Averaging of Non-Self Adjoint Parabolic Equations with Random Evolution (Dynamics)

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Abstract: The averaging problem for convection-diffusion non–stationary parabolic operator with rapidly oscillating coefficients is studied. Under the assumption that the coefficients are periodic in spatial variables and random stationary in time and that they possess certain mixing properties, we show that in appropriate moving coordinates the measures generated by the solutions of original problems converge weakly to a solution of limit stochastic PDE. The homogenized problem is well-posed and defines the limit measure uniquely.

Key-words: Homogenization, averaging, convection-diffusion, random operator, stochastic PDE

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Moyennisation d'équations paraboliques non auto–adjointes avec évolution aléatoire (dynamique)

Résumé : On étudie le problème de moyennisation pour des opérateurs de convection–diffusion, paraboliques, non–stationnaires avec des coefficients rapidement oscillants. Sous l'hypothèse que les coefficients sont périodiques en espace, aléatoires/stationnaires en temps et qu'ils possèdent certaines propriétés de mélange, nous montrons que pour un repère de coordonnées convenable les mesures générées par les solutions du problème d'origine convergent faiblement vers la solution d'une EDP stochastique limite. Le problème homogénéisé est bien posé et définit de façon unique la mesure limite.

Mots-clés : Homogénisation, moyennisation, convection–diffusion, opérateur aléatoire, EDP stochastique

Contents

1 Introduction

We study homogenization problem for parabolic non–selfadjoint operators with rapidly oscillating random in time and periodic in spatial variables coefficients. Assuming that all the coefficients depend on time via a stationary random process ξ_s with values in R^d, $d \geq 1$, we consider the corresponding Cauchy problem:

$$\frac{\partial}{\partial t}u^\varepsilon - \frac{\partial}{\partial x_i}a_{ij}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\Big)\frac{\partial}{\partial x_j}u^\varepsilon - \frac{1}{\varepsilon}b_i\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\Big)\frac{\partial}{\partial x_i}u^\varepsilon = 0, \qquad (1)$$
$$u^\varepsilon(x,0) = u_0(x),$$

and investigate the limit behavior of u^ε as $\varepsilon \to 0$.

Previously, parabolic equations of this type with symmetric elliptic part and diffusion process ξ_s were considered in [2], [1]. In [2] it was shown that for divergence form operators the classical homogenization result holds true, that is the family of solutions to Cauchy or boundary value problem for the original operators converges, as $\varepsilon \to 0$, to a solution of the corresponding homogenized Cauchy or boundary value problem involving a parabolic operator with constant nonrandom coefficients. The picture is rather different in the presence of large zero-order term (see [1]); here the limit dynamics remains, in general, random and a weaker averaging result holds. Namely, the family of measures generated by the solutions of original problems, converges weakly in an appropriate functional space to the limit measure that solves a limit martingale problem.

Homogenization problems for various elliptic and stationary parabolic operators with low order terms have been considered in [12], [13], [15], [17], [16], Special attention in the existing literature is paid to the operators with incompressible convection terms, see [18], [19], [20], [21], [14].

Non–stationary parabolic operators with large convection terms and coefficients periodic both in space and time variables, were studied in recent work [24], where the homogenization results were obtained under the assumption that the oscillation in time is "slower" than that in space variables.

The basic conceptions of homogenization theory can be found for instance in the books [3] and [9].

Like in deterministic case when studying problem (1) one should take into account a large convection term of order $1/\varepsilon$ that arises in the asymptotics of u^ε. In this connection the averaging results for problem (1) will be obtained in moving coordinates $x' = x - \bar{b}t/\varepsilon$ with constant vector $\bar{b}$. This change of variables will allow us to avoid the drift of order $1/\varepsilon$ in the limit dynamics.

In the present work we show that the family of measures generated by the solutions of (1), converges weakly, as $\varepsilon \to 0$, to a solution of the limit functional martingale problem. The latter problem involves finite dimensional diffusion with the coefficients of the form $\lambda \nabla u$, where $\lambda = \{\lambda_{ij}\}$ is a constant matrix, and homogenized drift operator being a second order elliptic operator with constant coefficients. The matrix λ is of special interest for in case of diffusive behavior of ξ_s this matrix cannot be expressed in terms of solutions of auxiliary local "cell" problems and requires more delicate technique.

In the paper we construct various correctors being usually solutions of auxiliary PDE problems, prove several a priory estimates and combine this technique with some ideas developed in [10], [11]. Some of the corrector terms we use are not standard, they are random and nonlocal in time.

In the first section we pose the problem and provide all the conditions on the coefficients and the process ξ_s.

The second section is aimed at proving the tightness results for the corresponding families of measures.

Section 3 deals with the passage to the limit; we construct the coefficients of the limit martingale problem and then, using the uniqueness of a solution, derive weak convergence of the distributions of u^ε.

In the last section we consider a particular case when the process ξ_s is diffusive.

2 Setting of the problem. Main results

We study the asymptotic behaviors of solutions to the following Cauchy problem

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) - \frac{\partial}{\partial x_i}a_{ij}\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}\Big)\frac{\partial}{\partial x_j}u^\varepsilon(x,t) - \frac{1}{\varepsilon}b_i\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^2}}\Big)\frac{\partial}{\partial x_i}u^\varepsilon(x,t) = 0, \quad (2)$$

$$u^\varepsilon(x,0) = u_0(x),$$

as $\varepsilon \downarrow 0$; here ξ_s is an ergodic stationary process with values in R^d, it is defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, the notation $\mathbf{E}$ will be used for the expectation. In an important particular case when the process ξ_s is diffusive, we denote by L its infinitesimal generator:

$$L = q_{ij}(y)\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} + B_i(y)\frac{\partial}{\partial y_i}.$$

Further assumptions on ξ_t will be made in terms of strong or uniform mixing coefficients or in terms of maximum correlation coefficient (see, for instance, [4], §9.2). For the reader's convenience we recall here the relevant definitions.

Denote by $\mathcal{F}_{\le t}$ and $\mathcal{F}_{\ge t}$, respectively, the σ-algebras $\sigma(\xi_s,\ s \le t)$ and $\sigma(\xi_s,\ s \ge t)$. The function $\alpha(s),\ s \ge 0$, with

$$\alpha(s) = \sup_{\mathcal{B}_1 \in \mathcal{F}_{\le 0},\, \mathcal{B}_2 \in \mathcal{F}_{\ge s}} |\mathbf{P}(\mathcal{B}_1 \cup \mathcal{B}_2) - \mathbf{P}(\mathcal{B}_1)\mathbf{P}(\mathcal{B}_2)|$$

is said to be a strong mixing coefficient of $\{\xi_\cdot\}$. The function $\phi(s),\ s \ge 0$, with

$$\phi(s) = \sup_{\mathcal{B}_1 \in \mathcal{F}_{\le 0},\, \mathcal{B}_2 \in \mathcal{F}_{\ge s}} |\mathbf{P}(\mathcal{B}_1 \mid \mathcal{B}_2) - \mathbf{P}(\mathcal{B}_1)|$$

is said to be a uniform mixing coefficient. Also a maximum correlation coefficient $\rho(s), s \ge 0$, is defined to be

$$\rho(s) = \sup \frac{|\mathrm{cov}(\eta_1,\eta_2)|}{\sqrt{\mathbf{E}(\eta_1 - \mathbf{E}(\eta_1)^2\mathbf{E}(\eta_2 - \mathbf{E}(\eta_2)^2}},$$

where sup is taken over all $\mathcal{F}_{\le 0}$-measurable η_1 and $\mathcal{F}_{\ge s}$-measurable η_2 with $\mathbf{E}(\eta_1)^2 \le \infty$ and $\mathbf{E}(\eta_2)^2 \le \infty$, and the notation cov is used for the covariance matrix

In what follows we always suppose that all the coefficients $a_{ij}(z,y)$ and $b_i(z,y)$ are periodic in z variable. Other assumptions are as follows:

A1 The function $a_{ij}(z,y)$ and $b_i(z,y)$ are uniformly bounded as well as their first order derivatives in z and y:

$$|a_{ij}(z,y)| + |\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \le c,$$

$$|b_i(z,y)| + |\nabla_z b_i(z,y)| + |\nabla_y b_i(z,y)| \le c;$$

here and afterwards c stands for a generic positive constant.

