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Abstract: We consider in this paper packets which arrive according to a Poisson process into a finite queue. Often, a group of consecutive packets forms a frame, and loss of one packet results in the loss of the whole frame. This is the case in ATM where a transport layer protocol (the AAL layer) is responsible for this grouping. Thus, packets of a frame, that arrive after a packet is lost from the same frame, have no use anymore, and it is advantageous to discard them. We derive exact expressions for the performance of such a policy and obtain simple approximations for the heavy traffic regime. Our analytical results may be quite useful in dimensioning the buffer size that is necessary for a given required bound on loss rate.

Key-words: PMD policy, Packet model, Goodput, Fluid queue

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Analyse de politique de rejet partiel de messages

Résumé : Nous considérons dans ce rapport des paquets qui arrivent à une file d'attente à capacité finie selon un processus de Poisson. Souvent dans les réseaux, un groupe de paquets contigus forment une trame, et la perte dún seul paquet dans la trame entraîne la perte de la trame entière. C'est le cas dans les réseaux ATM où un protocole de de la couche transport (dénommé AAL) est responsable de ce regroupement. Donc, les paquets d'une trame qui arrivent après une perte, qui a eu lieu dans cette même trame, sont inutiles et il est avantageux de les rejeter. Nous obtenons des des formules exactes pour les performances d'une telle politique de rejet et des approximations fluides pour le régime de forte charge. Nos résultats ont un intérêt pour dimensionner des tampons dans les réseaux qui garantiraient une borne sur le taux de pertes.

Mots-clés : Politique de rejet partiel, modèle paquets, files d'attente fluides

1 Introduction

Often, a set of consecutive packets are grouped into a frame, and loss of one packet results in the loss of the whole frame. This is the case in ATM where a transport layer protocol (known as AAL) is responsible for this grouping, see e.g. Chapter 5 in [7].

Thus, packets of a frame, that arrive after a packet is lost from the same frame, have no use anymore, and it is advantageous to discard them. This approach for discarding is known as the Partial Message Discard (PMD) policy. This policy as well as other discarding approaches have already been analysed in several previous papers [3, 4, 11, 6] and [9].

In [4, 11], the basic performance measure for the study of discarding policies is the *effective throughput*, which is the ratio of good packets on the outgoing link to the the total outgoing flow. However, as argued in [9], a more suitable performance measure is the *Goodput*, defined as the ratio of good packets out of the total number of packets that arrive at the network element's input.

The goal of this paper is to present *explicit expressions* for the goodput of the PMD policy. The first part of the paper considers a Markovian framework: a Poisson process of packet arrivals, geometrically distributed frame size, and exponentially distributed service times of packets. Explicit expressions for the queue size distribution and of the goodput are obtained based on recursions introduced in [9]. The second part of the paper derives approximations which are valid for heavy traffic conditions. The input process (which may be quite general) is approximated by a fluid with a constant rate. We obtain simple explicit expressions for the queue size distribution and of the goodput for the fluid approximation. Numerical examples for both the packet and the fluid analysis are provided.

Our analytical results may be quite useful in dimensioning the buffer size that should be used for a given required goodput. The explicit expressions are helpful in analytical studying the sensitivity of the goodput to different parameters for e.g., the message length, the buffer size, the load etc., which were studied numerically in earlier works [9], [6]. Also the fluid approximation is easily tractable and also has the potentials for similar analysis for more sophisticated policies, for e.g., Early Message Discard (EMD).

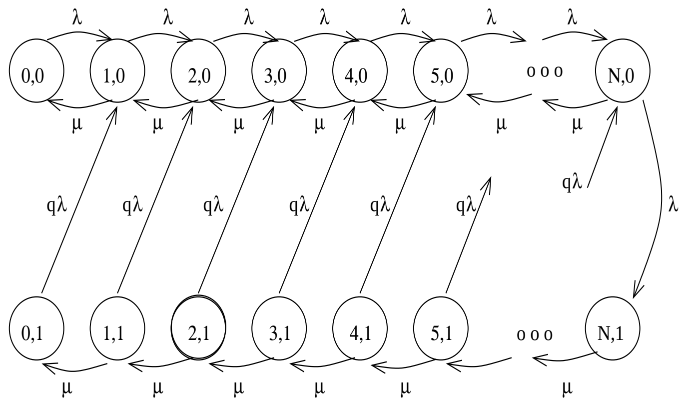

Figure 1: Transition structure under the PMD policy

2 Packet Model

The packet model is the same as the one proposed in [9]. We first describe the model in brief. In terms of packet the network element is a $M/M/1/N$ queue with the arrival rate λ and the service rate μ and the load $\rho = \lambda/\mu$. A message length (in terms of packets) is considered to be geometrically distributed with parameter q. In PMD policy, if a packet arrives when the queue is full, it is discarded and all the subsequent packets belonging to the same message are also discarded, irrespective of the state of the queue upon their arrival epochs, until the head-of-message packet (i.e., a new message) arrives. To model the policy, two modes for working of the network element are defined: the *normal mode*, in which packets are admitted, and the *discarding mode*, in which arriving packets are discarded. The state transition diagram for PMD policy under this model is shown in Figure (1). Let $P_{i,j}(0 \leq i \leq N, j = 0, 1)$ be the steady-state probability of having i packets in the system and the system is in mode $j (j = 0$ for normal; $j = 1$ for discarding). Thus, we have the following set of equations for the steady-state probabilities [9] from Figure (1).

$$\rho P_{0,0} = P_{1,0} \tag{1}$$

$$q\rho P_{0,1} = P_{1,1} \tag{2}$$

$$\begin{aligned} (\rho + 1)P_{i,0} &= \rho P_{i-1,0} + P_{i+1,0} + q\rho P_{i-1,1} \\ &\quad \text{for } 1 \leq i \leq N-1 \end{aligned} \tag{3}$$

$$
\begin{aligned}
(q\rho+1)P_{i,1} &= P_{i+1,1} \\
&\quad \text{for } 1 \le i \le N-1
\end{aligned} \tag{4}
$$

$$
(\rho+1)P_{N,0} = \rho P_{N-1,0} + q\rho P_{N-1,1} \tag{5}
$$

$$
P_{N,1} = \rho P_{N,0} \tag{6}
$$

$$
\sum_{i=0}^{N}(P_{i,0}+P_{i,1}) = 1 \tag{7}
$$

We define the transform function $Q_j(z)$($j = 0$ for normal mode and $j = 1$ for discarding mode) as follows,

$$
Q_j(z) = \sum_{i=0}^{N} z^i P_{i,j}
$$

and,

$$
Q(z) = Q_0(z) + Q_1(z) = \sum_{i=0}^{N}(P_{i,0}+P_{i,1})z^i
$$

2.1 PGF and distribution of the number of packets in the queue

Proposition 1 *The probability generating function $Q_j(z)$ is given by*

$$
\begin{aligned}
Q_0(z) &= P_{0,1}q\left[\rho^{-1}\frac{1-(\rho^{-1}(1+\rho q))^N}{1-\rho^{-1}(1+\rho q)}\right. \\
&\quad \left. + \sum_{i=1}^{N}\frac{\left[1-(\rho^{-1}(1+\rho q))^{N-i+1}\right](1+\rho q)^{i-1}}{1-\rho^{-1}(1+\rho q)} z^i\right]
\end{aligned} \tag{8}
$$

$$
Q_1(z) = P_{0,1}\frac{(1-z-z^{N+1}(q\rho+1)^N q\rho)}{1-z(q\rho+1)} \tag{9}
$$

with

$$
P_{0,1} = \frac{(1-\rho)(\rho(1-q)-1)}{D}
$$

and

$$
D = q(1-\rho^{-N}(1+\rho q)^N) + (\rho(1-q)-1)(1-\rho(1+\rho q)^N)
$$

The proof is given in the Appendix.

