American Option Prices as Unique Viscosity Solutions to Degenerated Hamilton-Jacobi-Bellman Equations

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Abstract: In this paper we show that the American price of standard (bounded) options in the Black-Scholes one-dimensional model, which is classically given by the value function of an optimal stopping problem, is also the value function of a degenerate stochastic control problem. As a byproduct we get that the American price u^* is the unique bounded and continuous viscosity solution of the fully non-linear parabolic equation $-\frac{\partial u^*}{\partial t}(t,x) = (Au^*)^+(t,x), t < T, x > 0, u^*(T,x) = \varphi(x)$ where A is the infinitesimal generator of the Black-Scholes model, T the maturity and φ the payoff of the option.

Key-words: American options, Hamilton-Jacobi-Bellman equation, Optimal stopping, Viscosity solutions.

(Résumé : tsvp)

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Les prix d'options Américaines comme unique solution viscosité d'une équation d'Hamilton-Jacobi-Bellman dégénérée

Résumé : On montre que le prix d'une option Américaine standard, de payoff borné, dans le modèle de Black-Scholes en dimension 1, qui est donné par la fonction valeur d'un problème d'arrêt optimal, est aussi donné par la fonction valeur d'un problème de contrôle stochastique dégénéré. On obtient ainsi que le prix Américain u^* est l'unique solution de viscosité continue et bornée de l'équation parabolique $-\frac{\partial u^*}{\partial t}(t,x) = (Au^*)^+(t,x), t < T, x > 0, u^*(T,x) = \varphi(x)$ où A est le générateur infinitésimal du modèle de Black-Scholes, T la maturité et φ le payoff de l'option.

Mots-clé : Arrêt optimal, Equation d'Hamilton-Jacobi-Bellman, Options Américaines, Solutions viscosité.

1 Introduction

Consider the standard one-dimensional Black-Scholes model:

$$S_u = x \exp\left(\rho u + \sigma B_u - \frac{\sigma^2}{2} u\right)$$

The price at time t of the so-called American option with maturity $T \geq t$ and payoff φ, which grants to its owner the right to get the amount of money $\varphi(S_\tau)$ at the time τ of his choice between t and T, is given by the value function of the associated optimal stopping problem, when $t \leq T$:

$$u^*(t,x) = \sup_\tau E\left[e^{-\rho\tau}\varphi(S_\tau)\right] \tag{1}$$

where τ runs across the set of stopping times of the filtration of B which are less than $T-t$ almost surely.

We shall introduce the following regularity assumption on φ :

$$\varphi \text{ is continuous and bounded from } \mathbb{R}_+^* \text{ to } \mathbb{R}_+ \tag{H}$$

Under (H) the function u^* is well-known to be continuous ([JLL], Proposition 1.2). From an analytical point of view, under additional smoothness assumptions on φ, u^* is the solution of the variational inequality

$$\max\left(\frac{\partial u^*}{\partial t} + Au^*, \varphi - u^*\right) = 0 \tag{2}$$

i.e. an evolution equation with an obstacle problem. In particular, it satisfies $u^*(t,x) = \varphi(x)$ in the so-called “exercice”, or “stopping” region, and $-\frac{\partial u^*}{\partial t}(t,x) = Au^*(t,x)$ in the complementary set (continuation or go-region), where A is the differential operator:

$$Af = \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2} + \rho\left(x\frac{\partial f}{\partial x} - f\right)$$

Notice also that (1) implies that u^* is greater than φ and that the function $t \mapsto u^*(t,x)$ is non-increasing.

After this listing of its most classical analytical property, it is apparent-since φ does not depend on t-that $u^*(t,x)$ should be a solution of the following fully non-linear PDE

$$-\frac{\partial v}{\partial t}(t,x) = (Av)^+(t,x)\,,\ t<T \qquad (3)$$
$$v(T,x) = \varphi(x)$$

at least in some adequate weak sense.

In this paper we show that under (H) u^* is the unique bounded continuous viscosity solution of (3).

For the definition of viscosity solutions of (3) we refer to [FS], chapter II.

In the first section, we show the easy fact that u^* is a continuous bounded viscosity solution of (3). Next we introduce a classical, yet degenerated, stochastic control problem (not an optimal stopping problem), the value function of which is known (cf [FS]) to be the unique continuous bounded viscosity solution of (3)-which maybe therefore written as a degenerated Hamilton-Jacobi-Bellman (henceforth HJB) equation. Lastly by a probabilistic time-change argument, we show directly that the value function of this problem is equal to u^*.