A2 Matrix $a_{ij}(z,y)$ is uniformly positive definite: there is $c > 0$ such that

$$a_{ij}\zeta_i\zeta_j \ge c|\zeta|^2, \quad \zeta \in R^n,$$

for all $(z,y) \in T^n \times R^d$.

A3 At least one of the following relations hold

$$\int_0^\infty (\alpha(s))^{1/2}\,ds < \infty; \quad \int_0^\infty (\phi(s))^{1/2}\,ds < \infty; \quad \int_0^\infty \rho(s)\,ds < \infty;$$

In case of diffusion process ξ_s it is natural to replace the above condition **A3** by a certain sufficient condition for **A3** that can be formulated in terms of coefficients of the corresponding infinitesimal generator. It turns out, however, that it is more convenient to formulate the said sufficient condition for the time-reversed process $\zeta_s = \xi_{-s}$ than for ξ_s itself. Namely, letting ζ_s be a diffusion process and denoting its generator by $\tilde{L}$ with

$$\tilde{L} = \tilde{q}_{kl}(y)\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l} + \tilde{B}_k(y)\frac{\partial}{\partial y_k},$$

we suppose the following condition instead of **A3**:

A3' The diffusion coefficients $\tilde{q}_{kl}(y)$ and their first order derivatives are uniformly bounded:

$$|\tilde{q}_{kl}(y)| + |\nabla_y \tilde{q}_{kl}(y)| \le c,$$

and the operator $\tilde{L}$ is uniformly elliptic:

$$\tilde{q}_{kl}\eta_k\eta_l \ge c|\eta|^2, \quad \eta \in R^d,$$

for all $y \in R^d$. Furthermore, there exist $\mu > -1$, $R > 0$ and $c > 0$ such that

$$\frac{\tilde{B}(y)\cdot y}{|y|} \le -c|y|^\mu$$

for all y from $\{y\ :\ |y|\geq R\}$. The vector function $\tilde B(y)$ admits polynomial estimate:

$$|\tilde B(y)|+|\nabla\tilde B(y)|\leq c(1+|y|^\kappa)$$

with some $\kappa>0$.

According to [5] under the above conditions **A3'** the process ζ_s (and thus ξ_s) possesses unique invariant measure with a smooth density $\rho(y)$; this density satisfies the equation

$$\tilde L^*\rho=0,\qquad \int\limits_{R^d}\rho(y)dy=1,$$

and decays at the infinity faster than any negative power of $|y|$; the notation $\tilde L^*$ is used for the adjoint operator. Moreover, for the stationary process ζ_s with generator $\tilde L$ the strong mixing coefficient $\alpha(s)$ decays at the infinity faster than any negative power of s. Therefore, condition **A3** holds for ξ_s.

The distribution of the process $\xi_s=\zeta_{-s}$ in the space of trajectories coincides with the corresponding distribution of the stationary diffusion process with generator

$$L=(\rho(y))^{-1}\tilde L^*(\rho(y)\cdot)=$$
$$=\tilde q_{kl}(y)\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}+\Big((\rho(y))^{-1}\frac{\partial}{\partial y_i}[\rho(y)\tilde q_{ki}(y)]-\tilde B_k(y)\Big)\frac{\partial}{\partial y_k};$$

from now on we identify these processes.

Under assumptions A1–A3, for any initial condition $u_0\in L^2(R^n)$ and any $\varepsilon>0$, problem (2) almost surely has unique solution $u^\varepsilon\in L^2(0,T;H^1(R^n))\cup C(0,T;L^2(R^n))$; for each $\varepsilon>0$ the family of solutions generates a Radon probability measure in the space

$$V=L^2_w(0,T;H^1(R^n))\cup C(0,T;L^2_w(R^n))$$

equiped with the Borel σ-algebra; symbol w stands for the weak topology. We denote this measure defined as the distribution of u^ε in V, by Q^ε.

3 Tightness

In the section we establish the tightness results for the family of distributions Q^ε defined above. To this end we obtain a number of a priori estimates for the solution u^ε of problem (2) and then derive the tightness in a standard way by means of Prokhorov criterium (see [10], [11]).

In general, the family Q^ε is not tight. In order to make it tight we introduce moving coordinates $(x', t) = (x - \bar{b}/\varepsilon, t)$ with a constant vector $\bar{b}$, consider $u^\varepsilon(x', t)$ and show that, under proper choice of $\bar{b}$, the family of distributions obtained is tight.

Proposition 3.1 *Uniformly in $\varepsilon > 0$ the estimate holds*

$$\sup_{t \le T} \Big(\|u^\varepsilon(t)\|^2 + \int_0^t \|\nabla u^\varepsilon(s)\|^2 ds \Big) \le c. \tag{3}$$

There exist a constant vector $\bar{b}$ such that for any $\varphi(x) \in C_0^\infty(R^n)$ and any $\gamma > 0$ the relation holds

$$\lim_{\nu \to 0} \sup_{\varepsilon > 0} \mathbf{P} \left\{ \sup_{|t-s|<\nu} \Big| (u^\varepsilon(t), \varphi(\cdot + \frac{\bar{b}}{\varepsilon} t)) - (u^\varepsilon(s), \varphi(\cdot + \frac{\bar{b}}{\varepsilon} s)) \Big| \ge \gamma \right\} = 0. \tag{4}$$

Proof Consider an auxiliary problem

$$\frac{\partial}{\partial s} p(z,s) + \frac{\partial}{\partial z_i}(a_{ij}(z, \xi_s) \frac{\partial}{\partial z_j} p(z,s)) - \frac{\partial}{\partial z_i}(b_i(z, \xi_s) p(z,s)) = 0, \tag{5}$$

$$(z, t) \in T^n \times (0, (T+1)/\varepsilon^2),$$

$$p(z,s)|_{s=(T+1)/\varepsilon^2)} = 1.$$

By the maximum principle (see [7]) the solution $p(z,s)$ is strictly positive; moreover, considering the structure of the equation and integrating by parts over a set $T^n \times (s, (T+1)/\varepsilon^2)$ we get

$$\int_{T^n} p(z,s) dz = 1.$$

>From the Harnack inequality (see [6]) it follows that

$$0 < c_1 \le p(z,s) \le c_2 < \infty, \tag{6}$$

where c_1 and c_2 depend neither on ε nor on a trajectory of ξ_s.

Now, if we multiply (2) by $p(x/\varepsilon, s/\varepsilon^2)u^\varepsilon(x,s)$ and integrate it over $R^n \times (0,t)$, then after simple transformations we find

$$\frac{1}{2}\int_{T^n}(u^\varepsilon(x,t))^2 p(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})dx - \frac{1}{2}\int_{T^n}(u^\varepsilon(x,t))^2 p(\frac{x}{\varepsilon},0)dx+$$

$$+\int_0^t\int_{T^n} a_{ij}(\frac{x}{\varepsilon},\xi_{s/\varepsilon^2})p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\frac{\partial}{\partial x_i}u^\varepsilon(x,s)\frac{\partial}{\partial x_j}u^\varepsilon(x,s)dxds-$$

$$-\frac{1}{2}\int_0^t\int_{T^n}(u^\varepsilon)^2\left[\frac{\partial}{\partial t}p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})dxds + \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}(a_{ij}(\frac{x}{\varepsilon},\xi_{s/\varepsilon^2})p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}))-\right.$$

$$\left.-\frac{\partial}{\partial x_i}(b_i(\frac{x}{\varepsilon},\xi_{s/\varepsilon^2})p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2}))\right]dxds \;=\; 0$$

According to (5) the last integral here is equal to 0. Combining this with (6) we derive (3).

Now we proceed to (4). The technique we use relies on the following statement.

Lemma 3.2 *Let the initial condition $u_0(x)$ in (2) belong to C_0^∞. Then there is a nonrandom function $\kappa_1(\varepsilon)$ vanishing as $\varepsilon \downarrow 0$, such that*

$$\sup_{(x,t)\in R^n\times(0,T)} \operatorname*{osc}_{Q_x^\varepsilon} u^\varepsilon \le \kappa_1(\varepsilon), \tag{7}$$

where we denote $Q_x^\varepsilon = x + (-\varepsilon/2, \varepsilon/2)^n$.