By taking the inverse *z-transform* of Equation (8) and (9) we obtain:

Corollary 1 *The steady state probabilities are given by,*

$$P_{0,0} = \frac{P_{0,1}q}{\rho}\frac{1-\left(\rho^{-1}(1+\rho q)\right)^N}{1-\rho^{-1}(1+\rho q)}$$

for $1 \le i \le N$,

$$P_{i,0} = P_{0,1}q\frac{\left[1-\left(\rho^{-1}(1+\rho q)\right)^{N-i+1}\right](1+\rho q)^{i-1}}{1-\rho^{-1}(1+\rho q)}$$

and,

$$P_{0,1} = \frac{(1-\rho)(\rho(1-q)-1)}{D}$$

and, for $1 \le i \le N$,

$$P_{i,1} = q\rho(1+q\rho)^{i-1}P_{0,1}$$

2.2 An equivalent vacation model

Next, we propose an interpretation of our model (denoted by P) as equivalent to a dual vacation model (denoted by P_d). This interpretation will especially be useful later, in considering the case of large buffers. Let $A_{t,N}$ be the number of packets in the P model at time t. We define a random variable, Y_t as,

$$Y_{t,N} = N - A_{t,N}.$$

$Y_{t,N}$[1] will be the number of packets in the equivalent vacation model. In other words, the number of packets in the vacation model equals to the number of vacant places in the original queue model. It thus follows that the service times in the vacation model are i.i.d. exponentially distributed with parameter λ, and that the arrival process to the vacation model has Poisson distributed with parameter μ. We note, however, that in the original process, arrivals are stopped during the "discarding period". This "discarding period" will correspond to a "vacation period" in the dual model. More precisely, we define a "discarding period" in the original model P as

[1] N in subscript is for the buffer size N

the duration from the instant that the buffer fills, till the next time the event B^1 occurs, where $B^1 = \{$ service occurs and the next packet to arrive after that service is not discarded$\}$.

Note that with this definition, the number of packets discarded during a "discarding period" may be zero. Indeed, with probability $\mu/(\lambda+\mu) = 1/(1+\rho)$, the first packets that arrives after the queue fills, will find the queue non-full and will not be discarded, and there are zero discarded packets in the "discarding period".

A particular important quantity in the equivalent vacation model is the distribution of the number of arrivals during a vacation denoted by V, or alternatively, the number of service times during a discarding period in the original model. Note that by definition of the original "discarding period", $V \geq 1$.

Let T denote the instant of the beginning of a discarding period, and let S denote the instant when the first service time occurs after T.

Let V_1 (V_2, resp.) be a RV distributed as V given that at time S, the message that is being transmitted is bad (good, resp.). In other words, $V =_d V_2$ iff the next packet to arrive after time S is not discarded. The latter occurs iff the following event B^2 occurs; $B^2 = \{$either there is no arrival during the interval (T, S), or there is at least one arrival but the last arrival in that interval is the last packet of a message$\}$. Let

$$\beta := \frac{(1-q)\rho}{1+\rho} \text{ thus } 1-\beta = \frac{1+q\rho}{1+\rho}$$

Thus $V =_d V_1$ with probability β and $V =_d V_2 = 1$ with probability $1-\beta$.

Next we study the distribution of V_1. V_1 equals in distribution to one plus $\mathcal{A} :=$ the number of services that occur during the duration of M arrivals, where M is distributed according to a geometric distribution with parameter q.

Let $B(L)$ be the number of services in a random duration L. Then $B_L^*(z)$, the PGF of $B(L)$, is given by

$$\begin{aligned} B_L^*(z) &= E[\sum_{i=0}^{\infty} e^{-\mu L} \frac{(\mu L)^i}{i!} z^i] \\ &= E[e^{-\mu L(1-z)}] = L^*(\mu(1-z)) \end{aligned}$$

where $L^*(.)$ is the LST of L. Thus we need to evaluate $L^*(.)$.

$$\begin{aligned} L^*(s) = E[e^{-sL}] &= E[E[e^{-sT_i}]^M] \\ &= E[(T^*(s))^M] = \mathcal{N}(T^*(s)) \end{aligned}$$

where $T^*(.)$ is LST of an interarrival time ($\sim \exp\lambda$) and $\mathcal{N}(.)$ is the PGF of a geometrically distributed r.v. with parameter q. Observe that,

$$T^*(s) = \frac{\lambda}{\lambda + s} \text{ and } \mathcal{N}(z_1) = \frac{qz_1}{1-(1-q)z_1}$$

Thus, if we denote by $\alpha(z)$ the PGF of $\mathcal{A}$, then

$$\begin{aligned} \alpha(z) = &= \mathcal{N}(T^*(\mu(1-z))) = \mathcal{N}\left(\frac{\lambda}{\lambda+\mu(1-z)}\right) \\ &= \frac{q}{\rho^{-1}(1-z)+q} \end{aligned}$$

The PGF of V is given by $z((1-\beta)+\beta\alpha(z))$.

2.3 The case of large buffer

We use now the interpretation proposed in Subsection 2.2 as the dual of a vacation model in order to study the behavior of our system as the buffer size N becomes large. Clearly, non-trivial distribution of $Y_{t,N}$ is obtained in the limit $N \to \infty$ only in the heavy traffic regime $\rho > 1$.

Observe that,

$$\lim_{N\to\infty} P(N - A_{t,N} = k) = \lim_{N\to\infty} P(Y_{t,N} = k) = P(Y_t = k).$$

In [5] (see also [10]) the authors have shown that the stationary number of customers present in a M/G/1 queueing system with generalized server vacation is a convolution of the distribution function of two independent positive random variables (*stochastic decomposition*), one of which being the stationary distribution of the number of customers in an ordinary M/G/1 queueing system without server vacations. The other corresponds to the PGF of the number of packets during an arbitrary moment in a vacation. Let $\phi(.)$ and $\pi(.)$ be the p.g.f. for the stationary distribution of the number of customers at a random point in time in the in the vacation system and in the standard M/G/1 queueing system, respectively. Also, let $\hat{\alpha}(.)$ denote the PGF of the random variable V (i.e., the number of customers that arrive during a vacation). Then, with arrival rate μ (service rate in PMD queue) and departure rate λ (arrival rate in PMD queue), and $\rho = \frac{\lambda}{\mu}$, we have from [5],

$$\phi(z) \quad = \quad \frac{1-\hat{\alpha}(z)}{\dot{\hat{\alpha}}(1)(1-z)}\pi(z)$$

with,

$$\pi(z) \;=\; \frac{(1-\rho^{-1})(1-z)B^*(\mu-\mu z)}{B^*(\mu-\mu z)-z}$$

where, $B^*(.)$ is the Laplace transform of the service time pdf. For our $M/M/1/V$ case, $\pi(z)$ simplifies to,

$$\pi(z) \;=\; \frac{1-\rho^{-1}}{1-\rho^{-1}z}$$

Using the result of the previous subsection, and inverting the PGF $\phi(z)$, we get finally:

$$P(Y=k) \;=\; \frac{q(\rho-1)}{(\rho(1-q)-1)}(\rho(1-q)(1+\rho q)^{-(k+1)}-\rho^{-(k+1)})$$

One can now check that this is indeed the limit obtained as $\lim_{N\to\infty} P(A_N = N-k)$ from Corollary 1. We have, for $1 \le k \le N-1$,

$$\begin{aligned}
\lim_{N\to\infty} P(A_N = N-k) &= \lim_{N\to\infty}(P_{N-k,0}+P_{N-k,1}) \\
&= \lim_{N\to\infty}\frac{1}{D}\left[\rho(\rho-1)q(1+q\rho)^{N-(k+1)}(\rho(1-q)-\rho^{-(k+1)}(1+\rho q)^{-(k+1)})\right] \\
&= \frac{q(\rho-1)}{(\rho(1-q)-1)}(\rho(1-q)(1+\rho q)^{-(k+1)}-\rho^{-(k+1)})
\end{aligned}$$

2.4 Goodput Ratio: $\mathcal{G}$

The goodput is defined in [9] as the ratio between total packets comprising good messages exiting the system and the total arriving packets at its input. Let $\mathcal{W}$ be the random variable that represents the length(number of packets) of an arriving message. Let $\mathcal{V}$ be the random variable representing the success of a message, $\mathcal{V}=1$ for a good message, and $\mathcal{V}=0$ for a message which has one or more dropped packets. Then $\mathcal{G}$ can be expressed as (see [9])

$$\mathcal{G} \;=\; q\sum_{n=1}^{\infty} nq(1-q)^{n-1}\sum_{i=0}^{N} P(\mathcal{V}=1|\mathcal{W}=n,\mathcal{Q}=i)P(\mathcal{Q}=i) \tag{10}$$

