcal{M}$. The $k$-dimensional linear space $T_{\boldsymbol{x}}\mathcal{M}$ is represented as a $k$-dimensional *multivector* of $T_{\boldsymbol{x}}\mathbf{S}_i$. The represented vector and multivectors form an orthonormal basis of $\mathbb{R}^n$.

surfaces of $\mathbb{R}^3$ would never be a single point. For this simple reason, two immersed curves intersecting at a point cannot be represented as the intersection of 2 surfaces when it is possible in the approach of Ruuth *et al.* [22] (thanks to the fact their describing function is not smooth). For the same reason, it is not possible to represent manifolds which have borders since the borders are $(k-1)$-dimensional manifolds.

It appears the moral of this story is that some important features of an ideal practical implicit representation should be

*i)* Smooth in the vicinity of the embedded manifold.

*ii)* Intrinsic to the manifold.

*iii)* Able to represent, in an homogeneous way, any manifold of dimension $1, \cdots, n-1$.

Future work in this direction should be to design a $n$-dimensional smooth vector function related (intrinsically) to the distance function of the embedded manifold.

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