Proof Let us consider an auxiliary periodic Cauchy problem:

$$\frac{\partial}{\partial s}v(z,s) - \frac{\partial}{\partial z_i}a_{ij}(z,\xi_s)\frac{\partial}{\partial z_j}v(z,s) - b_i(z,\xi_s)\frac{\partial}{\partial z_i}v(z,s) = 0, \quad z\in T^n,\ s>s_0,$$

$$v|_{s=s_0} = v_0(z).$$

Its solution $v(z,s)$ converges at the exponential rate to a constant as $s - s_0 \to \infty$; in particular,

$$\operatorname*{osc}_{T^n}\ v(\cdot, s_0+\varepsilon^{-1/2}) \le c' \exp(-c\varepsilon^{-1/2})|v_0|_{L^\infty} \tag{8}$$

with c' and c independent of s_0 and of a trajectory $\xi_\cdot$. Indeed, under assumptions **A1** and **A2** above the solution $v(z,s)$ satisfies uniform in s_0 and $\{\xi_\cdot\}$ Harnack inequality (see, for instance, [6]) which, in turn, implies the bound required.

The operator $A^\varepsilon = \frac{\partial}{\partial x_i} a_{ij}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\Big)\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon} b_i\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\Big)\frac{\partial}{\partial x_i}$ commutes with any shift operator of the form $S_j u(x) = u(x+\varepsilon j)$, $j \in Z^n$. Therefore, by the maximum principle,

$$|u^\varepsilon(x+\varepsilon j, t) - u^\varepsilon(x,t)| \le M\varepsilon|j|, \qquad j \in Z^n, \tag{9}$$

where $M = \max\limits_x |\nabla u_0(x)|$.

Next we fix arbitrary $t_0 \ge 0$ and $x_0 \in R^n$, restrict $u^\varepsilon(x,t_0)$ onto $Q^\varepsilon_{x_0}$ and denote the periodic in x extension of this function, with period ε in all the coordinate directions, by $\tilde v^\varepsilon_0$. By virtue of (9) we have

$$|\tilde v^\varepsilon_0(x) - u^\varepsilon(x,t_0)| \le M\varepsilon^{1/4}, \tag{10}$$

for all x such that $|x - x_0| \le \varepsilon^{1/4}$.

Let $p(t,t',x,x')$ be a fundamental solution of problem (2). According to [8], under assumptions **A1**-**A2** for any x, x' such that $|x-x'| > \varepsilon^{1/4}$, the following estimate holds

$$p(t_0, t_0+\varepsilon^{3/2}, x, x') \le c' \exp(-c|x-x'|^2/\varepsilon^{3/2}); \tag{11}$$

where c and c' only depend on the constants in **A1**-**A2**. Thus, integrating (11) over $\{x\,:\,|x-x'|>\varepsilon^{1/4}\}$, we get

$$\int\limits_{|x-x'|>\varepsilon^{1/4}} p(t_0, t_0+\varepsilon^{3/2}, x, x')dx \le c\varepsilon \tag{12}$$

Denoting by $\tilde v^\varepsilon(x,t)$ a solution of the equation $\partial_t \tilde v^\varepsilon - A^\varepsilon \tilde v^\varepsilon = 0$ with the initial condition $\tilde v^\varepsilon|_{t=t_0} = \tilde v_0(x)$, the estimate (8) reads

$$\operatorname*{osc}_{R^n} \tilde v^\varepsilon(\cdot, t_0+\varepsilon^{3/2}) \le c' \exp(-c\varepsilon^{-1/2})|\tilde v_0|_{L^\infty}.$$

On the other hand, for any $x \in Q^\varepsilon_{x_0}$, from (12) and (10) it follows

$$|u^\varepsilon(x, t+\varepsilon^{3/2}) - \tilde v^\varepsilon(x+\varepsilon^{3/2})| \le c\varepsilon|u_0|_{L^\infty} + cM\varepsilon^{1/4}.$$

The last two inequalities imply the desired statement for all $t \geq \varepsilon^{3/2}$. For small t this statement is a trivial consequence of (12). This completes the proof of Lemma.

Next, we consider a sequence of problems

$$\begin{gathered} \frac{\partial}{\partial s} p^N(z,s) + A^* p^N = 0, \quad (z,s) \in T^d \times (-\infty, N), \\ p^N(z,N) = 1; \end{gathered} \tag{13}$$

here and later on we use the notation A and A^* for the operators $\frac{\partial}{\partial z_i} a^{ij}(z,\xi_s) \frac{\partial}{\partial z_j} + b^i(z,\xi_s)\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial z_i} a^{ij}(z,\xi_s)\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i}(b^i(z,\xi_s)\cdot)$ respectively.

By the same arguments as above one can show that the functions p^N satisfy the bounds (6) uniformly in N and s, $-\infty < s \leq N$.

Lemma 3.3 *The sequence p^N converges, as $N \to \infty$, to a stationary ergodic process taking on values in $C(T^d)$ and satisfying the equation*

$$\frac{\partial}{\partial s} p + A^* p = 0, \quad \int\limits_{T^d} p(z,s)dz = 1, \qquad s \in (-\infty, +\infty). \tag{14}$$

Moreover, there are nonrandom constants $c > 0$ and $c_1 > 0$ such that

$$\max_{\substack{z \in T^d \\ k \leq s \leq k+1}} |p^N(z,s) - p(z,s)| \leq c_1 \exp(-c(N-k)). \tag{15}$$

Proof To obtain the convergence and the estimate (15) consider the following problem

$$\frac{\partial}{\partial s} q + A^* q = 0, \quad q|_{s=N} = q_0, \tag{16}$$

with $q_0 \in L^2(T^d)$ subject to

$$\int\limits_{T^d} q_0(z)dz = 0. \tag{17}$$

It suffices to show that a solution to the latter problem decays exponentially as $(N-s) \to \infty$:

$$|q(z,s)| \leq c_1 \|q_0\|_{L^2(T^d)} \exp(-c(N-s)). \tag{18}$$

Indeed, it is obvious that for any $N > 0$ and $k > 0$ the difference $(p^{N+k} - p^N)$ obeys the equation

$$\frac{\partial}{\partial s}(p^{N+k} - p^N) + A^*(p^{N+k} - p^N) = 0, \quad s < N,$$

and, also, that the relation

$$\int\limits_{T^d} (p^{N+k}(z,s) - p^N(z,s))dz = 0, \qquad \|(p^{N+k} - p^N)\|_{L^\infty} \leq c,$$

holds for any $s \leq N$. Therefore, by (18) we get

$$\|(p^{N+k} - p^N)\|_{C([k,k+1]\times T^d)} \leq c_1 \exp(-c(N-k))$$

By the Cauchy criterion p^N converges to a limit continuous function $p(z,s)$ and (15) holds. Then, passing to the limit in (13), we obtain (14).

To establish (18) consider an adjoint problem

$$\begin{gathered} \frac{\partial}{\partial s}\nu(z,s) - A\nu(z,s) = 0, \quad (z,s) \in T^d \times (s_0, +\infty), \\ \nu(z,s_0) = \varphi(z), \end{gathered} \tag{19}$$

and let φ be an arbitrary $L^2(T^d)$-function. Multiplying this equation by $q(z,s)$, taking the integral over $T^d \times [s_0, N]$ and integrating by parts, we have

$$\int\limits_{T^d} \nu(z,N)q_0(z)dz = \int\limits_{T^d} \varphi(z)q(z,s_0)dz. \tag{20}$$

By the Nash inequality (see [7])

$$\max|\nu(z,s_0+1)| \leq c\|\varphi\|_{L^2}. \tag{21}$$

Denote by $\nu^+(z,s_1)$ and $\nu^-(z,s_1)$ the positive and negative parts of $\nu(z,s_1)$ respectively:

$$\nu^+(z,s_1) = \max(\nu(z,s_1),0); \qquad \nu^-(z,s_1) = -\min(\nu(z,s_1),0).$$

