Optimality Conditions for Piecewise-Convex Maximization Problems

Ider Tsevendorj

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Optimality Conditions for Piecewise-Convex Maximization Problems

Ider Tsevendorj

N° 3941

THÈME 4

ISSN 0249-6399 ISRN INRIA/RR--3941--FR+ENG

Rapport de recherche

Optimality Conditions for Piecewise-Convex Maximization Problems

Ider Tsevendorj

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Adopt

Rapport de recherche n° 3941 — — 20 pages

Abstract: A function $F : \mathbb{R}^n \to \mathbb{R}$ is called a *piecewise convex*-function if it decomposes as

$$F(x) = \min\{f_j(x) \mid j \in M = \{1, 2..., m\}\},$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ is convex for all $j \in M$.

Let D be a nonempty, compact, and convex subset of $\mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator and $F : \mathbb{R}^n \to \mathbb{R}$ be piecewise convex, the purpose of this article is twofold:

1. to extend necessary and sufficient optimality conditions for convex maximization problem to piecewise convex maximization problem:

$$\text{maximize} \quad F(x), \text{ subject to } x \in D \tag{PCMP}$$

2. to apply (PCMP) to a class of nonmonotone variational inequality problem, i.e. find a vector $z \in D$ such that

$$\langle T(z), x - z\rangle \leq 0, \quad \text{for all } x \in D. \tag{VIP(T,D)}$$

Both problems have many practical and theoretical applications. Solution for the latter has been extensively considered under monotonicity or pseudomonotonicity of the operator $-T(\cdot)$; here we further study the nonmonotone case.

Key-words: nonconvex and nonsmooth problem, sufficient and necessary optimality conditions, piecewise convex function, variational inequality, nonmonotone operator

(Résumé : tsvp)

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : 01 39 63 55 11 - International : +33 1 39 63 55 11
Télécopie : (33) 01 39 63 53 30 - International : +33 1 39 63 53 30

Conditions d'optimalité pour des Problèmes de maximisation convexe par morceaux[1]

Résumé : Une fonction $F : \mathbb{R}^n \to \mathbb{R}$ est dite *convexe par morceaux* si elle se décompose de la façon suivante

$$F(x) = \min\{f_j(x) \mid j \in M = \{1, 2..., m\}\},$$

où $f_j : \mathbb{R}^n \to \mathbb{R}$ est convexe pour tout $j \in M$.

Étant donnés un sous-ensemble D compact et convexe de $\mathbb{R}^n$, un opérateur continu $T : \mathbb{R}^n \to \mathbb{R}^n$ et une fonction convexe par morceaux $F : \mathbb{R}^n \to \mathbb{R}$, le but de cet article est double:

1. étendre les conditions nécessaires et suffisantes d'optimalité pour les problèmes de maximisation convexe au cas convexe par morceaux:

$$\max \quad F(x), \text{ pour } x \in D \tag{PCMP}$$

2. appliquer (PCMP) à une classe d'inégalités variationnelles, c'est à dire trouver un vecteur $z \in D$ tel que

$$\langle T(z), x - z \rangle \leq 0, \quad \text{pour tout } x \in D. \tag{VIP(T,D)}$$

Ces deux problèmes ont de nombreuses applications tant théoriques que pratiques.

Pour les inégalités variationnelles, de nombreuses solutions ont été étudiées pour des conditions particulières de monotonie ou quasi-monotonie de l'opérateur $-T(\cdot)$; dans cet article, nous étudions plus particulièrement le cas non monotone.

Mots-clé : problèmes non convexes et non différentiables, conditions nécessaires et suffisantes d'optimalité, fonction convexe par morceaux, inégalités variationnelles, opérateur non monotone

[1] This research was supported by Direction des transports terrestres, Ministère de l'Equipement des Transports et du Logement (projet N 226 75 01 63-41 50)

1. INTRODUCTION

The purpose of this article is twofold: first, to extend necessary and sufficient optimality conditions for convex maximization problem to piecewise convex maximization problem (PCMP); second, to apply (PCMP) to a class of nonmonotone variational inequality problem (VIP).

Both problems have many practical and theoretical applications [HPT95, HT90, Ben95] and [Ham84, LPR96, KS80]. Solution for the latter has been extensively considered in the literature (see in [Ham84]) under monotonicity or quasimonotonicity; here we further study the nonmonotone case.

The present paper is organized as follows. First we will take a short look on some local and global necessary and sufficient conditions for convex maximization in section 2. As a result of the section we will derive optimality conditions for nonsmooth convex maximization problem; in section 3 we extend previous result to (PCMP); in section 4 we apply (PCMP) result to some variational inequality problem.

2. CONVEX MAXIMIZATION PROBLEM

Let $D \subset \mathbb{R}^n$ be a convex and compact set and $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function; in this section, we consider the global optimization (*convex maximization*) problem:

$$\begin{aligned} &\text{maximize} \quad f(x) \\ &\text{subject to} \;\; x \in D. \end{aligned} \tag{CMP}$$

The state-of-the-art in convex maximization including many algorithms and abundant applications, is extensively described in text books [HPT95, HT90], in papers [Str98, Str93, HU95, DHL98] and surveys [Ben95].

In recent years, several interesting necessary and sufficient optimality conditions characterizing a global maximum of (CMP) have been proposed:

Strekalovsky's necessary and sufficient condition ([Str87])

$$\partial f(y) \subset N(D, y) \quad \text{for all } y \;\; : \;\; f(y) = f(z), \tag{SgNS}$$

Hiriart-Urruty's necessary and sufficient condition ([HU89])

$$\partial_\varepsilon f(z) \subset N_\varepsilon(D, z) \quad \text{for all } \varepsilon \geq 0, \tag{HUgNS}$$

and Flores-Bazan's necessary and sufficient condition ([FB97])

$$\partial_\gamma f(z) \subset \partial_\gamma \delta(\cdot|D)(z), \tag{FBgNS}$$

where $\partial f(y)$ and $N(D, y)$ are subdifferential of a function $f(\cdot)$ and normal cone to a set D at point y:

$$\begin{aligned} \partial f(y) &= \{y^* \in \mathbb{R}^n \mid \;\; f(x) - f(y) \geq \langle y^*, x - y\rangle \text{ for all } x \in \mathbb{R}^n\}, \\ N(D, y) &= \{y^* \in \mathbb{R}^n \mid \;\; \langle y^*, x - y\rangle \leq 0 \text{ for all } x \in D\}, \end{aligned}$$