Denote the conditional probabilities $S_{n,i} \stackrel{\triangle}{=} P(\mathcal{V}=1|\mathcal{W}=n,\mathcal{Q}=i)$. In [9], recursions for evaluating these probabilities and hence $\mathcal{G}$ were given. We will present here an explicit expression for $\mathcal{G}$. To do this we will use the multidimensional generating function for probabilities $S_{n,i}$ which was obtained in a different context in [1] and also in [8]. By some abuse of notation let us denote by $\bar{S}_i(x)$ as the generating function for probabilities $S_{n,i}$, $1 \leq n \leq \infty$ for a fixed i and by $\bar{S}_n(y)$ as the generating function for probabilities $S_{n,i}$, $0 \leq i \leq N$ for a fixed n, i.e.,

$$\bar{S}_i(x) = \sum_{n=1}^{\infty} S_{n,i} x^{n-1} \quad \text{and} \quad \bar{S}_n(y) = \sum_{i=0}^{N} S_{n,i} y^i$$

We define the generating function of $S_{n,i}$ for $1 \leq n \leq \infty$ and $0 \leq i \leq N$, as $\bar{S}(x,y)$, i.e.,

$$\begin{aligned}
\bar{S}(x,y) &= \sum_{i=0}^{N} \bar{S}_i(x) y^i = \sum_{i=0}^{N} y^i \sum_{n=1}^{\infty} S_{n,i} x^{n-1} \\
&= \sum_{i=0}^{N} \sum_{n=1}^{\infty} S_{n,i} x^{n-1} y^i = \sum_{n=1}^{\infty} \sum_{i=0}^{N} S_{n,i} y^i x^{n-1} \\
&= \sum_{n=1}^{\infty} \bar{S}_n(y) x^{n-1}
\end{aligned}$$

Proposition 2 *The probability generating function $\bar{S}(x,y)$ can be expressed as*

$$\bar{S}(x,y) = \sum_{i=0}^{N} c_i(x) y^i$$

where, for $0 \leq i \leq N-1$,

$$\begin{aligned}
c_i(x) &= 1 + K_3 \left(A_1 - A_2 y_1^{N-(i+1)} - A_3 y_2^{N-(i+1)} \right) \\
&\quad + K_4 \left(B_1 y_1^{N-i} + B_2 y_2^{N-i} \right)
\end{aligned}$$

and for $i = N$,

$$c_i(x) = 0$$

and,

$$
\begin{aligned}
y_{1,2} &= \frac{1+\rho \pm \sqrt{(1+\rho)^2 - 4\rho x}}{2} \\
K_3 &= -x\rho \\
K_4 &= \frac{x\rho(y_1^N - y_2^N)}{y_2^{N+1}(y_1-\rho) - y_1^{N+1}(y_2-\rho)} \\
A_1 &= \frac{1}{(1-y_1)(1-y_2)} \\
A_2 &= \frac{1}{(1-y_1)(y_1-y_2)} \\
A_3 &= \frac{1}{(1-y_2)(y_2-y_1)} \\
B_1 = -B_2 &= \frac{1}{y_1-y_2}
\end{aligned}
$$

Proof: From [1] and [8], we have,

$$
\begin{aligned}
[(1-\alpha y)\alpha y - x\rho\alpha^2]\bar{S}(x,y) &= \frac{1-y^N}{1-y}(1-\alpha y)\alpha y - x\rho\alpha^2(\alpha y)^{N+1}K_1 \\
&\quad + x\alpha^2(y-\rho)K_2
\end{aligned} \tag{11}
$$

with,

$$
K_1 = \frac{\alpha^{-(N+1)}(y_1^N - y_2^N)}{y_2^{N+1}(y_1-\rho) - y_1^{N+1}(y_2-\rho)}
$$

$$
\begin{aligned}
K_2 = \frac{1}{(y_1-\rho)(y_2-\rho)} \Big[&-1 + y_1^N \\
&+ \frac{\rho y_1^{N+1}(y_2-\rho)(y_1^N - y_2^N)}{y_2^{N+1}(y_1-\rho) - y_1^{N+1}(y_2-\rho)} \Big]
\end{aligned}
$$

where, y_1 and y_2 are the roots of the equation $(1-\alpha y)\alpha y - x\rho\alpha^2 = 0$ i.e.,

$$
y_{1,2}(x) = \frac{1+\rho \pm \sqrt{(1+\rho)^2 - 4\rho x}}{2}
$$

Also, observe that $y_1 + y_2 = 1 + \rho$ and $y_1 y_2 = \rho x$. We will now represent $\bar{S}(x, y)$ as $\sum_{i=0}^{N} c_i(x) y^i$. Thus,

$$\begin{aligned}
\bar{S}(x,y) &= \frac{1-y^N}{1-y} - x\rho\frac{(1-y^N)}{(1-y)(y-y_1)(y-y_2)} \\
&+ x\rho\alpha^{N+1}K_1\frac{y^{N+1}}{(y-y_1)(y-y_2)} \\
&- xK_2\frac{(y-\rho)}{(y-y_1)(y-y_2)}
\end{aligned}$$

Again we denote $-x\rho = K_3$, $x\rho\alpha^{N+1}K_1 = K_4$ and $-xK_2 = K_5$. We shall now apply the partial fraction method and express the right side of the last equation in the form of $\frac{y^k - a^k}{(y-a)}$ for some k and a. Also, denote

$$G_1(y) = \frac{1-y^N}{1-y}, \quad G_2(y) = \frac{(1-y^N)}{(1-y)(y-y_1)(y-y_2)},$$

$$G_4(y) = \frac{(y-\rho)}{(y-y_1)(y-y_2)}, \quad G_3(y) = \frac{y^{N+1}}{(y-y_1)(y-y_2)},$$

Now, by partial fraction approach,

$$\begin{aligned}
G_2(y) &= (1-y^N)\left(\frac{A_1}{1-y} + \frac{A_2}{y-y_1} + \frac{A_3}{y-y_2}\right) \\
G_3(y) &= y^{N+1}\left(\frac{B_1}{y-y_1} + \frac{B_2}{y-y_2}\right) \\
G_4(y) &= \frac{C_1}{y-y_1} + \frac{C_2}{y-y_2}
\end{aligned}$$

where,

$$\begin{aligned}
A_1 &= \tfrac{1}{(1-y_1)(1-y_2)} \quad & A_2 &= \tfrac{1}{(1-y_1)(y_1-y_2)} \quad & A_3 &= \tfrac{1}{(1-y_2)(y_2-y_1)} \\
B_1 &= \tfrac{1}{y_1-y_2} & B_2 &= \tfrac{1}{y_2-y_1} \\
C_1 &= \tfrac{y_1-\rho}{y_1-y_2} & C_2 &= \tfrac{y_2-\rho}{y_2-y_1}
\end{aligned}$$

Thus we can write,

$$\bar{S}(x,y) = G_1(y) + K_3G_2(y) + K_4G_3(y) + K_5G_4(y)$$

$$
\begin{aligned}
&= (1+K_3A_1)\frac{1-y^N}{1-y} - K_3A_2\frac{y_1^N - y^N}{y_1-y} \\
&\quad -K_3A_3\frac{y_2^N-y^N}{y_2-y} + K_4B_1\frac{y_1^{N+1}-y^{N+1}}{y_1-y} \\
&\quad +K_4B_2\frac{y_2^{N+1}-y^{N+1}}{y_2-y} + \big(K_3A_2(1-y_1^N) \\
&\quad + K_4B_1y_1^{N+1} + K_5C_1\Big)\frac{1}{y-y_1} \\
&\quad + \Big(K_3A_3(1-y_2^N) + K_4B_2y_2^{N+1} + K_5C_2\Big)\frac{1}{y-y_2}
\end{aligned}
$$

But,

$$
K_3A_2(1-y_1^N) + K_4B_1y_1^{N+1} + K_5C_1 \quad = \quad K_3A_3(1-y_2^N) + K_4B_2y_2^{N+1} + K_5C_2 = 0
$$

This is because $\bar{S}(x,y)$ is analytic in y, the left-hand side of Equation (11) vanishes at $y = y_i$, $i = 1, 2$. Hence, the above equation can be written as

$$
\begin{aligned}
\bar{S}(x,y) &= (1+K_3A_1)\frac{1-y^N}{1-y} - K_3A_2\frac{y_1^N-y^N}{y_1-y} \\
&\quad -K_3A_3\frac{y_2^N-y^N}{y_2-y} + K_4B_1\frac{y_1^{N+1}-y^{N+1}}{y_1-y} \\
&\quad +K_4B_2\frac{y_2^{N+1}-y^{N+1}}{y_2-y}
\end{aligned}
$$

Again, recalling that

$$
\frac{a^k - y^k}{a-y} = a^{k-1} + a^{k-2}y + a^{k-3}y^2 + \ldots + ay^{k-2} + y^{k-1}
$$

and grouping the coefficients of the same power of y we get,

$$
\bar{S}(x,y) = \sum_{i=0}^{N} c_i(x)y^i
$$

$\diamond$

Having expressed $\bar{S}(x,y)$ as $\sum_{i=1}^{N} c_i(x)y^i$ (in Proposition 2) we now proceed to obtain the expression for $\mathcal{G}$ using $\bar{S}(x,y)$.