Subtracting if necessary an appropriate constant we may assume that $\max \nu(z, s_1) = -\min \nu(z, s_1)$ and hence $\|\nu^+(\cdot, s_1)\|_{L^\infty} = \|\nu^-(\cdot, s_1)\|_{L^\infty}$. Then for a solution to the problem

$$\frac{\partial}{\partial s}\nu^1(z, s) + A\nu^1(z, s) = 0, \quad (z, s) \in T^d \times (s_1, +\infty),$$
$$\nu^1(z, s_1) = \nu^+(z, s_1)$$

the Harnack inequality implies

$$\max \nu^1(z, s_1 + 1) \leq c_2 \min \nu^1(z, s_1 + 1),$$

where c_2 only depends on the constants from **A1** and **A2**. Combining this bound with evident estimate

$$0 \leq \nu^1 \leq \|\nu^+(\cdot, s_1)\|_{L^\infty},$$

we find

$$0 < c_3 \leq \nu^1(z, s_1 + 1) \leq \|\nu^+(\cdot, s_1)\|_{L^\infty}.$$

Similarly, ν^2 defined as a solution to the problem

$$\frac{\partial}{\partial s}\nu^2(z, s) + A\nu^2(z, s) = 0, \quad \nu^2(z, s_1) = \nu^-(z, s_1),$$

can be estimated as follows

$$0 < c_3 \leq \nu^2(z, s_1 + 1) \leq \|\nu^-(\cdot, s_1)\|_{L^\infty}.$$

$>$From the last two inequalities it follows that there is a constant $c_4 > 0$ such that for all $s > s_0 + 1$

$$\operatorname*{osc}_{T^d} \nu(z, s + 1) \leq (1 - c_4) \operatorname*{osc}_{T^d} \nu(z, s). \tag{22}$$

Considering (21) and (22) we derive

$$\operatorname*{osc}_{T^d} \nu(\cdot, s) \leq c_1 \exp(-c(s - s_0)) \|\varphi\|_{L^2(T^d)}$$

for all $s \geq s_0 + 1$. Finally, by (20), (17) and the latter bound we get

$$\left| \int\limits_{T^d} \varphi(z) q(z, s_0) dz \right| = \left| \int\limits_{T^d} \nu(z, N) q_0(z) dz \right| \leq$$

$$c_1 \exp(-c(N - s_0)) \|q_0\|_{L^2(T^d)} \|\varphi\|_{L^2(T^d)},$$

for any $\varphi \in L^2(T^d)$. Therefore,

$$\|q(\cdot, s_0)\|_{L^2(T^d)} \leq c_1 \exp(-c(N - s_0)) \|q_0\|_{L^2(T^d)}$$

and

$$|q(z, s_0)| \leq c_1 \exp(-c(N - s_0)) \|q_0\|_{L^2(T^d)};$$

the latter bound follows from the preceding one by virtue of the Nash estimate.

Next, we define functions $\tilde{p}^N(z, s)$ to be equal to $p^{N+s}(z, s)$ for each $s \in R$, where p^N are solutions to (13). Clearly, for each $N > 0$, $\tilde{p}^N$ is a stationary ergodic process with values in $C(T^d)$. As was proved above, $\tilde{p}^N(z, s)$ converges at the exponential rate, as $N \to \infty$, to $p(z, s)$. Consequently, $p(\cdot, s)$ is also stationary ergodic process. This completes the proof of Lemma.

Let us now introduce the following constant vector

$$\bar{b}^i = \mathbf{E} \int\limits_{T_d} \left(\frac{\partial}{\partial z_j} a^{ij}(z, \xi_s) + b^i(z, \xi_s)\right) p(z, s) dz,$$

and stationary random process with values in R^d

$$\eta^i(s) = \int\limits_{T^d} \left(\frac{\partial}{\partial z_j} a^{ij}(z, \xi_s) + b^i(z, \xi_s) - \bar{b}^i\right) p(z, s) dz. \tag{23}$$

All the processes involved in the definition of $\bar{b}$ are ergodic stationary thus by the Birkhoff theorem we have

$$\bar{b}^i = \lim_{T \to \infty} \frac{1}{T} \int\limits_0^T \int\limits_{T_d} \left(\frac{\partial}{\partial z_j} a^{ij}(z, \xi_s) + b^i(z, \xi_s)\right) p(z, s) dz ds \tag{24}$$

for a.a. $\omega \in \Omega$.

Lemma 3.4 *The process $\eta(s)$ satisfies functional Central Limit Theorem (invariance principle) with the covariance matrix given by*

$$\sigma^2 = \int_0^\infty \mathbf{E}\eta(s)\eta(0)ds.$$

That is

$$\varepsilon \int_0^{\cdot/\varepsilon^2} \eta(s)ds \xrightarrow[\varepsilon\to 0]{\mathcal{L}} \sigma w.,$$

where $w.$ is a standard d-dimensional Brownian motion.

Proof First we are going to show that there are constants $c > 0$ and $c_1 > 0$ such that

$$\|\mathbf{E}\{\eta_0|\mathcal{F}_{\geq T}\}\|_{L^2(\Omega)} \leq c_1(\exp(-cT) + \rho(T/2)) \tag{25}$$

for all $T > 0$. To this end we represent the function $p(z,s)$, $0 \leq s \leq T/2$, as a sum $p(z,s) = p^1(z,s) + p^2(z,s)$, where both p^1 and p^2 satisfy the equation $\frac{\partial}{\partial s}p^i(z,s) + A^* p^i(z,s) = 0$, $s < T/2$, with the initial conditions $p^1|_{s=T/2} = 1$ and $p^2|_{s=T/2} = p(z,T/2) - 1$ respectively. Then, $\eta(0) = \eta^1(0) + \eta^2(0)$ with

$$\eta^{m,i}(0) = \int_{T^d} ((\frac{\partial}{\partial z_j} a^{ij}(z,\xi_0) + b^i(z,\xi_0))p^m(z,0)dz, \quad m = 1,\, 2.$$

By the definition, $p^1(0)$ is measurable with respect to $\mathcal{F}_{\leq T/2}$, so is $\eta^1(0)$. Taking into account the mixing condition **A3**, we obtain

$$\|\mathbf{E}\{\eta^1(0)|\mathcal{F}_{\geq T}\|_{L^2(\Omega)} \leq \rho(T/2)\|\eta^1(0)\|_{L^2(\Omega)} \leq C\rho(T/2) \tag{26}$$

It follows from (15) that

$$|p^2(z,0)| = |p(z,0) - p^{T/2}(z,0)| \leq c_1 \exp(-cT/2),$$

and, therefore,

$$\|\mathbf{E}\{\eta^2(0)|\mathcal{F}_{\geq T}\|_{L^2(\Omega)} \leq \|\eta^2(0)\|_{L^2(\Omega)} \leq C\exp(-cT/2).$$

Finally, (25) is a consequence of the latter bound and (26).

According to [4], Chapter 9, §2, Theorem 1, the relation (25) and Assumption **A3** ensure the function CLT for the process $\eta(-s)$, which, in turn, implies the function CLT for $\eta(s)$. Indeed, the compactness of the family $\left\{\varepsilon \int_0^{t/\varepsilon^2} \eta(s)ds\right\}$ in $(C([0,T])^d$ follows from the compactness of $\left\{\varepsilon \int_0^{t/\varepsilon^2} \eta(-s)ds\right\}$ and from the stationarity, by virtue of Prokhorov's theorem (see [25]). The fact that the convergence of finite dimensional distributions of $\left\{\varepsilon \int_0^{t/\varepsilon^2} \eta(-s)ds\right\}$ implies the corresponding convergence of finite dimensional distributions of $\left\{\varepsilon \int_0^{t/\varepsilon^2} \eta(s)ds\right\}$ is trivial.

Next, we consider an auxiliary problem

$$\frac{\partial}{\partial s}\psi(z,s) + A^*\psi(z,s) = \frac{\partial}{\partial z}(a(z,\xi_s)p(z,s)) + a(z,\xi_s)\frac{\partial}{\partial s}p(z,s) + \tag{27}$$

$$+b(z,\xi_s)p(z,s) - \bar{b}p(z,s) - \eta(s)p(z,s), \quad (z,s) \in T^d \times (-\infty,+\infty).$$

Lemma 3.5 *Problem (27) has a stationary ergodic solution. Under the normalization*

$$\int\limits_{T^d} \psi(z,s)dz = 0$$

the solution is unique and ergodic.