This connection between variational inequalities, or optimal stopping problems, and degenerated HJB equations seems to be new.

In the rest of the paper we work under assumption (H).

2 u^* is a viscosity solution of (3)

We shall need the following classical properties of u^* as the value function of an optimal stopping problem, which follows from the continuity of u^* and of the process S^x:

Theorem 1 *([KS], Appendix A, Theorem D.9 and Corollary D.15) For any $(t,x)\,,\ t<T$, the process $r\mapsto e^{-\rho r}u^*(t+r,S_r^x)\,,\ 0\le r\le T-t$ is a supermartingale. Let*

$$\tau^* = \inf\left\{r>0,\ u^*(t+r,S_r^x) = \varphi(S_r^x)\right\}$$

Then $\tau^* \leq T$ *almost surely, the process* $r \mapsto e^{-\rho(r\wedge\tau^*)}u^*\left(t+r\wedge\tau^*, S^x_{r\wedge\tau^*}\right),\ 0 \leq r \leq T-t$ *is a martingale, and*

$$u^*(t,x) = E\left[e^{-\rho\tau^*}u^*\left(t+\tau^*, S^x_{\tau^*}\right)\right]$$

We can now prove:

Proposition 2 *u^* is a (continuous and bounded) viscosity solution of (3).*

By definition this splits in the sub-and super- viscosity solution property.

In both the following lemmas we shall consider for a given point (t,x) such that $t<T$ a smooth (i.e. $C^{1,2}$) function w such that $w(t,x)=u^*(t,x)$. Let

$$\tau_1 = \inf\left\{ r>0, \left| \int_0^r \frac{\partial w}{\partial x}(t+s,S^x_s)\, d\left(e^{-\rho s}S^x_s\right) \right| \geq 1 \right\}$$

Then by Ito's formula for any stopping time τ less than $T-t$

$$\begin{aligned} e^{-\rho(\tau\wedge\tau_1)}w\left(t+\tau\wedge\tau_1, S^x_{\tau\wedge\tau_1}\right) &= u^*(t,x) + \int_0^{\tau\wedge\tau_1} \frac{\partial w}{\partial x}(t+s,S^x_s)\, d\left(e^{-\rho s}S^x_s\right) \\ &\quad + \int_0^{\tau\wedge\tau_1} e^{-\rho s}\left[\frac{\partial w}{\partial t}+Aw\right](t+s,S^x_s)\,ds \qquad (4) \end{aligned}$$

Lemma 3 *u^* is a viscosity subsolution of (3).*

By definition this amounts to show that $w \leq u^*$ entails $\left(\frac{\partial w}{\partial t} + (Aw)^+\right)(t,x) \ \leq\ 0.$ So assume $w \leq u^*$. Then (4) gives

$$\begin{aligned} u^*(t,x) + E\left[\int_0^{\tau\wedge\tau_1} e^{-\rho s}\left[\frac{\partial w}{\partial t}+Aw\right](t+s,S^x_s)\,ds\right] &\leq E\left[e^{-\rho(\tau\wedge\tau_1)}u^*\left(t+\tau\wedge\tau_1, S^x_{\tau\wedge\tau_1}\right)\right] \\ &\leq u^*(t,x) \end{aligned}$$

for any τ by the supermartingale property. Therefore

$$E\left[\int_0^{\tau\wedge\tau_1} e^{-\rho s}\left[\frac{\partial w}{\partial t}+Aw\right](t+s,S^x_s)\,ds\right] \leq 0$$

By choosing $\tau = r$ for $r>0$, dividing by $E\left[\int_0^{r\wedge\tau_1} ds\right]$ and letting r to zero we get $\left(\frac{\partial w}{\partial t}+Aw\right)(t,x) \leq 0.$

Observe now for $\varepsilon > 0$ small enough that since $r \mapsto u^*(r,x)$ is non-increasing

$$w(t+\varepsilon,x) - w(t,x) \leq u^*(t+\varepsilon,x) - u^*(t,x) \leq 0$$

whence $\frac{\partial w}{\partial t} \leq 0$. Therefore either $Aw(t,x) \geq 0$ and $\left(\frac{\partial w}{\partial t} + (Aw)^+\right)(t,x) = \left(\frac{\partial w}{\partial t} + Aw\right)(t,x) \leq 0$, or $Aw(t,x) \leq 0$ and then $\left(\frac{\partial w}{\partial t} + (Aw)^+\right)(t,x) = \frac{\partial w}{\partial t}(t,x) \leq 0$, whence the result.