Proposition 3 *The Goodput ratio, $\mathcal{G}$ can be written as,*

$$\begin{aligned}\mathcal{G} &= q^2 \sum_{i=0}^{N} \left(\frac{d(xc_i(x))}{dx} \right)_{x=(1-q)} P(Q=i) \\ &= q^2 \left[\frac{d}{dx} \left(\sum_{i=0}^{N} xc_i(x) P(Q=i) \right) \right]_{x=(1-q)}\end{aligned}$$

Proof: We know by Equation (10),

$$\begin{aligned}\mathcal{G} &= q \sum_{n=1}^{\infty} nq(1-q)^{n-1} \sum_{i=0}^{N} S_{n,i} P(\mathcal{Q}=i) \\ &= q^2 \sum_{i=0}^{N} \sum_{n=1}^{\infty} S_{n,i} n(1-q)^{n-1} P(\mathcal{Q}=i)\end{aligned}$$

Also,

$$\bar{S}(x,y) = \sum_{i=0}^{N} y^i \sum_{n=1}^{\infty} S_{n,i} x^{n-1} = \sum_{i=0}^{N} c_i(x) y^i$$

Thus, $c_i(x) = \sum_{n=1}^{\infty} S_{n,i} x^{n-1}$ and

$$\left(\frac{d(xc_i(x))}{dx} \right)_{x=(1-q)} = \sum_{n=1}^{\infty} n S_{n,i} (1-q)^{n-1}$$

Thus,

$$\begin{aligned}\mathcal{G} &= q^2 \sum_{i=0}^{N} \sum_{n=1}^{\infty} S_{n,i} n(1-q)^{n-1} P(\mathcal{Q}=i) \\ &= q^2 \sum_{i=0}^{N} \left(\frac{d(xc_i(x))}{dx} \right)_{x=(1-q)} P(Q=i)\end{aligned}$$

$\diamond$

Thus, we can obtain the exact expression for the *Goodput ratio* by knowing the coefficients $c_i(x)$ and $P(Q=i) (= P_{i,0} + P_{i,1})$, for $0 \le i \le N$ (both being previously obtained in Corollary (1) and Proposition (2) respectively). Since the derivation as well as the final result are complex, we delay these to the appendix.

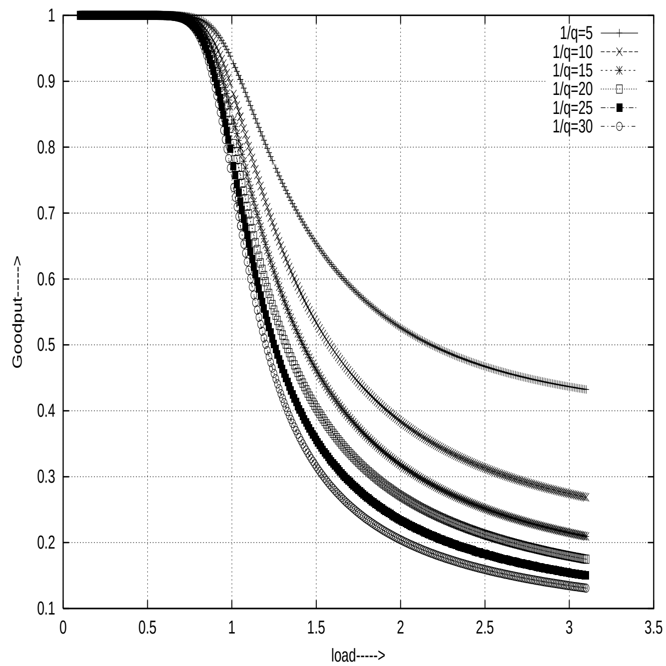

Figure 2: $\mathcal{G}$ Vs ρ for $1/q = 5, 10, 15, 20, 25, 30$ with $N = 20$

2.5 Numerical Examples

We now work with numerical examples and plot the $\mathcal{G}$ obtained with our explicit formula from Equation 31 (in appendix) with increasing load ρ. We first keep N fixed at 20 and plot for $1/q = 5, 10, 15, 20, 25, 30$ with ρ varying from 0.1 to 3.0 (in steps of 0.01) in Figure 2. Next we keep $1/q$ fixed at 20 and plot for $N = 5, 10, 15, 20, 25, 30$, again with ρ varying from 0.1 to 3.0 (in steps of 0.01) in Figure 3.

We have observed in Subsection 2.3 that there exists a nontrivial limiting behavior as N becomes large, while keeping all other parameters the same. This observation is supported by Figure 3 that shows that there is a limit of $\mathcal{G}$ as N tends to infinity.

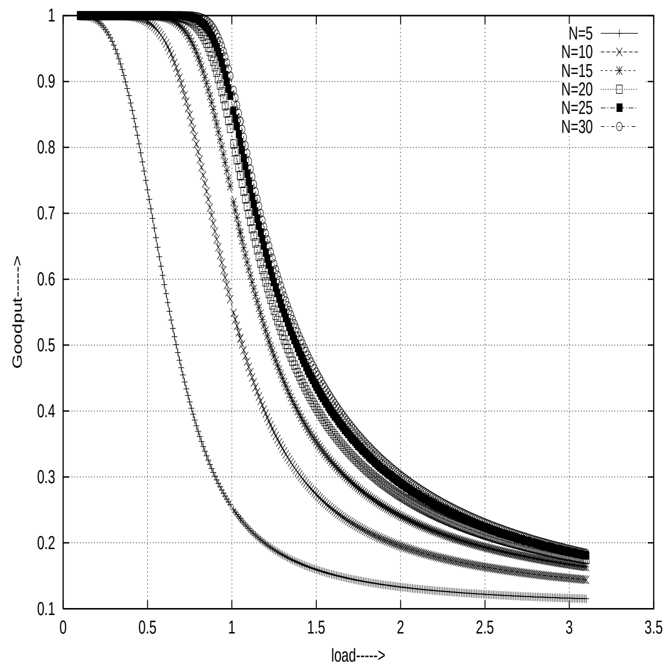

Figure 3: $\mathcal{G}$ Vs ρ for $N = 5, 10, 15, 20, 25, 30$ with $1/q = 20$

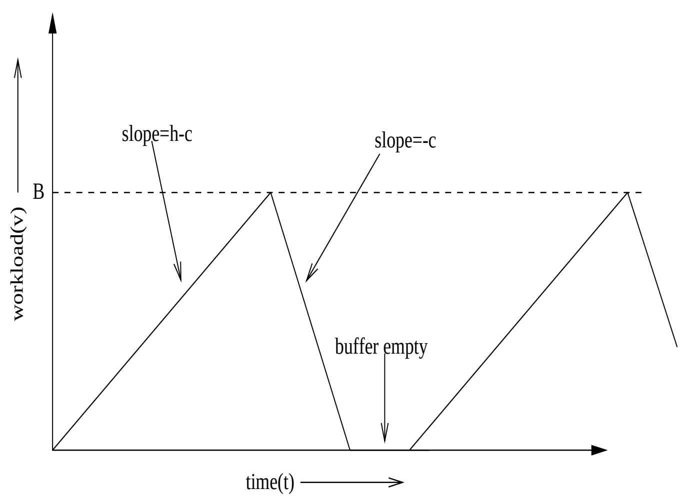

Figure 4: A typical evolution of the workload process, $V(t)$ in our fluid model

3 Fluid Approximation

3.1 The workload process

Our fluid source always has messages to send and the capacity of the fluid buffer is finite, say B. The fluid buffer is served with a capacity c. The length of a message is assumed to be exponentially distributed with parameter η. If during the arrival of a message, the workload process $V(t)$ (alternatively the queue length, i.e., amount of fluid in the fluid buffer) reaches B, then all the remaining fluid corresponding to this message is dropped. Let the fluid arrival rate be h.