$\partial_\varepsilon f(z)$ and $N_\varepsilon(D, z)$ are ε-subdifferential of a function $f(\cdot)$ and the set of ε-normal directions to a set D at a point z:

$$\begin{aligned} \partial_\varepsilon f(z) &= \{z^* \in \mathbb{R}^n \mid \;\; f(x) - f(z) \geq \langle z^*, x - z\rangle - \varepsilon \quad \text{for all} \quad x \in \mathbb{R}^n\}, \\ N_\varepsilon(D, z) &= \{z^* \in \mathbb{R}^n \mid \;\; \langle z^*, x - z\rangle \leq \varepsilon \text{ for all } x \in D\}, \end{aligned}$$

$\delta(\cdot|D)$ is the indicator function of D and $\partial_\gamma f(z)$ is γ-subdifferential of a function $f(\cdot)$ at point z:

$$\partial_\gamma f(z) = \{\phi(x) \text{ is continuous } \mid \;\; f(x) - f(z) \geq \phi(x) - \phi(z) \text{ for all } \quad x \in \mathbb{R}^n\}$$

It is worthwhile to notice that above conditions generalized the Rockafellar's local necessary and sufficient optimality condition

$$\partial f(z) \subset N(D, z). \tag{RℓNS}$$

It is not difficult to see that (SgNS) with ($y = z$), (HUgNS) with ($\varepsilon = 0$) and (FBgNS) with (linear ϕ) all imply (RℓNS).

The purpose of this section is to improve the (*classical*) local necessary optimality condition

$$\partial f(z) \bigcap N(D, z) \neq \emptyset \tag{CℓN}$$

in order to fully describe a global maximum; we could notice that the classical condition (CℓN) is not sufficient even for a local maximum.

Theorem 2.1. *[Tse98b] A necessary and sufficient condition for $z \in D$ to be a global maximum for (CMP) is:*

$$\begin{cases} \partial f(y) \bigcap N(D, y) \neq \emptyset \textit{ for all } y \textit{ s.t. } f(y) = f(z) \\ \textit{there exists } v \in \mathbb{R}^n \textit{ s.t. } f(v) < f(z) \end{cases} \tag{gNS}$$

Proof.

($\Rightarrow$) Let z solve (CMP), in other words $f(z) \geq f(x)$ for all $x \in D$. Then, due to convexity of the function $f(\cdot)$ and the definition of subdifferential, for all y such that $f(y) = f(z)$

$$0 \geq f(x) - f(z) = f(x) - f(y) \geq \langle y^*, x - y \rangle,$$

holds for all $y^* \in \partial f(y)$ and $x \in D$ so that $y^* \in \partial f(y) \bigcap N(D, y)$.

($\Leftarrow$) By contradiction, let z not a global maximum of (CMP). Thus, there is

$$u \in D \text{ such that } f(u) > f(z).$$

Then, let us consider a convex combination of u and a point v such that $f(v) < f(z)$

$$y(\alpha) = \alpha v + (1 - \alpha)u. \tag{1}$$

There is a number $\alpha_0 \in]0, 1[$ such that $f(y(\alpha_0)) = f(z)$ since $f(\cdot)$ is continuous and $f(v) < f(z) < f(u)$. Now, one shows $\partial f(y(\alpha_0)) \not\subset N(D, y(\alpha_0))$.

For all subgradient y_0^* of $f(\cdot)$ at $y(\alpha_0)$ satisfying $f(y(\alpha_0)) = f(z)$ and for $u \in D$, it holds

$$\langle y_0^*, u - y(\alpha_0) \rangle = \langle y_0^*, \frac{y(\alpha_0) - \alpha_0 v}{1 - \alpha_0} - y(\alpha_0) \rangle \geq \frac{\alpha_0}{\alpha_0 - 1}(f(v) - f(y(\alpha_0))) > 0.$$

proving $y_0^* \notin N(D, y(\alpha_0))$ for all $y_0^* \in \partial f(y(\alpha_0))$ ■

Remark.

- classical local necessary condition (CℓN) compared to necessary part of global optimality condition (gNS), only considers z instead of all points on the level set $f(z)$.
- when $f(\cdot)$ is a differentiable function, we retrieve the Strekalovsky's condition (SgNS) since $\partial f(y)$ consists of a single element $\nabla f(y)$.
- in case of nondifferentiable function $f(\cdot)$ one can see the difference between (gNS) and (SgNS) since the latter is in general intractable to check (see example).

EXAMPLE 2.1. Consider the problem in $\mathbb{R}^2$ to maximize piecewise linear convex function (polyhedral) defined by (see Fig. 2.1) :

$$f(x_1, x_2) = max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

subject to

$$D = \{x \in \mathbb{R}^2 \;/\; -3 \le x_i \le 3,\ i = 1, 2\}.$$

- At point $z = (3, -3)^\top$, **the classical optimality condition** (CℓN) is satisfied: for $z^* = (3, -1)^\top \in \partial f(z)$, $\langle z^*, x - z\rangle \le 0$ holds for all $x \in D$; however, z is not local maximum since **the Rockafellar's condition** (RℓNS) is violated (for instance, the subgradient $\overline{z}^* = (-1, -3)^\top \in \partial f(z)$, but $\overline{z}^* \notin N(D, z)$).
- The point $z' = (3, 3)^\top$ is a local maximum since sufficient local optimality condition is easily checked. But sufficient local optimality could not decide whether it is a global maximum.

Let us denote $U(z) = \{y \in \mathbb{R}^2 \;/\; f(y) = f(z)\}$ and $U(z) = U_1 \cup U_2 \cup U_3 \cup U_4$, where
$U_1(z) = \{y \;/\; 2y_1 + 3y_2 = f(z)\}$,
$U_2(z) = \{y \;/\; 3y_1 - y_2 = f(z)\}$,
$U_3(z) = \{y \;/\; -2y_1 + y_2 = f(z)\}$,
$U_4(z) = \{y \;/\; -2y_1 - 6y_2 = f(z)\}$.