Proof. Denote the right hand side in (27) by $F(z,s)$ and consider a sequence of problems

$$\frac{\partial}{\partial s}\psi^N(z,s) + A^*\psi^N(z,s) = \mathbf{1}_{\{N-1<s\le N\}}F(z,s), \quad (z,s) \in T^d \times (-\infty,N)$$

$$\psi^N|_{s=N} = 0.$$

Taking into account the fact that $\int_{T^d} F(z,s)dz \equiv 0$, one can easily verify that

$$\int\limits_{T^d} \psi^N(z,s) = 0$$

for all s, where we put $\psi^N = 0$ for $s > N$. By the estimate (18) we get

$$\|\psi^N\|_{L^\infty(T^d\times(k,k+1))} \le c_1 \exp(-c(N-k))$$

for all $k \leq N$. Summing up in N, $-\infty < n < +\infty$, we obtain a stationary solution $\psi = \sum\limits_{N=-\infty}^{+\infty} \psi^N$ to problem (27). Moreover,

$$\|\psi\|_{L^\infty(T^d\times(-\infty,+\infty))} \leq C \tag{28}$$

for some nonrandom constant C. The uniqueness and ergodicity can now be proved in the same way as in Lemma 3.3.

To complete the proof of the compactness of $\{u^\varepsilon\}$ we consider, for arbitrary test function $\varphi \in C_0^\infty(R^d)$, the expression

$$(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon \varphi) + \varepsilon(\tilde{u}^\varepsilon, \tilde{\psi}^\varepsilon \nabla_x \varphi),$$

where

$$\tilde{u}^\varepsilon(x,t) = u^\varepsilon(x - \varepsilon^{-1}\bar{b}t - \varepsilon^{-1}\int_0^t \eta(s/\varepsilon^2)ds, t),$$

$$\tilde{p}^\varepsilon(x,t) = p(x/\varepsilon - \varepsilon^{-2}\bar{b}t - \varepsilon^{-2}\int_0^t \eta(s/\varepsilon^2)ds, t/\varepsilon^2)$$

and

$$\tilde{\psi}^\varepsilon(x,t) = \psi(x/\varepsilon - \varepsilon^{-2}\bar{b}t - \varepsilon^{-2}\int_0^t \eta(s/\varepsilon^2)ds, t/\varepsilon^2).$$

Considering the above equations and using the notation $\tilde{\varphi} = \varphi(x + \varepsilon^{-1}\bar{b}t + \varepsilon^{-1}\int_0^t \eta(s/\varepsilon^2)ds)$, $\nabla_x^\varepsilon r(x) = \nabla_z r(z)|_{z=x/\varepsilon}$ and $\partial_t^\varepsilon r(t) = \frac{\partial}{\partial s} r(s)|_{s=t/\varepsilon^2}$ for a generic function r, after integration by parts and simple transformations we obtain

$$\begin{aligned}
\frac{d}{dt}[(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon \varphi) + \varepsilon(\tilde{u}^\varepsilon, \tilde{\psi}^\varepsilon \nabla_x \varphi)] &= \frac{d}{dt}[(u^\varepsilon, p^\varepsilon \tilde{\varphi}) + \varepsilon(u^\varepsilon, \psi^\varepsilon \nabla_x \tilde{\varphi})] = \\
&= (A^\varepsilon u^\varepsilon, p^\varepsilon \tilde{\varphi}) + (u^\varepsilon, \tilde{\varphi}\frac{\partial}{\partial t}p^\varepsilon) + \varepsilon^{-1}(\bar{b} + \eta(t/\varepsilon^2)) \cdot (u^\varepsilon, p^\varepsilon \nabla_x \tilde{\varphi}) + \\
+\varepsilon(A^\varepsilon u^\varepsilon, \psi^\varepsilon \nabla_x \tilde{\varphi}) &+ \varepsilon(u^\varepsilon, \nabla_x \tilde{\varphi}\frac{\partial}{\partial t}\psi^\varepsilon) + (\bar{b} + \eta(t/\varepsilon^2)) \cdot (u^\varepsilon, \psi^\varepsilon \nabla_x \nabla_x \tilde{\varphi}) = \quad (29) \\
= (u^\varepsilon, p^\varepsilon a^\varepsilon \nabla_x \nabla_x \tilde{\varphi}) &+ (u^\varepsilon \nabla_x \nabla_x \tilde{\varphi}, [\nabla_x^\varepsilon(a\psi) + a^\varepsilon \nabla_x^\varepsilon \psi + b^\varepsilon \psi^\varepsilon - \bar{b}\psi^\varepsilon - \eta^\varepsilon \psi^\varepsilon]) + \\
&+\varepsilon(u^\varepsilon \psi^\varepsilon, a^\varepsilon \nabla_x \nabla_x \nabla_x \tilde{\varphi}).
\end{aligned}$$

Hence, due to (3) and Lemma 3.3 and 3.5, the estimate

$$|(\tilde{u}^\varepsilon(t), \tilde{p}^\varepsilon(t)\varphi) - (\tilde{u}^\varepsilon(s), \tilde{p}^\varepsilon(s)\varphi)| \le c|t-s|\|\varphi\|_{C^3} \tag{30}$$

holds. Approximating if necessary $u_0 \in L^2(R^d)$ by a sequence $u_0^n \in C_0^\infty(R^d)$ and quoting (3), we may assume that $u_0 \in C_0^\infty(R^d)$.

Then, by (30) and Lemma 3.2 we get

$$|(\tilde{u}^\varepsilon(t), \varphi) - (\tilde{u}^\varepsilon(s), \varphi)| \le c|t-s|\|\varphi\|_{C^3} + c\kappa(\varepsilon)\|\varphi\|_{C^1}. \tag{31}$$

If we denote $\hat{u}^\varepsilon(x,t) = u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t)$ then

$$\begin{gathered}
|(\hat{u}^\varepsilon(t), \varphi) - (\hat{u}^\varepsilon(s), \varphi)| = |(\tilde{u}^\varepsilon(t), \varphi(\cdot + \varepsilon^{-1}\int_0^t \eta(\tau/\varepsilon^2)d\tau)) - \\
-(\tilde{u}^\varepsilon(s), \varphi(\cdot + \varepsilon^{-1}\int_0^s \eta(\tau/\varepsilon^2)d\tau))| \le \\
\le |(\tilde{u}^\varepsilon(t), \varphi(\cdot + \varepsilon^{-1}\int_0^s \eta(\tau/\varepsilon^2)d\tau)) - (\tilde{u}^\varepsilon(s), \varphi(\cdot + \varepsilon^{-1}\int_0^t \eta(\tau/\varepsilon^2)d\tau))| + \qquad (32) \\
+|(\tilde{u}^\varepsilon(t), \varphi(\cdot + \varepsilon^{-1}\int_0^t \eta(\tau/\varepsilon^2)d\tau) - \varphi(\cdot + \varepsilon^{-1}\int_0^s \eta(\tau/\varepsilon^2)d\tau))| \le \\
\le c|t-s|\|\varphi\|_{C^3} + c\kappa(\varepsilon)\|\varphi\|_{C^1} + \|\varphi\|_{C^1}\left|\varepsilon^{-1}\int_s^t \eta(\tau/\varepsilon^2)d\tau\right|.
\end{gathered}$$

For any $\varphi \in C_0^\infty(R^d)$ the first two terms on the r.h.s. of the latter formula go to zero, as $\varepsilon \to 0$ and $|t-s| \to 0$, uniformly in t, s and $\omega \in \Omega$. By Lemma 3.4 the integral $\varepsilon^{-1}\int_0^t \eta(\tau/\varepsilon^2)d\tau$ satisfies the function CLT in $(C[0,T])^d$ for any finite interval $[0,T]$. Thus, applying twice the Prokhorov criterion of compactness, we derive the compactness of the family $(\hat{u}^\varepsilon, \varphi)$ in $C(0,T)$.

4 Passage to the limit

The aim of this section is to show that the family $\{\tilde{u}^\varepsilon\}$ converges in probability and a.s. towards the unique solution to a certain Cauchy problem for deterministic parabolic operator with constant coefficients. First we justify the convergence in the space V defined above. Then, in the next section, we will prove that the convergence in the space with a stronger topology does also take place.

Since the compactness of $\{\tilde{u}^\varepsilon\}$ has been proved, in order to characterize the limit distribution it suffices to pass to the limit in the expressions of the form $(\tilde{u}^\varepsilon, \varphi)$, with $\varphi \in C_0^\infty$.