Remark 1 *The fluid limit can be seen as a weak limit of the original model through a standard scaling. More precisely, consider n models, and add n as a superscript to the parameters of the nth model. Then the scaling is obtained as follows.*

- *Arrival rate:* $\lambda^{(n)} := n\lambda$;
- *Service rate:* $\mu^{(n)} := n\mu$;
- *Size of messages: geometrically distributed with parameter* $q^{(n)} := q/n$;
- *Buffer size:* $N^{(n)} = nN$.

Let $X^{(n)}(t)$ be the queue length process of the nth model. Then, as $n \to \infty$, the process $X^{(n)}(t)/n$ weakly converges to our fluid process $V(t)$, with $h = \lambda$, $c = \mu$, and with $\eta = q\lambda$.

A typical evolution of $V(t)$ in our model is shown in Figure 4 Also, let A be the event that the incoming fluid is accepted. To remove trivialities we assume that $c < h$. Let V_n be the random variable denoting the queue length at the end of nth non-acceptance period. The dynamics of V_n can be written as,

$$V_{n+1} = V_n + (h-c)A_n - cA_n^c \tag{12}$$

where $A_n = (B - V_n)/(h-c)$ is the duration of the $n+1$st acceptance event and $A_n^c = \min(X_n, B/c)$ is the duration upto which the process $V(t)$ will have a negative slope where X_n is the remaining length of the current incoming message at the epoch when $V(t)$ hits B, i.e., at the end of the $n+1$th acceptance event and the start of the $n+1$th non-acceptance event. Let T_n be the epoch of the commencement of the $n+1$st acceptance event. Thus, $T_{n+1} = T_n + A_n + X_n$ and $(V_n, A_n + X_n)$ can be viewed as a marked point process [2]. Thus,

$$V_{n+1} = B - cA_n^c$$

Denote the Laplace Stieltjis Transform (LST) of V_n by $V_p(s)$ [2] in steady state. Let ρ_p be the probability density of V_n in steady state.

Lemma 1 *The LTS and the probability density of V_n in steady state are given by:*

$$V_p(s) = e^{-sB}\eta\left[\frac{1-e^{-\frac{B}{c}(\eta-sc)}}{\eta-sc}+\frac{e^{sB}e^{\frac{-\eta B}{c}}}{\eta}\right] \tag{13}$$

$$\rho_p(v) = \frac{\eta}{c}e^{\frac{\eta}{c}(v-B)}+e^{-\frac{\eta B}{c}}\delta(v) \tag{14}$$

(for $v\in[0,B)$).

Proof:

$$\begin{aligned} V_p(s) &= E[e^{-s(B-cA_n^c)}] \\ &= e^{-sB}E[e^{sc\left(X_nI(X_n<\frac{B}{c})+\frac{B}{c}I(X_n>\frac{B}{c})\right)}] \\ &= e^{-sB}\eta[\int_0^{\frac{B}{c}} e^{scx}e^{-\eta x}dx+\int_{\frac{B}{c}}^{\infty} e^{sB}e^{-\eta x}dx] \end{aligned}$$

which gives (13). The inversion of the LTS implies (14). ◇

In the above Lemma we obtained the workload LTS and probability density at the end of non-acceptance periods. Next we shall obtain these quantities at an arbitrary time, i.e. the LTS and the probability density of the time stationary workload.

Proposition 4 *The LTS $V(s)$ and the probability density ρ of $V(t)$ in stationary regime are given by:*

$$V(s) = \frac{\eta h}{(h-ce^{\frac{-B\eta}{c}})}e^{-sB}\left(\frac{1-e^{-\frac{B(\eta-sc)}{c}}}{\eta-sc}\right)-\frac{h(1-e^{\frac{-\eta B}{c}})}{(h-ce^{\frac{-B\eta}{c}})}+1 \tag{15}$$

For $v\in[0,B)$, $\rho(v)=\left(1-\frac{h(1-e^{\frac{-\eta B}{c}})}{(h-ce^{\frac{-B\eta}{c}})}\right)\delta(v)+\frac{\eta h}{(h-ce^{\frac{-B\eta}{c}})}\left(\frac{e^{\frac{\eta}{c}(v-B)}}{c}\right)$
and for $v\geq B$

$$\rho(v) = 0$$

Finally, the mean stationary workload M_f is given by

$$M_f=\frac{h}{h-ce^{\frac{-B\eta}{c}}}\left(B-\frac{c}{\eta}(1-e^{\frac{-B\eta}{c}})\right). \tag{16}$$

[2]The subscript p indicates the point process

Proof: We shall now use the following inversion formula (see e.g., [2], Ch.1, Sec.4) to obtain the LST for the workload process $V(t)$ which we will then invert to get the probability density function.

$$E[e^{-sV(t)}] = \frac{E^0[\int_0^{T_1} e^{-sV(t)}]}{E^0[T_1]}$$

Thus,

$$\begin{aligned}
V(s) &= \frac{E^0[\int_0^{\frac{B-V_0}{h-c}} e^{-s(V_0+(h-c)t)}dt}{E^0[\frac{B-V_0}{h-c} + X_0]} + \frac{\int_{\frac{B-V_0}{h-c}}^{\frac{B-V_0}{h-c}+A_0^c} e^{-s(B-c(t-\frac{B-V_0}{h-c}))}dt + X_0 - A_0^c]}{E^0[\frac{B-V_0}{h-c} + X_0]} \\
&= \frac{E^0[\frac{e^{-sV_0}-e^{-sB}}{s(h-c)} + e^{-sB}\left(\frac{e^{scA_0^c}-1}{sc}\right) + X_0 - A_0^c]}{\frac{(h-ce^{\frac{-B\eta}{c}})}{(h-c)\eta}} \\
&= \frac{(h-c)\eta}{(h-ce^{\frac{-B\eta}{c}})}\left[\frac{E^0[e^{-sV_0}] - e^{-sB}}{s(h-c)} + \frac{e^{-sB}(E^0[e^{scA_0^c}] - 1)}{sc} + E[X_0] - E[A_0^c]\right]
\end{aligned}$$

Observe that we have expressions for $E[e^{-sV_0}]$ (i.e., $V_p(s)$), $E[V_0]$ (can be obtained from $V_p(s)$) and the expression for $E[e^{scA_0^c}]$ and hence for $E[A_0^c]$ can be easily obtained by the definition of A_0^c. Thus, we get after some calculations (15). The inverse of the LST of the last equation gives $\rho(v)$.

Having obtained the density function we can evaluate many performance metrics. In particular all the moments of the process $V(t)$ (these can also easily be calculated from the *LST*, $V(s)$), the mean queue length and the *Goodput Ratio*.

Finally, M_f is obtained by the integration:

$$M_f = \int_0^B v\rho(v)dv = \int_0^B v\frac{\eta h}{(h-ce^{\frac{-B\eta}{c}})}\left(\frac{e^{\frac{\eta}{c}(v-B)}}{c}\right)dv$$

which implies (16). $\diamond$

3.2 The Goodput Ratio $\mathcal{G}_f$

We proceed with the model from the previous subsection and in particular, we continue to assume that $c < h$ in order to avoid trivialities. We define, the fluid analog of the *Goodput Ratio*, $\mathcal{G}_f$ as the ratio of the total fluid comprising good messages (i.e., messages which do not suffer any fluid loss due to Buffer overflow) exiting the node to the total arriving fluid at its input. Let $\mathcal{V}_f$ be the random variable representing the success of a message, $\mathcal{V}_f = 1$ for a good message, and $\mathcal{V}_f = 0$ for a message which has lost some fluid. Let us define the sub-distribution function $F(w,1)$ as the probability that a message is of length $\leq w$ and is good, i.e,

$$F(w,1) = P(W \leq w, \mathcal{V}_f = 1)$$

Then we can write the goodput ratio as

$$\mathcal{G}_f = \frac{\int_0^\infty w dF(w,1)}{\int_0^\infty w dF(w)}$$

where, $F(w)$ is the message length distribution ($\sim \exp \lambda$). Again, writing $F(w,1)$ as,

$$\begin{aligned} F(w,1) &= P(\mathcal{V}_f = 1 | W \leq w) P(W \leq w) \\ &= \int_0^B P(\mathcal{V}_f = 1 | W \leq w, V = v) \rho(v) dv \int_0^w f(u) du \end{aligned}$$

where, $\rho(v)$ is the queue length density and V is the queue length at the epoch of the arrival of the message [3] and $f(x)$ is the message length density.