Using theorem 2.1, it is easy to see that necessary condition is violated at point $z' = (3, 3)^\top$ since for all $y \in U_4(z')$ $y^* = (-1, -3)^\top \in \partial f(y)$ but $y^* \notin N(D, y)$; therefore, z' is not a global maximum.

- Now, let consider the point $z'' = (-3, -3)^\top$; in order to conclude for a global maximum, one has to check sufficient part of global optimality condition (gNS). According to **theorem 2.1**, it amounts to check whether y_i^* belongs to $N(D, y^i)$, for all $y^i \in U_i,\ \ i = 1, 2, 3, 4$.

Notice that using **the Strekalovsky's condition** (SgNS) instead, requires to check in addition $y^* \in \partial f(y)$, for all $y \in U_k(z) \cap U_l(z)$ and $(k,l) \in \{(1,2),(2,3),(3,4),(4,1)\}$, which is an intractable problem in general.

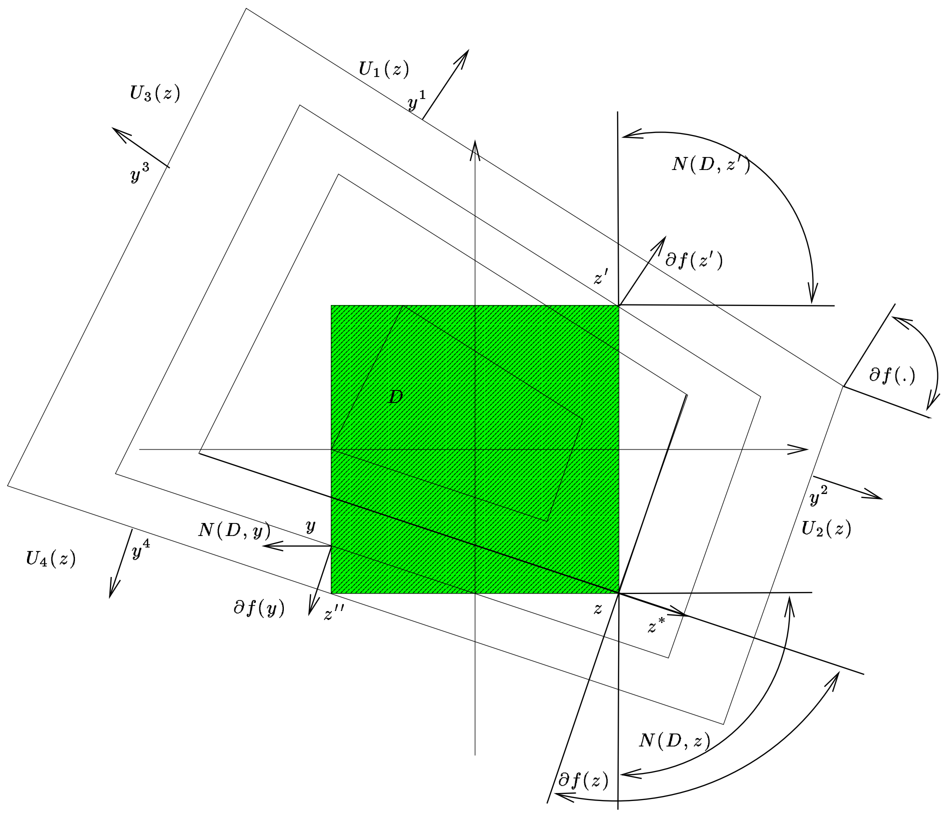

FIG. 1. Example 2.1

3. PIECEWISE CONVEX MAXIMIZATION PROBLEM

We call a function $F : \mathbb{R}^n \to \mathbb{R}$ a *piecewise convex*-function if it decomposes as

$$F(x) = \min\{f_j(x) \quad | \quad j \in M = \{1, 2..., m\}\},$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ is convex for all $j \in M$

In this section, we consider the nonconvex and nonsmooth *piecewise convex maximization problem* (also known as discrete maxmin problem)

$$\begin{array}{ll} \text{maximize} & F(x) \\ \text{subject to} & x \in D, \end{array} \qquad \text{(PCMP)}$$

where $D \in \mathbb{R}^n$ is a convex and compact set and $F(\cdot)$ is piecewise convex.

Obviously, if $m = 1$ or all functions $f_j(\cdot)$ are affine then the problem (PCMP) turns out to be a convex maximization problem (CMP).

An important property of convex maximization problem is that every local (and in particular global) solution is achieved at an extreme point of the feasible domain. In general, this property does not hold for (PCMP) as a large number of local optima could lie anywhere in D.

We will use further notations, $clco(D)$ as closure of convex hull of set D and :

$$\begin{aligned} I(z) &= \{i \in M \ / \ f_i(z) = F(z)\}, \\ \mathcal{L}_g^{\leq}(\alpha) &= \{x \in \mathbb{R}^n \quad | \quad g(x) \leq \alpha\}, \\ \mathcal{L}_g^{>}(\alpha) &= \{x \in \mathbb{R}^n \quad | \quad g(x) > \alpha\}, \\ D_k(z) &= D \cap (\bigcap_{j \neq k, j \in M} \mathcal{L}_{f_j}^{>}(F(z))) = D \cap \{x \mid f_j(x) > F(z) \text{ for all } j \in M \setminus \{k\}\} \end{aligned}$$

for respectively, set of active function at z, Lebesgue set and its complement for function $g(\cdot)$ and special subdomain.

LEMMA 3.1. *If for a point $z \in D$, both $F(z) \geq F(x)$ for all $x \in D$ and $f_k(z) = F(z)$ for some $k \in M$ hold, then $f_k(z) \geq f_k(x)$ for all $x \in D_k(z)$.*

Proof. Let us assume that there exists some $u \in D_k(z)$ such that $f_k(u) > f_k(z)$.

Then, from $u \in D_k(z)$ we get $f_j(u) > F(z)$ for all $j \in M \setminus \{k\}$ so that $F(u) = \min\{f_j(u) \mid j \in M\} > F(z)$, a contradiction to $F(z) \geq F(x)$ for all $x \in D$. ■

Lemma 3.1 together with necessary part of (gNS) provides a necessary condition for global solution to (PCMP).