Lemma 4 *u^* is a viscosity supersolution of (3).*

Assume now $w \geq u^*$. Suppose first that $u^*(t,x) = \varphi(x)$. Then since $w(r,x)$ is greater than $\varphi(x)$ for any r, $\frac{\partial w}{\partial t}(t,x) = 0$ whence $\left(\frac{\partial w}{\partial t} + (Aw)^+\right)(t,x) \geq 0$.

If $u^*(t,x) > \varphi(x)$, then $\tau^* > 0$ almost surely. (4) gives then for any τ less than τ^*:

$$u^*(t,x) + E\left[\int_0^{\tau\wedge\tau_1} e^{-\rho s}\left[\frac{\partial w}{\partial t} + Aw\right](t+s,S_s^x)\,ds\right] \geq E\left[e^{-\rho(\tau\wedge\tau_1)}u^*\left(t+\tau\wedge\tau_1, S_{\tau\wedge\tau_1}^x\right)\right]$$

Now by the martingale property of $r \mapsto e^{-\rho(r\wedge\tau^*)}u^*(t+r\wedge\tau^*, S_{r\wedge\tau^*}^x)$,

$$E\left[e^{-\rho(\tau\wedge\tau_1)}u^*\left(t+\tau\wedge\tau_1, S_{\tau\wedge\tau_1}^x\right)\right] = u^*(t,x)$$

so that $E\left[\int_0^{\tau\wedge\tau_1} e^{-\rho s}\left[\frac{\partial w}{\partial t} + Aw\right](t+s,S_s^x)\,ds\right] \geq 0$. By taking $\tau = r\wedge\tau^*$ we get as above $\left(\frac{\partial w}{\partial t} + Aw\right)(t,x) \geq 0$, whence $\left(\frac{\partial w}{\partial t} + (Aw)^+\right)(t,x) \geq 0$.

3 The stochastic control problem

In order to work within a classical framework we work with $\ln(S)$ instead of S. We set for $y \in \mathbb{R}$

$$\psi(y) \stackrel{def}{=} \varphi(e^y)$$

Start now from (3). After the change of variable $y = \ln(x)$ it is easily seen to write:

$$\begin{aligned} -\frac{\partial w}{\partial t}(t,y) &= (\mathcal{A}w)^+(t,y),\ t < T \\ w(T,y) &= \psi(y) \end{aligned}$$

where

$$\begin{aligned}\mathcal{A}g &= \frac{\sigma^2}{2}\left(\frac{\partial^2 g}{\partial y^2}-\frac{\partial g}{\partial y}\right)+\rho\left(\frac{\partial g}{\partial y}-g\right)\\ &= \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial y^2}+\left(\rho-\frac{\sigma^2}{2}\right)\frac{\partial g}{\partial y}-\rho g\end{aligned}$$

Introduce now the family of operators

$$\mathcal{A}_u g=\frac{\sigma^2}{2}u^2\frac{\partial^2 g}{\partial y^2}+\left(\rho u^2-\frac{\sigma^2}{2}u^2\right)\frac{\partial g}{\partial y}-\rho u^2 g$$

where $0\le u\le 1$. Then

$$(\mathcal{A}w)^+(t,y)=\sup_{0\le u\le 1}\mathcal{A}_u w(y)$$

We normalize a bit further the situation: by time-scaling we can always assume that $\sigma^2=1$. Let

$$\alpha=\tfrac{\rho}{\sigma^2}$$

Then $\mathcal{A}_u g$ may be rewritten $\mathcal{A}_u g=\frac{1}{2}u^2\frac{\partial^2 g}{\partial y^2}+\left(\alpha-\frac{1}{2}\right)u^2\frac{\partial g}{\partial y}-\alpha u^2 g$. The associated controlled stochastic process will be

$$\begin{aligned}dX_s^{(y,u.)} &= u_s dB_s+\left(\alpha-\tfrac{1}{2}\right)u_s^2 ds,\ s>0\\ X_0^{(y,u.)} &= y\end{aligned}$$

where u is a progressive (with respect to the filtration of the underlying Brownian motion B) process with values in $[0,1]$. The corresponding reward function is given for any $t\le T$ by

$$J(t,y,u.)=E\left[e^{-\alpha\int_0^{T-t}u_s^2 ds}\psi\left(X_{T-t}^{(y,u.)}\right)\right]$$