Proposition 5 *The Goodput is given by*

$$\begin{aligned} \mathcal{G}_f &= \frac{c}{(h - ce^{\frac{-B\eta}{c}})} \left[e^{\frac{-B\eta h}{(h-c)c}} (1 - \frac{c}{h}) + (\frac{c}{h} - e^{\frac{-B\eta}{c}}) \right] \\ &\quad + \frac{\eta^2 B e^{\frac{-B\eta h}{h-c}}}{h - ce^{\frac{-B\eta}{c}}} \end{aligned} \tag{17}$$

Proof: Observe that, for $w \in [0, \frac{B-v}{h-c}]$,

$$P(\mathcal{V}_f = 1 | W \leq w, V = v) = 1$$

and, for $w > \frac{B-v}{h-c}$,

$$P(\mathcal{V}_f = 1 | W \leq w, V = v) = P(W < \frac{B-v}{h-c} | W < w)$$

[3] due to PASTA the queue length distribution at the arrival epochs of messages, which come as a Poisson stream, is same as the stationary queue length distribution

Or in other words, for $w \in [0, \frac{B}{h-c}]$, if $v \in [0, B - w(h-c)]$,

$$P(\mathcal{V}_f = 1 | W \leq w, V = v) \quad = \quad 1$$

else,

$$P(\mathcal{V}_f = 1 | W \leq w, V = v) \quad = \quad \frac{P(W < \frac{B-v}{h-c})}{P(W < w)}$$

And for $w > \frac{B}{h-c}$,

$$P(\mathcal{V}_f = 1 | W \leq w, V = v) \quad = \quad \frac{P(W < \frac{B-v}{h-c})}{P(W < w)}$$

Thus we write, for $w \in [0, \frac{B}{h-c}]$, $F(w, 1) = F_1(w) + F_2(w)$, where,

$$\begin{aligned}
F_1(w) = (1 - e^{-\eta w}) \int_0^{B-w(h-c)} \rho(v) dv \\
&= (1 - e^{-\eta w}) \int_0^{B-w(h-c)} \left[\left(1 - \frac{h(1 - e^{\frac{-\eta B}{c}})}{(h - ce^{\frac{-B\eta}{c}})} \right) \delta(v) \right. \\
&\quad \left. + \frac{\eta h}{(h - ce^{\frac{-B\eta}{c}})} \left(\frac{e^{\frac{\eta}{c}(v-B)}}{c} \right) \right] dv \\
&= (1 - e^{-\eta w}) \left[1 - \frac{h(1 - e^{\frac{-w\eta(h-c)}{c}})}{(h - ce^{\frac{-B\eta}{c}})} \right]
\end{aligned}$$

Further,

$$\begin{aligned}
F_2(w) &= \int_{B-w(h-c)^+}^{B} (1 - e^{-\frac{B-v}{h-c}\eta}) \rho(v) dv \\
&= \int_{(B-w(h-c))^+}^{B} (1 - e^{-\frac{B-v}{h-c}\eta}) \frac{\eta h}{(h - ce^{\frac{-B\eta}{c}})} \left(\frac{e^{\frac{\eta}{c}(v-B)}}{c} \right) dv \\
&= \frac{h}{(h - ce^{\frac{-B\eta}{c}})} \left[(1 - e^{\frac{-\eta w(h-c)}{c}}) - \frac{(h-c)}{h} (1 - e^{\frac{-\eta h w}{c}}) \right]
\end{aligned}$$

and for $w > B/(h-c)$

$$F(w) \quad = \quad \int_0^B (1 - e^{-\frac{(B-v)\eta}{h-c}}) \rho(v) dv$$

Thus, we get,

$$\begin{aligned}
dF_1(w) &= \frac{-\eta h(h-c)(e^{\frac{-w\eta(h-c)}{c}} - e^{\frac{-w\eta h}{c}})}{c(h-ce^{\frac{-\eta B}{c}})}dw \\
&\quad + \left[1 - \frac{h(1-e^{\frac{-w\eta(h-c)}{c}})}{(h-ce^{\frac{-B\eta}{c}})}\right]\eta e^{-\eta w}dw \\
&= \frac{\eta h}{(h-ce^{\frac{-B\eta}{c}})}\Big((e^{\frac{-w\eta h}{c}} - \frac{c}{h}e^{-\frac{\eta B}{c}}e^{-\eta w}) \\
&\quad +(1-\frac{h}{c})(e^{\frac{-w\eta(h-c)}{c}} - e^{\frac{-w\eta h}{c}})\Big)\,dw
\end{aligned}$$

And,

$$dF_2(w) = \frac{h\eta(h-c)}{c(h-ce^{\frac{-B\eta}{c}})}\left[e^{\frac{-\eta(h-c)w}{c}} - e^{\frac{-\eta hw}{c}}\right]dw$$

Thus, for $w \in [0, \frac{B}{h-c})$,

$$\begin{aligned}
dF(w,1)) &= dF_1(w) + dF_2(w) \\
&= \frac{\eta h}{(h-ce^{\frac{-B\eta}{c}})}\left((e^{\frac{-w\eta h}{c}} - \frac{c}{h}e^{-\frac{\eta B}{c}}e^{-\eta w})\right)dw
\end{aligned}$$

and for $w > \frac{B}{h-c}$,

$$dF(w,1) = 0$$

However, at $w = \frac{B}{h-c}$, the function $dF(w,1)$ is discontinuous. As,

$$\lim_{\epsilon\to 0} dF(\frac{B}{h-c} - \epsilon, 1) = \frac{\eta e^{\frac{-B\eta h}{(h-c)c}}(h-c)}{(h-ce^{\frac{-B\eta}{c}})}$$

$$\lim_{\epsilon\to 0} dF(\frac{B}{h-c} + \epsilon, 1) = 0$$

Hence we obtain

$$\mathcal{G}_f = \frac{\eta^2 h}{(h-ce^{\frac{-B\eta}{c}})}\int_0^{\frac{B}{h-c}} w(e^{\frac{-w\eta h}{c}} - \frac{c}{h}e^{-\frac{\eta B}{c}}e^{-\eta w})dw + \frac{\eta^2 e^{\frac{-B\eta h}{(h-c)c}}B}{(h-ce^{\frac{-B\eta}{c}})}$$

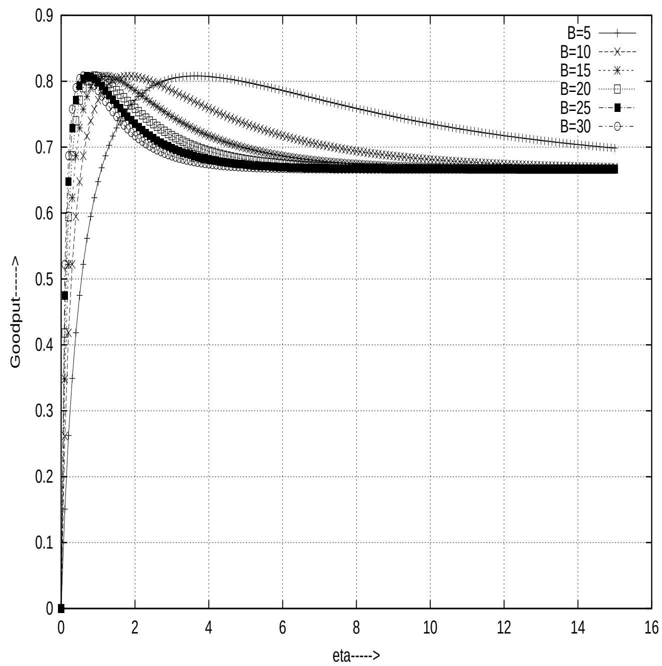

Figure 5: $\mathcal{G}_f$ Vs η for $\eta \in [0, 15]$ for different values of B with $c = 8$ and $h = 12$

from which we obtain(17). $\diamond$

Let us now observe the behavior of $\mathcal{G}_f$ for extreme values of η, keeping all other parameters fixed. As η tends to zero we see from expression (17) that $\mathcal{G}_f$ tends to zero. This can be explained by the fact that small η corresponds to very long frames, so that the probability of that the queue will fill during the arrival of a message tends to one (since $h > c$).