PROPOSITION 3.1. *If $z \in D$ is a global maximum of (PCMP) then for all $k \in I(z)$*

$$\partial f_k(y) \bigcap N(D_k(z), y) \neq \emptyset \quad \textit{for all} \quad y \textit{ s.t. } f_k(y) = F(z). \tag{gN}$$

Proof. By definition of the set $I(z)$ we have $f_k(z) = F(z)$ for all $k \in I(z) \subset M$. Then by lemma 3.1, if z solves (PCMP) then z is maximum for $f_k(\cdot)$ over $D_k(z)$ for all $k \in I(z)$. Using necessary part of theorem 2.1 and definition of $I(z)$, it leads to

$$\partial f_k(y) \bigcap N(D_k(z), y) \neq \emptyset \text{ for all } y \text{ s.t. } f_k(y) = f_k(z) = F(z).$$

■

In order to strengthen this necessary condition and achieve a sufficient condition for (PCMP), we first prove the following two lemmas.

LEMMA 3.2. *Given vectors $c \in \mathbb{R}^n$ and $u \in clco(D_k(z))$, then there exists $w \in D_k(z)$ such that $\langle c, u \rangle \leq \langle c, w \rangle$.*

Proof. Let us assume that there exists some $c \in \mathbb{R}^n$ and $u \in clco(D_k(z))$ such that $\langle c, u \rangle > \langle c, w \rangle$ for all $w \in D_k(z)$.

Then, by Caratheodory's theorem [Roc70] (p. 155) $u \in clco(D_k(z))$ implies that there are $x^1, x^2, ..., x^{n+1} \in D_k(z)$ and nonnegative $\alpha_i \in \mathbb{R}$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$ and $u = \sum_{i=1}^{n+1} \alpha_i x^i$. From assumption, we have $\langle c, u \rangle > \langle c, x^i \rangle$ for all $i = 1, 2, ..., n+1$. Now, multiplying previous inequalities by corresponding α_i and summing yields

$$\sum_{i=1}^{n+1} \alpha_i \langle c, u \rangle > \langle c, \sum_{i=1}^{n+1} \alpha_i x^i \rangle = \langle c, u \rangle$$

a contradiction. ■

LEMMA 3.3. *Given continuous functions $g(\cdot), h(\cdot)$, let $\varphi(\cdot) = \min\{g(\cdot), h(\cdot)\}$. If for all $x \in D, h(x) > \varphi(z) \geq g(x)$ for some z then $\varphi(z) \geq \varphi(x)$ for all $x \in D$.*

Proof. We will use the simple observation in $\mathbb{R}$ that for any a, b, c, if $a \leq b$ then $\min\{a, c\} \leq \min\{b, c\}$. Decompose D as disjoint union $D = D^+ \cup D^-$, where $D^+ = D \cap \{x \mid h(x) > \varphi(z)\}$ and $D^- = D \cap \{x \mid h(x) \leq \varphi(z)\}$.

For $x \in D^+$, from $g(x) \leq \varphi(z)$ one has $\min\{g(x), h(x)\} \leq \min\{\varphi(z), h(x)\}$ and hence $\varphi(z) \geq \varphi(x)$ for all $x \in D^+$.

For $x \in D^-$, from $h(x) \leq \varphi(z)$ one has $\min\{h(x), g(x)\} \leq h(x) \leq \varphi(z)$ and hence $\varphi(z) \geq \varphi(x)$ for all $x \in D^-$. ■

We are now in a position to establish the main result of this section.

THEOREM 3.1. *A sufficient condition for $z \in D$ to be a global maximum for (PCMP) is:*

$$\begin{cases} \textit{there exist } k \in I(z) \textit{ and } v \in \mathbb{R}^n \textit{ s.t. } f_k(v) < f_k(z) \textit{ and} \\ \partial f_k(y) \bigcap N(clco(D_k(z)), y) \neq \emptyset \textit{ for all } y \textit{ s.t. } f_k(y) = F(z) \end{cases} \tag{gS}$$

Proof. By sufficient part of theorem 2.1 applied to v of (gNS) condition above, we have

$$f_k(z) \geq f_k(x) \text{ for all } x \in clco(D_k(z))$$

and hence

$$f_k(z) \geq f_k(x) \text{ for all } x \in D_k(z). \tag{2}$$

Denoting $\psi_k(x) = min\{f_j(x) \mid j \in M \setminus \{k\}\}$ then $x \in D_k(z)$ implies

$$\psi_k(x) > F(z) \text{ for all } x \in D.$$

On the other hand, $F(z) = f_k(z)$ holds, since $k \in I(z)$ therefore (2) is equivalent to $F(z) \geq f_k(x)$ for all $x \in D$ such that $\psi_k(x) > F(z)$.

Finally, using lemma 3.3 we get z is a global maximum since

$$F(z) \geq F(x) = \min\{f_k(x), \psi_k(x)\} \text{ for all } x \in D.$$

■

Remark.

- The assumption that there are $k \in I(z)$ and $v \in \mathbb{R}^n$ such that $f_k(v) < f_k(z)$ means that z is not local minimum of $F(\cdot)$ in $\mathbb{R}^n$. If this assumption is violated, in other words for all $k \in I(z)$ one has $z = \arg\min\{f_k(x) \mid x \in \mathbb{R}^n\}$ then $F(z) = f_k(z) \leq f_k(x)$ $(k \in I(z))$ and $F(z) < f_j(z)(j \in M \setminus I(z))$. Hence there exists some neighborhood say a ball around z of radius $\varepsilon > 0$ such that for all $x \in B(z, \varepsilon) \cap D$ we have $F(z) \leq F(x)$.

On the other hand, for all $k \in I(z)$ $0 \in \partial f_k(z) \cap N(clco(D_k(z), z)$. In that case, a local search search improve $F(z)$ since any feasible direction gives better point with respect to (PCMP).