Let also

$$V(t,y)\overset{def}{=}\sup_{u.}J(t,y,u.)$$

Then we have the following:

Theorem 5 *([FS], section V.9) The function $(t,y) \mapsto V(t,y)$ is continuous and bounded. It is the only continuous and bounded viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} -\frac{\partial w}{\partial t}(t,y) &= \sup_{0\le u\le 1} \mathcal{A}_u w(t,y)\,,\ t<T \\ w(T,y) &= \psi(y) \end{aligned} \tag{5}$$

or yet

$$\begin{aligned} -\frac{\partial w}{\partial t}(t,y) &= (\mathcal{A}w)^+(t,y)\,,\ t<T \\ w(T,y) &= \psi(y) \end{aligned} \tag{6}$$

Notice that the proof given in [FS] deals with the non-discounted case where the value function is given by $\sup_\tau E\left[\psi\left(Y(W)_\tau\right)\right]$ or yet where the Hamiltonian $\sup_{0\le u\le 1}\mathcal{A}_u w(t,y)$ does not depend on the function w. Nevertheless the extension to our case is straightforward ([FS], section V, Remark 8.2).

Let us stress that the proof of the unicity ([FS], section V, Theorem 9.1) is purely analytic and relies on Ishii's lemma altogether with the semi-convex/concave approximation of Lasry and Lions ([FS], Chapter V), even if the assumptions of this theorem are formulated in terms of the family of controlled diffusions associated to the HJB equation at hand.

4 The American Connection

4.1 The easy way

Let us remark first that (remember that $\sigma = 1$):

$$v^*(t,y) \overset{def}{=} u^*(t,e^y) = \sup_\tau E\left[e^{-\alpha\tau}\psi\left(Y(W)_\tau\right)\right] \tag{7}$$

where W is any $P-$ Brownian motion,

$$Y(W)_s = y + W_s + \left(\alpha - \frac{1}{2}\right)s,\ s>0$$

and τ runs across the stopping times of the filtration of W less than $T-t$.

Come back now to the controlled processes X. Observe that for any stopping time τ of the filtration of the driving Brownian motion B which is less than $T-t$, the choice of the control process

$$\left(u\left(\tau\right)_s = 1\left(s<\tau\right)\right)_{0\leq s\leq T-t}$$

yields $X_{T-t}^{(y,u(\tau).)} = Y\left(B\right)_\tau$ altogether with $\int_0^{T-t} u_s^2 ds = \int_0^{T-t} 1\left(s<\tau\right) ds = \tau$ so that for any (t,y) we get

$$V\left(t,y\right) \geq v^*\left(t,y\right)$$

4.2 The reverse inequality

The reverse inequality is more interesting. The idea is to use the super-hedging property of the American price.

More precisely we shall need the following which is a direct consequence of Doob's decomposition of a supermartingale and of the predictable representation property of the Brownian filtration:

Lemma 6 *For any complete filtered probability space $(\Omega, F, (F_t), P)$, β $(F_t)-P$ Brownian motion, there is a predictable (with respect to the filtration of β) process $\Delta\left(\beta\right)$, such that $\int_0^{T-t}\Delta\left(\beta\right)_r^2 dr$ is finite almost surely and such that, for any $0\leq s\leq T-t$*

$$v^*\left(t,y\right) + \int_0^s \Delta\left(\beta\right)_r d\beta_r - e^{-\alpha s} v^*\left(t+s, Y\left(\beta\right)_s\right) \geq 0 \ a.s. \tag{8}$$

where $Y\left(\beta\right)_u = y + \beta_u + \left(\alpha - \frac{1}{2}\right) u$. Moreover (8) holds for any random time R (not necessarily a stopping time) such that $R\leq T-t$ a.s.

Notice that the last assertion directly follows from the continuity of the process

$$s \mapsto v^*\left(t,y\right) + \int_0^s \Delta\left(\beta\right)_r d\beta_r - e^{-\alpha s} v^*\left(t+s, Y\left(\beta\right)_s\right)$$

Let us turn now to the time-change trick:

Lemma 7 *For any controlled process $X^{(y,u.)}$, let β a Brownian motion such that for any $r > 0$*

$$\int_0^r u_s dB_s = \beta_{\int_0^r u_s^2 ds}$$

Then

$$v^*(t,y) + \int_0^{T-t} \Delta(\beta)_{\int_0^s u_r^2 dr} \left(dX_s^{(y,u.)} - \left(\alpha - \tfrac{1}{2}\right) u_s^2 ds\right) \tag{9}$$
$$-e^{-\alpha \int_0^{T-t} u_s^2 ds} v^* \left(t + \int_0^{T-t} u_s^2 ds, X_{T-t}^{(y,u.)}\right) \geq 0 \ a.s.$$

where $\Delta(\beta)$ is defined in the previous lemma.