For the other extreme, i.e. $\eta \to \infty$, the length of a message is very short; one could then expect that the Goodput would equal to the relative amount of fluid that is lost, since a message corresponds to an infinitesimal amount of fluid. This would give a Goodput of c/h. This is however not the real limiting value of the Goodput: we see, in fact, that as $\eta \to \infty$, we get $\mathcal{G}_f \to \frac{c^2}{h^2}$ from expression (17).

Also, observe that as B tends to 0, $\mathcal{G}_f$ tends to 0 and when B tends to ∞, $\mathcal{G}_f$ tends to $\frac{c^2}{h^2}$.

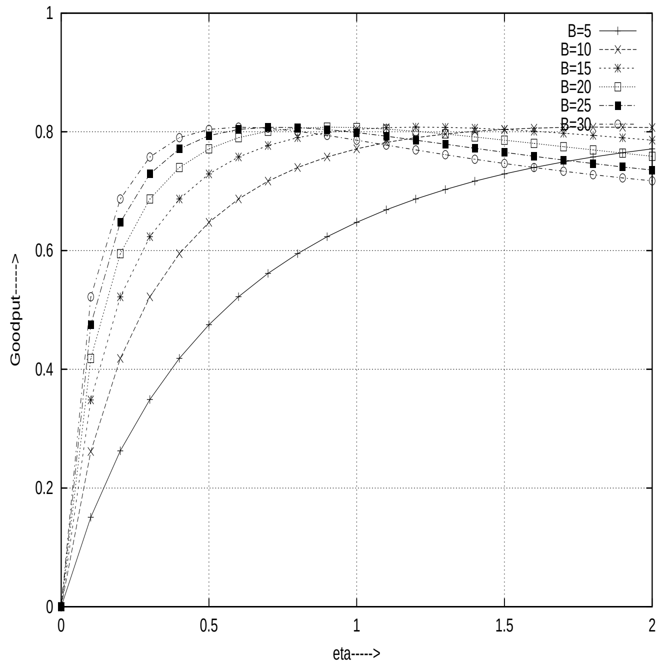

Figure 6: $\mathcal{G}_f$ Vs η for $\eta \in [0, 2]$ for different values of B with $c = 8$ and $h = 12$

3.3 Numerical Examples

We now work with some numerical examples and show that behavior of $\mathcal{G}_f$ as we increase η. We take $c = 8$, $h = 12$ and observe the behavior of $\mathcal{G}_f$ for $B = 5, 10, 15, 20, 25, 30$ as η increases from 0 to 15 in Figure 5 and further an enlarged portion of Figure 5 in Figure 6 for $\eta \in [0, 2]$. The limiting behavior ($\lim_{\eta\to\infty} \mathcal{G}_f = \frac{c^2}{h^2} = 0.6667$) as η increases is seen even at low values of η as B increases.

4 Conclusion

We have provided explicit expressions for the queue size distribution and for the goodput for the packet model based on recursions introduced in [9]. We then provided an alternative fluid approximation for studying the PMD policy and obtained the queue size distribution and goodput in this framework of fluid queue. Our analytical results will be quite useful in dimensioning the buffer size that should be used for a given required goodput under the PMD policy. We are currently studying the potentials of the fluid model as an alternative to the packet model. Our future work is directed towards analytical study of the sensitivity of the goodput to different parameters, generalizing the fluid model to include Markovian fluids, deriving explicit expressions for the queue size distribution and the goodput for other queue-management policies, in particular, Early Message Discard (EMD) and Random Early Discard (RED) and extending the single node analysis to multiple nodes.

5 Appendix

5.1 Proof of Proposition 1

From Equation (3) and Equation (5) by applying *z-transform* we get

$$\begin{aligned} & Q_0(z)[(\rho+1) - \rho z - z^{-1}] - q\rho z Q_1(z) \\ = \;& P_{0,0}[1 + \rho - z^{-1}] - P_{1,0} \\ & -\rho z^{N+1}[P_{N,0} + qP_{N,1}] \end{aligned} \tag{18}$$

Similarly, by applying *z-transform* to Equation (4), we get

$$Q_1(z) = P_{0,1}\frac{(1 - z - z^{N+1}(q\rho+1)^N q\rho)}{1 - z(q\rho+1)} \tag{19}$$

Now from Equations (1) (2) (6) and Equation (18), we get on solving for $Q_0(z)$,

$$Q_0(z) = \frac{P_{0,0}[1 - z^{-1}] - \rho z^{N+1} P_{N,0}[1 + \rho q] + Q_1(z) q\rho z}{[(\rho + 1) - \rho z - z^{-1}]} \tag{20}$$

The zeros of the denominator of Equation (20) are $z = 1, \rho^{-1}$. At these values of z, the numerator of Equation (20) should be equal to 0 because of the analyticity of $Q_0(z)$ (being a polynomial in z of degree atmost N). Thus, substituting $z = 1$ in the numerator of Equation (20) and equating it to 0 we get an equation

$$P_{N,0} = (1 + q\rho)^{N-1} q P_{0,1} \tag{21}$$

Also, substituting $z = \rho^{-1}$ in the numerator of Equation (20) and equating it to 0 we get another equation

$$P_{0,0}(1 - \rho) - \rho^{-N} P_{N,0}(1 + \rho q) + Q_1(\rho^{-1}) q = 0 \tag{22}$$

From Equations (19), (21) and (22), we get,

$$P_{0,0} = P_{0,1} \frac{q(1 - (p^{-1} + q)^N)}{[p(1 - q) - 1]} \tag{23}$$

We shall interpret Equation (7) as

$$Q_0(1) + Q_1(1) = 1 \tag{24}$$

From Equation (19)

$$Q_1(1) = (q\rho + 1)^N P_{0,1} \tag{25}$$

and

$$\dot{Q}_1(1) = \frac{P_{0,1}\left[1 + (\rho q + 1)^N (N\rho q - 1)\right]}{q\rho} \tag{26}$$

From Equation (20) differentiating the numerator and denominator and taking limit as $z \to 1$, we get

$$\begin{aligned} Q_0(1) &= \lim_{z \to 1} Q_0(z) \\ &= \lim_{z \to 1} \frac{z^{-2} P_{0,0} - \rho(N+1) z^N P_{N,0}(1 + \rho q)}{[-\rho + z^{-2}]} \\ &\quad + \frac{q\rho(z\dot{Q}_1(z) + Q_1(z))}{[-\rho + z^{-2}]} \end{aligned} \tag{27}$$

Thus from Equations (25) (26) (21) (23) and Equation (27) we get

$$Q_0(1) = \frac{P_{0,1}}{(1-\rho)}\left[\frac{q(1-(p^{-1}+q)^N)}{[p(1-q)-1]} + 1 - (1+\rho q)^N\right] \tag{28}$$

Substituting Equation (28) and (25) in Equation (24), and solving for $P_{0,1}$, we get

$$P_{0,1} \;\; = \;\; \frac{(1-\rho)(\rho(1-q)-1)}{q(1-\rho^{-N}(1+\rho q)^N) + (\rho(1-q)-1)(1-\rho(1+\rho q)^N)} \tag{29}$$

Having known $P_{0,1}$ we have obtained the generating functions $Q_0(z)$ and $Q_1(z)$. However, we can further modify the expression for $Q_1(z)$ to a more meaningful form. From Equation (20) we write, after some algebraic manipulations,

$$\begin{aligned} Q_0(z) \;\; &= \;\; P_{0,1}q\Big[(1+\rho q)^{N-1}\left(\frac{z^{N+1}-\rho^{-(N+1)}}{z-\rho^{-1}}\right) \\ &+ \;\; (1+\rho q)^{N-2}\left(\frac{z^N-\rho^{-N}}{z-\rho^{-1}}\right) + \ldots + \frac{z^2-\rho^{-2}}{z-\rho^{-1}}\Big] \end{aligned}$$

Observe that, each fraction inside the bracket on the right hand side of the last equation is of the form $(x^k - a^k)/(x-a)$ which simplifies to

$$\frac{x^k - a^k}{x-a} = x^{k-1} + x^{k-1}a + x^{k-2}a^2 + \ldots + xa^{k-2} + a^{k-1}$$

thus we get,

$$\begin{aligned} Q_0(z) = P_{0,1}q\rho\Big[&(1+\rho q)^{N-1}z^{N+1}\sum_{j=1}^{N+1}\left(\frac{1}{\rho z}\right)^j \\ &+(1+\rho q)^{N-2}z^N\sum_{j=1}^{N}\left(\frac{1}{\rho z}\right)^j + \ldots + z^2\sum_{j=1}^{2}\left(\frac{1}{\rho z}\right)^j\Big] \end{aligned}$$

Thus

$$\begin{aligned}
Q_0(z) &= P_{0,1}q\rho\left[\sum_{j=1}^{2}\left(\frac{1}{\rho z}\right)^j\left[z^2+z^3(1+\rho q)+\ldots+(1+\rho q)^{N-1}z^{N+1}\right]\right.\\
&+ \left(\frac{1}{\rho z}\right)^3\left[z^3+z^4(1+\rho q)^2+\ldots+(1+\rho q)^{N-1}z^{N+1}\right]\\
&+ \ldots+\left(\frac{1}{\rho z}\right)^N\left[z^N(1+\rho q)^{N-2}+z^{N+1}(1+\rho q)^{N-1}\right]\\
&+ \left.\left(\frac{1}{\rho z}\right)^{N+1}(1+\rho q)^{N-1}z^{N+1}\right]
\end{aligned}$$

Grouping the coefficients of the powers of z we get (8).