- The sufficient global optimality condition (gS) could be written as follows

$$\begin{cases} \text{there exist } k \in I(z) \text{ and } v \in \mathbb{R}^n \text{ s.t. } f_k(v) < f_k(z) \\ \text{and there exists also } y_k^* \in \partial f_k(y) \text{ s.t. } \langle y_k^*, x - y \rangle \leq 0, \\ \text{for all } x \in clco(D_k(z)) \text{ and } y \text{ s.t. } f_k(y) = F(z) \end{cases}$$

• Let (gS) be violated at z, in other words for all $k \in I(z)$ there are y^k, u^k fullfilling respectively $f_k(y^k) = F(z)$ and $u^k \in clco(D_k(z))$ and such that for all $y^*_k \in \partial f_k(y^k)$ the inequality $0 < \langle y^*_k, u^k - y^k \rangle$ holds.

Then by lemma 3.2 there exists $w^k \in D_k(z)$ such that $\langle y^*_k, u^k \rangle \leq \langle y^*_k, w^k \rangle$. So due to convexity of all functions $f_k(\cdot)$ we have $0 < \langle y^*_k, u^k - y^k \rangle \leq \langle y^*_k, w^k - y^k \rangle \leq f_k(w^k) - f_k(y^k)$, that implies $F(z) < f_k(w^k)$. On the other hand, by definition of $D_k(z)$, $w^k \in D_k(z)$ implies $w^k \in D$ and $F(z) < f_j(w^k)$ for all $j \in M \setminus \{k\}$. As a result, we have a better point $w^k \in D$.

• By proposition 3.1 and lemma 3.2 it is easy to see that (gS) is not only sufficient, it is necessary and sufficient condition for global maximum to (PCMP).

Example 3.1. Consider the problem in $\mathbb{R}^2$ to maximize piecewise convex function :

$$F(x) = min\{f_j(x) \mid j = 1,2,3,4,5\},$$

where

$$\begin{aligned} f_1(x) &= x_1^2 + (x_2+4)^2 - 36, \\ f_2(x) &= (x_1+8)^2 + (x_2-3)^2 - 36, \\ f_3(x) &= x_1^2 + (x_2-8)^2 - 16, \\ f_4(x) &= (x_1-8)^2 + (x_2-3)^2 - 49, \\ f_5(x) &= (x_1-10)^2 + (x_2+10)^2 - 4 \end{aligned}$$

subject to

$$D = \{x \in \mathbb{R}^2 \mid -4 \leq x_1 \leq 10, \ \ -6 \leq x_2 \leq 8, \ \ x_1 - x_2 \leq 10\}.$$

The point $z = (6,-4)^\top$ is a local maximum with $F(z) = 0$. One wonders if it is global maximum?

The Lebesque set $\mathcal{L}^{\leq}_F(F(z))$ contains two nonconnected sets and one of them is set-difference of two nonconvex sets (see Fig. 2.). Functions $f_1(\cdot)$ and $f_4(\cdot)$ are active at z.

Let us consider the function $f_1(\cdot)$. According to our notations $D_1(z) = D \cap \{x \mid f_2(x) > 0, \ \ f_3(x) > 0, \ \ f_4(x) > 0\}$ since $f_5(x) > 0$ for all $x \in D$ and $F(z) = 0$.

In order to use the necessary global optimality condition (gN), one has to check the inclusion $D_1(z) \subset \mathcal{L}^{\leq}_{f_1}(F(z))$. It is easy to see that $D_1(z)$ is not included to $\mathcal{L}^{\leq}_{f_1}(F(z))$ since at $y = (0,2)^\top \ : \ f_1(y) = f(z)$ and $u = (0,3)^\top \in D_1(z)$ the inequality $\langle \nabla f_1(y), u - y \rangle > 0$ holds. Therefore z is not a global maximum.

Now, we consider another point $z^* = (0,3)^\top$ which is local maximum too. At the point, functions $f_1(\cdot), f_2(\cdot), f_3(\cdot), f_4(\cdot)$ are active. And one can see that $D_1(z^*)$ is included to

$\mathcal{L}^{\leq}_{f_1}(13)$, $(F(z^*) = 13)$. That is enough, according to sufficient global optimality conditions (gS), to say that z^* is global maximum.

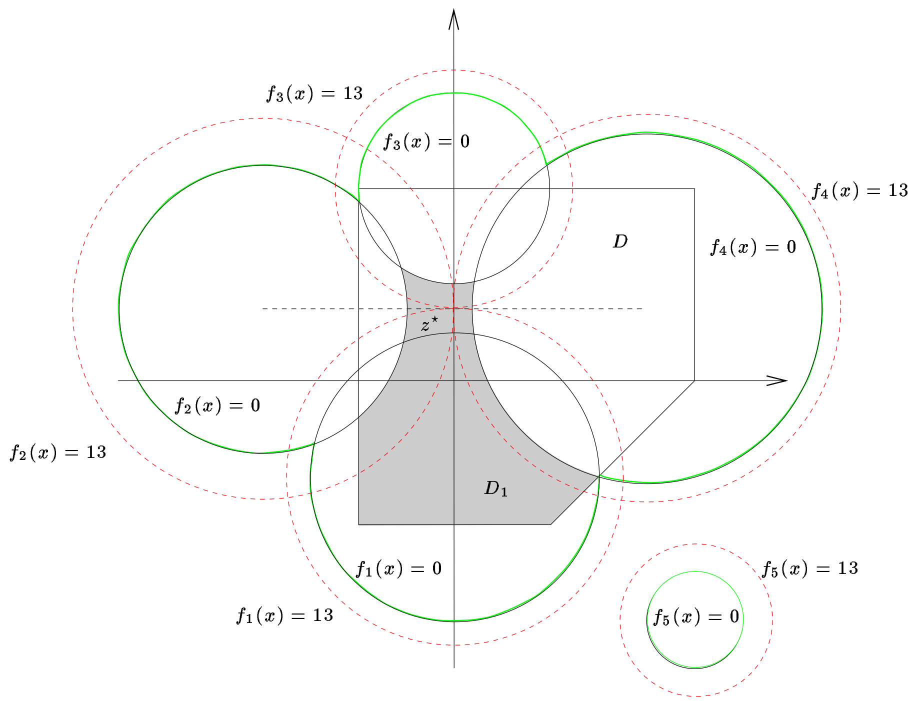

FIG. 2. a simple example 3.1

EXAMPLE 3.2. Here we consider a piecewise convex maximization problem in $\mathbb{R}^2$ with functions (see Fig. 3.)

$$
\begin{aligned}
f_1(x) &= x_1^2 + (x_2+2)^2 - 9,\\
f_2(x) &= 9(x_1+3)^2 + 4x_2^2 - 36,\\
f_3(x) &= (x_1+1)^2 + (x_2-4)^2 - 4,\\
f_4(x) &= \frac{1}{9}(x_1-3)^2 + \frac{1}{36}(x_2-4)^2 - 1,\\
f_5(x) &= (x_1-5)^2 + (x_2+5)^2 - 1
\end{aligned}
$$

subject to box constraint:

$$D = \{x \in \mathbb{R}^2 \mid -2 \le x_1 \le 5, \;\; -3 \le x_2 \le 4\}.$$

We consider it just to show some difficulty of solving (PCMP). It seems to us, even this two dimensional problem is not trivial to solve.