By (29) and Lemma 3.2 we have

$$(\tilde{u}^\varepsilon(t), \varphi) - (u_0, \varphi) = (\tilde{u}^\varepsilon(t), p^\varepsilon(t)\varphi) + \varepsilon(\tilde{u}^\varepsilon(t), \psi^\varepsilon(t)\nabla_x\varphi) -$$

$$-(u_0, p^\varepsilon(0)\varphi) - \varepsilon(u_0, \psi^\varepsilon(0)\nabla_x\varphi) + O(\varepsilon) =$$

$$= \int_0^t (u^\varepsilon(t), a^\varepsilon p^\varepsilon(t)\nabla_x\nabla_x\tilde{\varphi})dt + \int_0^t (u^\varepsilon(t), a^\varepsilon \nabla_x^\varepsilon \psi^\varepsilon(t)\nabla_x\nabla_x\tilde{\varphi})dt + \tag{33}$$

$$+ \int_0^t (u^\varepsilon(t), \nabla_x^\varepsilon(a^\varepsilon\psi^\varepsilon(t))\nabla_x\nabla_x\tilde{\varphi})dt + \int_0^t (u^\varepsilon(t), b^\varepsilon\psi^\varepsilon(t)\nabla_x\nabla_x\tilde{\varphi})dt -$$

$$- \int_0^t (u^\varepsilon(t), \bar{b}\psi^\varepsilon(t)\nabla_x\nabla_x\tilde{\varphi})dt - \int_0^t (u^\varepsilon(t), \eta^\varepsilon(t)\psi^\varepsilon(t)\nabla_x\nabla_x\tilde{\varphi})dt + O(\varepsilon).$$

The following statement allows to pass to the limit in the above formula.

Lemma 4.1 *Let $\zeta(z,s)$ be a stationary ergodic process with values in $L^2(T^d)$ ($C(T^d)$), such that*

$$\|\zeta\|_{L^2(T^d\times(0,1))} \le C$$

for a nonrandom constant C. Then for any C_0^∞-function φ the relation holds true a.s.:

$$\lim_{\varepsilon\to 0} \left| \int_0^t (u^\varepsilon(t), \zeta^\varepsilon(t)\tilde{\varphi})dt - \overline{<\zeta>} \int_0^t (u^\varepsilon(t), \tilde{\varphi})dt \right| = 0 \tag{34}$$

where

$$\overline{<\zeta>} = \mathbf{E} \int_{T^d} \zeta(z,s)dz.$$

Proof. Without loss of generality one can assume $\overline{<\zeta>} = 0$. Denote $\mu(s) = \int_{T^d} \zeta(z,s)dz$. Then, by Lemma 3.2

$$\sup_{t\le T}\left|\int_0^t [(u^\varepsilon(\tau),\zeta^\varepsilon(\tau)\tilde\varphi) - \mu(\frac{\tau}{\varepsilon^2})(u^\varepsilon(\tau),\tilde\varphi)]d\tau\right| \le C\kappa(\varepsilon)$$

with nonrandom C. Thus, it is sufficient to show that

$$\lim_{\varepsilon\to 0}\sup_{t\le T}\left|\int_0^t \mu(\frac{\tau}{\varepsilon^2})(u^\varepsilon(\tau),\tilde\varphi)d\tau\right| = 0 \tag{35}$$

It follows from (31) that for any $\varphi\in C_0^\infty(R^d)$ there is a compact subset $K\subset C(0,T)$ such that $(\tilde u^\varepsilon(\cdot),\varphi)\in K$ a.s. The rest of the proof is standard, we consider a finite δ-net consisting of step functions and use the Birkhoff theorem to obtain the limit relation

$$\lim_{\varepsilon\to 0}\left|\int_0^t \mu(\frac{\tau}{\varepsilon^2})(u^\varepsilon(\tau),\tilde\varphi)d\tau\right| = 0$$

valid for all $t\le T$. It remains to combine this relation with the estimate

$$\left|\int_s^t \mu(\frac{\tau}{\varepsilon^2})(u^\varepsilon(\tau),\tilde\varphi)d\tau\right| \le c\|\mu(\frac{\cdot}{\varepsilon^2})\|_{L^2(0,T)}|t-s|^{1/2},$$

and (35) follows.

Denote by $\bar a^{ij}$ the "homogenized" matrix:

$$\bar a^{ij} = \mathbf{E}\int_{T^d}[a^{ij}(z,\xi_s)p(z,s) + a^{ik}(z,\xi_s)\frac{\partial}{\partial z_k}\psi^j(z,s) + b^i(z,\xi_s)\psi^j(z,s)]dz;$$

this definition is natural since

$$\int_{T^d}\left(\frac{\partial}{\partial z_k}(a^{ik}(z,\xi_s)\psi^j(z,s)) - \bar b^i\psi^j(z,s) - \eta^i(s)\psi^j(z,s)\right)dz = 0.$$

By Lemma 4.1, considering (33), we obtain

$$\lim_{\varepsilon\to 0}\sup_{t\le T}\left|(\tilde u^\varepsilon(t),\varphi)-\int\limits_0^t(\tilde u^\varepsilon(s),a^\varepsilon(s)\nabla_x\nabla_x\varphi)ds\right|=0$$

a.s. for any $\varphi\in C_0^\infty$. Thus, for a typical realization of $\xi_\cdot$ any accumulation point of $\{\tilde u^\varepsilon\}$ in V satisfies the equation

$$\frac{\partial}{\partial t}u^0-\bar A u^0=0,\quad u^0|_{t=0}=u_0, \tag{36}$$

where $\bar A=\bar a^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$. Due to the uniqueness of solution of the latter problem the family $\{\tilde u^\varepsilon\}$ converges a.s. in V, as $\varepsilon\to 0$, towards u^0.

In fact, the above convergence can be improved. Namely, we are going to show that under a proper choice of stationary ergodic process $\chi(z,s)$ with values in $C(T^d)$, the function

$$\begin{gathered}v^\varepsilon(x,t)\equiv u^\varepsilon(x,t)-u^0\Big(x-\bar b\tfrac{t}{\varepsilon}-\varepsilon^{-1}\int\limits_0^t\eta(\tfrac{\tau}{\varepsilon^2})d\tau,t\Big)-\\-\varepsilon\nabla_x u^0\Big(x-\bar b\tfrac{t}{\varepsilon}-\varepsilon^{-1}\int\limits_0^t\eta(\tfrac{\tau}{\varepsilon^2})d\tau,t\Big)\chi(\tfrac{x}{\varepsilon},\tfrac{t}{\varepsilon^2})\end{gathered} \tag{37}$$

converges to 0 in $L^\infty(R^d\times[0,T])\cap L^2(0,T;H^1(R^d))$-norm. To this end we substitute the expression (37) in the original problem (1), collect the terms of like powers of ε and equate the obtained expressions to 0. The first equation related to ε^{-1}, reads

$$\left(\frac{\partial}{\partial s}\chi(z,s)-A\chi(z,s)\right)\Big|_{z=\frac{x}{\varepsilon},\,s=\frac{t}{\varepsilon^2}}\nabla_x u^0(x,t)=$$

$$=\Big[-\nabla_z a(z,\xi_s)-b(z,\xi_s)-\bar b-\eta(s)\Big]\Big|_{z=\frac{x}{\varepsilon},\,s=\frac{t}{\varepsilon^2}}\nabla_x u^0(x,t)$$

Hence, we want $\chi(z,s)$ to satisfy the equation

$$\frac{\partial}{\partial s}\chi^j(z,s)-A\chi^j(z,s)-\nabla_{z_i}a^{ij}(z,\xi_s)-b^j(z,\xi_s)-\bar b^j-\eta^j(s) \tag{38}$$

Let us study the properties of this equation in detail. Multiplying it by $p(z,s)$ and integrating by parts over a cylinder $T^d \times [s_1, s_2]$, it is easy to see that

$$\int\limits_{T^d} \chi(z,s)p(z,s)dz = const;$$

the definitions of $p(z,s)$, $\bar{b}$ and $\eta(s)$ have also been used here.

Then, arguing like in the proof of Lemma 3.5 one can show that the equation (38) has unique up to an additive constant stationary solution $\chi(z,s)$; moreover, $\chi(\cdot,s)$ is an ergodic process with values in $C(T^d)$.