5.2 Exact Expression for $\mathcal{G}$

We shall first obtain an expression for $\sum_{i=0}^{N} c_i P(Q=i)$.

$$\begin{aligned}
&\sum_{i=0}^{N} c_i(x)P(Q=i) = \\
&\qquad c_0(x)P(Q=0)+\sum_{i=1}^{N-1} c_i(x)P(Q=i)\\
&= (1+K_3A_1)+(c_0-(1+K_3A_1))P(Q=0)+\\
&\qquad [K_4B_1y_1^N-K_3A_2y_1^{N-1}]\sum_{i=1}^{N-1}P(Q=i)y_1^{-i}+\\
&\qquad [K_4B_2y_2^N-K_3A_3y_2^{N-1}]\sum_{i=1}^{N-1}P(Q=i)y_2^{-i}-\\
&\qquad (1+K_3A_1)P(Q=N) \qquad (30)
\end{aligned}$$

Observe that,

$$\begin{aligned}
\sum_{i=1}^{N-1} P(Q=i)y_1^{-i} &= Q_0(y_1^{-1}) + Q_1(y_1^{-1}) - P(Q=0) \\
&\quad -P(Q=N)y_1^{-N} \\
\sum_{i=1}^{N-1} P(Q=i)y_2^{-i} &= Q_0(y_2^{-1}) + Q_1(y_2^{-1}) - P(Q=0) \\
&\quad -P(Q=N)y_2^{-N}
\end{aligned}$$

Writing $Q(z) = Q_1(z)+Q_2(z)$, from the above Equation the expression for $\sum_{i=1}^{N} c_i P(Q = i)$ simplifies to

$$\begin{aligned}
\sum_{i=0}^{N} c_i(x)P(Q=i) &= (1+K_3A_1)(1-P(Q=N)) \\
&\quad +[K_4B_1y_1^N - K_3A_2y_1^{N-1}]Q(y_1^{-1}) \\
&\quad +[K_4B_2y_2^N - K_3A_3y_2^{N-1}]Q(y_2^{-1}) \\
&\quad +K_3(A_2y_1^{-1} + A_3y_2^{-1})P(Q=N)
\end{aligned}$$

And by Proposition 3 we write,

$$\begin{aligned}
\mathcal{G} &= q^2\left[(1-q)\sum_{i=0}^{N}\left(\frac{dc_i(x)}{dx}\right)_{x=(1-q)} P(Q=i) + \sum_{i=0}^{N} c_i(1-q)P(Q=i)\right] \\
&= q^2\left[(1-q)\left(\frac{d}{dx}\left(\sum_{i=0}^{N} c_i(x)P(Q=i)\right)\right)_{x=(1-q)} + \right. \\
&\quad \left. \sum_{i=0}^{N} c_i(1-q)P(Q=i)\right] \qquad (31)
\end{aligned}$$

Thus we need to evaluate $\frac{d}{dx}\left(\sum_{i=0}^{N} c_i(x)P(Q=i)\right)$. From the expression for $\sum_{i=0}^{N} c_i(x)P(Q = i)$ from Equation (30), we find that it involves the evaluation of differential with respect to x of some functions of x.

$$
\begin{aligned}
&\frac{d}{dx}\left(\sum_{i=0}^{N} c_i P(Q=i)\right) \\
&= \frac{d}{dx}K_3A_1(1-P(Q=N)) + [K_4B_1y_1^N - K_3A_2y_1^{N-1}]\frac{d}{dx}Q(y_1^{-1}) \\
&\quad +Q(y_1^{-1})\frac{d}{dx}[K_4B_1y_1^N - K_3A_2y_1^{N-1}] + [K_4B_2y_2^N - K_3A_3y_2^{N-1}]\frac{d}{dx}Q(y_2^{-1}) \\
&\quad +Q(y_2^{-1})\frac{d}{dx}[K_4B_2y_2^N - K_3A_3y_2^{N-1}] \\
&\quad +\frac{d}{dx}\left(K_3(A_2y_1^{-1} + A_3y_2^{-1})\right)P(Q=N)
\end{aligned}
$$

Thus, we need to obtain the differential terms on the right side of the last equation. The final expressions are provided below with $\delta_K = y_1^K - y_2^K$ and $\phi_{K+1} = x\delta_K - \delta_{K+1}$ for $K \geq 1$

$$
\begin{aligned}
\frac{d}{dx}K_3A_1 &= \frac{1}{q^2} \\
(K_4B_1y_1^N - K_3A_2y_1^{N-1}) &= \frac{y_1^{N-1}x\rho}{(1-y_1)\phi_{N+1}\delta}\left(\rho\phi_N - \delta_N(1-y_1)\right) \\
(K_4B_2y_2^N - K_3A_2y_2^{N-1}) &= -\frac{y_2^{N-1}x\rho}{(1-y_2)\phi_{N+1}\delta}\left(\rho\phi_N - \delta_N(1-y_2)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx}\left(K_4B_1y_1^N - K_3A_2y_1^{N-1}\right) \\
&= \frac{y_1^{N-1}x\rho}{(1-y_1)\phi_{N+1}\delta}\left[\rho\left(x(N-1)\beta_{N-2}\frac{dy_1}{dx} + \delta_{N-1} - N\beta_{N-1}\frac{dy_1}{dx}\right)\right. \\
&\quad -(1-y_1)N\beta_{N-1}\frac{dy_1}{dx} + \delta_N\frac{dy_1}{dx} + (\rho\phi_N - \delta_N(1-y_1)) \times \\
&\quad \left.\left(\frac{1}{x} + \frac{1}{(1-y_1)}\frac{dy_1}{dx} + \frac{(N-1)}{y_1}\frac{dy_1}{dx} - \frac{1}{\delta}2\frac{dy_1}{dx} - \frac{1}{\phi_{N+1}}\frac{d\phi_{N+1}}{dx}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx}\left(K_4 B_2 y_2^N - K_3 A_3 y_2^{N-1}\right)\\
&= \quad -\frac{y_2^{N-1} x\rho}{(1-y_2)\phi_N \delta}\left[\rho\left(x(N-1)\beta_{N-2}\frac{dy_2}{dx} + \delta_{N-1} - N\beta_{N-1}\frac{dy_2}{dx}\right)\right.\\
&\quad -(1-y_2)N\beta_{N-1}\frac{dy_2}{dx} + \delta_N \frac{dy_2}{dx} + (\rho\phi_N - \delta_N(1-y_2)) \times\\
&\quad \left.\left(\frac{1}{x} + \frac{1}{(1-y_2)}\frac{dy_2}{dx} + \frac{(N-1)}{y_2}\frac{dy_2}{dx} - \frac{1}{\delta}2\frac{dy_2}{dx} - \frac{1}{\phi_{N+1}}\frac{d\phi_{N+1}}{dx}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
Q(y) \quad &= \quad (P_{0,0} + P_{0,1}) + \frac{qP_{0,1}}{(1+q\rho)(1-\rho^{-1}(1+\rho q))}\left[\rho(1-q)(1+\rho q)y\right.\\
&\quad \left.\frac{1-(1+\rho q)^N y^N}{1-(1+\rho q)y} - \frac{(1+\rho q)^{N+1}}{\rho^N} y \frac{1-(y\rho)^N}{1-(y\rho)}\right]
\end{aligned}
$$

Thus, having obatined all the terms in Equation 31 we have the explicit expression for $\mathcal{G}$

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