It is easy to see that there is number of local maxima on vertices, edges and interior of the box. In other words, global maximum could be anywhere in the box.

As in the example (3.1) using (gN) we can escape from local maxima which are on box vertices and edges. And we look for it inside the box. Unlike (3.1), there is no point in the box where all functions $f_1(\cdot), f_2(\cdot), f_3(\cdot), f_4(\cdot)$ are active.

Here the point $(-1.3286, 1.7381)^\top$ is global maximum with value $f(z) = 1.2240$. This solution was found by enumerating tree of curve intersection vertices under Maple.

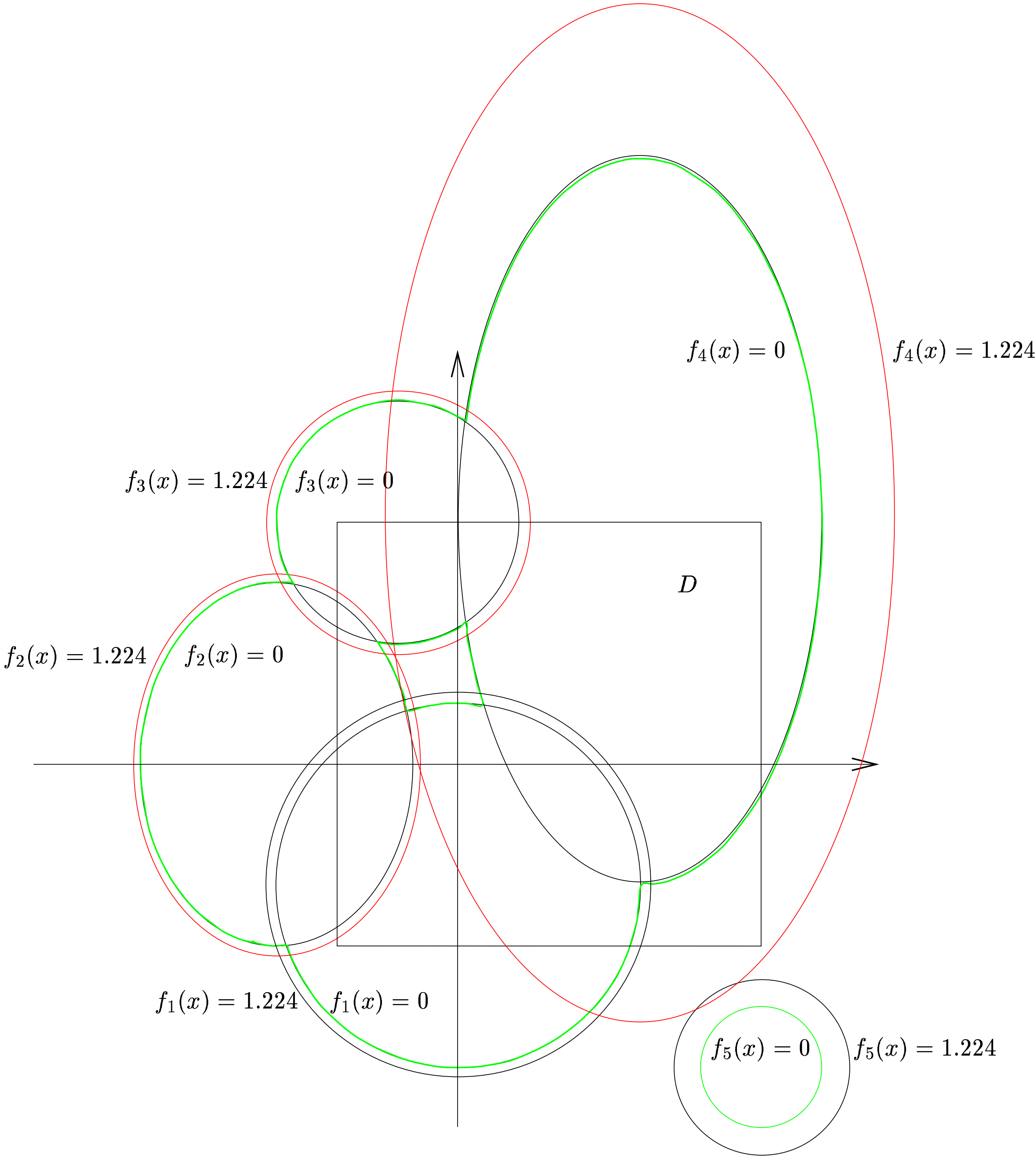

FIG. 3. a non trivial example 3.2

4. AN APPLICATION TO VARIATIONAL INEQUALITY PROBLEMS

Let D be a nonempty, compact, and convex subset of $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. This section deals with the variational inequality problem, denoted by $VIP(T, D)$, i.e. to find a vector $z \in D$ such that

$$\langle T(z), x - z\rangle \le 0, \quad \text{for all } x \in D. \qquad \text{(VIP(T,D))}$$

The case of VIP(T,D) when the operator $-T(\cdot)$ is monotone (pseudomonotone) on D is well studied and there exist methods solving it (see [Ham84]). In practical application like traffic assignement problem, usually the operator $-T(\cdot)$ is not monotone and therfore we consider this case below.

Our approach in this paper is based on global optimality conditions for piecewise convex maximization problem.

Given a starting point $y^0 \in D$ and some index set $M_0 = \{0\}$. At iteration y^k $(k = 0, 1, 2, ..)$ information $(T(y^k), \nabla T(y^k))$ collected so far is used to build up a model of some implicit function connected to $VIP(T, D)$. In the sequel, we use $\nabla(\cdot)$ for both gradient and Hessian for function and vector function.