Under the above choice of χ we have

$$(\frac{\partial}{\partial t} - A^\varepsilon)v^\varepsilon = (\frac{\partial}{\partial t} - \bar{A})u^0 +$$

$$+a^\varepsilon \nabla_x \nabla_x u^0 + \nabla_x^\varepsilon(a^\varepsilon \chi^\varepsilon) + a^\varepsilon \nabla_x^\varepsilon \chi^\varepsilon + b^\varepsilon \chi^\varepsilon - \bar{b}\chi^\varepsilon - \eta(\frac{t}{\varepsilon^2})\chi^\varepsilon - \check{a} + O(\varepsilon), \quad (39)$$

where $O(\varepsilon)$ is vanishing in the uniform metric uniformly in $\omega \in \Omega$, and

$$\check{a}^{ij} = \mathbf{E}\int\limits_{T^d} \Big(a^{ij}(z,\xi_s) + \nabla_{z_k}(a^{ik}(z,\xi_s)\chi^j(z,s)) + a^{ik}(z,\xi_s)\nabla_{z_k}\chi^j(z,s) +$$

$$+ b^i(z,\xi_s)\chi^j(z,s)\Big)\, p(z,s)dz$$

For brevity we will use the following notation

$$\check{a}^\varepsilon_{\rm aux}(x,t) = \{\check{a}^{\varepsilon,ij}_{\rm aux}(x,t)\} = a(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}) + \nabla_x^\varepsilon(a(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2})\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})) +$$

$$+a(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2})\nabla_x^\varepsilon \chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) + b(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2})\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}) - (\bar{b} + \eta(\frac{t}{\varepsilon^2}))\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$$

and

$$< \check{a}_{\rm aux} > (s) = \int\limits_{T^d} \Big(a^{ij}(z,\xi_s) + \nabla_{z_k}(a^{ik}(z,\xi_s)\chi^j(z,s)) +$$

$$+a^{ik}(z,\xi_s)\nabla_{z_k}\chi^j(z,s) + b^i(z,\xi_s)\chi^j(z,s)\Big)\, p(z,s)dz$$

It will be clear later that $\check{a} = \bar{a}$. Thus the first term on the r.h.s. of (39) is equal to 0. Now, to achieve an energy estimate, we multiply (39) by $p^\varepsilon v^\varepsilon$ and integrate over $R^d \times (0,T)$. After simple transformations we get

$$\int\limits_{T^d}(v^\varepsilon(x,t))^2 p(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})dx + \int\limits_0^t\int\limits_{T^d} p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})a^{ij}(\frac{x}{\varepsilon},\xi_{s/\varepsilon^2})\frac{\partial}{\partial x_i}v^\varepsilon(x,s)\frac{\partial}{\partial x_j}v^\varepsilon(x,s)dxds$$

$$= \varepsilon^2\int\limits_{T^d} p(\frac{x}{\varepsilon},0)(\chi^i(\frac{x}{\varepsilon},0)\frac{\partial}{\partial x_i}u_0(x))^2dx+$$

$$+\int\limits_0^t\int\limits_{T^d}[\check{a}^{\varepsilon,ij}_{\rm aux}(x,s) - \check{a}^{ij}]p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0(x,s)v^\varepsilon(x,s)dxds =$$

$$= \varepsilon^2\int\limits_{T^d} p(\frac{x}{\varepsilon},0)(\chi^i(\frac{x}{\varepsilon},0)\frac{\partial}{\partial x_i}u_0(x))^2dx+$$

$$+\int\limits_0^t\int\limits_{T^d}[\check{a}^{\varepsilon,ij}_{\rm aux}(x,s) - <\check{a}_{\rm aux}>^{ij}(\frac{s}{\varepsilon^2})]p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0(x,s)v^\varepsilon(x,s)dxds+$$

$$+\int\limits_0^t\int\limits_{T^d}[<\check{a}_{\rm aux}>^{ij}(\frac{s}{\varepsilon^2}) - \check{a}^{ij}]p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0(x,s)v^\varepsilon(x,s)dxds.$$

The second integral on the r.h.s. can be estimated as follows:

$$\left|\int\limits_0^t\int\limits_{T^d}[\check{a}^{\varepsilon,ij}_{\rm aux}(x,s) - <\check{a}_{\rm aux}>^{ij}(\frac{s}{\varepsilon^2})]p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0(x,s)v^\varepsilon(x,s)dxds\right| =$$

$$= \varepsilon\left|\int\limits_0^t\int\limits_{T^d}\Psi^{ij,k}(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\nabla_{x_k}(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u^0(x,s)v^\varepsilon(x,s))dxds\right| \le c\varepsilon$$

with nonrandom c; here $\Psi^{ij,k}(z,s)$ stands for the functions that satisfy the relation

$$\mathrm{div}\Psi^{ij}(z,s) = (\check{a}^{\varepsilon,ij}_{\rm aux}(z,s) - <\check{a}_{\rm aux}>^{ij})p(z,s)$$

We proceed by estimating the last term. Clearly,

$$v^\varepsilon(x,t) = u^\varepsilon(x,t) - u^0(x - \bar b \frac{t}{\varepsilon} - \varepsilon^{-1}\int\limits_0^t \eta(\frac{\tau}{\varepsilon^2})d\tau, t) + O(\varepsilon)$$

uniformly in x, t and ω. Then, by (31), the family

$$(u^\varepsilon(t) - \tilde u^0(t), \nabla_x\nabla_x \tilde u^0(t)) = (\tilde u^\varepsilon(t) - u^0(t), \nabla_x\nabla_x u^0(t))$$

is compact in $(C[0,T])^{d^2}$ uniformly in ω, and the Birkhoff theorem implies the a.s. relation

$$\lim_{\varepsilon\to 0}\sup_{t\le T}\left|\int\limits_0^t\int\limits_{R^d}[< \breve a_{\rm aux} >^{ij}(\frac{s}{\varepsilon^2}) - \breve a^{ij}]p(\frac{x}{\varepsilon},\frac{s}{\varepsilon^2})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\tilde u^0(x,s)v^\varepsilon(x,s)dxds\right| = 0$$

We arrive at the following statement.

Lemma 4.2 *The difference $(u^\varepsilon - \tilde u^0)$ converges a.s., as $\varepsilon \to 0$, towards 0 in $L^\infty(0,T; L^2(R^d))$-norm. So does $(\tilde u^\varepsilon - u^0)$.*

Now we proceed with the main results. Denote by Q^ε the distribution of $\hat u^\varepsilon(x,t) = u^\varepsilon(x - \varepsilon^{-1}\bar b t, t)$ in V.

Theorem 4.3 *Let $\{u^\varepsilon\}$ be a family of solutions to problem (1), and assume $u_0 \in L^2(R^d)$. Then the distributions Q^ε converge weakly in V to the unique solution of the following martingale problem*

$$du(t) = (\bar A + \frac{1}{2}(\Lambda^2)^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j})u(t)dt + \Lambda\nabla_x u(t)dw_t \qquad (40)$$

$$u|_{t=0} = u_0,$$

where

$$(\Lambda^2)^{ij} = \int\limits_0^\infty \mathbf{E}(\eta^i(0)\eta^j(s) + \eta^j(0)\eta^i(s))ds$$

and w_s is a standard d-dimensional Wiener process.

Proof. It is easy to verify by virtue of Ito's formula that the function $u^0(x - \Lambda w_t, t)$ solves problem (40); according to [22] this problem is well-posed and, in particular, has unique solution. We then represent $u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t)$ as

$$u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t) = u^0\Big(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2})ds\,,\,t\Big) + u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t) -$$

$$-u^0\Big(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2})ds\,,\,t\Big).$$

As was proved in Lemma 4.2 the $L^\infty(0,T;L^2(R^d))$-norm of the second term on the r.h.s. vanishes as $\varepsilon \to 0$. Thus for any $\varphi \in C_0^\infty$ the families $(u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t, \varphi)$ and $(u^0(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2}ds, t), \varphi)$ converge in law in $C(0,T)$ to the same limit distribution.

The mapping $F_\varphi\,:\,(C(0,T))^d \to C(0,T)$ defined by

$$F(\theta(\cdot)) = (u^0(\cdot + \theta(t), t), \varphi)$$

is continuous, therefore the convergence in law of $\{\varepsilon^{-1}\int_0^t \eta(s/\varepsilon^2)ds\}$ to Λw_t in $(C(0,T))^d$ implies the convergence in law of $(u^0(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2}ds, t), \varphi)$ towards $(u^0(x - \Lambda w_t, t), \varphi)$ and the theorem is proved.