This entails:

Corollary 8 *For any (t,y), $v^*(t,y) \geq V(t,y)$*

Indeed since $dX_s^{(y,u.)} - \left(\alpha - \frac{1}{2}\right) u_s^2 ds = u_s dB_s$ the stochastic integral in (9) is a local martingale, and by Fatou's lemma since v^* is bounded one gets easily

$$E\left[\int_0^{T-t} \frac{\partial v^*\left(t + \int_0^s u_a^2 da, X_s^{(y,u.)}\right)}{\partial y} \left(dX_s^{(y,u.)} - \left(\alpha - \tfrac{1}{2}\right) u_s^2 ds\right)\right] \leq 0$$

This entails $v^*(t,y) \geq E\left[e^{-\alpha \int_0^{T-t} u_s^2 ds} v^* \left(t + \int_0^{T-t} u_s^2 ds, X_{T-t}^{(y,u.)}\right)\right]$. Now since v^* is greater than ψ :

$$v^*(t,y) \geq E\left[e^{-\alpha \int_0^{T-t} u_s^2 ds} \psi\left(X_{T-t}^{(y,u.)}\right)\right] = J(t,y,u.)$$

whence by taking the supremum over the controls, $v^*(t,y) \geq V(t,y)$.

Let us now prove the lemma.

The trick to prove (9) is to use a time-change technique. By the Dambins-Dubins-Schwarz theorem, there is some Brownian motion β defined for any time v between 0 and $\int_0^{T-t} u_s^2 ds$ such that for any r between 0 and $T-t$:

$$\int_0^r u_s dB_s = \beta_{\int_0^r u_s^2 ds}$$

Introduce the related time-change C_r defined by

$$\int_0^{C_r} u_s^2 ds = r$$

Now $X_{C_r}^{(y,u.)} = y + \left(\alpha - \frac{1}{2}\right) \int_0^{C_r} u_s^2 ds + \beta_{\int_0^{C_r} u_s^2 ds} = y + \left(\alpha - \frac{1}{2}\right) r + \beta_r = Y(\beta)_r$ so that the left-hand-side of (9) may be re-written by setting $s = C_r$

$$\begin{aligned} & v^*(t,y) + \int_0^{C^{-1}(T-t)} \Delta(\beta)_r \, d\beta_r \\ & -e^{-\alpha \int_0^{T-t} u_s^2 ds} v^* \left(t + \int_0^{T-t} u_s^2 ds, Y(\beta)_{\int_0^{T-t} u_s^2 ds} \right) \end{aligned}$$

where $C^{-1}(T-t) = \int_0^{T-t} u_s^2 ds$, or yet

$$\begin{aligned} & v^*(t,y) + \int_0^{\int_0^{T-t} u_s^2 ds} \Delta(\beta)_r \, d\beta_r \\ & -e^{-\alpha \int_0^{T-t} u_s^2 ds} v^* \left(t + \int_0^{T-t} u_s^2 ds, Y(\beta)_{\int_0^{T-t} u_s^2 ds} \right) \end{aligned}$$

Observe now that $\int_0^{T-t} u_s^2 ds$ is a stopping time of the filtration with respect to which β is a Brownian motion, which is usually greater than that of β-thanks to lemma 6 this does not matter, and lemma 7 is proved.

Remark 9 *Of course this extends to the Black-Scholes model with a constant instantaneous dividend rate δ where*

$$\begin{aligned} S_u &= x \exp\left((\rho - \delta) u + \sigma B_u - \frac{\sigma^2}{2} u \right) \\ u^*(t,x) &= \sup_\tau E\left[e^{-\rho\tau} \varphi(S_\tau) \right] \end{aligned}$$

Also by working under the probability measure $dP^* = \frac{(1+S_{T-t})}{1+e^{\rho(T-t)}}$ and replacing $x \mapsto \varphi(x)$ by $x \mapsto \frac{\varphi(x)}{1+x}$ one can handle in exactly the same way the linear growth case, i.e.: the function $x \mapsto \frac{\varphi(x)}{1+x}$ is continuous and bounded.

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