Now we construct some function such that its gradient equals T at y^k

$$\mathcal{T}_k(x) = \frac{1}{2}\langle \nabla T(y^k)x, x\rangle + \langle T(y^k) - \nabla T(y^k)y^k, x\rangle, \tag{3}$$

$$\begin{aligned} \nabla \mathcal{T}_k(x) &= T(y^k) + \nabla T(y^k)(x - y^k), \\ \nabla \mathcal{T}_k(y^k) &= T(y^k). \end{aligned}$$

It is well known that there are two positive semidefinite matrices $\nabla T(y^k)^+$ and $\nabla T(y^k)^-$ such that

$$\nabla T(y^k) = \nabla T(y^k)^+ - \nabla T(y^k)^-. \tag{4}$$

Using this one writes function $\mathcal{T}_k(x)$ as d.c. (difference of two convex) function

$$\mathcal{T}_k(x) = g_k(x) - h_k(x), \tag{5}$$

where

$$\begin{aligned} g_k(x) &= \frac{1}{2}\langle \nabla T(y^k)^+ x, x\rangle + \langle T(y^k) - \nabla T(y^k)y^k, x\rangle, \\ h_k(x) &= \frac{1}{2}\langle \nabla T(y^k)^- x, x\rangle. \end{aligned}$$

Due to convexity of $h_k(x)$, convex tangent approximation by above of $\mathcal{T}_k(x)$ at y^{k+1} is

$$f_k(x) = g_k(x) - [h(y^k) + \langle \nabla h_k(y^k), x - y^k \rangle]$$

since

$$\begin{aligned} f_k(x) &\geq \mathcal{T}_k(x), \text{ for all } x \in \mathbb{R}^n, \\ f_k(y^k) &= \mathcal{T}_k(y^k). \end{aligned}$$

Now we could define a piecewise convex function as some model-function at each iteration

$$F_k(x) = \min\{f_j(x) \mid j \in M_k = \{0, 1, 2, ..., k\}\}. \tag{6}$$

Obviously $F_k(\cdot)$ does not copy the desired implicit function. To master this lack of information one enriches the model by finding one more point y^{k+1} and by including one more function $f_{k+1}(\cdot)$. So, we obtain the following **iteration** $y^k \longmapsto y^{k+1}$.

ALGORITHM 1 (PCMP FOR VIP(T,D)).

1. for k from 0 while $y^k \notin \arg\max\{F_k(x) \mid x \in D\}$
2. Construct piecewise convex function $F_k(x) = \min\{F_{k-1}(x), f_k(x)\}$
3. Compute $y^{k+1} = \arg\max\{F_k(x) \mid x \in D\}$
4. $M_{k+1} := M_k \cup \{k+1\}$
5. endfor;

Let us assume that piecewise convex maximization problem can been solved at each iteration. Then under this assumption we are in position to formulate some convergence results.

LEMMA 4.1. *The numerical sequence of* $\{F_k(y^{k+1})\}$ *has a limit.*

Proof. First, we prove that $\{F_k(y^{k+1})\}$ is decreasing.
By definition of the function $F_k(\cdot)$ we have

$$F_k(y^{k+1}) = \min\{f_j(y^{k+1}) \mid j \in M_k\} = \min\{F_{k-1}(y^{k+1}), f_k(y^{k+1})\} \leq F_{k-1}(y^{k+1}).$$

Since $F_{k-1}(x) \leq F_{k-1}(y^k)$ for all $x \in D$, as a particular case we have

$$F_{k-1}(y^{k+1}) \leq F_{k-1}(y^k)$$

and therefore $F_k(y^{k+1}) \leq F_{k-1}(y^k)$.

Then we prove that this sequence is bounded by below. From definition of convex tangent approximation

$$f_j(x) \geq \mathcal{T}_j(x) \text{ for all } x \in \mathbb{R}^n, \quad j \in M_k$$
and
$$F_k(x) = \min\{f_j(x) \mid j \in M_k\} \geq \min\{\mathcal{T}_j(x) \mid j \in M_k\} \tag{7}$$

By the Weierstrass's theorem the continuous function $\min\{\mathcal{T}_j(x) \mid j \in M_k\}$ has a minimum over compact set D. Therefore

$$F_k(y^{k+1}) \geq \min\{\min\{\mathcal{T}_j(x) \mid j \in M_k\} \mid x \in D\} \tag{8}$$

Hence convergence of the sequence $F_k(y^{k+1})$. ■

Corollary 4.1. *Sequence $\{y^k\}$ has a accumulation point.*

Proof. Since the function $F_k(\cdot)$ is continuous and D is compact, lemma 4.1 implies convergence of the sequence $\{y^k\}$. ■

Proposition 4.1. *As stopping criterion of the algorithm 1, if $\mathcal{T}_{k+1}(y^{k+1}) > F_k(y^{k+1})$ then $y^{k+1} = argmax\{F_{k+1}(x) \mid x \in D\}$*

Proof. By assumption of the proposition and by definition of function $f_{k+1}(\cdot)$ it holds

$$f_{k+1}(y^{k+1}) \geq \mathcal{T}_{k+1}(y^{k+1}) > F_k(y^{k+1}).$$

Therefore

$$F_{k+1}(y^{k+1}) = \min\{F_k(y^{k+1}), f_{k+1}(y^{k+1})\} = F_k(y^{k+1}) \geq F_k(x) \text{ for all } x \in D,$$

when last inequality comes from y^{k+1} as maximum of $F_k(x)$. On the other hand

$$F_k(x) \geq \min\{F_k(x), f_{k+1}(x)\} = F_{k+1}(x) \text{ for all } x \in D.$$

Last two inequalities prove the result

$$F_{k+1}(y^{k+1}) \geq F_{k+1}(x) \text{ for all } x \in D.$$

■

Proposition 4.2. *If at some accumulation point y^{k+1} we have $I(y^{k+1}) = \{k\}$ then y^{k+1} solves VIP(T,D).*

Proof. By definition of the set of active functions at y^{k+1}, $I(y^{k+1}) = \{k\}$ implies $F_k(y^{k+1}) = f_k(y^{k+1})$ and $F_k(y^{k+1}) < f_j(y^{k+1})$ for all $j \in M_k \setminus \{k\}$. Hense there exists some neighborhood of y^{k+1} of radius $\delta > 0$ such that for all $x \in B(y^{k+1}, \delta) \cap D$ we have $f_k(x) < f_j(x)$ for $j \in M_k \setminus \{k\}$ and therefore $F_k(x) = f_k(x)$. On the other hand $y^{k+1} = \arg\max\{F_k(x) \mid x \in D\}$ implies $f_k(x) \leq f_k(y^{k+1})$ for all $x \in D \cap B(y^{k+1}, \delta)$ that means by optimality condition $\langle \nabla f_k(y^{k+1}), x - y^{k+1} \rangle \leq 0$ for all $x \in D \cap B(y^{k+1}, \delta)$.