The topology of V is quite poor. In fact, more strong convergence takes place.

Theorem 4.4 *The family $\hat{u}^\varepsilon$ converges in law in the functional space $V_1 = L^\infty(0,T;L^2(R^d))$ endowed with the topology of convergence in norm (strong topology).*

Proof. As was already proved, the difference $u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t) - u^0(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2}ds, t)$ converges a.s. to zero in V^1, i.e. in $L^\infty(0,T;L^2(R^d))$-norm. According to the Prokhorov theorem and Lemma 3.4 there is a compact subset $K \in (C(0,T))^d$ such that

$$\sup_{\varepsilon>0} \mathbf{P}\left\{\frac{1}{\varepsilon}\int_0^t \eta(\frac{s}{\varepsilon^2})ds \notin K\right\} < \delta.$$

The mapping $\Phi\,:\,(C(0,T))^d \longrightarrow V^1$ defined by

$$\Phi(\theta(\cdot)) = u^0(x + \theta(t), t),$$

is continuous. Therefore, for any $\delta > 0$ there is a compact subset K^1 of V^1 such that

$$\sup_{\varepsilon>0} \mathbf{P}\left\{ u^0(x + \frac{1}{\varepsilon}\int_0^t \eta(\frac{s}{\varepsilon^2})ds, t) \notin K^1 \right\} < \delta.$$

In view of vanishing the difference $u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t) - u^0(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2}ds, t)$, as $\varepsilon \to 0$, this implies the fact that the families $\{u^\varepsilon(x - \varepsilon^{-1}\bar{b}t, t)\}$ and $\{u^0(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2}ds, t)\}$ have the same limit law in V^1. The convergence in law of $u^0(x - \varepsilon^{-1}\int_0^t \eta(\frac{s}{\varepsilon^2}ds, t)$ towards $u^0(x - \Lambda w_t, t)$ follows from Lemma 3.4 and continuity of the mapping Φ defined above.

5 Operators with diffusive coefficients.

In this section we assume that the process $\xi_\cdot$ used in the definition of the coefficients of equation (1), is a diffusion process. In this particular case all the coefficients of effective martingale problem (40) can be found in terms of solutions to auxiliary deterministic PDE problems. Also, various sufficient conditions insuring the mixing property **A3** of $\xi_\cdot$ can be formulated in terms of coefficients of its generator.

Let us recall the notation $\zeta_s = \xi_{-s}$. If we define $(\tilde{X}_s, \zeta_s)$ to be a diffusion process with values in $T^n \times R^d$, governed by the operator

$$A + \tilde{L} = \frac{\partial}{\partial z_i} a_{ij}(z, y)) \frac{\partial}{\partial z_j} + b_i(z, y)\frac{\partial}{\partial z_i} + \tilde{q}_{kl}(y)\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l} + \tilde{B}_k(y)\Big)\frac{\partial}{\partial y_k},$$

then, according to [5], under the assumptions **A1, A2 and A3'** the process $(\tilde{X}_s, \zeta_s)$ has unique invariant measure whose density satisfies the equation

$$(A^* + \tilde{L}^*)\tilde{\rho}(z, y) = 0, \qquad \int_{T^n}\int_{R^d} \tilde{\rho}(z, y)dzdy = 1. \tag{41}$$

Furthermore, $\tilde{\rho}(z, y)$ decays faster than any negative power of $|y|$, as $|y| \to \infty$, uniformly in z, and for any $f(z, y)$ of polynomial growth in y the equation

$$(A + \tilde{L})\tilde{\chi}(z, y) = f(z, y)$$

is solvable if and only if

$$\int_{T^d}\int_{R^n} f(z,y)\tilde\rho(z,y)dzdy = 0; \tag{42}$$

the solution possesses polynomial growth as well.

Denote by $p(z,s)$ the density of conditional distribution of X_s given ξ_τ, $\tau\in[s,+\infty)$. Then $p(z,s)$ is a solution to the equation (14) (see [26]), and we have

$$\eta(s) = \mathbf{E}\left\{\mathrm{div}[a(X_s,\xi_s)] + b^i(X_s,\xi_s) - \bar b^i \Big| \sigma(\xi_\tau,\,\tau\ge s)\right\}.$$

where the process η has been defined in (23).

In what follows we always suppose $(\tilde X_s,\zeta_s)$ is a stationary process distributed with the density $\tilde\rho(z,y)$. Then (42) is equivalent to the relation $\mathbf{E}f(\tilde X_s,\zeta_s)=0$. In particular, the solvability condition is satisfied for the function $f^b(z,y) = (\mathrm{div_z}[a(z,y)] + b^i(z,y) - \bar b^i)$. We use the notation $\chi_b(z,y)$ for the corresponding solution. Applying Ito's formula to the function $\chi_b(\tilde X_s,\zeta_s)$ we get after simple transformation:

$$\int_0^t f^b(\tilde X_s,\zeta_s)ds = \chi_b(\tilde X_t,\zeta_t) + \int_0^t \nabla_z\chi_b(\tilde X_s,\zeta_s)dw_s^1 + \int_0^t \nabla_y\chi_b(\tilde X_s,\zeta_s)dw_s^2;$$

here w^1 and w^2 are independent Wiener process of dimensions d and n respectively. Taking in the preceding formula the conditional expectation w.r.t. $\tilde{\mathcal F}_{-\infty,t} = \sigma\{\zeta_\tau\,,\,-\infty<\tau\le t\} = \mathcal F_{-t,+\infty}$ and considering the independence of $\{\zeta_\cdot\}$ and w^1 we obtain

$$\frac{1}{\sqrt t}\int_0^t \eta_{-s}ds = \frac{1}{\sqrt t}\mathbf{E}\{\chi_b(\tilde X_t,\zeta_t)\}|\tilde{\mathcal F}_{-\infty,t}\} + \frac{1}{\sqrt t}\mathbf{E}\left\{\int_0^t \nabla_y\chi_b(\tilde X_s,\zeta_s)dw_s^2\Big|\tilde{\mathcal F}_{-\infty,t}\right\}$$

$$= \frac{1}{\sqrt t}\mathbf{E}\{\chi_b(\tilde X_t,\zeta_t)\}|\tilde{\mathcal F}_{-\infty,t}\} + \frac{1}{\sqrt t}\int_0^t \mathbf{E}\{\nabla_y\chi_b(\tilde X_s,\zeta_s)|\tilde{\mathcal F}_{-\infty,s}\}dw_s^2,$$

where the process η_s has been defined in(23). The first term on the r.h.s. vanishes as $t\to\infty$. Calculating the quadratic characteristics of Ito's integral in the second term gives

$$\frac{1}{t}\int\limits_0^t \mathbf{E}\{\nabla_y\chi_b^i(\tilde{X}_s,\zeta_s)|\tilde{\mathcal{F}}_{-\infty,s}\}\mathbf{E}\{\nabla_y\chi_b^j(\tilde{X}_s,\zeta_s)|\tilde{\mathcal{F}}_{-\infty,s}\}ds \longrightarrow$$

$$\mathbf{E}\Big(\mathbf{E}\{\nabla_y\chi_b^i(\tilde{X}_0,\zeta_0)|\tilde{\mathcal{F}}_{-\infty,0}\}\mathbf{E}\{\nabla_y\chi_b^j(\tilde{X}_0,\zeta_0)|\tilde{\mathcal{F}}_{-\infty,0}\}\Big) =$$

$$= \mathbf{E}\Big(\mathbf{E}\{\nabla_y\chi_b^i(\tilde{X}_0,\xi_0)|\mathcal{F}_{0,+\infty}\}\mathbf{E}\{\nabla_y\chi_b^j(\tilde{X}_0,\xi_0)|\mathcal{F}_{0,+\infty}\}\Big) = (\Lambda^2)^{ij},$$

the Birkhoff theorem has been used here. Finally, by [4] Theorem 9.1, 9.2, we have

$$\frac{1}{\sqrt{\tau}}\int\limits_0^{t\tau}\eta_{-s}ds \ \xrightarrow[\tau\to\infty]{\mathcal{L}}\ \Lambda W_t.$$

where W_t is a d-dimensional Wiener process.

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