Since y^{k+1} is an accumulation point, we have $y^k = y^{k+1}$ and $\langle \nabla f_k(y^k), x - y^k \rangle \leq 0$ for all $x \in D \cap B(y^k, \delta)$. Hense easy to see that $\langle \nabla f_k(y^k), x - y^k \rangle \leq 0$ for all $x \in D$ since for $w \notin B(y^k, \delta)$ (in other words $\|w - y^k\| > \delta$) one considers $v = y^k + \frac{\delta}{\|w-y^k\|}(w - y^k) \in D$ and $0 \geq \langle \nabla f_k(y^k), v - y^k \rangle = \frac{\delta}{\|w-y^k\|} \langle \nabla f_k(y^k), w - y^k \rangle$.

Now remember that $f_k(y^k) = \mathcal{T}_k(y^k)$ and $\nabla \mathcal{T}_k(y^k) = T(y^k)$ we can conclude $\langle T(y^k), x - y^k \rangle \leq 0$ for all $x \in D$. ■

5. CONCLUDING REMARKS

In this paper we have shown how global optimality conditions for (CMP) carries over (PCMP) and we gave an algorithm for some VIP application.

As a by-product of main theorem 2.1 we add a further item to a well known result [Roc70, Str98] from convex analysis:
Given two convex sets $A, B \in \mathbb{R}^n\,(B = cl(B))$. Inclusion $A \subset B$ is true if one of the following equivalent conditions is satisfied

1. $(B-y)^\circ \in (A-y)^\circ$ for $y \in (A \cap B)$;
2. $\sigma(y \mid A) \leq \sigma(y \mid B)$ for all $y \in \mathbb{R}^n$;
3. $N(y \mid B) \subset N(y \mid A)$ for all $y \in bd(B)$;

namely

4. $N(y \mid B) \cup N(y \mid A) \neq \emptyset$ for all $y \in bd(B)$

where $cl(D)$, D°, $\sigma(\cdot \mid D)$, $N(\cdot \mid D)$, $bd(D)$ are used for closure, polar, support function, normal cone and boundary of D respectively.

In section 2, for (CMP), $A = D$, $B = \mathcal{L}_f^{\leq}(f(z))$ and in section 3, for (PCMP), $A = clco(D_k(z))$, $B = \mathcal{L}_{f_k}^{\leq}(F(z))$.

We could notice however, that to turn (gS) (section 3) into an effective algorithm, requires to compute $w^k \in D_k(z)$ from $u^k \in clco(D_k(z)$ since Caratheodory's theorem only provides existence of such a w^k.

For (VIP), further research should be done on model-function $\min\{\mathcal{T}_j(x) \mid j \in M_k\}$ in order to better describe solution space.

Acknowledgements

The author expresses his thanks to Professor A.S. Strekalovsky from Irkutsk, whose encouragement and support has been a mainstay since his student days. Further deep thanks are due to D. Fortin of INRIA (France), who looked over the manuscript and gave some helpful suggestions, discussions and advice.

REFERENCES

Ben95. H. P. Benson. *Concave Minimization: Theory, applications and algorithms.* Kluwer Academic, Dordrecht/Boston/London, 1995.

DHL98. M. Dur, R. Horst, and M. Locatelli. Necessary and sufficient global optimality conditions for convex maximization revisited. *Journal of Mathematical Analysis and Applications*, (217):637–649, 1998.

FB97. F. Flores-Bazan. On minima of the difference of functions. *J. Optim. Theory Appl.*, 93(3):525–531, 1997.

Ham84. Janice H. Hammond. *Solving Asymmetric Variational Inequality Problems and System of Equations with Generalized Nonlinear Programming Algorithms.* PhD thesis, Massachusetts Institute of Technology, 1984.

HPT95. R. Horst, P.M. Pardalos, and N. Van Thoai. *Introduction to Global Optimization.* Kluwer Academic, 1995.

HT90. R. Horst and H. Tuy. *Global Optimization.* Springer-Verlag, 1990.

HU89. J. B. Hiriart-Urruty. *From Convex Optimization to Nonconvex Optimization, Part I. Necessary and Sufficient Conditions for Global Optimality.* Plenum, New York, 1989.

HU95. J. B. Hiriart-Urruty. *Conditions for Global Optimality.* Kluwer Academic, Dordrecht/Boston/London, 1995.

KS80. D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications.* Academic Press, 1980.

LPR96. Z.-Q. Luo, J.-S. Pang, and D Ralph. *Mathematical Programs with Equilibrium Constraints.* Cambridge University Press, 1996.

Roc70. R. T. Rockafellar. *Convex Analysis.* Princeton University Press, 1970.

Str87. A. S. Strekalovskii. On the global extremum problem. *Soviet Dokl.*, (292):1062–1066, 1987.

Str93. A. S. Strekalovsky. The search for a global maximum of a convex functional on an admissibleset. *Computational Mathematics and Mathematical Physics*, 33(3):315–328, 1993.

Str98. A. S. Strekalovsky. Global optimality conditions for nonconvex optimization. *J. Global Optim.*, 12(4):415–434, 1998.

Tse98a. I. Tsevendorj. To a global solution characterization of nonconvex problems. In *11-th Baikal International School-Seminar on Optimization Methods and its applications*, pages 212–215, Irkutsk, Russia, 1998.

Tse98b. Ider Tsevendorj. On the conditions for global optimality. *Journal of the Mongolian Mathematical Society*, 1(2):58–61, 1998.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399